

Scalar model of SU(N) glueball à la Heisenberg

Vladimir Dzhunushaliev*

*Dept. Theor. and Nucl. Phys., KazNU, Almaty, 050040, Kazakhstan and
IETP, Al-Farabi KazNU, Almaty, 050040, Kazakhstan*

Vladimir Folomeev†

*Institute of Physicotechnical Problems and Material Science of the NAS of the Kyrgyz Republic,
265 a, Chui Street, Bishkek 720071, Kyrgyz Republic*

Nonperturbative model of glueball is studied. The model is based on the nonperturbative quantization technique suggested by Heisenberg. 2- and 4-point Green functions for a gauge potential are expressed in terms of two scalar fields. The first scalar field describes quantum fluctuations of the subgroup $SU(n) \subset SU(N)$, and the second one describes quantum fluctuations of the coset $SU(N)/SU(n)$. An effective Lagrangian for the scalar fields is obtained. The coefficients for all terms in the Lagrangian are calculated, and it is shown that they depend on $\dim SU(n)$, $\dim SU(N)$. It is demonstrated that a spherically symmetric solution describing the glueball does exist.

I. INTRODUCTION

Glueball is a hypothetical particle appearing in SU(3) quantum gauge field theory only. Roughly speaking, one can say that the glueball is a proton (or a neutron) with remote quarks. The existence of glueballs is a consequence of the self-interaction of gluons within QCD. Fritzsche and Minkowski [1] developed a model of glueball on the basis of the quark-gluon field theory by assuming an analogy between massless gluons and photons. Jaffe and Johnson [2] investigated the bag model of glueball. At the present time, several candidates for low-mass glueballs with quantum numbers 0^{++} , 2^{++} , 0^{-+} and 1^{--} are under discussion [3–6].

Historically, the idea of studying QCD in the large- N limit was first put forward in 1974 by 'tHooft [7], who proposed to consider $1/N$ as an expansion parameter. Some nonperturbative features of QCD can be understood in such a large- N limit [8, 9].

Here we would like to calculate the mass of a glueball in SU(N) gauge theory by using the nonperturbative quantization technique à la Heisenberg [10]. The essence of such a technique consists in writing equation(s) for field operator(s). In fact, such equation(s) is(are) equivalent to an infinite set of equations for all Green functions [11]. Mathematically, such procedure is similar to writing down of infinite system of equations for all cumulants in turbulence modeling [12, 13]. Since it is impossible to solve such an infinite set of equations, one can proceed similarly to the case of turbulence modeling when one has to cut off the set of equations using some assumptions concerning Green functions (the closure problem). For example, this can be done by introducing some approximate connection between higher- and lower-order Green functions. In addition, we will use the following assumptions: (a) 2-point Green functions of SU(N) gauge potentials can approximately be expressed through scalar functions; (b) 2-point Green functions for the subgroup SU(n) and the coset SU(N)/SU(n) are different. Namely, we employ χ for SU(n) 2,4-point Green functions and ϕ for SU(N)/SU(n) coset 2,4-point Green functions; (c) there is an anisotropy in the color space in the sense that Green functions are not symmetric under exchange of color indices.

Based on the above assumptions and ideas, we have the following approximate model of a glueball: (a) it is a ball filled with fluctuating SU(N) quantum gauge fields; (b) quantum fields are approximately described by two scalar fields; (c) one of the fields describes the fluctuating SU(n) components, and another one – the coset SU(N)/SU(n) components; (d) 2- and 4-point Green functions are nonsymmetric in the color space under exchange of color indices; (e) the approximate description of the glueball is therefore carried out by using nonlinear equations for two scalar fields; (f) the glueball is described by spherically symmetric solutions of these equations; (g) using the expression for the scalar fields energy density, we can calculate the mass of the glueball within this model; (h) all components of the subgroup SU(n) give similar contributions to a Green function; the same holds true for the coset SU(N)/SU(n).

It will be shown below that equations for the scalar fields contain parameters $\lambda_{1,2,3}$, describing the self-interaction of the scalar fields (gauge potentials), and the value $\chi(0)$ of one of the scalar fields at the origin (dispersion of fluctuations of the gauge potentials). After fixing $\lambda_{1,2,3}$, one can show that the dimensionless glueball mass for SU(N) gauge field

*Electronic address: v.dzhunushaliev@gmail.com

†Electronic address: vfolomeev@mail.ru

theory depends on the value of one scalar field at the origin and the dimensions of the subgroup $SU(n)$ and the gauge group $SU(N)$: $n = \dim SU(n)$, $N = \dim SU(N)$.

II. SCALAR MODEL OF A GLUEBALL

Our approximation is based on the main assumption that 2- and 4-points Green functions are described in terms of some scalar fields ϕ and χ according to the following relations (which are some variations of ansatz from [11]):

$$(G_2)_{\mu\nu}^{ab}(x, x) = \langle A_\mu^a(x) A_\nu^b(x) \rangle \approx C^{ab} \tilde{b}_{\mu\nu} \tilde{\chi}^2(x), \quad (1)$$

$$(G_2)_{\mu\nu; \alpha\beta}^{ab}(x, x) = \langle \partial_\alpha A_\mu^a(x) \partial_\beta A_\nu^b(x) \rangle \approx C^{ab} \tilde{b}_{\mu\nu} \partial_\alpha \tilde{\chi}(x) \partial_\beta \tilde{\chi}(x), \quad (2)$$

$$(G_4)_{\mu\nu\rho\sigma}^{abcd}(x, x, x, x) = \langle A_\mu^a(x) A_\nu^b(x) A_\rho^c(x) A_\sigma^d(x) \rangle \approx \\ \left(\langle A_\mu^a(x) A_\nu^b(x) \rangle - m_{\mu\nu}^{ab} \right) \left(\langle A_\rho^c(x) A_\sigma^d(x) \rangle - m_{\rho\sigma}^{cd} \right) - m_{\mu\nu}^{ab} m_{\rho\sigma}^{cd} = \\ C^{ab} \tilde{b}_{\mu\nu} C^{cd} \tilde{b}_{\rho\sigma} \left[(\tilde{\chi}^2(x) - \tilde{m}_2^2)^2 - \tilde{m}_2^4 \right], \quad (3)$$

$$(G_2)_{\mu\nu}^{mn}(x, x) = \langle A_\mu^m(x) A_\nu^n(x) \rangle \approx C^{mn} \tilde{b}_{\mu\nu} \left(\tilde{m}_1^2 - \tilde{\phi}^2(x) \right), \quad (4)$$

$$(G_2)_{\mu\nu; \alpha\beta}^{mn}(x, x) = \langle \partial_\alpha A_\mu^m(x) \partial_\beta A_\nu^n(x) \rangle \approx C^{mn} \tilde{b}_{\mu\nu} \partial_\alpha \tilde{\phi}(x) \partial_\beta \tilde{\phi}(x), \quad (5)$$

$$(G_4)_{\mu\nu\rho\sigma}^{mnpq}(x, x, x, x) = \langle A_\mu^m(x) A_\nu^n(x) A_\rho^p(x) A_\sigma^q(x) \rangle \approx \langle A_\mu^m(x) A_\nu^n(x) \rangle \langle A_\rho^p(x) A_\sigma^q(x) \rangle = \\ C^{mn} \tilde{b}_{\mu\nu} C^{pq} \tilde{b}_{\rho\sigma} \left(\tilde{\phi}^2(x) - \tilde{m}_1^2 \right)^2, \quad (6)$$

$$(G_4)_{\mu\nu\rho\sigma}^{abmn}(x, x, x, x) = \langle A_\mu^a(x) A_\nu^b(x) A_\rho^m(x) A_\sigma^n(x) \rangle \approx \langle A_\mu^a(x) A_\nu^b(x) \rangle \langle -A_\rho^m(x) A_\sigma^n(x) + C_{\mu\nu}^{mn} m_1^2 \rangle = \\ C^{ab} \tilde{b}_{\mu\nu} C^{mn} \tilde{b}_{\rho\sigma} \tilde{\phi}^2(x) \tilde{\chi}^2(x), \quad (7)$$

where $a, b, c, d = 1, 2, \dots, n$ are the $SU(n)$ indices, $m, n, p, q = 4, 5, \dots, N$ are the coset $SU(N)/SU(n)$ indices, $C^{ab, mn}$, $\tilde{b}_{\mu\nu}$, $\tilde{b}_{\mu\nu}$, and $m_{\mu\nu}^{ab, mn}$ are the closure constants. We see that, similarly to turbulence modeling, we have to introduce some closure constants.

III. ESTIMATION OF NUMERICAL FACTORS DEPENDING ON $\dim SU(N)$, $\dim SU(n)$

To begin with, let us consider the $SU(N)$ Lagrangian

$$-\mathcal{L}_{SU(N)} = \frac{1}{4} F_{\mu\nu}^B F^{B\mu\nu} = \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{4} F_{\mu\nu}^m F^{m\mu\nu}, \quad (8)$$

where $B = 1, 2, \dots, \dim SU(N)$ is the $SU(N)$ index, $F_{\mu\nu}^B = \partial_\mu A_\nu^B - \partial_\nu A_\mu^B + g f^{BCD} A_\mu^C A_\nu^D$ is the field strength operator, g is the coupling constant, and

$$F_{\mu\nu}^a = f_{\mu\nu}^a f^{a\mu\nu} + g f^{abc} A_\mu^b A_\nu^c + g f^{amn} A_\mu^m A_\nu^n, \quad (9)$$

$$F_{\mu\nu}^m = f_{\mu\nu}^m f^{m\mu\nu} + g f^{mpq} A_\mu^p A_\nu^q + g f^{mna} (A_\mu^n A_\nu^a - A_\nu^n A_\mu^a). \quad (10)$$

Here $f_{\mu\nu}^B = \partial_\mu A_\nu^B - \partial_\nu A_\mu^B$; $a, b, c = 1, 2, \dots, n$ are the $SU(n)$ indices; m, n, p, q are the coset indices $\in SU(N)/SU(n)$. In order to obtain an effective Lagrangian, we have to average the $SU(N)$ Lagrangian (8) over a quantum state $|Q\rangle$. In this case we will have the following expressions for both terms from (8):

$$\langle F_{\mu\nu}^a F^{a\mu\nu} \rangle = \langle f_{\mu\nu}^a f^{a\mu\nu} \rangle + g^2 f^{ab_1 c_1} f^{ab_2 c_2} \langle A_\mu^{b_1} A_\nu^{c_1} A^{b_2 \mu} A^{c_2 \nu} \rangle + g^2 f^{am_1 n_1} f^{am_2 n_2} \langle A_\mu^{m_1} A_\nu^{n_1} A^{m_2 \mu} A^{n_2 \nu} \rangle + \\ g^2 f^{abc} f^{amn} \langle A_\mu^b A_\nu^c A^{m\mu} A^{n\nu} \rangle, \quad (11)$$

$$\langle F_{\mu\nu}^m F^{m\mu\nu} \rangle = \langle f_{\mu\nu}^m f^{m\mu\nu} \rangle + g^2 f^{mp_1 q_1} f^{mp_2 q_2} \langle A_\mu^{p_1} A_\nu^{q_1} A^{p_2 \mu} A^{q_2 \nu} \rangle + g^2 f^{mp_1 a_1} f^{mp_2 a_2} \left(\langle A_\mu^{p_1} A_\nu^{a_1} A^{p_2 \mu} A^{a_2 \nu} \rangle + \right. \\ \left. \langle A_\nu^{a_1} A_\mu^{p_1} A^{a_2 \nu} A^{p_2 \mu} \rangle - \langle A_\mu^{p_1} A_\nu^{a_1} A^{a_2 \nu} A^{p_2 \mu} \rangle - \langle A_\nu^{a_1} A_\mu^{p_1} A^{p_2 \mu} A^{a_2 \nu} \rangle \right). \quad (12)$$

Here we assume that all terms having odd number of potential components are zero: $\langle A_\mu^{a, m} \rangle = \langle A_\mu^{a, m} A_\nu^{b, n} A_\rho^{c, p} \rangle = 0$ and $\langle A_\mu^a A_\nu^m A_\nu^n \rangle \approx \langle A_\mu^a \rangle \langle A_\nu^m A_\nu^n \rangle = 0$.

Our goal is to calculate the effective Lagrangian

$$\mathcal{L}_{\text{eff}} \approx \langle \mathcal{L}_{SU(N)} \rangle. \quad (13)$$

To do this, let us consider first terms on the right-hand sides of Eqs. (11) and (12):

$$\langle f_{\mu\nu}^a f^{a\mu\nu} \rangle = 2 \langle \partial_\mu A_\nu^a \partial^\mu A^{a\nu} \rangle - 2 \langle \partial_\mu A_\nu^a \partial^\nu A^{a\mu} \rangle. \quad (14)$$

Substituting here the expression (1), we obtain

$$\begin{aligned} & \langle \partial_\mu A_\nu^a \partial^\mu A^{a\nu} \rangle - \langle \partial_\mu A_\nu^a \partial^\nu A^{a\mu} \rangle = C^{aa} (b_\alpha^\alpha \eta^{\mu\nu} - b^{\mu\nu}) \partial_\mu \chi \partial_\nu \chi = \\ & C^{aa} \left[- (b^{11} + b^{22} + b^{33}) \partial_0 \chi \partial_0 \chi + (-b^{00} + b^{22} + b^{33}) \partial_1 \chi \partial_1 \chi + (-b^{00} + b^{11} + b^{33}) \partial_2 \chi \partial_2 \chi + \right. \\ & \left. (-b^{00} + b^{11} + b^{22}) \partial_3 \chi \partial_3 \chi - \sum_{\mu \neq \nu} b^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right]. \end{aligned} \quad (15)$$

It is seen that the structure of the term $\langle f_{\mu\nu}^a f^{a\mu\nu} \rangle$ is very confusing. As a toy model, consider the case $b_{\mu\nu} = -\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is Minkowski metric. Let us note that in this case $\langle (A_0^a)^2 \rangle < 0$. This is the price for using such an approximation.

As a first approximation we assume that all numbers C^{ab} in Eq. (2) have the same order, and this factor can be estimated as $C^{ab} \approx C_2$. In this case (14) has the form

$$\langle f_{\mu\nu}^a f^{a\mu\nu} \rangle \approx -6C_2 (n^2 - 1) \partial_\mu \tilde{\chi} \partial^\mu \tilde{\chi}. \quad (16)$$

Here we took into account that the summation over a in $\sum_a^{\dim SU(n)}$ has $(n^2 - 1)$ summands, $C^{aa} \approx C_2 (n^2 - 1)$, and C_2 is a constant describing the dispersion of one component of the gauge potential A_μ^a . Using the same approximation, we obtain

$$\langle f_{\mu\nu}^m f^{m\mu\nu} \rangle \approx -6C_1 (N^2 - n^2) \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}. \quad (17)$$

Here we took into account that the summation over index m has $\dim SU(N) - \dim SU(n) = N^2 - n^2$ summands, $C^{mm} = C_1 (N^2 - n^2)$, and C_1 is a constant describing the dispersion of one component of the gauge potential A_μ^m .

We assume that our physical situations can be described by the ansatz (3) for the 4-point Green function, and consequently

$$f^{ab_1c_1} f^{ab_2c_2} \langle A_\mu^{b_1} A_\nu^{c_1} A^{b_2\mu} A^{c_2\nu} \rangle \approx f^{ab_1c_1} C^{b_1c_1} f^{ab_2c_2} C^{b_2c_2} \left[(\chi^2 - \tilde{m}_2^2)^2 - \tilde{m}_2^4 \right], \quad b_{1,2} \neq c_{1,2}. \quad (18)$$

Let us consider the summands with fixed a in more detail:

$$\begin{aligned} & f^{ab_1c_1} f^{ab_2c_2} C^{b_1c_1} C^{b_2c_2} + f^{ab_1c_1} f^{ac_2b_2} C^{b_1c_1} C^{c_2b_2} + f^{ac_1b_1} f^{ab_2c_2} C^{c_1b_1} C^{b_2c_2} + \\ & f^{ac_1b_1} f^{ac_2b_2} C^{c_1b_1} C^{c_2b_2} = f^{ab_1c_1} f^{ab_2c_2} (C^{b_1c_1} - C^{c_1b_1}) (C^{b_2c_2} - C^{c_2b_2}) = f^{ab_1c_1} f^{ab_2c_2} \Delta C^{b_1c_1} \Delta C^{b_2c_2} \end{aligned} \quad (19)$$

(no summation over repeated indices). The number of such terms is equal to the number of pairs of the structure constants $f^{ab_1c_1}, f^{ab_2c_2}$ with $a, b_{1,2}, c_{1,2} \in SU(n)$ and different a . Now we want to estimate the term from (18) as

$$f^{ab_1c_1} f^{ab_2c_2} \langle A_\mu^{b_1} A_\nu^{c_1} A^{b_2\mu} A^{c_2\nu} \rangle \approx \frac{(\tilde{\lambda}_2)^{n,n,n}}{4} \left[(\tilde{\chi}^2 - \tilde{m}_2^2)^2 - \tilde{m}_2^4 \right], \quad (20)$$

where $(\tilde{\lambda}_2)^{n,n,n}_{n,n,n} = 4 f^{ab_1c_1} f^{ab_2c_2} \Delta C^{b_1c_1} \Delta C^{b_2c_2}$.

Next term is

$$f^{am_1n_1} f^{am_2n_2} \langle A_\mu^{m_1} A_\nu^{n_1} A^{m_2\mu} A^{n_2\nu} \rangle \approx f^{am_1n_1} C^{m_1n_1} f^{am_2n_2} C^{m_2n_2} \left(\tilde{\phi}^2 - m_1^2 \right)^2, \quad m_{1,2} \neq n_{1,2}. \quad (21)$$

Proceeding as in (18), we have

$$\begin{aligned} & f^{am_1n_1} f^{am_2n_2} C^{m_1n_1} C^{m_2n_2} + f^{am_1n_1} f^{an_2m_2} C^{m_1n_1} C^{m_2m_2} + f^{an_1m_1} f^{am_2n_2} C^{n_1m_1} C^{m_2n_2} + \\ & f^{an_1m_1} f^{an_2m_2} C^{n_1m_1} C^{n_2m_2} = f^{am_1n_1} f^{am_2n_2} (C^{m_1n_1} - C^{n_1m_1}) (C^{m_2n_2} - C^{n_2m_2}) = \\ & f^{am_1n_1} f^{am_2n_2} \Delta C^{m_1n_1} \Delta C^{m_2n_2} \end{aligned} \quad (22)$$

(again no summation over repeated indices). Now we can estimate this term as

$$f^{am_1 n_1} f^{am_2 n_2} \langle A_\mu^{m_1} A_\nu^{n_1} A^{m_2 \mu} A^{n_2 \nu} \rangle \approx \frac{\left(\tilde{\lambda}_1\right)_{n, N/n, N/n}}{4} \left(\tilde{\phi}^2 - \tilde{m}_1^2\right)^4, \quad (23)$$

where $\left(\tilde{\lambda}_1\right)_{n, N/n, N/n}^{n, N/n, N/n} = 4 f^{am_1 n_1} f^{am_2 n_2} \Delta C^{m_1 n_1} \Delta C^{m_2 n_2}$.

Next term in Eq. (11) is

$$f^{abc} f^{amn} \langle A_\mu^b A_\nu^c A^{m\mu} A^{n\nu} \rangle \approx f^{abc} \langle A_\mu^b A_\nu^c \rangle f^{amn} \langle A^{m\mu} A^{n\nu} \rangle = f^{abc} f^{amn} C^{bc} C^{mn} \tilde{\phi}^2 \tilde{\chi}^2, \quad (24)$$

for which we have

$$\begin{aligned} & f^{abc} f^{amn} C^{ab} C^{mn} + f^{abc} f^{anm} C^{ab} C^{nm} + f^{acb} f^{amn} C^{cb} C^{mn} + \\ & f^{acb} f^{anm} C^{cb} C^{nm} = f^{abc} f^{amn} (C^{bc} - C^{cb}) (C^{mn} - C^{nm}) = f^{abc} f^{amn} \Delta C^{ab} \Delta C^{mn}. \end{aligned} \quad (25)$$

Consequently,

$$f^{abc} f^{amn} \langle A_\mu^b A_\nu^c A^{m\mu} A^{n\nu} \rangle \approx \frac{\left(\tilde{\lambda}_3\right)_{n, n, n}}{2} \tilde{\phi}^2 \tilde{\chi}^2, \quad (26)$$

where $\left(\tilde{\lambda}_3\right)_{n, n, n}^{n, N/n, N/n} = 2 f^{abc} f^{amn} \Delta C^{ab} \Delta C^{mn}$.

Similarly, we can estimate the second term in Eq. (12) as follows:

$$f^{mp_1 q_1} f^{mp_2 q_2} \langle A_\mu^{p_1} A_\nu^{q_1} A^{p_2 \mu} A^{q_2 \nu} \rangle \approx f^{mp_1 q_1} f^{mp_2 q_2} C^{p_1 q_1} C^{p_2 q_2} \left[\left(\tilde{\phi}^2 - \tilde{m}_1^2\right)^2 \right], p_{1,2} \neq q_{1,2}. \quad (27)$$

Proceeding as in (18), we have

$$\begin{aligned} & f^{mp_1 q_1} f^{mp_2 q_2} C^{p_1 q_1} C^{p_2 q_2} + f^{mp_1 q_1} f^{mq_2 p_2} C^{p_1 q_1} C^{q_2 p_2} + f^{mq_1 p_1} f^{mp_2 q_2} C^{q_1 p_1} C^{p_2 q_2} + \\ & f^{mq_1 p_1} f^{mq_2 p_2} C^{q_1 p_1} C^{q_2 p_2} = f^{mp_1 q_1} f^{mp_2 q_2} (C^{p_1 q_1} - C^{q_1 p_1}) (C^{p_2 q_2} - C^{q_2 p_2}) = f^{mp_1 q_1} f^{mp_2 q_2} \Delta C^{p_1 q_1} \Delta C^{p_2 q_2} \end{aligned} \quad (28)$$

(no summation over indices). Then

$$f^{mp_1 q_1} f^{mp_2 q_2} \langle A_\mu^{p_1} A_\nu^{q_1} A^{p_2 \mu} A^{q_2 \nu} \rangle \approx \frac{\left(\tilde{\lambda}_1\right)_{N/n, N/n, N/n}}{4} \left(\tilde{\phi}^2 - \tilde{m}_1^2\right)^2, \quad (29)$$

where $\left(\tilde{\lambda}_1\right)_{N/n, N/n, N/n}^{N/n, N/n, N/n} = 4 f^{mp_1 q_1} f^{mp_2 q_2} \Delta C^{p_1 q_1} \Delta C^{p_2 q_2}$.

Finally, the last term in Eq. (12) can be estimated as

$$f^{mp_1 a_1} f^{mp_2 a_2} \langle A_\mu^{p_1} A_\nu^{a_1} A^{p_2 \mu} A^{a_2 \nu} \rangle \approx f^{mp_1 a_1} f^{mp_2 a_2} \langle A_\nu^{a_1} A^{a_2 \nu} \rangle \langle A_\mu^{p_1} A^{p_2 \mu} \rangle = f^{mp_1 a_1} f^{mp_2 a_2} C^{p_1 a_1} C^{p_2 a_2} \tilde{\phi}^2 \tilde{\chi}^2. \quad (30)$$

Let us now consider the summands with fixed m in more detail:

$$C^{a_1 a_2} C^{p_1 p_2} (f^{mp_1 a_1} f^{mp_2 a_2} + f^{mp_1 a_1} f^{ma_2 p_2} + f^{ma_1 p_1} f^{mp_2 a_2} + f^{ma_1 p_1} f^{ma_2 p_2}) = 0 \quad (31)$$

(no summation over indices). Consequently, in our approximation

$$f^{mp_1 a_1} f^{mp_2 a_2} \langle A_\mu^{p_1} A_\nu^{a_1} A^{p_2 \mu} A^{a_2 \nu} \rangle = 0. \quad (32)$$

Notice that in Eqs. (18)-(30) we have used the assumption that C^{AB} [where $A, B \in SU(N)$] are not symmetric, i.e., $C^{AB} - C^{BA} \neq 0$.

Combining all expressions for 2- and 4-point Green functions, we obtain the following approximate Lagrangian:

$$\mathcal{L}_{eff} = \frac{1}{2} (N^2 - n^2) \partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + \frac{1}{2} (n^2 - 1) \partial_\mu \bar{\chi} \partial^\mu \bar{\chi} - \frac{\bar{\lambda}_1}{4} (\bar{\phi}^2 - \bar{m}_1^2)^2 - \frac{\bar{\lambda}_2}{4} [(\bar{\chi}^2 - \bar{m}_2^2)^2 - \bar{m}_2^4] - \frac{\bar{\lambda}_3}{2} \bar{\phi}^2 \bar{\chi}^2, \quad (33)$$

where the coefficients $C_{1,2}$ have been eliminated by redefining $\tilde{\chi}, \tilde{\phi}$ and $\tilde{m}_{1,2}$:

$$\bar{\phi} = \tilde{\phi}\sqrt{12C_1}, \quad (34)$$

$$\bar{\chi} = \tilde{\chi}\sqrt{12C_2}, \quad (35)$$

$$\bar{m}_1 = \tilde{m}_1\sqrt{12C_1}, \quad (36)$$

$$\bar{m}_2 = \tilde{m}_2\sqrt{12C_2}. \quad (37)$$

Also, for brevity, we have introduced

$$\bar{\lambda}_1 = \frac{\left(\tilde{\lambda}_1\right)_{n,N/n,N/n}^{n,N/n,N/n} + \left(\tilde{\lambda}_1\right)_{N/n,N/n,N/n}^{N/n,N/n,N/n}}{(12C_1)^2}, \quad (38)$$

$$\bar{\lambda}_2 = \frac{\left(\tilde{\lambda}_2\right)_{n,n,n}^{n,n,n}}{(12C_2)^2}, \quad (39)$$

$$\bar{\lambda}_3 = \frac{\left(\tilde{\lambda}_3\right)_{n,n,n}^{n,N/n,N/n}}{144C_1C_2}. \quad (40)$$

Entities entering the Lagrangian (33) have the following meanings and origins:

- the scalar fields χ and ϕ describe the nonperturbatively quantized $SU(n)$ and coset $SU(N)/SU(n)$ degrees of freedom, respectively;
- the terms $(\nabla_\mu\phi)^2$ and $(\nabla_\mu\chi)^2$ are the result of the nonperturbative quantum averaging of $(\nabla_\mu A_\nu^B)^2$ in the initial $SU(N)$ Lagrangian;
- the terms $(\phi^2 - m_1^2)^2$ and $[(\chi^2 - m_2^2)^2 - m_4^4]$ are the result of the nonperturbative quantum averaging of $f^{ABC}f^{AMN}A_\mu^B A_\nu^C A^{M\mu} A^{N\nu}$;
- the term $\phi^2\chi^2$ is the result of the nonperturbative quantum averaging of $f^{Aab}f^{Amn}A_\mu^a A_\nu^b A^{m\mu} A^{n\nu}$;
- the closure coefficients $\lambda_{1,2}$ and $m_{1,2}$ appear;
- $C_{1,2}$ are free parameters;
- $m_{1,2}$ are eigenvalues obtained in solving the field equations (41) and (42).

IV. FIELD EQUATIONS

Using the Lagrangian (33), one can derive the corresponding field equations describing a gluon condensate in the following form:

$$\partial_\mu\partial^\mu\phi = -\phi[\lambda_3\chi^2 + \lambda_1(\phi^2 - m_1^2)], \quad (41)$$

$$\partial_\mu\partial^\mu\chi = -\chi[\lambda_3\phi^2 + \lambda_2(\chi^2 - m_2^2)], \quad (42)$$

where $\phi = \sqrt{N^2 - n^2}\bar{\phi}$, $\chi = \sqrt{n^2 - 1}\bar{\chi}$, $m_1 = \sqrt{N^2 - n^2}\bar{m}_1$, $m_2 = \sqrt{n^2 - 1}\bar{m}_2$, $\lambda_1 = \bar{\lambda}_1/(N^2 - n^2)^2$, $\lambda_2 = \bar{\lambda}_2/(n^2 - 1)^2$, $\lambda_3 = \bar{\lambda}_3/[(N^2 - n^2)(n^2 - 1)]$. The coefficients $\lambda_{1,2,3}$ depend on the dimensions $n = \dim SU(n)$ and $N = \dim SU(N)$, where $SU(n) \subset SU(N)$.

Now let us consider one special case.

A. $SU(2) \subset SU(3)$ glueball

In this case we have $n = 2, N = 3$. The effective Lagrangian will then be

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{\lambda_1}{4}(\phi^2 - m_1^2)^2 - \frac{\lambda_2}{4}[(\chi^2 - m_2^2)^2 - m_4^4] - \frac{\lambda_3}{2}\phi^2\chi^2, \quad (43)$$

where $\phi = \tilde{\phi}\sqrt{60C_1}$, $\chi = \tilde{\chi}\sqrt{36C_2}$, $\lambda_1 = (\Delta C_\phi)^2 / (600C_1^2)$, $\lambda_2 = (\Delta C_\chi)^2 / (108C_2^2)$, and $\lambda_3 = \Delta C_\phi \Delta C_\chi / (1080C_1C_2)$; $|\Delta C^{mn}| \approx \Delta C_\phi$ for all m, n ; $|\Delta C^{ab}| \approx \Delta C_\chi$ for all a, b . If we assume that $\Delta C_\phi \approx \Delta C_\chi = \Delta C$ and $C_1 = C_2 = C$ then

$$\lambda_1 = 9\lambda/5, \quad \lambda_2 = \lambda/10, \quad \lambda_3 = \lambda = (\Delta C)^2 / (1080C^2). \quad (44)$$

For such a case the field equations are

$$\partial_\mu \partial^\mu \phi = -\phi [\lambda_3 \chi^2 + \lambda_1 (\phi^2 - m_1^2)], \quad (45)$$

$$\partial_\mu \partial^\mu \chi = -\chi [\lambda_3 \phi^2 + \lambda_2 (\chi^2 - m_2^2)] \quad (46)$$

with $\lambda_{1,2,3}$ from (44).

We seek a glueball solution as a spherically symmetric solution with $\phi(r), \chi(r)$. In this case we have the following ordinary differential equations:

$$\phi'' + \frac{2}{r}\phi' = \phi [\lambda_3 \chi^2 + \lambda_1 (\phi^2 - m_1^2)], \quad (47)$$

$$\chi'' + \frac{2}{r}\chi' = \chi [\lambda_3 \phi^2 + \lambda_2 (\chi^2 - m_2^2)], \quad (48)$$

where the prime denotes differentiation with respect to r . Notice that here one can eliminate λ_3 by redefining the radial coordinate. Then, taking into account (44), the system (47) and (48) takes the form

$$\phi'' + \frac{2}{x}\phi' = \phi \left[\chi^2 + \frac{9}{5} (\phi^2 - m_1^2) \right], \quad (49)$$

$$\chi'' + \frac{2}{x}\chi' = \chi \left[\phi^2 + \frac{1}{10} (\chi^2 - m_2^2) \right], \quad (50)$$

where $x = \sqrt{\lambda}r$ and the prime denotes now differentiation with respect to x .

These equations are to be solved subject to the boundary conditions given in the neighborhood of the center by the following expansions:

$$\phi \approx \phi_0 + \frac{1}{2}\phi_2 x^2, \quad \chi \approx \chi_0 + \frac{1}{2}\chi_2 x^2, \quad (51)$$

where ϕ_0, χ_0 are central values of the scalar fields and the expansion coefficients ϕ_2, χ_2 are determined from Eqs. (49) and (50).

For given values of ϕ_0, χ_0 , the system of equations (49) and (50) has regular solutions only for certain values of the masses of the scalar fields m_1, m_2 . As a result, the problem reduces to a search for *eigenvalues* of the parameters m_1, m_2 and for the corresponding *eigenfunctions* ϕ and χ of the nonlinear system of differential equations (49) and (50). We will seek the specified eigenvalues by using the shooting method. A step-by-step description of the procedure for finding solutions can be found, e.g., in Ref. [14].

Proceeding in this way, we have obtained the results shown in Figs. 1 and 2. It is seen that $\phi \rightarrow m_1$ and $\chi \rightarrow 0$ at large distances, i.e., the solutions approach asymptotically the local minimum of the potential energy from the Lagrangian (43). One can also see from Fig. 2 that the scalar fields energy density

$$\varepsilon = \frac{1}{2}\phi'^2 + \frac{1}{2}\chi'^2 + \frac{\lambda_1}{4}(\phi^2 - m_1^2)^2 + \frac{\lambda_2}{4}[(\chi^2 - m_2^2)^2 - m_2^4] + \frac{\lambda_3}{2}\phi^2\chi^2 \quad (52)$$

goes to zero as $x \rightarrow \infty$.

The asymptotic behavior of the scalar fields can be found in the following form:

$$\phi = m_1 - \delta\phi, \quad \chi = \delta\chi, \quad (53)$$

where $\delta\phi, \delta\chi \ll 1$ behave as

$$\delta\phi \approx k_\phi \frac{\exp\left(-3\sqrt{2m_1^2/5}x\right)}{x}, \quad \delta\chi \approx k_\chi \frac{\exp\left(-\sqrt{m_1^2 - m_2^2/10}x\right)}{x}, \quad (54)$$

where k_ϕ, k_χ are integration constants, and the values of the masses are $m_1 \approx 1.216$ and $m_2 \approx 3.809$ for the case of $\phi_0 = \chi_0 = 1$ shown in Figs. 1 and 2.

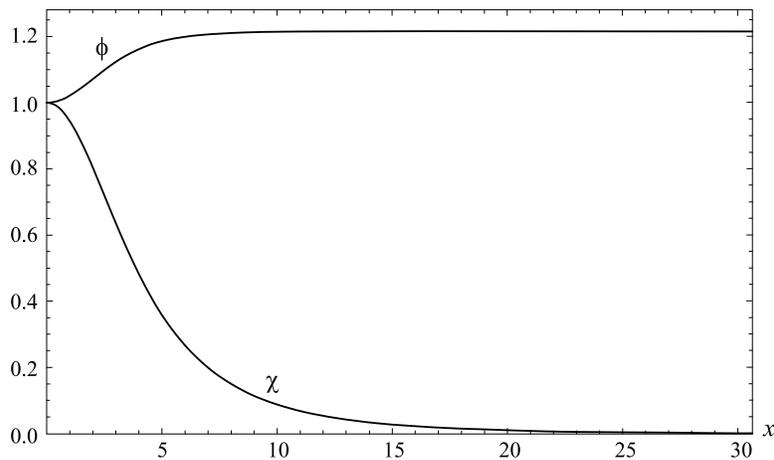


FIG. 1: The typical behavior of the scalar fields. Asymptotically, as $x \rightarrow \infty$, the field ϕ tends to m_1 and χ goes to 0 [see Eqs. (53)-(54)].

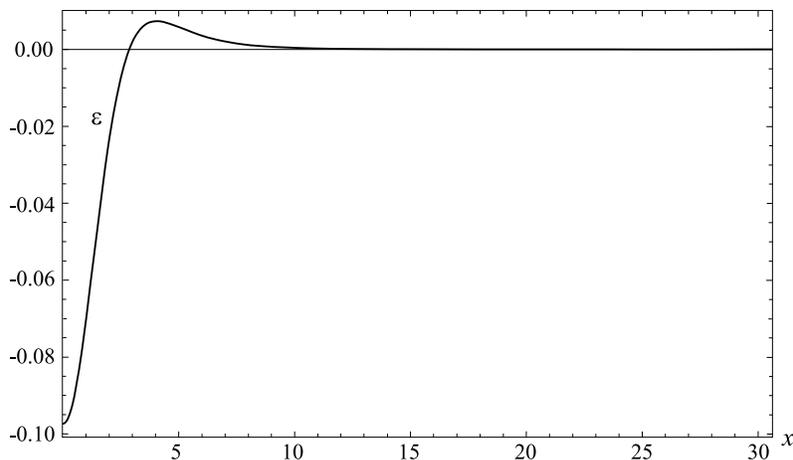


FIG. 2: The scalar fields energy density ε from (52).

Using Eq. (52), one can also find the mass of the glueball:

$$M = 4\pi \int_0^\infty \varepsilon r^2 dr. \quad (55)$$

Taking into account that $r = \lambda^{-1/2}x$, this expression gives the following numerical value: $M \approx 7.4/\sqrt{\lambda}$. Using (44), we have for the coupling constant $\lambda = (\Delta C_\phi)^2 / (1080C_1^2) = (\Delta C_\chi)^2 / (1080C_2^2)$.

Another possible type of solutions of the system (49)-(50) could be solutions for which $\phi \rightarrow 0$ and $\chi \rightarrow m_2$ asymptotically. However, we have not been successful in obtaining such type of solutions that perhaps indicates their absence.

V. CONCLUSION AND FURTHER PROBLEMS

Here we have investigated the scalar model of a glueball proceeding from the nonperturbative quantization ideas à la Heisenberg. Basic features of the model are:

- 2- and 4-point Green functions of quantum fields are described by two scalar fields ϕ, χ .
- The quantum behavior of gauge fields belonging to the subgroup $SU(n) \subset SU(N)$ and $SU(N)$ is different. The scalar field χ describes quantum fluctuations of $A_\mu^a \in SU(n)$, and ϕ describes quantum fluctuations of $A_\mu^m \in SU(N)/SU(n)$.

- The color space is anisotropic in the sense that $\langle A_{\dots}^B A_{\dots}^C \rangle \neq \langle A_{\dots}^C A_{\dots}^B \rangle$ and similarly for 4-point Green functions.
- The anisotropy $\langle A_{\dots}^b A_{\dots}^b - A_{\dots}^c A_{\dots}^b \rangle = C^{bc} \approx C_{\chi}$ and $\langle A_{\dots}^m A_{\dots}^n - A_{\dots}^n A_{\dots}^m \rangle = C^{mn} \approx C_{\phi}$. The constants $C_{\phi, \chi}$ do not depend on color indices.
- The dispersion $\langle (A_{\dots}^B)^2 \rangle$ does not depend on the color index B , but may be different for $SU(n)$ and coset $SU(N)/SU(n)$.
- 4-point Green functions are some bilinear combinations of 2-point Green functions.
- For every group $SU(N)$ there exist different glueballs with different subgroups $SU(n) \subset SU(N)$.

There are a few further problems which could be addressed within the framework of the model under discussion:

- The comparison with lattice calculations.
- The toy model with $b_{\mu\nu} = -\eta_{\mu\nu}$ has negative norm $\langle A_{\mu} A_{\nu} \rangle$ for $\mu, \nu = 0$. Consideration of more plausible model will be much more complicated.
- Study of the asymptotic behavior of the coefficients $\lambda_{1,2,3}$.
- Study of the existence of solutions to the field equations with different λ .

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