Leptonic CP phases near the $\mu - \tau$ symmetric limit

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Abstract

The neutrino masses and mixings indicated by current neutrino oscillation experiments suggest that the neutrino mass matrix possesses an approximate $\mu - \tau$ exchange symmetry. In this study, we explore the neutrino parameter space and show that if a small $\mu - \tau$ symmetry breaking is considered, the Majorana CP phases must be unequal and non-zero independently of the neutrino mass scale. Moreover, a small $\mu - \tau$ symmetry breaking favors quasi-degenerate masses. We also show that Majorana phases are strongly correlated with the Dirac CP violating phase. Within this framework, we obtain robust predictions for the values of the Majorana phases when the experimental indications for the Dirac CP phase are used.

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I. INTRODUCTION

Neutrino oscillation experiments performed in the last decades have provided remarkable information about neutrino mixing parameters. Global fits obtained with all three standard flavor neutrinos indicate that [1, 2], within the 3σ range, the mixing angles are given by solar $\sin^2\theta_{12} \equiv \sin^2\theta_\odot \approx 0.308^{+0.064}_{-0.055}$, atmospheric $\sin^2\theta_{23} \equiv \sin^2\theta_{ATM} \approx 0.437^{+0.189}_{-0.063}$ ($0.455^{+0.186}_{-0.075}$), and reactor $\sin^2\theta_{13} \approx 0.0234^{+0.0061}_{-0.0058}$ ($0.0240^{+0.0058}_{-0.0062}$), for the normal (inverted) hierarchy. The squared mass differences are also known with very high accuracy, where these are given by the so-called solar scale $\Delta m^2_{21} = m^2_2 - m^2_1 \equiv \Delta m^2_{sol} \approx 7.54 \times 10^{-5} eV^2$ and the atmospheric scale $|\Delta m^2_{31}| \sim \Delta m^2_{ATM} \sim 2.4 \times 10^{-3} eV^2$. However, the sign in $\Delta m^2_{31} = m^2_3 - m^2_1$, and thus the neutrino mass hierarchy pattern, is still unknown. A very recent global analysis of neutrino oscillation data was provided by [3].

Unlike the quark sector where the mixing angles are all small, the mixings measured in oscillation experiments are large, except for θ_{13}, which was found to be rather small but certainly non-zero. In the standard parameterization, the mixings are given by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [4, 5],

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - c_{23}s_{12}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix} \cdot K , \quad (1.1)$$

where c_{ij} and s_{ij} denote $\cos\theta_{ij}$ and $\sin\theta_{ij}$, respectively, and the mixing angles are given as θ_{12}, θ_{13}, and θ_{23}, where δ_{CP} is the Dirac CP-violating phase, whereas $K = \text{Diag}(e^{-i\beta_1/2}, e^{-i\beta_2/2}, 1)$ is a diagonal matrix containing two Majorana CP-violating phases, β_1 and β_2, which do not contribute to neutrino oscillations. In addition, although δ_{CP} has not been determined well, several suggestions from global fits [1–3, 6] support $\delta_{CP} \sim -\pi/2$. A number of ongoing and future oscillation neutrino experiments aim to determine the neutrino mass hierarchy and the Dirac CP-violating phase. Determining these parameters will be very important for identifying the flavor symmetry that underlies the pattern of lepton flavor mixing.

After considering a basis where the charged lepton masses and weak interactions are simultaneously diagonal, the labels associated with weak flavors, $\alpha, \beta = e, \mu, \tau$, are transparent and the PMNS matrix also becomes that which diagonalizes the neutrino mass terms, where they are generally given by the effective operator

$$\bar{\nu}_{\alpha L}(M_\nu)_{\alpha\beta}\nu^c_{\beta L} + h.c., \quad (1.2)$$

such that $U_{PMNS} = U_\nu \cdot K$. Therefore, the neutrino mass matrix can be written in terms of a diagonal (complex) mass matrix, $M_{diag} = \text{Diag}\{m_1 e^{-i\beta_1}, m_2 e^{-i\beta_2}, m_3\}$, simply as

$$M_\nu = U_\nu^* \cdot M_{diag} \cdot U_\nu^\dagger. \tag{1.3}$$

In the following, we employ this formulation and we denote $m_j \equiv |m_j| e^{-i\beta_j}$, for $j = 1, 2$.

It should be noted that while the observed θ_{13} is small but not zero, θ_{ATM} is close to its maximal value, $\pi/4$. Clearly, neither of the central values of these angles are the exact critical values ($\theta_{13} = 0$ and $\theta_{ATM} = \pi/4$), but it is intriguing to observe that it is possible to establish an approximate empirical relation regardless of the hierarchy,

$$1/2 - \sin^2 \theta_{ATM} \approx \sin \theta_{13}/\text{few} , \tag{1.4}$$

which suggests that the deviation of θ_{ATM} from its maximal value, $\Delta\theta = \pi/4 - |\theta_{ATM}|$, could be correlated to the deviation from zero present in θ_{13} [7]. In these terms, we note that the empirical relation given above can be simply rewritten as $\sin \Delta\theta \approx \sin \theta_{13}/\text{few}$. Theoretically, this may facilitate an understanding of the observed mixings as possibly having a common physical origin by highlighting a well-defined flavor symmetry. In fact, it is easy to see that null values of $\Delta\theta$ and θ_{13} do increase the symmetry of the neutrino sector. These values imply that the mixing matrix U_ν will take the bimaximal form

$$U_{BM} = \begin{pmatrix} \cos\varphi_{12} & \sin\varphi_{12} & 0 \\ \frac{-\sin\varphi_{12}}{\sqrt{2}} & \frac{\cos\varphi_{12}}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-\sin\varphi_{12}}{\sqrt{2}} & \frac{\cos\varphi_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{1.5}$$

where the only undefined angle is φ_{12}, which could be taken as the solar mixing. If Eq. (1.5) is combined with Eq. (1.3) and the mass matrix elements are defined as $m_{\alpha\beta}^0 = (M_\nu)_{\alpha\beta}$, then we obtain

$$m_{ee}^0 = m_1 \cos^2 \varphi_{12} + m_2 \sin^2 \varphi_{12},$$
$$m_{e\mu}^0 = m_{e\tau}^0 = \frac{\sin 2\varphi_{12}}{\sqrt{8}} (m_2 - m_1),$$
$$m_{\mu\tau}^0 = \frac{1}{2} \left(m_1 \sin^2 \varphi_{12} + m_2 \cos^2 \varphi_{12} - m_3 \right),$$
$$m_{\mu\mu}^0 = m_{\tau\tau}^0 = \frac{1}{2} \left(m_1 \sin^2 \varphi_{12} + m_2 \cos^2 \varphi_{12} + m_3 \right) . \tag{1.6}$$

The two general conditions given as $m_{e\mu}^0 = m_{e\tau}^0$ and $m_{\mu\mu}^0 = m_{\tau\tau}^0$ reduce the number of free mass parameters to four and give rise to the so-called $\mu - \tau$ symmetry [8]. As a consequence,

the observed values of θ_{13} and $\Delta\theta$ can be understood as a result of the breaking of $\mu - \tau$ symmetry. This issue has inspired many theoretical studies in recent years [7–10]. A very recent review of the status of the $\mu - \tau$ flavor symmetry was provided by [11].

However, there have been very few studies of the possible values for CP violating phases ($CPVP$) in this context. In particular, [12] provides different values for the Dirac phase δ_{CP} considering small deviations from the $\mu - \tau$ symmetry, although the Majorana phases were not considered. To the best of our knowledge, the conditions under which $\mu - \tau$ symmetry breaking is small have not been explored. This could be relevant because small symmetry breaking is considered as the starting point in this type of study. In this study, we provide a detailed analysis of the magnitude of the $\mu - \tau$ symmetry breaking using the current data for neutrino mixing parameters, and we identify the mass spectrum and $CPVP$ required to obtain small symmetry breaking.

The remainder of this paper is organized as follows. First, we parameterize the breaking of $\mu - \tau$ symmetry. Next, we use experimental results obtained for neutrino masses and mixings to explore the breaking parameter space but without assuming any special values for $CPVP$. We then show that relatively small parameters, and thus perturbative approximations, are still allowed by the data only in specific cases. In particular, we show that $\mu - \tau$ symmetry breaking can be small (less than 10%) when the neutrino masses are almost degenerate and for specific values of $CPVP$ that are strongly correlated. We also show that this correlation indicates a well-defined region in the parameter space for $CPVP$, which is a testable feature for models near the $\mu - \tau$ symmetric limit. Finally, we present our conclusions.

II. $\mu - \tau$ SYMMETRY BREAKING PARAMETERS

As mentioned above, experimental results show that θ_{13} and $\Delta\theta$ are non-zero, and thus the $\mu - \tau$ symmetry is broken. However, this breaking can actually be small and in some cases, the $\mu - \tau$ symmetry may be considered as an approximate symmetry. In fact, [13] found that very small symmetry breaking can be possible in the quasi-degenerate hierarchy. In this study, we consider the breaking of $\mu - \tau$ to identify the hierarchy, as well as extracting some useful information about $CPVP$, which supports small symmetry breaking. In general, any generic neutrino mass matrix can always be parameterized in terms of a symmetric part plus a correction that explicitly breaks the symmetry by $M_\nu = M_{\mu-\tau} + \delta M$, where $M_{\mu-\tau}$

4

has a $\mu - \tau$ symmetry and δM is defined by only two non-zero elements,

$$\delta M = \begin{pmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \\ \delta & 0 & \epsilon \end{pmatrix}, \tag{2.1}$$

where the breaking parameters are clearly defined as $\delta = m_{e\tau} - m_{e\mu}$ and $\epsilon = m_{\tau\tau} - m_{\mu\mu}$. Of course, there is no a priori reason why these parameters should be small compared with $M_{\mu-\tau}$ mass elements. In order to deal with dimensionless parameters, we define

$$\hat{\delta} \equiv \frac{\delta}{m_{e\mu}} = \frac{\sum_i (U_{ei}^* U_{\tau i}^* - U_{ei}^* U_{\mu i}^*) m_i}{\sum_i U_{ei}^* U_{\mu i}^* m_i},$$

$$\hat{\epsilon} \equiv \frac{\epsilon}{m_{\mu\mu}} = \frac{\sum_i (U_{\tau i}^* U_{\tau i}^* - U_{\mu i}^* U_{\mu i}^*) m_i}{\sum_i U_{\mu i}^* U_{\mu i}^* m_i}, \tag{2.2}$$

where the right-hand-sides are written according to Eq. (1.3). As usual, we take $i = 1, 2, 3$. The latter expressions give the complex parameters, $\hat{\delta}$ and $\hat{\epsilon}$, in terms of three observed mixing angles, three absolute masses, $|m_{1,2,3}|$, and three $CPVP$.

For the purposes of calculation, we rewrite the absolute masses in terms of the two observed mass squared differences involved in neutrino oscillations, Δm_{sol}^2 and Δm_{ATM}^2, and the lightest absolute neutrino mass, i.e., m_0, as

$$|m_2| = \sqrt{m_0^2 + \Delta m_{sol}^2} \text{ and } |m_3| = \sqrt{m_0^2 + |\Delta m_{ATM}^2|} \text{ for NH.} \tag{2.3}$$

$$|m_1| = \sqrt{m_0^2 + |\Delta m_{ATM}^2|} \text{ and } |m_2| = \sqrt{m_0^2 + |\Delta m_{ATM}^2| + \Delta m_{sol}^2} \text{ for IH.}$$

Note that m_0 becomes $|m_1|$ for the normal mass hierarchy (NH) and m_3 for the inverted mass hierarchy (IH). Thus, the breaking parameters, $\hat{\delta}$ and $\hat{\epsilon}$, depend on nine observables: three mixing angles, two mass squared differences, the lightest neutrino mass, and three $CPVP$, where neutrino oscillation experiments have already provided accurate values for the first five, so only the last four are unknown. Hence, in practical terms, the breaking parameter space $\hat{\delta} - \hat{\epsilon}$ that can accommodate neutrino mixings and oscillation mass scales within experimental uncertainties should in principle remain undetermined due to the arbitrariness of the m_0 and CP phases. However, we suggest that by assuming the smallness of $|\hat{\delta}|$ and $|\hat{\epsilon}|$, the resulting bounded parameter space will not be consistent with any arbitrary values of the neutrino observables, but instead it will yield specific predictions for the lightest neutrino mass and CP phases. We elaborate on this idea in the following.

We perform a general scan of the absolute values of $\hat{\delta}$ and $\hat{\epsilon}$ using the data obtained from oscillation experiments, where we allow m_0 and $CPVP$ to vary within $(0 - 0.4)$ eV

and $(0 - 2\pi)$, respectively. However, before presenting our numerical results, we perform an approximate analysis of the expressions in Eq. (2.2) in order to obtain some insights into the expected conditions required for small symmetry breaking. These expressions can be written in a suitable form as

$$
\hat{\delta} = \frac{y_- f s_{13} - y_+}{1 + f \ s_{13} \tan \ \theta_{23}} \ ,
$$
$$
\hat{\epsilon} = \frac{g \cos \ 2\theta_{23} - s_{13} h}{1 + g s_{23}^2 + s_{13} h/2} \ ,
$$

(2.4)

where

$$
y_\pm = \frac{c_{23} \pm s_{23}}{c_{23}},
$$
$$
f = \frac{(c_{12}^2 |m_1| e^{-i\beta_1} + s_{12}^2 |m_2| e^{-i\beta_2}) e^{-i\delta_{CP}} - m_3 e^{i\delta_{CP}}}{c_{12} s_{12} (|m_1| e^{-i\beta_1} - |m_2| e^{-i\beta_2})} \ ,
$$
$$
g = \frac{(c_{12}^2 s_{13}^2 - s_{12}^2)|m_1| e^{-i\beta_1} + (s_{12}^2 s_{13}^2 - c_{12}^2)|m_2| e^{-i\beta_2} + m_3 c_{13}^2}{s_{12}^2 |m_1| e^{-i\beta_1} + c_{12}^2 |m_2| e^{-i\beta_2}} \ ,
$$
$$
h = \frac{(|m_1| e^{-i\beta_1} - |m_2| e^{-i\beta_2}) \sin \ 2\theta_{23} \ \sin \ 2\theta_{12} e^{-i\delta_{CP}}}{s_{12}^2 |m_1| e^{-i\beta_1} + c_{12}^2 |m_2| e^{-i\beta_2}} \ .
$$

(2.5)

As shown by these expressions, $\hat{\delta}$ and $\hat{\epsilon}$ in Eq. (2.4) become zero when $\theta_{13} = 0$ and $\theta_{23} = -\pi/4$, as expected from the exact $\mu - \tau$ symmetry. Next, let us analyze the conditions for $|\hat{\delta}|, |\hat{\epsilon}| \ll 1$. According to the three possible approaches given by NH and IH, the almost degenerate limit where all neutrino masses are about the same order, and using the central values for the current mixing parameters [1], we have the following.

- For NH, $|m_1| \ll |m_2| \approx \sqrt{\Delta m_{sol}^2} \ll m_3 \approx \sqrt{\Delta m_{ATM}^2}$, and thus

$$
f \approx \frac{e^{i(\delta_{CP} + \beta_2)}}{s_{12} \ c_{12}} \sqrt{\frac{\Delta m_{ATM}^2}{\Delta m_{sol}^2}} \left(1 - \mathcal{O}\left(\frac{\Delta m_{sol}^2}{\Delta m_{ATM}^2}\right)\right) \ , \ |f| \sim 12.5 \ ,
$$

(2.6)

  which implies that $|\hat{\delta}| \sim 3.26$. Thus, in NH, the breaking of $\mu - \tau$ symmetry by $\hat{\delta}$ is always large, and thus there is no need to examine $\hat{\epsilon}$. We note that this conclusion is independent of the values of $CPVP$. These results are confirmed by the numerical analysis presented below.

- For IH, we have $|m_1| \approx \sqrt{\Delta m_{ATM}^2}$, $|m_2| \approx \sqrt{\Delta m_{sol}^2 + \Delta m_{ATM}^2} \gg m_3$, which gives

$$
f \approx \frac{-e^{-i\delta_{CP}}(c_{12}^2 e^{-i(\beta_1 - \beta_2)} + s_{12}^2 + \frac{s_{12}^2}{2} \frac{\Delta m_{sol}^2}{\Delta m_{ATM}^2})}{s_{12} \ c_{12} \left(1 - e^{-i(\beta_1 - \beta_2)} + \frac{1}{2} \frac{\Delta m_{sol}^2}{\Delta m_{ATM}^2}\right)} \ .
$$

(2.7)

In this case, we can see that $|f|$ is very large when $\beta_1 - \beta_2 = 0$, whereas $|f| \sim 1$ when $\beta_1 - \beta_2 = \pm\pi$, and thus $|\hat{\delta}| \sim 0.1$, which is desirable. In addition, $\hat{\epsilon}$ is given in terms of g and h, as in Eq. (2.4), which can now be approximated as

$$g \approx \frac{(c_{12}^2 s_{13}^2 - s_{12}^2)e^{-i(\beta_1-\beta_2)} + (s_{12}^2 s_{13}^2 - c_{12}^2)}{s_{12}^2 e^{-i(\beta_1-\beta_2)} + c_{12}^2} \ , \qquad (2.8)$$

$$h \approx \frac{(e^{-i(\beta_1-\beta_2)} - 1)\sin 2\theta_{23} \ \sin 2\theta_{12} e^{-i\delta_{CP}}}{s_{12}^2 e^{-i(\beta_1-\beta_2)} + c_{12}^2} \ . \qquad (2.9)$$

From the expressions above, we can see that $\beta_1 - \beta_2 = \pm\pi$ gives $g \approx -1$ and $h \approx 4e^{-i\delta_{CP}}$, from which we obtain $|\hat{\epsilon}| \sim 0.6$. Hence, the largest contribution to $\mu - \tau$ breaking in this case comes from $|\hat{\epsilon}|$ rather than $|\hat{\delta}|$.

- Finally, in the degenerate hierarchy (DH) limit, where $|m_1| \approx |m_2| \approx m_3$, we obtain

$$f \approx \frac{-e^{-i\delta_{CP}}(c_{12}^2 e^{-i(\beta_1-\beta_2)} + s_{12}^2) - e^{i\delta_{CP}} e^{i\beta_1}}{s_{12} \ c_{12}(1 - e^{-i(\beta_1-\beta_2)})} \ . \qquad (2.10)$$

$$g \approx \frac{(c_{12}^2 s_{13}^2 - s_{12}^2)e^{-i(\beta_1-\beta_2)} + (s_{12}^2 s_{13}^2 - c_{12}^2) + c_{13}^2 e^{-i\beta_1}}{s_{12}^2 e^{-i(\beta_1-\beta_2)} + c_{12}^2} \ , \qquad (2.11)$$

$$h \approx \frac{(e^{-i(\beta_1-\beta_2)} - 1)\sin 2\theta_{23} \ \sin 2\theta_{12} e^{-i\delta_{CP}}}{s_{12}^2 e^{-i(\beta_1-\beta_2)} + c_{12}^2} \ . \qquad (2.12)$$

From these equations, we again note that $|f|$ is strongly enhanced for $\beta_1 - \beta_2 = 0$, which implies that $|\hat{\delta}| > 1$, e.g., for $\beta_1 = \beta_2 = \pi$ and $\delta_{CP} = -\pi/2$, we obtain $|f| \sim 68$, which leads to $|\hat{\delta}| \sim 2$. However, other specific values of $CPVP$ may lead to small breaking parameters, e.g., for $\beta_1 = \pi$, $\beta_2 = \pi/2$ and $\delta_{CP} = -\pi/2$, we obtain $|f| \sim 0.6$, $|g| \sim 2$, and $|h| \sim 1.7$, such that $|\hat{\delta}| \sim 0.2$ and $|\hat{\epsilon}| \sim 0.1$.

Based on the previous analysis, we can conclude that the case of equal Majorana phases is strongly disfavored in any hierarchy. In addition, we can easily determine that the symmetry is strongly broken for the NH. This is consistent with the previous results presented by [7], where only CP conserving situations were analyzed. It is also consistent with the numerical analysis presented in the following.

In Fig. 1, we show the allowed region for the symmetry breaking parameters obtained by varying the Dirac phase, δ_{CP}, and Majorana phases, β_1 and β_2, within the $(0 - 2\pi)$ interval, but taking the mixing parameters within a $3\,\sigma$ level. We depict the maximum value between $|\hat{\delta}|$ and $|\hat{\epsilon}|$, i.e., $\text{Max}[|\hat{\delta}|, |\hat{\epsilon}|]$, as a function of the lightest neutrino mass. According to this
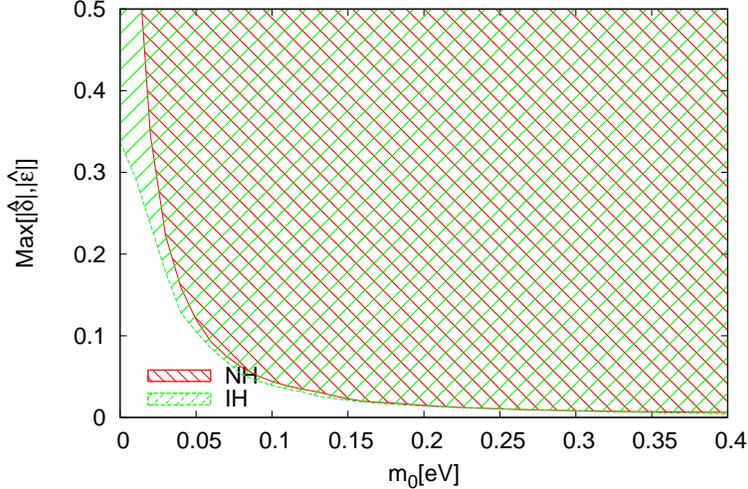
FIG. 1. Allowed region (hashed area) based on the experimental data for Max[$|\hat{\delta}|$, $|\hat{\epsilon}|$], and the maximum between $|\hat{\delta}|$ and $|\hat{\epsilon}|$ as a function of the lightest neutrino mass, m_0, for NH and IH. The regions are obtained by varying δ_{CP} and $\beta_{1,2}$ within the interval $[0, 2\pi]$.

figure, we can easily see that for $m_0 \sim 0$ eV in the NH, the breaking parameters are very large regardless of the $CPVP$ values, as found in the previous analysis. However, for $m_0 \sim 0$ eV in the IH, the breaking is at least of 30%. Finally, and more interestingly, we can see that for $m_0 \gtrsim 0.05$ eV, which corresponds to the almost DH, the breaking can be less than 10%. This indicates that DH is the best hierarchy for regarding $\mu - \tau$ as a good approximate symmetry, which is consistent with the results obtained by [13] using a different approach.

However, it should be noted that even larger values for the breaking parameters are possible in the DH case, which is clear from the figure. Imposing the phenomenological requirement of a very small amount of breaking, as suggested by the theoretical indication of a perturbative origin, has the effect of cutting off all the parameter space above Max[$|\hat{\delta}|, |\hat{\epsilon}|$] > 0.1, or any other selected small value. Of course, this can be achieved by excluding the values for $CPVP$ that are consistent only with larger breaking of the symmetry. Therefore, we may conclude that a small $\mu - \tau$ symmetry breaking implies more specific values for the CP phases, as discussed in the following.
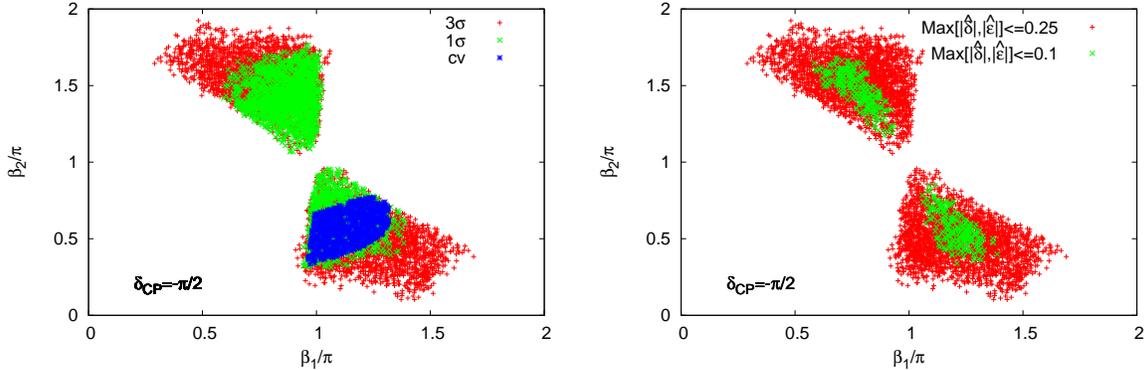
FIG. 2. Allowed regions for the two Majorana phases for $\delta_{CP} = -\pi/2$. In the left-hand plot, the green color (cross signs) corresponds to the mixing parameters varied up to $1\,\sigma$ error, the red color (plus signs) corresponds to the $3\,\sigma$ interval, and the blue color (stars) corresponds to the central values (cv) of the mixing parameters, which match with Max$[|\hat{\delta}|,\ |\hat{\epsilon}|] \leq 0.25$. In the right-hand plot, the green color (cross signs) corresponds to Max$[|\hat{\delta}|,\ |\hat{\epsilon}|] \leq 0.1$ and the red color (plus signs) corresponds to Max$[|\hat{\delta}|,\ |\hat{\epsilon}|] \leq 0.25$, where the experimental values are varied within $3\,\sigma$.

III. MAJORANA PHASES AND $\mu - \tau$ SYMMETRY

In the previous section, we showed that the current experimental data indicate that the requirement for small symmetry breaking favors the quasi-degenerate mass hierarchy, i.e., $m_0 \gtrsim 0.05$ eV. Accordingly, we take $m_0 = 0.1$ eV in the following. However, we note that our results do not change significantly if we consider other values for m_0 within that range. This can be understood as a consequence of the fact that Eqs. (2.10) to (2.12), which ultimately define the breaking parameters (see Eq. 2.4), have no explicit dependence on the absolute neutrino mass scale in this limit. Nevertheless, we use the exact expressions in (2.4) and (2.5) for our numerical analysis, thereby avoiding any further approximation.

After selecting m_0, $\hat{\delta}$ and $\hat{\epsilon}$ only depend on 3 $CPVP$. Thus, by selecting a suitable value of the Dirac phase, δ_{CP}, we can obtain some information about the two remaining Majorana phases that fulfill the requirement for small symmetry breaking. In summary, any two given values for the phases will provide a unique set of absolute breaking parameters, and vice versa. Therefore, only a well-defined allowed region in the CP phase space will be consistent with our symmetry breaking requirement. The results of our numerical analysis show this
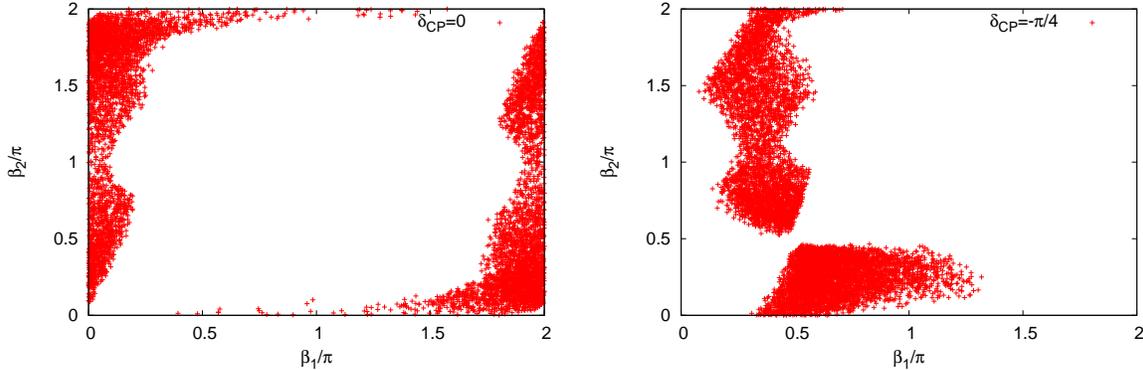
FIG. 3. 2σ allowed regions for the Majorana phases, β_1 and β_2, for $\text{Max}[|\hat{\delta}|, |\hat{\epsilon}|] \leq 0.25$. The left-hand plot corresponds to $\delta_{CP} = 0$ and the right-hand plot to $\delta_{CP} = -\pi/4$.

and they are depicted in Fig. 2, where the left-hand side shows the allowed region for the two Majorana phases, β_1 and β_2, which match with the condition that $\text{Max}[|\hat{\delta}|, |\hat{\epsilon}|] \leq 0.25$. To explain this plot, we select $\delta_{CP} = -\pi/2$, which is near to the best fit value reported by [1–3, 6]. The other mixing parameters are varied up to $1\,\sigma$ and $3\,\sigma$ from their central values, respectively. The corresponding regions are indicated in the figure. We can easily see that there is a strong correlation between the two Majorana phases, which can be inferred from the expressions in Eqs. (2.10 –2.12), where the relative phase $\beta_1 - \beta_2$ appears. This figure shows that the values of $\beta_1 = \pi$ and $\beta_2 = \pi/2$ discussed in the previous section, which lead to $\text{Max}[|\hat{\delta}|, |\hat{\epsilon}|] \approx 0.2$, are contained well within the allowed regions. An interesting outcome that we want to stress is that null Majorana phases are excluded completely at the 3σ level. Furthermore, as shown by the same plot, the allowed region is not modified greatly when we move the experimental values from 1σ to 3σ deviation limits. Thus, the correlation is only slightly sensitive to small mixing angle variations. By contrast, as shown in the right-hand plot in Fig. 2, when we restrict the upper bound condition even more on $\text{Max}[|\hat{\delta}|, |\hat{\epsilon}|]$, then the allowed region for β_1 and β_2 is reduced greatly. Therefore, in this framework, it is possible to obtain some indications about the Majorana phases, as well as the pattern of masses, according to the analysis in the last section. We consider that this would be useful for future research based on an approximate $\mu - \tau$ symmetry.

Finally, in order to study the dependence of our results on the value selected for the Dirac phase, we consider different values in our analysis, thereby demonstraing that there

is a strong correlation between the allowed Majorana phase space and the Dirac CP phase. Two distinctive output examples are presented in Fig. 3, where the values of β_1 and β_2 are shown that match the conditions $\text{Max}[|\hat{\delta}|, |\hat{\epsilon}|] \leq 0.25$, for $\delta_{CP} = 0$ and $\delta_{CP} = -\pi/4$ (left- and right-hand sides, respectively). From Figs. 2 and 3, we can seen that in any of these cases, equal Majorana phases are totally disfavored for small breaking, as discussed in Sec. II. In particular, simultaneously zero Majorana phases remain excluded in any of the cases considered. In addition, we note that the two Majorana phases are sensitive to variations in the Dirac CP phase. This is expected from Eqs. (2.10-2.12), where δ_{CP} enters as a global phase factor that cannot cancel out. Nonetheless, the regions associated with different values of the Dirac CP phase are completely different, which supports an interesting conclusion within the small $\mu - \tau$ breaking hypothesis, so in the near future, determining the Dirac CP phase will provide specific predictions for the allowed values of the Majorana phases, thereby making this hypothesis testable.

IV.   CONCLUDING REMARKS

In this study, we provided a complete analysis of the two dimensionless parameters $\hat{\delta}$ and $\hat{\epsilon}$, which encode $\mu - \tau$ symmetry breaking. By taking neutrino oscillation mixings and mass scales according to the current experimental data, we showed that the breaking parameters depend only on four free variables: the lightest neutrino mass, m_0, the Dirac CP phase, δ_{CP}, and the two Majorana phases. First, we studied these parameters in an analytical manner under the three approximations given by the hierarchies. We found that equal Majorana phases lead to strong breaking of $\mu - \tau$ symmetry regardless of the hierarchy. We also verified the result reported by [13], thereby indicating that small symmetry breaking does prefer a quasi-degenerate mass hierarchy. Next, by allowing all three $CPVP$ to vary within the $[0 - 2\pi]$ interval, we numerically studied the absolute values of both the $\hat{\delta}$, and $\hat{\epsilon}$ parameters in order to quantify the breaking of $\mu - \tau$ as a function of m_0. Based on this analysis, we found that for $m_0 \lesssim 0.02$ eV in the NH, the $\mu - \tau$ symmetry will be broken by more than a 50%. By contrast, in the case of IH, we found that for $m_0 = 0$ eV, the breaking is always larger than a 30%. However, this limit decreases when m_0 increases regardless of the hierarchy. Clearly, this indicates that in only the almost degenerate neutrino masses limit, when $m_0 \gtrsim 0.05$ eV, we can obtain breaking as small as 10%. Therefore, we may conclude

that although current neutrino data are consistent with an approximate $\mu - \tau$ symmetry, the very near to symmetry limit prefers the quasi-degenerate active neutrino spectrum.

Based on the above, we restricted the analysis to the large absolute neutrino mass range, i.e., for $m_0 \gtrsim 0.1$ eV, and we then focused only on the parameter space where $|\hat{\delta}|$, $|\hat{\epsilon}|$ are effectively small to explore the allowed CP phases. Remarkably, our analysis showed that the smallness of the breaking parameters is fairly independent of the specific value taken by m_0, but their size is strongly governed by the $CPVP$. The latter appear to be strongly correlated among themselves up to the point that a given value of the Dirac phase should predict a well-bounded allowed parameter space for the Majorana phases. The allowed values for the latter are slightly sensitive to the variations in oscillation mixings and mass scales up to 3σ level, but as expected, they are highly dependent on the allowed maximum values for the breaking parameters.

Finally, we can make two important conclusions based on our analysis. First, if we regard $\mu - \tau$ as a good approximate symmetry, then the Majorana phases will definitely be non-zero. Second, a future determination of the Dirac phase will greatly restrict the Majorana phases that may be compatible with the condition of a weak breaking of $\mu - \tau$. It is interesting that a recent global analysis provided by [3] confirmed the previous intriguing preference for negative values of $\sin(\delta_{CP})$ with a best fit for δ_{CP} near $-\pi/2$, which makes the robust prediction of the Majorana phases obtained for this value more substantial. Experimental searches of the Dirac phase, neutrinoless double beta decay, and determinations of the mixing angles with improved precision would provide useful information to support or exclude the scenario for small $\mu - \tau$ symmetry breaking.

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