

# Improved analyses for $\mu^-e^- \rightarrow e^-e^-$ in muonic atoms by contact interactions

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The charged lepton flavor violating (CLFV) processes of  $\mu^-e^- \rightarrow e^-e^-$  decay by four Fermi contact interactions in a muonic atom for various atoms are investigated. The wave functions of bound and scattering state leptons are properly treated by solving Dirac equations with Coulomb interaction of the finite nuclear charge distributions. This new effect contributes significantly in particular for heavier atoms, where the obtained decay rate is about one order of magnitude larger than the previous estimation for  $^{208}\text{Pb}$ . We find that, as the atomic number  $Z$  increases, the  $\mu^-e^- \rightarrow e^-e^-$  decay rates increase more rapidly than the result of the previous work of  $Z^3$ , suggesting this decay as one of the promising processes to search for CLFV interaction.

## I. INTRODUCTION

The charged lepton flavor violating (CLFV) processes are known to provide important signals on physics beyond the Standard Model (SM). The analysis of search for  $\mu^+ \rightarrow e^+\gamma$  decays in the cosmic-ray muons by Hincks and Pontecorvo in 1947 [1] has given the first upper limit on the branching ratio of CLFV processes. Since then, the upper limits of the branching ratios of CLFV processes have been improved and now reach around the orders of  $10^{-12} \sim 10^{-13}$  [2, 3]. These upper limits put stringent constraints on various theoretical models beyond the SM. These CLFV searches include the processes such as  $\mu^+ \rightarrow e^+\gamma$ ,  $\mu^+ \rightarrow e^+e^-e^+$  and  $\mu^-N \rightarrow e^-N$  conversion in a muonic atom. Recently, another process of  $\mu^-e^- \rightarrow e^-e^-$  decay in a muonic atom has been proposed by Koike *et al.* in 2010 [4]. A unique feature of this process is that both photonic and contact leptonic interaction can be proved, and an experimentally clean signal is expected because the sum of the energies of two electrons is restricted to the muon mass minus the binding energy of the muon in a muonic atom. The measurement of this process is planned in the COMET Phase-I experiment in J-PARC [5].

In Ref. [4], the decay rate of muonic atom was evaluated by using the non-relativistic bound state wave functions of muon and electron and the plane wave approximation of the final electrons. It was shown the decay rate increases with the atomic number  $Z$  as  $\Gamma \sim Z^3$ . Therefore heavy muonic atoms would provide a great opportunity of CLFV search. However, as is well known, the effects of the Coulomb interaction is significant for the ordinary decay of bound muons in heavy nuclei [6, 7]. Since the quantitative evaluation of the decay process is needed in order to disentangle the mechanism of CLFV interaction, it is important to update the estimations of Ref. [4] by taking into account the effects of the Coulomb interactions for the relativistic leptons. The importance of the Coulomb distortion for the  $\mu^- - e^-$  conversion process in a muonic atom has been reported in Refs. [8–10]. For  $\mu^- - e^-$  conversion process where the nucleus stays intact, it is sufficient to consider the s-wave muon and electron states. For  $\mu^-e^- \rightarrow e^-e^-$  decay of muonic atom, on the other hand, two electrons with the energy of approximately a half of muon mass are emitted in the final state. The angular momentum of the each electrons is not limited in this process. A formalism of the  $\mu^-e^- \rightarrow e^-e^-$  decay with the partial wave expansion of leptons is necessary, as has been common in the nuclear beta decay and muon capture reactions [12].

In Section II, we summarize the relevant effective Lagrangian for the  $\mu^-e^- \rightarrow e^-e^-$  process and develop a formula of the decay rate using the partial wave expansion of the lepton wave function. Our refined estimations of the  $\mu^-e^- \rightarrow e^-e^-$  decay rate for the muonic atom is presented in Section III. Finally our conclusion is given in Section IV.

## II. FORMULATION

The effective Lagrangian of the CLFV process  $\mu^- e^- \rightarrow e^- e^-$  is given as,

$$\mathcal{L}_I = \mathcal{L}_{\text{photo}} + \mathcal{L}_{\text{contact}}, \quad (1)$$

$$\mathcal{L}_{\text{photo}} = -\frac{4G_F}{\sqrt{2}} m_\mu [A_R \bar{e}_L \sigma^{\mu\nu} \mu_R + A_L \bar{e}_R \sigma^{\mu\nu} \mu_L] F_{\mu\nu} + [h.c.], \quad (2)$$

$$\begin{aligned} \mathcal{L}_{\text{contact}} = & -\frac{4G_F}{\sqrt{2}} [g_1 (\bar{e}_L \mu_R) (\bar{e}_L e_R) + g_2 (\bar{e}_R \mu_L) (\bar{e}_R e_L) \\ & + g_3 (\bar{e}_R \gamma_\mu \mu_R) (\bar{e}_R \gamma^\mu e_R) + g_4 (\bar{e}_L \gamma_\mu \mu_L) (\bar{e}_L \gamma^\mu e_L) \\ & + g_5 (\bar{e}_R \gamma_\mu \mu_R) (\bar{e}_L \gamma^\mu e_L) + g_6 (\bar{e}_L \gamma_\mu \mu_L) (\bar{e}_R \gamma^\mu e_R)] + [h.c.], \end{aligned} \quad (3)$$

where  $G_F = 1.166 \times 10^{-5} \text{GeV}^{-2}$  is the Fermi coupling constant, and  $A_{R,L}$  and  $g_i$ 's ( $i = 1, 2, \dots, 6$ ) are dimensionless coupling constants. The left and right handed fields  $\psi_{L/R}$  are given as  $\psi_{L/R} = P_{L/R} \psi$  with  $P_{L/R} = (1 \mp \gamma_5)/2$ . The effective Lagrangian consists of two parts. The first part,  $\mathcal{L}_{\text{photo}}$ , represents the photonic interaction of  $\mu \rightarrow e\gamma$  types, which generates the long range  $\mu - e$  interaction with one photon exchange between a muon and an electron. The second part,  $\mathcal{L}_{\text{contact}}$ , is the four Fermi interaction. In this work, we concentrate on the contact interaction as our first attempt to examine the role of Coulomb interaction on the  $\mu^- e^- \rightarrow e^- e^-$  decay of muonic atoms.

We evaluate the decay rate of two-electron emission of the muonic atom within the independent particle picture of the muonic atom and the final state. The transition amplitude is given by the matrix element of the effective CLFV interaction in Eq. (3)

$$\begin{aligned} M(\mathbf{p}_1, s_1, \mathbf{p}_2, s_2; \alpha_\mu, s_\mu, \alpha_e, s_e) & \equiv \int d^3 r \langle e_{\mathbf{p}_1}^{s_1} e_{\mathbf{p}_2}^{s_2} | \mathcal{L}_{\text{contact}} | \mu_{\alpha_\mu}^{s_\mu} e_{\alpha_e}^{s_e} \rangle \\ & = -\frac{4G_F}{\sqrt{2}} \sum_{i=1}^6 g_i \left[ \int d^3 r \bar{\psi}_{\mathbf{p}_1, s_1}^{e(-)}(\mathbf{r}) O_i^A \psi_{\alpha_\mu, s_\mu}^\mu(\mathbf{r}) \bar{\psi}_{\mathbf{p}_2, s_2}^{e(-)}(\mathbf{r}) O_i^B \psi_{\alpha_e, s_e}^e(\mathbf{r}) - (1 \leftrightarrow 2) \right], \end{aligned} \quad (4)$$

where  $\psi_{\mathbf{p}, s}^{e(-)}(\mathbf{r})$  is the wave function of a scattering electron with its momentum  $\mathbf{p}$  and spin  $s$ . The superscript  $(-)$  represents the incoming wave boundary condition. The wave functions of bound leptons are denoted as  $\psi_{\alpha, s}^l$  with  $l = \mu, e$ , spin  $s$  and  $\alpha = n, \kappa$ . Here,  $\kappa$  represents both the orbital and the total angular momentum simultaneously [11, 13]. The second term ( $1 \leftrightarrow 2$ ) in Eq. (4) is the exchange term obtained from the first term by exchanging the quantum numbers of final electrons. The Dirac matrix  $O_i^A$  and  $O_i^B$  for each  $g_i$  in Eq. (3) are given as

$$\begin{aligned} O_1^A &= O_1^B = P_R, O_2^A = O_2^B = P_L, \\ O_3^A &= O_5^A = \gamma_\mu P_R, O_4^A = O_6^A = \gamma_\mu P_L, \\ O_3^B &= O_6^B = \gamma^\mu P_R, O_4^B = O_5^B = \gamma^\mu P_L. \end{aligned} \quad (5)$$

We assume that the muon bound state is in the  $n = 1, \kappa = -1$  state denoted simply by  $\alpha_\mu = 1S$ . Since orbit of the bound muon is about 200 times smaller than that of electron, the  $n = 1, \kappa = -1$  electron bound state gives the main contribution to the decay rate of the muonic atom as long as we consider the contact interaction in Eq. (3). The  $\mu^- e^- \rightarrow e^- e^-$  decay rate of a muonic atom is given, with possible contributions of electron bound states of  $\kappa_e = -1$  and any  $n$  included, as follows:

$$\begin{aligned} \Gamma &= \frac{1}{2} \left( \sum_{s_1, s_2} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^3 2E_1 (2\pi)^3 2E_2} \right) \left( \frac{1}{2} \sum_{s_\mu, s_e, n} \right) \\ & \times 2\pi \delta(E_{p_1} + E_{p_2} - m_\mu - m_e + B_\mu + B_e^n) |M(\mathbf{p}_1, s_1, \mathbf{p}_2, s_2; 1S, s_\mu, nS, s_e)|^2, \end{aligned} \quad (6)$$

where  $B_\mu$  and  $B_e^n$  are the binding energies of the muon and electron in a muonic atom and  $E_{p_i}$  is an energy of one of the electrons with its momentum  $p_i$ . Here the initial muon spins are averaged. The normalization of the bound state wave function is given as

$$\int d^3 \mathbf{r} \psi_{\alpha, s}^l \dagger(\mathbf{r}) \psi_{\alpha, s'}^l(\mathbf{r}) = \delta_{\alpha, \alpha'} \delta_{s, s'}, \quad (7)$$

and the scattering wave function is normalized as

$$\int d^3 \mathbf{r} \psi_{\mathbf{p}, s}^{e(-)\dagger}(\mathbf{r}) \psi_{\mathbf{p}', s'}^{e(-)}(\mathbf{r}) = 2E_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{s, s'}. \quad (8)$$

The double differential decay rate with respect to the electron energy and the angle  $\theta$  between emitted electrons is given as

$$\frac{d^2\Gamma_n}{dE_{p_1}d\cos\theta} = \frac{4\pi \cdot 2\pi}{8(2\pi)^5} |\mathbf{p}_1||\mathbf{p}_2| \sum_{s_1, s_2, s_\mu, s_e} |M(\mathbf{p}_1, s_1, \mathbf{p}_2, s_2; 1S, s_\mu, nS, s_e)|^2, \quad (9)$$

where  $E_{p_2} = -E_{p_1} + m_\mu + m_e - B_\mu - B_e^n$  and related to the total decay rate as

$$\Gamma = \frac{1}{2} \sum_n \int_{m_e}^{E_{\max}^n} dE_{p_1} \int_{-1}^1 d\cos\theta \frac{d^2\Gamma_n}{dE_{p_1}d\cos\theta}, \quad (10)$$

where  $E_{\max}^n = m_\mu - B_\mu - B_e^n$ . The transition matrix element  $M$  is evaluated by using the partial wave expansion of the electron scattering state. The electron scattering state with the incoming boundary condition is expressed as

$$\psi_{\mathbf{p},s}^{e(-)}(\mathbf{r}) = \sum_{\kappa, \nu, m} 4\pi i^{l_\kappa} (l_\kappa, m, 1/2, s | j_\kappa, \nu) Y_{l_\kappa, m}^*(\hat{\mathbf{p}}) e^{-i\delta_\kappa} \psi_{p,\nu}^\kappa(\mathbf{r}), \quad (11)$$

where  $\delta_\kappa$  is a phase shift for partial wave  $\kappa$ .  $(l_\kappa, m, 1/2, s | j_\kappa, \nu)$  and  $Y_{l_\kappa, m}(\hat{\mathbf{p}})$  are Clebsch-Gordan coefficients and spherical harmonics, respectively. Furthermore, the wave function  $\psi_{p,\nu}^\kappa(\mathbf{r})$ , where the subscripts  $p, \nu$  mean a momentum of the electron and a spin of the partial wave, is written with the radial part  $g_p^\kappa(r), f_p^\kappa(r)$  and the angular-spin part  $\chi_\kappa$  [11, 13] as follows,

$$\psi_{p,\nu}^\kappa(\mathbf{r}) = \begin{pmatrix} g_p^\kappa(r) \chi_\kappa^\nu(\hat{\mathbf{r}}) \\ i f_p^\kappa(r) \chi_{-\kappa}^\nu(\hat{\mathbf{r}}) \end{pmatrix}. \quad (12)$$

Similarly, the bound state wave function is given as

$$\psi_{\alpha,s}^l(\mathbf{r}) = \begin{pmatrix} g_{n,l}^\kappa(r) \chi_\kappa^s(\hat{\mathbf{r}}) \\ i f_{n,l}^\kappa(r) \chi_{-\kappa}^s(\hat{\mathbf{r}}) \end{pmatrix}, \quad (13)$$

where the subscript  $l$  is for muon  $l = \mu$  or electron  $l = e$  and  $s$  is a spin of the lepton. The radial wave functions  $g^\kappa(r)$  and  $f^\kappa(r)$  are obtained by solving following Dirac equation with the Coulomb potential  $V_C(r)$  for appropriate boundary condition,

$$\frac{dg^\kappa(r)}{dr} + \frac{1+\kappa}{r} g^\kappa(r) - (E + m + eV_C(r)) f^\kappa(r) = 0, \quad (14)$$

$$\frac{df^\kappa(r)}{dr} + \frac{1-\kappa}{r} f^\kappa(r) - (E - m + eV_C(r)) g^\kappa(r) = 0. \quad (15)$$

Using the partial wave expansion of the scattering wave function the transition amplitude can be written as follows

$$\begin{aligned} M(\mathbf{p}_1, s_1, \mathbf{p}_2, s_2; 1S, s_\mu, nS, s_e) &= 2\sqrt{2}G_F \sum_{\kappa_1, \kappa_2, \nu_1, \nu_2, m_1, m_2} (4\pi)^2 (-i)^{l_{\kappa_1} + l_{\kappa_2}} e^{i(\delta_{\kappa_1} + \delta_{\kappa_2})} \\ &\times Y_{l_{\kappa_1}, m_1}(\hat{\mathbf{p}}_1) Y_{l_{\kappa_2}, m_2}(\hat{\mathbf{p}}_2) (l_{\kappa_1}, m_1, 1/2, s_1 | j_{\kappa_1}, \nu_1) (l_{\kappa_2}, m_2, 1/2, s_2 | j_{\kappa_2}, \nu_2) \\ &\times \sum_{J, M} (j_{\kappa_1}, \nu_1, j_{\kappa_2}, \nu_2 | J, M) (j_{-1}, s_\mu, j_{-1}, s_e | J, M) \\ &\times \frac{2\sqrt{(2j_{\kappa_1} + 1)(2j_{\kappa_2} + 1)}}{4\pi} \sum_{i=1}^6 g_i W_i(J, \kappa_1, \kappa_2, E_{p_1}). \end{aligned} \quad (16)$$

Here  $W_i(J, \kappa_1, \kappa_2, E_{p_1})$  is the transition matrix element for the  $g_i$  term that includes both direct and exchange terms. We introduce the function  $Z_{ABCD}(L, S, J)$ , which consists of the radial overlap integral, 9j and parity Clebsch-Gordan coefficients as

$$\begin{aligned} Z_{ABCD}(L, S, J) &= \int_0^\infty dr r^2 A_{p_1}^{\kappa_1}(r) B_{1,\mu}^{-1}(r) C_{p_2}^{\kappa_2}(r) D_{n,e}^{-1}(r) \\ &\times \sqrt{(2l_{\kappa_1}^A + 1)(2l_{-1}^B + 1)(2l_{\kappa_2}^C + 1)(2l_{-1}^D + 1)} \\ &\times (l_{\kappa_1}^A, 0, l_{\kappa_2}^C, 0 | L, 0) (l_{-1}^B, 0, l_{-1}^D, 0 | L, 0) \\ &\times \begin{Bmatrix} l_{\kappa_1}^A & 1/2 & j_{\kappa_1} \\ l_{\kappa_2}^C & 1/2 & j_{\kappa_2} \\ L & S & J \end{Bmatrix} \begin{Bmatrix} l_{-1}^B & 1/2 & 1/2 \\ l_{-1}^D & 1/2 & 1/2 \\ L & S & J \end{Bmatrix}. \end{aligned} \quad (17)$$

Here  $A$  and  $C$  represent the electron scattering states with momentum  $p_1$  and  $p_2$  and  $B$  and  $D$  represent the bound states of muon and electron. The radial wave functions  $A(r), B(r), C(r)$  and  $D(r)$  are either  $g(r)$  or  $f(r)$  introduced in Eqs. (12) and (13). The angular momentum  $l_\kappa^h$  is defined as

$$l_\kappa^h = \begin{cases} l_{+\kappa} & \text{for } h = g, \\ l_{-\kappa} & \text{for } h = f. \end{cases} \quad (18)$$

The amplitude  $W_i$  for  $i = 1, \dots, 6$  are written by using linear combination of  $Z$  as

$$W_1(J) = \frac{1}{2} \left\{ X_\alpha^-(J, 0, J) - X_\beta^+(J, 0, J) + i \left[ Y_\alpha^+(J, 0, J) + Y_\beta^+(J, 0, J) \right] \right\}, \quad (19)$$

$$W_2(J) = \frac{1}{2} \left\{ X_\alpha^-(J, 0, J) - X_\beta^+(J, 0, J) - i \left[ Y_\alpha^+(J, 0, J) + Y_\beta^+(J, 0, J) \right] \right\}, \quad (20)$$

$$W_3(J) = 2 \left\{ X_\alpha^-(J, 0, J) + X_\beta^+(J, 0, J) - i \left[ Y_\alpha^+(J, 0, J) - Y_\beta^+(J, 0, J) \right] \right\}, \quad (21)$$

$$W_4(J) = 2 \left\{ X_\alpha^-(J, 0, J) + X_\beta^+(J, 0, J) + i \left[ Y_\alpha^+(J, 0, J) - Y_\beta^+(J, 0, J) \right] \right\}, \quad (22)$$

$$W_5(J) = 3 \sum_{L=|J-1|}^{J+1} X_\beta^-(L, 1, J) - X_\alpha^+(J, 0, J) + i \left[ 3 \sum_{L=|J-1|}^{J+1} Y_\alpha^-(L, 1, J) + Y_\beta^-(J, 0, J) \right], \quad (23)$$

$$W_6(J) = 3 \sum_{L=|J-1|}^{J+1} X_\beta^-(L, 1, J) - X_\alpha^+(J, 0, J) - i \left[ 3 \sum_{L=|J-1|}^{J+1} Y_\alpha^-(L, 1, J) + Y_\beta^-(J, 0, J) \right], \quad (24)$$

with

$$X_\alpha^\pm(L, S, J) = Z_{gggg}(L, S, J) + Z_{ffff}(L, S, J) \pm [Z_{gfgf}(L, S, J) + Z_{fgfg}(L, S, J)], \quad (25)$$

$$X_\beta^\pm(L, S, J) = Z_{ggff}(L, S, J) + Z_{ffgg}(L, S, J) \pm [Z_{gfgg}(L, S, J) + Z_{fggf}(L, S, J)], \quad (26)$$

$$Y_\alpha^\pm(L, S, J) = Z_{ggfg}(L, S, J) - Z_{ffgf}(L, S, J) \pm [Z_{fggg}(L, S, J) - Z_{gfff}(L, S, J)], \quad (27)$$

$$Y_\beta^\pm(L, S, J) = Z_{gggf}(L, S, J) - Z_{fffg}(L, S, J) \pm [Z_{gfgg}(L, S, J) - Z_{fgff}(L, S, J)]. \quad (28)$$

Since we assume the bound states of muon and electron are both in the  $\kappa = -1$  state, the total angular momentum  $J$  can be  $J = 0$  or  $1$  and  $X_{\alpha(\beta)}^\pm$  and  $Y_{\alpha(\beta)}^\pm$  are non zero only for even  $L$  and odd  $L$  respectively. It is noticed that only  $S = 0$  term contributes for  $W_1, W_2, W_3$  and  $W_4$ , while both  $S = 0$  and  $1$  terms contribute for  $W_5$  and  $W_6$ .

After summing the spins of leptons, we yield the differential transition rate,

$$\begin{aligned} \frac{d^2\Gamma_n}{dE_{p_1} d\cos\theta} &= \frac{G_F^2}{\pi^3} |\mathbf{p}_1| |\mathbf{p}_2| \sum_{\kappa_1, \kappa_2, \kappa'_1, \kappa'_2, J, l} (2J+1)(2j_{\kappa_1}+1)(2j_{\kappa_2}+1)(2j_{\kappa'_1}+1)(2j_{\kappa'_2}+1) \\ &\times \frac{1 + (-1)^{l_{\kappa_1} + l_{\kappa'_1} + l}}{2} \frac{1 + (-1)^{l_{\kappa_2} + l_{\kappa'_2} + l}}{2} i^{-l_{\kappa_1} - l_{\kappa_2} + l_{\kappa'_1} + l_{\kappa'_2}} e^{i(\delta_{\kappa_1} + \delta_{\kappa_2} + \delta_{\kappa'_1} + \delta_{\kappa'_2})} \\ &\times (j_{\kappa_1}, 1/2, j_{\kappa'_1}, -1/2 | l, 0) (j_{\kappa_2}, 1/2, j_{\kappa'_2}, -1/2 | l, 0) W(j_{\kappa_1} j_{\kappa_2} j_{\kappa'_1} j_{\kappa'_2}; J l) \\ &\times (-1)^{J - j_{\kappa_2} - j_{\kappa'_2}} \sum_{i=1}^6 g_i W_i(J, \kappa_1, \kappa_2, E_{p_1}, n) \sum_{i'=1}^6 g_{i'}^* W_{i'}^*(J, \kappa'_1, \kappa'_2, E_{p_1}, n) P_l(\cos\theta), \end{aligned} \quad (29)$$

where  $P_l(x)$  is Legendre polynomial and  $W$  is Racah coefficient, and we can estimate the angular integral analytically and obtain the following formula for the decay rate of  $\mu^- e^- \rightarrow e^- e^-$  of a muonic atom,

$$\Gamma = \frac{G_F^2}{\pi^3} \sum_n \int_{m_e}^{E_{\max}^n} dE_{p_1} |\mathbf{p}_1| |\mathbf{p}_2| \sum_{J, \kappa_1, \kappa_2} (2J+1)(2j_{\kappa_1}+1)(2j_{\kappa_2}+1) \left| \sum_{i=1}^6 g_i W_i(J, \kappa_1, \kappa_2, E_{p_1}, n) \right|^2. \quad (30)$$

### III. NUMERICAL RESULTS

At first, we study the transition density  $\rho_{\text{tr}}(r)$  given by the product of lepton wave functions as

$$\rho_{\text{tr}}(r) = g_{p_1}^{-1}(r) g_\mu^{-1}(r) g_{p_2}^{-1}(r) g_e^{-1}(r), \quad (31)$$

to find the role of the Coulomb interaction on the lepton wave function. Here we take the most important transition matrix element of  $1S$  electron and muon to the  $\kappa = -1$  electrons ( $\mu^-(1S) + e^-(1S) \rightarrow e^-(\kappa = -1) + e^-(\kappa = -1)$ ), where the two electrons are equally sharing the energy  $E_{p_1} = E_{p_2} = (m_\mu + m_e - B_\mu^{1S} - B_e^{1S})/2$ . We examine four models for the lepton wave functions shown in Table I. In the model I, the bound state wave functions of muon and electron are calculated in the non-relativistic approximation with Coulomb interaction of point nuclear charge and the electron scattering states are in the plane wave approximation. Then the wave function of the scattering state are replaced by the solution of Dirac equation in the model II. In the model III, both of the bound state and the scattering state lepton wave functions are calculated from the Dirac equation with point nuclear charge. Finally, we used the uniform nuclear charge distribution in the model IV.

The transition densities of the four models for the  $\mu^-e^- \rightarrow e^-e^-$  decay of  $^{208}\text{Pb}$  muonic atom are shown in FIG. 1. The dashed curve shows transition density in the model I which simulates the previous analysis. By including the Coulomb attraction for scattering electrons in the model II, the transition density is enhanced around the muon Bohr radius as shown by the dash-dotted curve. Further we use the consistent lepton wave functions of the Dirac equation with point nuclear charge in the model III. The transition density becomes very large as shown by the dash-two-dotted curve, which is  $1/3$  of the actual transition density. However, the use of point nuclear charge would not be appropriate for atom of large  $Z$ , where the Bohr radius of muon can be comparable to the nuclear radius. The solid curve shows our final result by using finite charge distribution of nucleus in the model IV. The peak position of the transition density is shifted toward larger  $r$  compared with that of point nuclear charge. Here the charge distribution of nucleus is taken as uniform distribution as

$$\rho_C(r) = \frac{3Ze}{4\pi R^3}\theta(R-r). \quad (32)$$

We use  $R = 1.2A^{1/3}$  for mass number  $A$ . For each  $Z$ , we take the mass number  $A$  of the most abundant isotope [14], e.g.  $A = 208$  for  $Z = 82$ .

TABLE I. Models for the lepton wave functions.

Model	Bound state	Scattering state
I	Non. Rel./Point Coul.	Rel./PLW
II	Non. Rel./Point Coul.	Rel./Point Coul.
III	Rel./Point Coul.	Rel./Point Coul.
IV	Rel./Uniform Coul.	Rel./Uniform Coul.

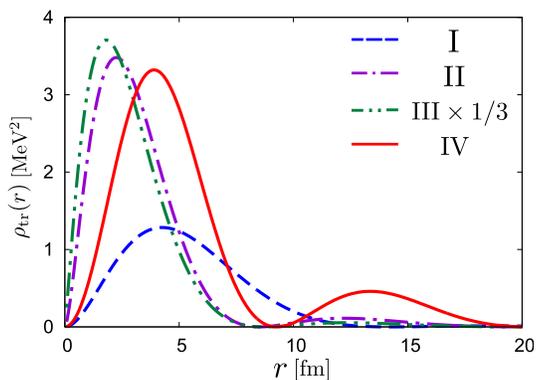


FIG. 1. The transition density  $\rho(r)r^2$  for the  $\mu^-(1S) + e^-(1S) \rightarrow e^-(\kappa = -1) + e^-(\kappa = -1)$ . The dashed, dash-dotted, dash-two-dotted and solid curves show the transition density in the model I, II, III and IV, respectively.

An analytic formula of the  $\mu^-e^- \rightarrow e^-e^-$  decay rate of muonic atom is given in the previous work[4] as

$$\Gamma_0 = \frac{m_\mu}{8\pi^2}(Z-1)^3\alpha^3(G_F m_\mu^2)^2\left(\frac{m_e}{m_\mu}\right)^3 G, \quad (33)$$

where  $G \equiv G_{12} + 16G_{34} + 4G_{56} + 8G'_{14} + 8G'_{23} - 8G'_{56}$  with  $G_{ij} \equiv |g_i|^2 + |g_j|^2$  and  $G'_{ij} \equiv \text{Re}(g_i^* g_j)$ . The formula shows that the decay rate is proportional to  $(Z - 1)^3$ . The formula was obtained by using the non-relativistic bound state of the muon and  $1S$  electron with a point nuclear charge and the plane wave approximation for the final electrons.

The decay rate  $\Gamma$  obtained in this work is shown in FIG. 2. Here the ratios  $\Gamma/\Gamma_0$  are plotted. We retain only the term of  $g_1$  and set the other  $g$ 's zero. The contribution of the dominant  $1S$  bound electron is included. The dashed

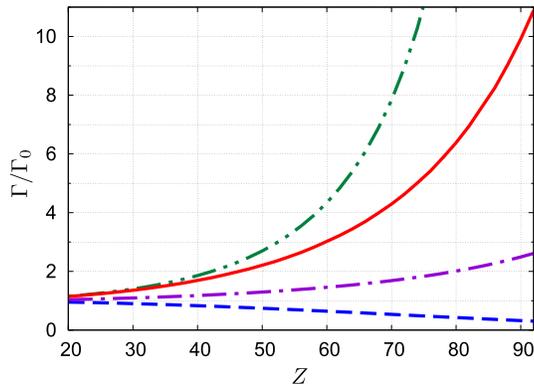


FIG. 2. The atomic number ( $Z$ ) dependence of the ratio of the decay rate  $\Gamma/\Gamma_0$ . The dashed, dash-dotted, dash-two-dotted and solid curves show the transition density in the model I, II, III and IV, respectively. We note that the factor  $1/3$  has not been multiplied to the dash-two-dotted curve.

curve in Fig 2 shows the decay rate evaluated with model I. The ratio for model I deviates from the unity for large  $Z$  because of using finite size bound muon wave function instead of using plane wave in the previous estimation. When we replace the plane wave electrons with the Dirac wave function for point nuclear charge(II), the decay rate increases as shown in the dash-dotted curve. When both bound and scattering states are described by the Dirac equation that includes the Coulomb interaction of point nuclear charge(III), the decay rate is even more enhanced as shown in the dash-two-dotted curve. A realistic description of  $\Gamma/\Gamma_0$  is obtained by using the uniform nuclear charge distribution in model IV as shown in the solid curve in Fig. 2.

The results show that, while  $\Gamma_0$  gives reasonable estimation for smaller  $Z \sim 20$ , the  $Z$  dependence of the  $\Gamma$  is stronger than  $(Z - 1)^3$ . The ratio  $\Gamma/\Gamma_0$  is about 7.0 for the  $^{208}\text{Pb}$ . We found slightly different  $Z$  dependence of  $\Gamma$  for two types of the effective CLFV contact interaction. The interaction of  $g_i$  term with  $i = 1, 2, 3, 4$ , which leads to the same helicity states of two electrons gives  $\Gamma/\Gamma_0 \sim 7.0(1.1)$  for  $^{208}\text{Pb}$  ( $^{40}\text{Ca}$ ). For  $g_i$  term with  $i = 5, 6$ , where the opposite helicity states of electrons are emitted, the decay rate is  $\Gamma/\Gamma_0 \sim 6.3(1.1)$  for  $^{208}\text{Pb}$  ( $^{40}\text{Ca}$ ). Therefore  $Z$  dependence of the decay rate for  $g_1 \sim g_4$  is slightly stronger than that of  $g_5$  and  $g_6$ .

All the results of the decay rate were obtained by including sufficiently large number of partial waves of final electrons. The convergence properties of the decay rate against the number of partial waves included is shown in Table II. The number of partial waves needed to obtain convergent results was  $|\kappa| \sim 6$  for Pb and Sn and  $|\kappa| \sim 13$  for Ca. This happens because the muon Bohr radius is increasing for decreasing  $Z$ .

TABLE II. The convergence property of the partial wave expansion of  $\Gamma/\Gamma_0$ .

nuclei	$ \kappa  \leq 1$	$ \kappa  \leq 5$	$ \kappa  \leq 10$	$ \kappa  \leq 20$
$^{40}\text{Ca}$	0.141	0.847	1.11	1.15
$^{120}\text{Sn}$	0.731	2.17	2.21	2.21
$^{208}\text{Pb}$	2.89	6.94	6.96	6.96

We have also examined realistic form of the distribution of nuclear charge using the Woods-Saxon form,

$$\rho_C(r) = \rho_0 \left[ 1 + \exp\left(\frac{r-c}{z}\right) \right]^{-1}, \quad (34)$$

for  $^{40}\text{Ca}$ ,  $^{120}\text{Sn}$  and  $^{208}\text{Pb}$ . The parameters,  $c$  and  $z$  and the ratio of the decay rate  $\Gamma/\Gamma_0$  are listed in TABLE III. The modification of the decay rate using Woods-Saxon form charge distribution in place of uniform distribution is less than 1%.

TABLE III. The parameters of the charge distribution of Woods-Saxon form and the ratio of the decay rates  $\Gamma/\Gamma_0$  for  $^{40}\text{Ca}$ ,  $^{120}\text{Sn}$  and  $^{208}\text{Pb}$  [15]. In the 4th (5th) column, the  $\Gamma/\Gamma_0$  including the contribution of the  $1S$  ( $1S$  and higher shells) is shown.

nuclei	$c$ [fm]	$z$ [fm]	$\Gamma/\Gamma_0$ (only $1S$ )	$1S + 2S + \dots$
$^{40}\text{Ca}$	3.51(7)	0.563	1.15	1.35
$^{120}\text{Sn}$	5.315(25)	0.576(11)	2.21	2.67
$^{208}\text{Pb}$	6.624(35)	0.549(8)	6.96	8.78

The results shown so far were obtained including only the main transitions where the initial electrons are bound in the  $1S$  state. The contributions of the electrons from the higher shell  $2S, 3S, \dots$  are estimated within the independent particle model for the atomic electrons. Contributions of higher shell electrons increase the transition rate by  $\sim 20\%$  as shown in Table III, which is consistent with the result of the previous work.

The energy and angular distribution of electron calculated from the double differential decay rate in Eq. (9) for the  $^{208}\text{Pb}$  is shown in FIG. 3. The two final electrons are mainly emitted with the same energy in a opposite direction, since the momentum carried by the bound two leptons is minimized in this configuration. The electron energy spectrum

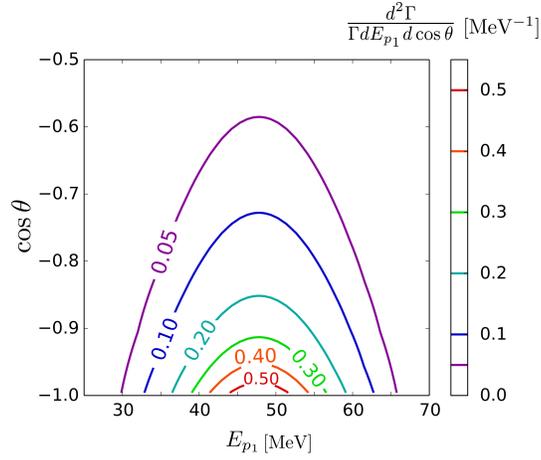


FIG. 3. The energy and angular distribution of emitted electrons for the  $^{208}\text{Pb}$  by using  $g_1$  type interaction.

normalized by decay rate  $d\Gamma/dE/\Gamma$  and the angular distribution between the two electrons are shown in FIG. 4 and in FIG. 5 for the models IV (solid) and I (dash). The maximum of the energy distribution is around half of the total energy  $m_\mu + m_e - B_\mu^{1S} - B_e^{1S}$ . Most of the final electrons are emitted in the opposite directions. The shapes of the energy distribution and the angular distribution are significantly different from the models I and IV. The angular and the energy distributions in model IV becomes narrower than those of model I. This is because the muon is less bound for finite range nuclear charge distribution, and therefore it has smaller high momentum component.

For the interaction which leads to the same chirality of final electrons, i.e.  $g_1 \sim g_4$  terms of Eq. (4), the Pauli principle prevents the final electron from having the same momentum. On the other hand, in  $g_5$  and  $g_6$  terms which leads to electrons with opposite chiralities, this does not apply. A difference between two interaction terms appears near  $\cos \theta = 1$  as seen in FIG. 6.

Finally, we evaluate upper limits for the branching ratio of the  $\mu^- e^- \rightarrow e^- e^-$  decay of a muonic atom. The branching ratio of  $\mu^- e^- \rightarrow e^- e^-$  is defined by using the  $\mu^- e^- \rightarrow e^- e^-$  decay rate of muonic atom  $\Gamma(\mu^- e^- \rightarrow e^- e^-)$  ( $\Gamma$  given in Eq. (6)) and the total decay rate of muonic atom  $1/\tilde{\tau}_\mu$ :

$$Br(\mu^- e^- \rightarrow e^- e^-) \equiv \tilde{\tau}_\mu \Gamma(\mu^- e^- \rightarrow e^- e^-). \quad (35)$$

We estimate the strength of the CLFV interaction from the current upper limit of the branching ratio of  $\mu^+ \rightarrow e^+ e^+ e^-$ . The branching ratio  $Br(\mu^+ \rightarrow e^+ e^+ e^-)$  is given as

$$Br(\mu^+ \rightarrow e^+ e^+ e^-) \equiv \tau_\mu \Gamma(\mu^+ \rightarrow e^+ e^+ e^-). \quad (36)$$

Here  $\Gamma(\mu^+ \rightarrow e^+ e^+ e^-)$  and  $1/\tau_\mu$  are the decay rate of  $\mu^+ \rightarrow e^+ e^+ e^-$  and total decay rate of free muon, respectively.

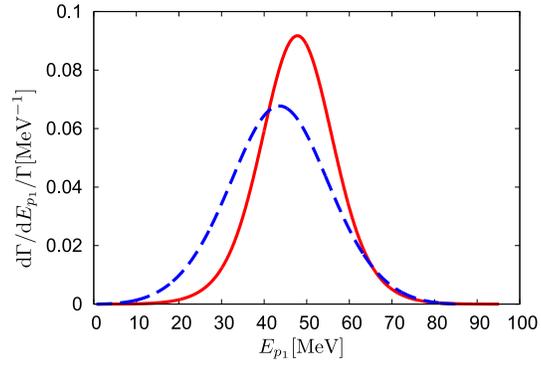


FIG. 4. The normalized energy distribution of electron for the  $^{208}\text{Pb}$ . The solid (red) curve and the dashed (blue) curve are obtained by using model IV and model I, respectively. The  $g_1$  term is included and all bound  $S$  state electrons are taken into account.

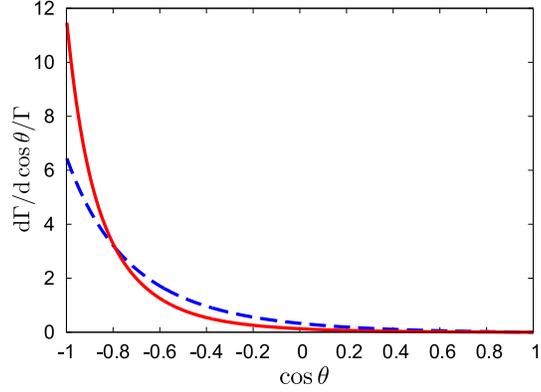


FIG. 5. The normalized angular distribution of emitted electrons for the  $^{208}\text{Pb}$ . The other features are the same as those in Fig. 4.

Using the contact CLFV interaction in Eq. (3), the branching ratio  $Br(\mu^+ \rightarrow e^+e^+e^-)$  is given as[16]

$$Br(\mu^+ \rightarrow e^+e^+e^-) = \frac{1}{8}(G_{12} + 16G_{34} + 8G_{56}). \quad (37)$$

Keeping only  $g_1$  term of CLFV interaction, we can express the branching ratio of  $\mu^-e^- \rightarrow e^-e^-$  decay of muonic

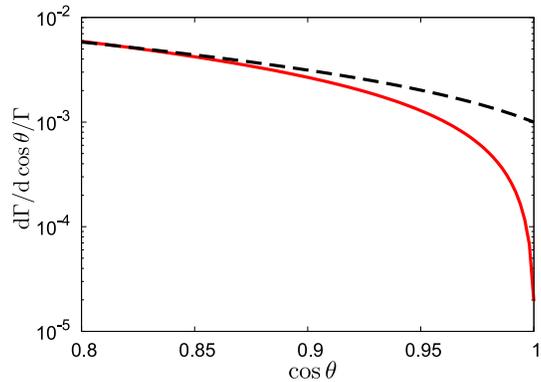


FIG. 6. The angular distribution of emitted electrons for the  $^{208}\text{Pb}$ . The angular distribution due to the  $(g_1-g_4)$  terms is shown in solid (red) curve, and that of  $(g_5-g_6)$  terms is shown in dashed (black) curve.

atom as[4]

$$Br(\mu^- e^- \rightarrow e^- e^-) \simeq 192\pi(Z-1)^3 \alpha^3 \left(\frac{m_e}{m_\mu}\right)^3 \frac{\tilde{\tau}_\mu \Gamma}{\tau_\mu \Gamma_0} Br(\mu^+ \rightarrow e^+ e^+ e^-). \quad (38)$$

Here we used  $\tau_\mu = 192\pi^3 / (G_F^2 m_\mu^5) = 2.197 \times 10^{-6}$  [s]. The upper limit of  $Br(\mu^- e^- \rightarrow e^- e^-)$  can be estimated by using the current upper limit of the branching ratio  $Br(\mu^+ \rightarrow e^+ e^+ e^-)$ .

The upper limits of branching ratio of the previous work(dashed curve) and our results with 1*S*(solid curve) and all *nS* electrons(dotted curve) are shown in FIG. 7. Here we used the result of the SINDRUM experiment  $Br(\mu^+ \rightarrow e^+ e^+ e^-) < 1.0 \times 10^{-12}$  [18] and the data of the lifetime of muonic atoms  $\tilde{\tau}_\mu$  given in [17]. For  $^{208}\text{Pb}$  ( $^{238}\text{U}$ ), the branching ratios  $Br(\mu^- e^- \rightarrow e^- e^-)$  considering only 1*S* electrons and all electrons are  $3.3 \times 10^{-18}$  ( $6.9 \times 10^{-18}$ ) and  $4.2 \times 10^{-18}$  ( $9.8 \times 10^{-18}$ ), respectively.  $Br(\mu^- e^- \rightarrow e^- e^-)$  reaches about  $10^{-17}$  for  $^{238}\text{U}$ .

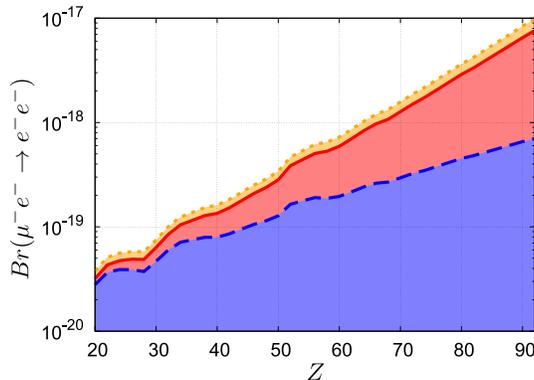


FIG. 7. Upper limits of  $Br(\mu^- e^- \rightarrow e^- e^-)$ . The dashed(blue) curve shows the result of previous work [4]. Our results including only 1*S* electrons and all 1*S* electrons are shown by the solid(red) curve and the dotted(orange) curve, respectively.

#### IV. CONCLUSION

We have made an improved study on the  $\mu^- e^- \rightarrow e^- e^-$  decay in muonic atoms. Coulomb interaction of leptons with finite nuclear charge distributions is taken into account by using the standard multipole expansion formalism and the numerical solutions of Dirac equations for both the electron and muon wave functions. The effects of Coulomb distortion of the emitted electron and relativistic treatments of the bound leptons are significantly important for quantitative estimations of the decay rate. Enhancements of the decay rates of about 9 and 14 times for  $^{208}\text{Pb}$  and  $^{238}\text{U}$  respectively compared with the previous analysis are obtained due to the enhanced overlap integrals of the lepton wave functions. We also found that different helicity structures of the CLFV interaction generate sizable difference in the  $Z$ -dependence of the decay rate and also the angular distribution of the emitted electrons. Finally, the upper limits of the branching ratio of the  $\mu^- e^- \rightarrow e^- e^-$  decay of muonic atom was estimated.

In this work we have included only the four Fermi CLFV interaction. It is important to estimate the photonic interaction which generates long range interactions between the bound muon and many electrons in an atom. In addition to the decay rates, it would be of great interest to find some other observables that may be useful to discriminate photonic and contact interactions and also various terms of the effective CLFV interactions. These issues are under progress and will be discussed in a separate paper.

#### ACKNOWLEDGMENTS

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