

# The relation between the NJL model and QCD with condensed gluons

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We try to find the relation between the three-flavor Nambu–Jona-Lasinio model and QCD based on the hypothesis that the gluon momenta are sharply condensed around the QCD scale,  $\mu_g$ . We find that the effective four- and six-fermion interactions,  $G_4$  and  $G_6$ , should be scaled by  $G_4 \propto \mu_g^{-2}$  and  $G_6 \propto \mu_g^{-5}$  being consistent with the mass dimension counting in the obtained effective Lagrangian. We then study the  $\mu_g$  dependence on the phase diagram of the chiral phase transition at finite temperature and chemical potential and the location of the critical point. We find that the location of the critical point are sensitively affected by the value of the introduced gluon energy scale.

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## I. INTRODUCTION

Hadrons are composite objects constructed from quarks and gluons via strong interaction. The first principle theory for quarks and gluons is quantum chromodynamics (QCD), and one of our goals is to describe observed properties of hadrons based on QCD. Although the perturbative method is nicely adopted for high energy phenomena, the investigation on the system at low energy, such as composite bound state of hadrons, is still challenging issue. There we often employ chiral effective models with four- and six-point interactions to study the hadron properties at low energy.

The Nambu–Jona-Lasinio (NJL) model is one of successful effective models of quantum chromodynamics (QCD) for describing hadron physics [1]. The three-flavor version of the model contains four- and six-fermion interactions in the Lagrangian density, and the latter is called the Kobayashi-Maskawa-'tHooft (KMT) term [2, 3]. This KMT term explicitly breaks the  $U_A(1)$  symmetry which is not realized in the real world, and the model incorporating the KMT term can describe the nonet meson properties in a satisfactory manner (for reviews, see, e.g., [4–9].)

Since the NJL model is regarded as an effective model for QCD, the first principle theory of quarks and gluons, we believe that the model should somehow be related to QCD. Recently, the relation between the four-fermion interaction and the original QCD Lagrangian has been discussed in [10], where it was found that the two-flavor NJL Lagrangian can be connected to QCD with hypothetical gluon condensate. It may also be interesting to test whether the six-fermion interaction with the determinant form suggested in [2] can be derived based on the same assumption that gluons are highly condensed in the momentum space. Then, in this paper, we shall try to find the relation between the effective six-fermion interaction in the three flavor NJL model and the original QCD with the gluon condensate.

The paper is organized as follows. We start from the partition function of QCD then apply the assumption of the gluon condensate in Sec. II. We then set the three-flavor NJL model and perform the numerical analyses in

Sec. III. The discussions on the effective six-fermion interaction and the concluding remark are given in Secs. IV and V.

## II. QCD WITH CONDENSED GLUONS

Following the prescription employed in [10], we first evaluate the partition function of QCD then introduce the effect of gluon condensate for the sake of making the trial to find the relation between the three-flavor NJL model and QCD.

### A. QCD partition function

We start from the following Lagrangian density

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_0 + \mathcal{L}_1, \quad (1)$$

$$\mathcal{L}_0 = \bar{q}(i\cancel{\partial} - m)q - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2, \quad (2)$$

$$\begin{aligned} \mathcal{L}_1 = & g\bar{q}\gamma^\mu t^a q A_\mu^a - g f^{abc}(\partial_\mu A_\nu^a)A^{\mu b}A^{\nu c} \\ & - \frac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{\mu c}A^{\nu d}), \end{aligned} \quad (3)$$

where  $\mathcal{L}_0$  and  $\mathcal{L}_1$  represent the free and interacting parts.  $q$  and  $m$  are the quark field and its current mass,  $A_\mu^a$  is the gluon field,  $g$  is the coupling constant for the strong interaction and  $t^a = \lambda^a/2$  with  $\lambda^a$  being the Gell-Mann matrices in the color space.

The partition function can be expanded by using the Taylor series as

$$\begin{aligned} Z_{\text{QCD}} &= \int \mathcal{D}q \int \mathcal{D}A \exp \left[ i \int d^4x \mathcal{L}_{\text{QCD}} \right] \\ &= \int \mathcal{D}q \int \mathcal{D}A e^{i \int d^4x \mathcal{L}_0} \sum_{n=0}^{\infty} \frac{1}{n!} \left( i \int d^4x \mathcal{L}_1 \right)^n. \end{aligned} \quad (4)$$

Here we study the terms up to the order of  $g^4$  to consider

the four- and six-fermion interactions, then we see

$$\begin{aligned} \mathcal{Z}_{\text{QCD}} \simeq & \int \mathcal{D}q \int \mathcal{D}A e^{i \int d^4x \mathcal{L}_0} \left[ 1 + \frac{1}{2} \left( ig \int d^4x \mathcal{L}_I \right)^2 \right. \\ & \left. + \frac{1}{3!} \left( ig \int d^4x \mathcal{L}_I \right)^3 + \frac{1}{4!} \left( ig \int d^4x \mathcal{L}_I \right)^4 \right]. \quad (5) \end{aligned}$$

Note the first linear term for  $\mathcal{L}_I$  is not required for our purpose, since it is expected to vanish. Expanding and retaining the relevant terms, we have

$$\begin{aligned} \mathcal{Z}_{\text{QCD}} \simeq & \int \mathcal{D}q \int \mathcal{D}A e^{i \int d^4x \mathcal{L}_0} \\ & \times \left[ 1 + \frac{1}{2} \left( ig \int d^4x \bar{q} \gamma^\mu t^a q A_\mu^a \right)^2 \right. \\ & + \frac{1}{3!} \cdot 3 \left( ig \int d^4x \bar{q} \gamma^\mu t^a q A_\mu^a \right)^3 \\ & \times \left( -i \frac{g^2}{4} \int d^4y (f^{hbc} A_\nu^b A_\rho^c) (f^{hde} A^{\nu d} A^{\rho e}) \right) \\ & + \frac{1}{4!} \cdot 4 \left( ig \int d^4x \bar{q} \gamma^\mu t^a q A_\mu^a \right)^4 \\ & \left. \times \left( ig \int d^4y f^{abcd} (\partial_\nu A_\rho^b) A^{\nu c} A^{\rho d} \right) \right]. \quad (6) \end{aligned}$$

This is the partition function we will consider in what follows.

## B. Treatment of the gluon condensate

Having aligned the relevant terms for the partition function, we now perform the functional integral of gluon under special condition. As discussed in [10], we assume that the gluon momenta are condensed around the specific scale,  $p^2 \sim \mu_g^2$ . Then, for the gluon propagator of the usual form,

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i g_{\mu\nu} \delta^{ab}}{p^2} e^{-ip \cdot (x-y)}, \quad (7)$$

we apply the following replacement

$$\frac{1}{p^2} \rightarrow \frac{1}{\mu_g^2}. \quad (8)$$

In more detail,  $p^\mu \rightarrow \mu_g^\mu$  which becomes important when we study the amplitudes at  $g^4$  order. With the replacement we obtain the Feynman rule shown below

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{-i g_{\mu\nu} \delta^{ab}}{\mu_g^2} \delta^{(4)}(x-y) \quad (9)$$

after performing the momentum integration, where the delta function induce the contact interaction for fermion fields.

We should note that the propagator becomes infinite when  $y = x$ , because the one-loop amplitude,

$$\phi_g^{\text{bare}} = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2}, \quad (10)$$

badly diverges. We then apply the renormalization so as to obtain finite prediction, and we set

$$\langle A_\mu^a(x) A_\nu^b(x) \rangle = g_{\mu\nu} \delta^{ab} \phi_g, \quad (11)$$

where  $\phi_g$  is finite renormalized quantity. Here we call  $\phi_g$  as the gluon condensate being reminiscent of the chiral condensate,  $\phi_q \equiv \langle \bar{q}q \rangle$ . One more attention should be paid when we consider the six-fermion interaction, there we apply the following rule,

$$\int \frac{d^4p}{(2\pi)^4} \frac{\bar{q} \not{\mu}_g q}{\mu_g^2} e^{-i(x-y)} \rightarrow \frac{\bar{q}q}{\mu_g} \delta^{(4)}(x-y), \quad (12)$$

with the relation  $\not{\mu}_g q = \mu_g q$  set by using the Dirac equation  $\not{p}q = \sqrt{p^2}q$ .

Based on the rules shown above, one can easily integrate out the gluon degree of freedom,

$$\begin{aligned} \mathcal{Z}_{\text{QCD}} \simeq & \mathcal{N}_A \int \mathcal{D}q e^{i \int d^4x \mathcal{L}_0} \\ & \times \left[ 1 + \frac{ig^2}{2\mu_g^2} \int d^4x (\bar{q} \gamma^\mu t^a q) (\bar{q} \gamma_\mu t^a q) \right. \\ & \left. + \frac{ig^4}{\mu_g^5} \int d^4x f^{abc} (\bar{q} \gamma_\mu t^a q) (\bar{q} \gamma^\mu t^c q) (\bar{q} t^b q) \right], \quad (13) \end{aligned}$$

where  $\mathcal{N}_A$  is the over all constant relating to the gluon functional integral, and we keep only the leading contribution for each term. Moving back the resulting terms inside the exponential using the relation  $1 + \epsilon \simeq e^\epsilon$ , we arrive at the following effective Lagrangian density,

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \bar{q}(i\cancel{\partial} - m)q + \frac{g^2}{2\mu_g^2} (\bar{q} \gamma^\mu t^a q) (\bar{q} \gamma_\mu t^a q) \\ & + \frac{g^4}{\mu_g^5} f^{abc} (\bar{q} \gamma_\mu t^a q) (\bar{q} \gamma^\mu t^c q) (\bar{q} t^b q). \quad (14) \end{aligned}$$

Thus we have obtained the four- and six-fermion contact interaction based on the hypothesis of the condensed gluons. It is interesting to note that the effective couplings are scaled by

$$G_4 \propto \frac{g^2}{\mu_g^2}, \quad G_6 \propto \frac{g^4}{\mu_g^5}, \quad (15)$$

as expected by the mass counting. It should also be noted that if one sets  $G_4, G_6$  to be constants as usually done in practical model studies, the model loses the renormalizability.

### C. Difficulty of deriving the KMT term

We have discussed how the forms of the four- and six-fermion contact interactions arise from the QCD with condensed gluons in the previous subsection. Here it may be important to note that the last term representing the six-fermion interaction vanishes due to the antisymmetric property of  $f^{abc}$ . This indicates that the  $U_A(1)$  breaking term does not appear through the procedure based on the assumed gluon condensate. We think this is natural consequence since the original QCD Lagrangian is symmetric under the  $U_A(1)$  transformation. Moreover, the fact that the KMT term includes the mixture of three flavors, up, down and strange, can not be related with our starting partition function since the original Lagrangian does not include flavor structure. We will present further discussions on this difficulty in Sec. IV.

## III. THE MODEL

We have checked the mass dimensions of the four- and six-point couplings in the previous section through seeing the relation between the effective couplings and the original QCD. We are now going to study the effect of the condensed energy scale  $\mu_g$  here.

### A. Three flavor NJL model

For the sake of testing the  $\mu_g$  dependence on the model predictions, we consider the following effective Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{NJL}} = & \bar{q}(i\partial - m)q + \frac{G_0}{\mu_g^2} [(\bar{q}q)^2 + (\bar{q}i\gamma_5 q)^2] \\ & + \frac{K_0}{\mu_g^5} [\det \bar{q}(1 - \gamma_5)q + \det \bar{q}(1 + \gamma_5)q], \end{aligned} \quad (16)$$

with the dimensionless couplings,  $G_0$  and  $K_0$ . Where the effective six-fermion term is introduced following [2], and the determinant runs over the flavor space. If we write two couplings as  $G_0/\mu_g^2 = G$  and  $K_0/\mu_g^5 = K$ , the model reduces to the usual NJL model. After the mean-field approximation, we have the linearized form,

$$\mathcal{L} = \bar{q}(i\partial - M)q - \frac{2G_0}{\mu_g^2} (\phi_u^2 + \phi_d^2 + \phi_s^2) + \frac{4K_0}{\mu_g^5} \phi_u \phi_d \phi_s, \quad (17)$$

where  $M$  represents the diagonal mass matrix for constituent quarks,  $M_i = m_i - 4G\phi_i + 2K\phi_j\phi_k$  ( $i \neq j \neq k$ ), and  $\phi_i$  indicate the chiral condensates,  $\phi_i \equiv \langle \bar{q}q \rangle$ . The expectation values of the chiral condensates are determined by the gap equations,

$$\phi_i = -\text{tr} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{\not{q} - M_i}, \quad (18)$$

which is derived under the stationary condition of the effective potential  $\mathcal{V} = -\ln Z/(V\beta)$  with the inverse temperature  $\beta = 1/T$ . The above expression quadratically diverges, then we will introduce the three-momentum cutoff  $\Lambda$  to obtain finite quantity.

The model has seven parameters, the three-momentum cutoff  $\Lambda$ , the four- and six point couplings  $G_0$ ,  $K_0$ , the current quark masses  $m_u$ ,  $m_d$ ,  $m_s$ , and the gluon condensate scale  $\mu_g$ . Following [6], we first set  $m_u = m_d = 5.5\text{MeV}$ , then determine the four parameters  $\Lambda$ ,  $G_0$ ,  $K_0$ ,  $m_s$  by using the four physical observables,  $m_\pi = 138\text{MeV}$ ,  $f_\pi = 92\text{MeV}$ ,  $m_K = 495\text{MeV}$ ,  $m_{\eta'} = 958\text{MeV}$ , at the scale  $\mu_g = 250\text{MeV}$ . The above condition leads the values,  $\Lambda = 631\text{MeV}$ ,  $G_0 = 0.288$ ,  $K_0 = 0.09$ ,  $m_s = 136\text{MeV}$ , with  $m_u = m_d = 5.5\text{MeV}$  and  $\mu_g = 250\text{MeV}$ .

### B. Phase diagram

To see the effect of the gluon condensate on the chiral phase transition, we draw the phase diagram on chemical potential ( $\mu$ )-temperature plane through changing the value of  $\mu_g$  with the other parameters fixed. The

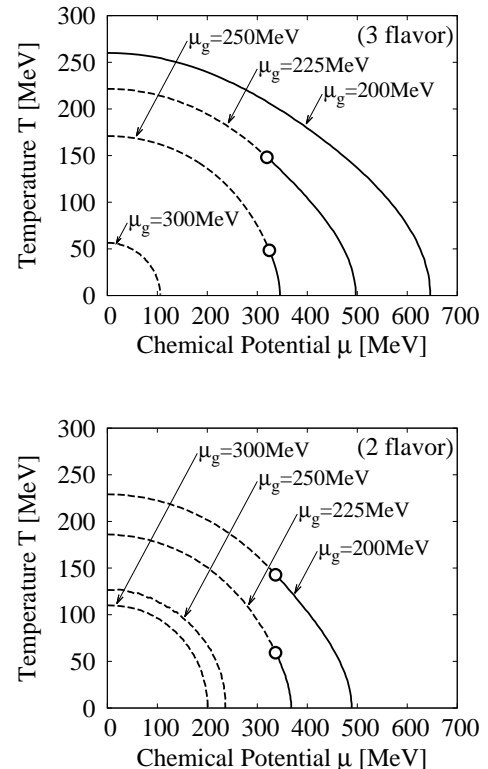


FIG. 1.  $\mu_g$  dependence on the phase diagram in the three- and two-flavor models. The solid (dashed) curves represent the first order (crossover) transition, and the circles do the critical points.

upper panel of Fig. 1 displays the numerical results on

the phase diagram in the three flavor case for  $\mu_g = 200, 225, 250$  and  $300\text{MeV}$ . To study the effect of the six-fermion interaction, we also show the phase diagram in the two-flavor model with basically the same parameters  $\Lambda = 631\text{MeV}$ ,  $G_0 = 0.288$  and  $m_u = m_d = 5.5\text{MeV}$ , but without the strange quark mass and six-point interaction. One sees that the region of the broken phase shrinks with increasing with  $\mu_g$ ; the tendency is common between three- and two-flavor cases. This is easy to understand, since the coupling strength for four-point interaction  $G_0/\mu_g^2$  becomes smaller when one choose larger value of  $\mu_g$ , then the symmetry tends to be restored at lower  $T$  and  $\mu$ . Quantitatively, the change of the area of the broken phase on  $\mu - T$  plane in the three-flavor model is more drastic than the two-flavor case. This comes from the six-fermion interaction,  $K = K_0/\mu_g^5$  which is more sensitively affected by the energy scale  $\mu_g$ ; then the total change enhances in the three-flavor case. We also note that the three-flavor case shows the stronger tendency of the first order phase transition, which is also understood by the above mentioned reasoning due to the six-fermion term. Thus the investigation by the three-flavor case is important when one studies the critical point, there the  $U_A(1)$  anomaly plays the crucial role on the phase transition. We will perform more detailed analysis on the location of the critical point in the following.

### C. Critical point

We see that the critical point drastically moves with changing the gluon energy scale in the previous subsection. It may also be interesting to discuss how the critical point moves with varying the gluon condensate scale  $\mu_g$  in more detail. Figure 2 shows how the location of the critical point ( $T_{cp}, \mu_{cp}$ ) changes with respect to  $\mu_g$ . One

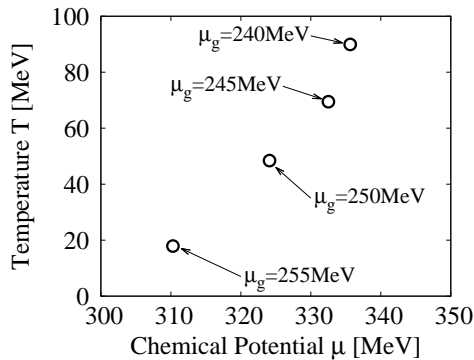


FIG. 2.  $\mu_g$  dependence on the critical point.

observes that the critical point moves towards the lower temperature direction when the gluon energy scale becomes larger. This is the straightforward consequence of the smaller  $K$ , because the KMT term is intimately re-

lated to the strength of the  $U_A(1)$  breaking and the term heightens the tendency of the first order phase transition [11]. Thus the location or the existence of the critical point is indeed sensitively affected by the value of the gluon condensate scale.

### IV. REMARK ON THE KMT TERM

We have seen that the KMT term can not be obtained by starting from the assumption of QCD with gluon condensate in Sec. II. We have also tried to obtain the term by following the discussion in the paper [3] where the determinant term with flavor structure is derived based on the  $SU(2)$  gauge theory in the standard model (SM). However, we also faced the difficulty on the derivation, since the extension from the  $SU(2)$  in the SM to the  $SU(3)$  in QCD contains a certain subtlety. In the SM case, the Pauli matrices with respect to the gauge connection are set in the flavor space, while in the QCD case the Gell-Mann matrices with respect to the gauge connection are defined in the color space. The difference is crucial when one tries to obtain the determinant term in the flavor space; in the QCD case the determinant structure can appear in the color space. Thus, the extension from the  $SU(2)$  in the SM to the  $SU(3)$  in QCD is not straightforward with respect to the flavor indices, so the derivation of the KMT term from the original QCD Lagrangian is highly non-trivial.

### V. SUMMARY AND CONCLUSION

We studied how the three-flavor NJL model can be related to the QCD based on the hypothesis of condensed gluons in this paper. There the correct mass dimensions on the effective four- and six-fermion interactions,  $G_4 \propto \mu_g^{-2}$  and  $G_6 \propto \mu_g^{-5}$ , are found. We also found that it is not possible to find the direct connection between the KMT term and the QCD based on the assumption of the gluon condensate. We think this is unavoidable consequence since the original QCD Lagrangian does not include any flavor structure in its form.

We then studied the gluon energy scale dependence on the phase structure of the chiral phase transition. We see that the tendency of the chiral symmetry breaking and the first order phase transition become stronger when  $\mu_g$  decreases. The tendency is easily understood because the phenomenon of the symmetry breaking is expected to be enhanced at low energy as the renormalization group analyses insists.

Finally, we believe that the current analysis has a lot of applicability on physics relating to gluons since the gluon degree of freedom is somehow incorporated to the effective model. Then we think it is interesting to perform further investigations based on the QCD with the gluon condensate.

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