

Hamiltonian YM 2+1: note on point splitting regularization

Hermann Schulz ¹

*Institut für Theoretische Physik, Leibniz Universität Hannover,
Appelstr. 2, D-30167 Hannover, Germany*

Abstract

The Hamiltonian of 2+1 dimensional Yang Mills theory was derived by Karabali, Kim and Nair by using point splitting regularization. But in calculating e.g. the vacuum wave functional this scheme was left in favour of arguments. Here we follow up a conjecture of Leigh, Minic and Yelnikov of how this gap might be filled by including all positive powers of the regularization parameter ($\epsilon \rightarrow +0$). Admittedly, though we concentrate on the ground state in the large N limit, only two such powers could be included due to the increasing complexity of the task.

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1 The Problem

This report is highly technical, treats a very special topic, can not even reach the final answer to and is therefore not considered for publication. Detailed Introductions are found in [1], [3], [2], [4], [5]. Let us jump over them here. Some are lengthy (e. g. 40 pages in [5]).

According to Karabali, Kim and Nair [1] the Hamiltonian can be prepared to act in the space of wave functionals depending on only the currents j^a . It reads

$$T + V = m \int \omega_r^a \delta_{j_r^a} + m \int \int' \Omega_{rr'}^{ab} \delta_{j_r^a} \delta_{j_{r'}^b} + V \quad , \quad V = \frac{N}{m\pi} \int (\bar{\partial} j_r^a) \bar{\partial} j_r^a \quad . \quad (1.1)$$

¹he.schulz38@web.de

Here a is the color index running from 1 to $n := N^2 - 1$, $m = e^2 N / 2\pi$ (e the coupling), and \int is shorthand for $\int d^2 r$ running over the space of YM 2+1. Also $\int' = \int d^2 r'$, $\int'' = \int d^2 r''$ etc. An Index r on a quantity means that it is a function of $\vec{r} = (x, y)$. Through $x - iy = z$, $x + iy = \bar{z}$ we have $r^2 = z\bar{z}$ and may define $\partial := d/dz$ and $\bar{\partial} := d/d\bar{z}$. By $:=$ or $=:$ the object near to the colon is defined.

In strict temporal gauge there are only $2 * n$ real gauge flds A_1^a , A_2^a , combined to antihermitean and traceless $N \times N$ fields by $A_j = -iA_j^a T^a$ or even to $A = (A_1 + iA_2)/2$. T^a are the n traceless generators of $SU(N)$: $[T^a, T^b] = if^{abc}T^c$, $\text{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$, $T^a T^a = n/(2N)$. Matrices $M \in SL(N, \mathbb{C})$ parametrize the gauge fields by

$$A = -(\partial M)M^{-1}, \quad A = -iT^a A^a, \quad A^a = i2\text{Tr}(T^a A), \quad A^{ab} = -f^{abc}A^c. \quad (1.2)$$

The above third relation rests on $2\text{Tr}(T^a T^b) = \delta^{ab}$. But it is also obtained by using $M^{ab} = 2\text{Tr}(T^a M T^b M^{-1})$ in $[(\partial M)M^{-1}]^{ab}$. The single-indexed fields A^a specify the functional derivatives in (2.3) below.

The arguments of the functional derivatives in (1.1) are the currents

$$j^a = 2\text{Tr}(T^a j) \quad \text{with} \quad j = (\partial H)H^{-1} = T^a j^a, \quad j^{ab} = -if^{abc}j^c, \quad j^a = \frac{i}{N}f^{abc}j^{bc}, \quad (1.3)$$

of the WZW model, where $H = M^\dagger M$ is a gauge invariant. The last relation in (1.3) derives via $H^{ab} = 2\text{Tr}(T^a H T^b H^{-1})$. Again a superscript r stands for \vec{r} but mostly H_r is considered as depending on z and \bar{z} : $H_r = H_{z\bar{z}}$, $H_{r'} = H_{z'\bar{z}'}$ etc. Round brackets are used to stop the action of a differentiation at the right bracket. To distinguish fundamental from adjoint traces we shall write Tr for the first and $(\dots)^{aa}$ for the latter. Under an adjoint trace inner color-index-pairs are often omitted as e.g. in $(j k \ell)^{aa} := j^{ab}k^{bc}\ell^{ca}$.

As long as the regularization parameter ϵ is kept non-zero positive the functional Schrödinger equation $(T + V)\psi[j_r^a] = E\psi[j_r^a]$ is fully regularized. There are no singularities (“no field theory” – apart from the continuum of variables). At first the solution ψ is to be obtained (ψ depending on ϵ , of course). The limit $\epsilon \rightarrow +0$ is allowed only afterwards. The expressions ω_r^a and $\Omega_{r'r'}^{ab}$ in (1.1) can be booked down in closed form (section 2) and can be fully expanded as power series in ϵ with all coefficients finite (section 3).

Former treatments peromed $\epsilon \rightarrow +0$ too early, so they either needed arguments or/and normal ordering or/and contact to known limiting cases. Fine. Note that, thereby, all these authors accepted a certain break in the consequent working through the Karabali–Nair setup. The reason clearly is in the overwhelming complications otherwise. At least Leigh, Minic and Yelnikov [2] made an attempt in their appendix A (*Regulated Computations*). They could not reach the expected result. However they raised a conjecture about, what would have to be done. In the present report we follow it up and can justify their idea. Admittedly the desired final result will be not reached here as well – though by quite different reasons.

Adopting the ansatz $\psi = e^P$ for the wave functional, with H from (1.1) and with $e^{-P}\delta_{ja}e^P = P'^a + \delta_{ja}$ in mind ($P'^a := \delta_{ja}P$) the Schrödinger equation turns into

$$m \int \omega_r^a P_r'^a + m \int \int' \Omega_{r'r'}^{ab} (\delta_{j_r^a} P_r'^a + P_r'^a P_r'^b) = E - V. \quad (1.4)$$

Neither ω nor Ω depend on m (see section 2). By organizing P in powers of $1/m^2$, i.e. $P = P_0 + P_1 + P_2 + P_3 + \dots$ with $P_n \sim 1/m^{2n}$, (1.4) decomposes in recursive equations [1] for P_n . In this note we concentrate on the leading nontrivial term P_1 towards large N , i.e. $m \rightarrow \infty$ (the constant P_0 may be set equal to zero). For this task the term quadratic in P in (1.4) may be deleted since $\sim 1/m^4$. The now linear equation for P_1 can simply be read off from (1.4). But in booking it down, let us multiply the P_1 -equation by $\pi m/N$ and rescale P_1 , E and V to reach a more convenient form:

$$\mathcal{T} \mathcal{P} = \mathcal{E} - \mathcal{V} \quad , \quad \text{where} \quad (1.5)$$

$$\mathcal{T} = \frac{T}{m} = \int \omega_r^a \delta_{j_r^a} + \iint' \Omega_{rr'}^{ab} \delta_{j_r^a} \delta_{j_{r'}^b} \quad \text{and} \quad (1.6)$$

$$\mathcal{P} = \frac{\pi m^2 P_1}{N} \quad , \quad \mathcal{E} = \frac{\pi m E}{N} \quad , \quad \mathcal{V} = \frac{\pi m V}{N} \int (\bar{\partial} j^a) (\bar{\partial} j^a) \quad . \quad (1.7)$$

All the experts [1] to [4] agree, that the solution to (1.5) at $\epsilon \rightarrow +0$ is given by

$$\mathcal{P} = -\frac{1}{2} \mathcal{V} = -\frac{1}{2} \int (\bar{\partial} j_r^a) \bar{\partial} j_r^a \quad \text{or} \quad P_1 = -\frac{\pi}{2Nm^2} \int (\bar{\partial} J_r^a) \bar{\partial} J_r^a \quad (1.8)$$

where P_1 is rewritten in terms of the original currents $J^a = Nj^a/\pi$ in [1] and [2] (but j^a is $-J^a$ in [4]).

For solving $\mathcal{T} \mathcal{P} = \mathcal{E} - \mathcal{V}$ we will first observe that \mathcal{T} can be written as a power series in ϵ . If the unknown \mathcal{P} is written as a power series aswell, a coefficient comparison yields to a set of equations, one for each given ϵ -power. We emphasize that **all** positive powers of ϵ are included in Sections 2 and 3. Also Section 4 does so – only the list (4.10), (4.11) of invariants remains too poor there.

To learn about the solution step by step, one can break apart at a given ϵ^n : “step n ” neglects powers higher than n . Each step ends up with a system of equations for the pefactors of the holomorphic invariants involved. Then $\epsilon \rightarrow 0$ always leads to $\mathcal{P} = c_0 \mathcal{V}$. As will be seen in step 2 the result for c_0 varies under change of the highest included power n . Ultimately $n \rightarrow \infty$ is required – a strong support of the conjecture in [2].

The solutions to step 1 and step 2 are obtained in the sequel. Also step 3 is attacked but will remain incomplete. This is all we were able to really work through.

2 Closed expressions for the kernels ω and Ω in (1.6)

To derive (rederive in essence) these kernels we go a few steps back in the Karabali–Kim–Nair analysis [1]. Thereby special attention is paid to maintain all ϵ -dependence. The regularised kinetic energy $\mathcal{T} := T/m$ originally reads

$$\mathcal{T} = \frac{\pi}{N} \int \int' \int'' H_r^{ab} \bar{\mathcal{G}}_{rr'}^{au} \bar{p}_r^u \mathcal{G}_{rr''}^{bv} p_{r''}^v \quad . \quad (2.1)$$

Here the regularized Green functions are

$$\begin{aligned}\overline{\mathcal{G}}_{rr'}^{au} &= \overline{G}_{rr'} \left(\delta^{au} - e^{-(\vec{r}-\vec{r}')^2/\epsilon} (H_{z\vec{z}'} H_{z'\vec{z}'}^{-1})^{au} \right) , \\ \mathcal{G}_{rr''}^{bv} &= G_{rr''} \left(\delta^{bv} - e^{-(\vec{r}-\vec{r}'')^2/\epsilon} (H_{z''\vec{z}''}^{-1} H_{z''\vec{z}''})^{bv} \right) ,\end{aligned}\quad (2.2)$$

and the operators \overline{p} , p are linear in functional differentiations with respect to A^{a*} and A^a , respectively:

$$\overline{p}_r^u = -\overline{\partial}' (M_r^\dagger)^{ud} \delta_{A^{d*}(\vec{r}')} , \quad p_{r''}^v = \partial'' (M_{r''}^{-1})^{vc} \delta_{A^c(\vec{r}'')} , \quad (2.3)$$

see (1.3). The prefactors \overline{G} and G in (2.2) are the bare Green functions

$$\overline{G}_{rr'} = \frac{1}{\pi} \frac{\vec{z} - \vec{z}'}{(\vec{r} - \vec{r}')^2 + \epsilon^2} , \quad G_{rr''} = \frac{1}{\pi} \frac{z - z''}{(\vec{r} - \vec{r}'')^2 + \epsilon^2} , \quad (2.4)$$

Since the singularity, which is controlled by ϵ here, is compensated by the round brackets in (2.2) (ϵ positive. Note that $\epsilon \neq \epsilon$), one may perform $\epsilon \rightarrow +0$ there. But otherwise it may happen that the limit $\epsilon \rightarrow +0$ must wait at least until $\overline{\partial} \overline{G}_{rr'} = \delta(\vec{r} - \vec{r}')$ or $\partial G_{rr''} = \delta(\vec{r} - \vec{r}'')$ are reached.²

Since the kinetic energy \mathcal{T} operates in the space of $\psi [j_r^a]$ it can be further reformulated. When applying (2.1)

$$p_{r''}^v \psi = i \partial'' H_{r''}^{ev} \delta_{j_{r''}^e} \psi \quad (2.5)$$

is needed first. Appendix A starts with a short derivation of (2.5). Hence over this ψ -space (2.1) becomes

$$\mathcal{T} = \frac{\pi}{N} \int \int' \int'' H_r^{ab} \overline{\mathcal{G}}_{rr'}^{au} \overline{p}_r^u \mathcal{G}_{rr''}^{bv} i \partial'' H_{r''}^{ev} \delta_{j_{r''}^e} . \quad (2.6)$$

\overline{p}_r^u does *not* commute with $\mathcal{G}_{rr''}^{bv}$, see (3.8) below. But note that even positive powers of ϵ are under study here. Hence by the chain rule the functional derivative in \overline{p}_r^u , (2.3) splits \mathcal{T} into three terms (acting only on \mathcal{G}^{bv} , only on H^{ev} and only at the right end (i.e. on $\delta_{j_{r''}^e} \psi$), respectively.

The first two terms are linear in δ , the third term is quadratic. Hence, with the notation of (1.6), there will be *two* contributions to ω_r^a – and hence in total three to \mathcal{T} :

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 , \quad \mathcal{T}_1 = \int \omega_r^{\heartsuit a} \delta_{j_r^a} , \quad \mathcal{T}_2 = \int \omega_r^{\diamond a} \delta_{j_r^a} , \quad \mathcal{T}_3 = \iint' \Omega_{rr'}^{ab} \delta_{j_r^a} \delta_{j_{r'}^b} . \quad (2.7)$$

\mathcal{T}_1 is studied with much detail in Appendix A since this has possibly never been done. But $\mathcal{T}_2, \mathcal{T}_3$ merely recapitulate [1] in Appendix A. Here we summarize the results:

$$\omega_r^{\heartsuit a} = -\frac{i}{N} [\partial - j_r]^{ab} f^{bu} \int' \frac{e^{-2(\vec{r}-\vec{r}')^2/\epsilon}}{\pi(\vec{r}-\vec{r}')^2} (H_{z\vec{z}'} H_{z'\vec{z}'}^{-1} H_{z'\vec{z}''} H_{z\vec{z}''}^{-1})^{uv} , \quad (2.8)$$

²Just to illustrate this let us ignore the above regularization details for a moment. It is a digression but one that could support confidence in the whole setup. We replace the round brackets in (2.2) by δ^{au} and δ^{bc} , respectively. Now the Delta functions just mentioned come into play by partial integrations in (2.1). $H^{ab} = M^{\dagger ag} M^{gb}$ combines with the M 's in (2.3) to δ^{cd} (M does not depend on A^*), and things turn into familiar quantum mechanics: End of the digression.
$$\mathcal{T} = -\frac{\pi}{N} \int \delta_{A^{c*}(\vec{r})} \delta_{A^c(\vec{r})} = \frac{1}{m} \frac{e^2}{2} \int \frac{1}{i} \delta_{A_j^a(\vec{r})} \frac{1}{i} \delta_{A_j^a(\vec{r})} .$$

$$\omega_r^{\diamond a} = \frac{i}{N} f^{auv} \Theta_{rr}^{uv} \quad , \quad (2.9)$$

$$\Omega_{rr}^{ab} = \frac{1}{N} [\partial - j_r]^{ac} \Theta_{rr}^{cb} \quad \text{with} \quad (2.10)$$

$$\Theta_{rr}^{ab} = -\pi \int'' \overline{\mathcal{G}}_{r,r}^{ua} H_r^{uv} (\partial' \mathcal{G}_{r,r}^{vc}) (H_r^{-1})^{cb} \quad . \quad (2.11)$$

By $\Theta = \pi\Lambda$ the notation of [1] is reached. There we are. And now?

3 Expansion of ω^\heartsuit , ω^\diamond and Ω in powers of ϵ

3.1 ω^\heartsuit

This is the sometimes omitted term of order $O(\epsilon)$. It is tempting to perform the shift of variables $\vec{r}' \rightarrow \vec{r}' + \vec{r}$ under the primed integral in (2.8) and then writing the shifts by \bar{z} or z in the H -arguments as Taylor expansions:

$$\omega_r^{\heartsuit a} = -\frac{i}{N} [\partial - j_r]^{ab} f^{buw} H_r^{vw} \int' \frac{e^{-2r'^2/\epsilon}}{\pi r'^2} \sum_{s,t=0}^{\infty} \frac{z'^s \bar{z}'^t}{s! t!} \hat{\partial}^s \bar{\partial}^t \left(\underline{H}_{z\bar{z}} \hat{H}_{z\bar{z}}^{-1} \hat{H}_{z\bar{z}} \right)^{uw} \quad . \quad (3.1)$$

Here $\hat{\partial}$ acts on \hat{H} only and $\bar{\partial}$ only on \underline{H} . The primed integral is nonzero only for $s = t$, and at $s = t = 0$ (3.1) vanishes due to $(HH^{-1}HH^{-1})^{vu} = \delta^{vu}$. Now the integration can be done. Using $\int e^{-2r^2/\epsilon} (r^2)^{s-1} / \pi = (\epsilon/2)^s (s-1)!$ we get

$$\omega_r^{\heartsuit a} = -\frac{i}{N} [\partial - j_r]^{ab} f^{buw} H_r^{vw} \sum_{s=1}^{\infty} \frac{(\epsilon/2)^s}{s s!} \hat{\partial}^s \bar{\partial}^s \left(\underline{H}_{z\bar{z}} \hat{H}_{z\bar{z}}^{-1} \hat{H}_{z\bar{z}} \right)^{uw} \quad . \quad (3.2)$$

$\hat{\partial}$ acts on two of the H 's. Correspondingly, $\hat{\partial}^s$ can be decomposed binomically by

$$\hat{\partial}^s = \sum_{k=0}^s \frac{s!}{k! (s-k)!} \partial_{\text{first}}^k \partial_{\text{second}}^{s-k} \quad , \quad \text{hence} \quad (3.3)$$

$$\sum_{s=1}^{\infty} \dots \stackrel{=}{=} \sum_{s=1}^{\infty} \sum_{k=0}^s \frac{(\epsilon/2)^s}{s k! (s-k)!} \left[\left(\bar{\partial}^s H (\partial^k H^{-1}) \right) \partial^{s-k} H \right]^{uw} \quad (3.4)$$

$$\stackrel{=}{=} \sum_{s=1}^{\infty} \sum_{k=0}^s \frac{(\epsilon/2)^s}{s k! (s-k)!} \left[\left(\bar{\partial}^s B(k) \right) A(s-k) H \right]^{uw} \quad (3.5)$$

with the definitions

$$B(p) := H \partial^p H^{-1} \quad \text{and} \quad A(p) := (\partial^p H) H^{-1} \quad , \quad B(p)^{ab} = A(p)^{ba} \quad , \quad (3.6)$$

since $(H^{-1})^{ab} = H^{ba}$ from $H^{ab} = 2 \text{Tr} (T^a H T^b H^{-1})$. Inserting (3.5) into (3.2), redefining $k \rightarrow s - k$ and using $B^{ab} = A^{ba}$ one obtains

$$\omega_r^{\heartsuit a} = \frac{i}{N} [\partial - j]^{ab} f^{buw} \sum_{s=1}^{\infty} \sum_{k=0}^s \frac{(\epsilon/2)^s}{s k! (s-k)!} \left(B(k) \bar{\partial}^s A(s-k) \right)^{uw} \quad . \quad (3.7)$$

For the final $\omega^{\heartsuit a}$ -version

$$\omega^{\heartsuit a} = \frac{i}{N} [\partial - j]^{ab} f^{buw} \sum_{s=1}^{\infty} \sum_{k=0}^s \frac{(\epsilon/2)^s}{s k! (s-k)!} M(k, s, s-k, 0)^{uw} \quad (3.8)$$

the more general definition

$$M(a, b, c, d) := B(a) \bar{\partial}^b A(c) B(d) \quad (3.9)$$

has been introduced. Note that $B(0) = 1$. With $j = (\partial H)H^{-1}$ it follows from (3.6) that $B(p) = [\partial - j]^p 1$ and in particular

$$\begin{aligned} B(1) &= -j, \quad B(2) = -\partial j + j j, \quad B(3) = -\partial^2 j + (\partial j) j + 2j \partial j - j j j \quad \text{and} \\ B(4) &= -\partial^3 j + (\partial^2 j) j + 3j \partial^2 j + 3(\partial j) \partial j - (\partial j) j j - 2j (\partial j) j - 3j j \partial j + j j j j. \end{aligned} \quad (3.10)$$

For special A 's use $A^{ab} = B^{ba}$ ($A(0) = 1$, $A(1) = j, \dots$). (3.8) shows that the contribution $\omega^{\heartsuit a}$ to ω^a would have been bypassed by a too rush $\epsilon \rightarrow +0$.

3.2 Θ

(2.9) and (2.10) show that both quantities, ω^{\diamond} and Ω , are traced back to the object $\Theta_{rr'}^{ab}$. The rhs of (2.11) has not quite the form of *something*^{ab}. But the indices on the troublemaker $\bar{\mathcal{G}}_{rr'}^{ua}$ (see (2.2)) are easily reversed by $(H_{z''\bar{z}} H_r^{-1})^{au} = H_{z''\bar{z}}^{ac} H_r^{uc} = (H_r H_{z''\bar{z}}^{-1})^{ua}$. In the sequel let us omit (but keep in mind) the superscripts ^{ab} on both sides. Also let $e_{r''r}$ stand for $\exp(-(r'' - \vec{r})^2/\epsilon)$. In otherwise full detail (2.11) now reads

$$\Theta_{rr'} = -\pi \int'' \bar{\mathcal{G}}_{r''r} (1 - e_{r''r} H_r H_{z''\bar{z}}^{-1}) H_{r''} (\partial' G_{r''r'} (1 - e_{r''r} H_{z'\bar{z}''}^{-1} H_{r'})) H_{r'}^{-1} \quad (3.11)$$

Now $\bar{\mathcal{G}}$ and G may be replaced by their bare versions ($\epsilon = 0$). Note that ∂' and $G_{r''r'}$ can be commuted since the Delta function would be multiplied by the vanishing round bracket. Hence

$$\begin{aligned} \Theta_{rr'} = \frac{1}{\pi} \int'' \frac{e_{r''r}}{(z'' - z)(\bar{z}'' - \bar{z}')} & \left(\frac{\bar{z}'' - \bar{z}'}{\epsilon} H_{r''} H_{z'\bar{z}''}^{-1} + H_{r''} (\partial' H_{z'\bar{z}''}^{-1} H_{r'}) H_{r'}^{-1} \right. \\ & \left. - e_{r''r} \frac{\bar{z}'' - \bar{z}'}{\epsilon} H_r H_{z''\bar{z}}^{-1} H_{r''} H_{z'\bar{z}''}^{-1} - e_{r''r} H_r H_{z''\bar{z}}^{-1} H_{r''} (\partial' H_{z'\bar{z}''}^{-1} H_{r'}) H_{r'}^{-1} \right) \quad (3.12) \end{aligned}$$

(3.12) is in perfect agreement with the four terms I to IV in [1] (the Λ there is $= \Theta/\pi$). In [1] I to IV are eqs. (4.7 b to e). (Some of their subsequent equations are not free from errors or typos). What follows here is new ground.

Towards evaluation of the integral the shift $\vec{r}'' \rightarrow \vec{r}'' + \vec{r}'$ is a first useful step:

$$\begin{aligned} e_{r''r} &\rightarrow \exp(-r''^2/\epsilon), \quad e_{r''r} \rightarrow \exp(-r''^2/\epsilon - \sigma \bar{z}''/\epsilon - \bar{\sigma} z''/\epsilon - \sigma \bar{\sigma}/\epsilon) \\ \text{where } \sigma &:= z' - z \quad \text{and} \quad \bar{\sigma} := \bar{z}' - \bar{z} \quad . \end{aligned} \quad (3.13)$$

$$\begin{aligned}
\Theta_{rr'} = & \frac{1}{\pi} \int'' \left(\frac{e^{-r''^2/\epsilon}}{\epsilon(z'' + \sigma)} H_{z'+z'' \bar{z}'+\bar{z}''} H_{z' \bar{z}'+\bar{z}''}^{-1} \right. \\
& + \frac{e^{-r''^2/\epsilon}}{\bar{z}''(z'' + \sigma)} H_{z'+z'' \bar{z}'+\bar{z}''} (\partial' H_{z' \bar{z}'+\bar{z}''}^{-1} H_{r'}) H_{r'}^{-1} \\
& - \frac{e^{-2r''^2/\epsilon}}{\epsilon(z'' + \sigma)} e^{-\sigma \bar{z}''/\epsilon} e^{-\bar{\sigma} z''/\epsilon} e^{-\sigma \bar{\sigma}/\epsilon} H_r H_{z'+z'' \bar{z}'+\bar{z}''}^{-1} H_{z'+z'' \bar{z}'+\bar{z}''} H_{z' \bar{z}'+\bar{z}''}^{-1} \\
& \left. - \frac{e^{-2r''^2/\epsilon}}{\bar{z}''(z'' + \sigma)} e^{-\sigma \bar{z}''/\epsilon} e^{-\bar{\sigma} z''/\epsilon} e^{-\sigma \bar{\sigma}/\epsilon} H_r H_{z'+z'' \bar{z}'+\bar{z}''}^{-1} H_{z'+z'' \bar{z}'+\bar{z}''} (\partial' H_{z' \bar{z}'+\bar{z}''}^{-1} H_{r'}) H_{r'}^{-1} \right) \quad (3.14)
\end{aligned}$$

now shows that all the z'' and \bar{z}'' in the H -arguments can be transformed into powers by means of Taylor expansions. Hence all the integrals to be done will be of the form

$$I_{p,q}(\epsilon, \sigma) = \frac{1}{\pi} \int'' \frac{e^{-r''^2/\epsilon}}{z'' + \sigma} z''^p \bar{z}''^{q-1} \quad . \quad (3.15)$$

The four lines of (3.14) become

$$\begin{aligned}
\Theta_{rr'} = & \frac{1}{\epsilon} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{1}{p!(q-1)!} I_{p,q}(\epsilon, \sigma) \bar{\partial}'_{\uparrow}{}^{q-1} \partial'_{\bullet}{}^p H_{r'} H_{r'}^{-1} \\
& + \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!q!} I_{p,q}(\epsilon, \sigma) \bar{\partial}'_{\uparrow}{}^q \partial'_{\bullet}{}^p H_{r'} (\partial' H_{r'}^{-1} H_{r'}) H_{r'}^{-1} \\
& - \frac{1}{\epsilon} e^{-\sigma \bar{\sigma}/\epsilon} H_r \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{1}{p!(q-1)!} I_{p,q}(\frac{\epsilon}{2}, \sigma) \left[\bar{\partial}'_{\uparrow} - \frac{\sigma}{\epsilon} \right]^{q-1} \left[\partial'_{\bullet} - \frac{\bar{\sigma}}{\epsilon} \right]^p H_{z' \bar{z}'+\bar{z}''}^{-1} H_{r'} H_{r'}^{-1} \\
& - e^{-\sigma \bar{\sigma}/\epsilon} H_r \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!q!} I_{p,q}(\frac{\epsilon}{2}, \sigma) \left[\bar{\partial}'_{\uparrow} - \frac{\sigma}{\epsilon} \right]^q \left[\partial'_{\bullet} - \frac{\bar{\sigma}}{\epsilon} \right]^p H_{z' \bar{z}'+\bar{z}''}^{-1} H_{r'} (\partial' H_{r'}^{-1} H_{r'}) H_{r'}^{-1} . \quad (3.16)
\end{aligned}$$

The meaning of the subscripts: a dotted ∂' acts *only* on dotted H -arguments and an arrowed only on arrowed. In particular, $\bar{\partial}'$ in the last two lines does *not* act on $\bar{\sigma}$ in the second square bracket.

The integral I , (3.15), is evaluated in Appendix B to be

$$I_{p,q}(\epsilon, \sigma) = (-\sigma)^{p-q} \int_{\sigma \bar{\sigma}}^{\infty} dt t^{q-1} e^{-t/\epsilon} - \theta_{q>p} (q-1)! (-\sigma)^{p-q} \epsilon^q \quad . \quad (3.17)$$

Here $\theta_{q>p}$ is 1 for $q > p$ and zero otherwise. Surprisingly, the left half of (3.17) does not contribute to (3.16) at all. The check to this nice outcome is left to the reader³. Hence for use in (3.16) the second term of (3.17) suffices.

Due to $\theta_{q>p}$ all q -sums in (3.16) become $\sum_{q=p+1}^{\infty}$. The factor $(q-1)!$ either compensates with the denominator or leaves a q there. For just one more detail note that the ends of

³ Perform each of the four p -sums. Note that the operator $\exp(-\sigma \partial')$ changes a dotted H -index z' into $z' - (z - z') = z$. Show that the third line of (3.16) is the negative of the first one. "But $I(\epsilon/2, \dots)$ contains the wrong $\exp(-2t/\epsilon)!$?" Well, use the q -sum to produce the missing $\exp(+t/\epsilon)$: $\sum_{q=0}^{\infty} \frac{1}{q!} \left[-t \bar{\partial}' / \sigma + t / \epsilon \right]^q = e^{[-t \bar{\partial}' / \sigma + t / \epsilon]} = \sum_{q=0}^{\infty} \frac{1}{q!} \left[-t \bar{\partial}' / \sigma \right]^q e^{+t/\epsilon}$. Show fourth line = -second.

the second and fourth line may be written as

$$(\partial' H_{\uparrow}^{-1} H_{r'}) H_{r'}^{-1} = H_{\uparrow}^{-1} (-j_{r'} + j_{r'}) \quad . \quad (3.18)$$

For the further lengthy analysis we denote the four lines of (3.16) by ① , ② , ③ , ④ . One obtains

$$\textcircled{1} + \textcircled{2} = \frac{1}{\sigma} + \mathcal{C} j_{r'} + \partial' \mathcal{C} \quad \text{with} \quad \mathcal{C} = - \sum_{p=1}^{\infty} \sum_{q=p+1}^{\infty} \frac{(-\sigma)^{p-q}}{p! q} \epsilon^q \bar{\partial}'^q \partial'^p H_{\uparrow} H_{r'} H_{r'}^{-1} \quad . \quad (3.19)$$

More laboriously one arrives at

$$\begin{aligned} \textcircled{3} + \textcircled{4} &= -\frac{1}{\sigma} e^{-\sigma \bar{\sigma} / \epsilon} H_r H_{z' \bar{z}}^{-1} + \mathcal{E} j_{r'} + \partial' \mathcal{E} \quad \text{with} \\ \mathcal{E} &= e^{-\sigma \bar{\sigma} / \epsilon} H_r \sum_{p=1}^{\infty} \sum_{q=p+1}^{\infty} \left(\frac{\epsilon}{2}\right)^q \left[\bar{\partial}' - \frac{\sigma}{\epsilon}\right]^q \left[\partial' - \frac{\bar{\sigma}}{\epsilon}\right]^p H_{z' \bar{z}}^{-1} H_{r'} H_{r'}^{-1} \quad . \end{aligned} \quad (3.20)$$

Through ① + ② + ③ + ④ and with $\mathcal{A} := \mathcal{C} + \mathcal{E}$ we obtain

$$\Theta_{rr'} = \frac{1}{\sigma} (1 - e^{-\sigma \bar{\sigma} / \epsilon} H_r H_{z' \bar{z}}^{-1}) + \partial' \mathcal{A}_{rr'} + \mathcal{A}_{rr'} j_{r'} \quad . \quad (3.21)$$

The double sum over p and q is common to \mathcal{C} and \mathcal{E} , hence also to \mathcal{A} . A rearrangement of this double sum comes into mind by

$$\sum_{p=0}^{\infty} \sum_{q=p+1}^{\infty} = \sum_{p=0}^{\infty} \left(\sum_{q=p+1}^{\infty} - \sum_{q=1}^{\infty} \right) + DS = - \sum_{p=1}^{\infty} \sum_{q=1}^p + DS \quad \text{with} \quad DS := \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \quad . \quad (3.22)$$

\mathcal{A} can be split into the part with the finite q -sum and the part with DS . Let the latter part be called \mathcal{F} . Then

$$\begin{aligned} \mathcal{F} &= \sum_{q=1}^{\infty} \frac{1}{q} \left[- \left(-\frac{\epsilon}{\sigma} \bar{\partial}' \right)^q + \left(-\frac{\epsilon}{2\sigma} \left[\bar{\partial}' - \frac{1}{2} \right]^q \right) \right] H_{z' \bar{z}} H_{r'}^{-1} \\ &= \left[\ln \left(1 + \frac{\epsilon}{\sigma} \bar{\partial}' \right) - \ln \left(1 + \frac{\epsilon}{2\sigma} \bar{\partial}' - \frac{1}{2} \right) \right] H_{z' \bar{z}} H_{r'}^{-1} = \ln(2) H_{z' \bar{z}} H_{r'}^{-1} \end{aligned} \quad (3.23)$$

and this drops out in (3.21) because $\partial' \mathcal{F} + \mathcal{F} j_{r'} = 0$.

Hence the suitable completion of (3.21) is

$$\begin{aligned} \mathcal{A}_{rr'} &= \sum_{p=1}^{\infty} \sum_{q=1}^p \frac{1}{p! q} (-\sigma)^{p-q} \left(\epsilon^q \bar{\partial}'^q (\partial'^p H_{r'}) H_{r'}^{-1} \right. \\ &\quad \left. - e^{-\sigma \bar{\sigma} / \epsilon} H_r \left(\frac{\epsilon}{2}\right)^q \left[\bar{\partial}' - \frac{\sigma}{\epsilon}\right]^q \left[\partial' - \frac{\bar{\sigma}}{\epsilon}\right]^p H_{z' \bar{z}}^{-1} H_{r'} H_{r'}^{-1} \right) \quad . \end{aligned} \quad (3.24)$$

Two agreeable properties of (3.19) might be emphasized. All dependences on ϵ are contained (all positive and negative ϵ powers – imagine the exponential functions be expanded). Secondly, there are no negative powers of σ or $\bar{\sigma}$ in $\Theta_{rr'}$. This is a welcome fact towards coincidence limits $\vec{r}' \rightarrow \vec{r}$ which make σ , $\bar{\sigma}$ vanish. Such limits occur when two functional derivatives – remember $\mathcal{T}_3 = \iint' \Omega_{rr'}^{ab} \delta_{j_r^a} \delta_{j_r^b}$, and $\Omega_{rr'}^{ab} = \frac{1}{N} [\partial - j_r]^{ac} \Theta_{rr'}^{cb}$ – are applied to holomorphic invariants. $\Theta_{rr'}$ is further processed in the subsections 4.2 and 4.3.

4 Coincidence limits

4.1 Holomorphic invariants

The mapping of A to the M -space is not unique since $A_r = -(\partial M_r)M_r^{-1}$ remains unchanged under

$$M \rightarrow Mh^\dagger(\bar{z}) \quad \curvearrowright \quad M_r^\dagger \rightarrow h(z)M_r^\dagger \quad \text{and} \quad H = M^\dagger M \rightarrow h(z)Hh^\dagger(\bar{z}) \quad . \quad (4.1)$$

Physics must not depend on the choice of h . The solution \mathcal{P} to $\mathcal{TP} = \mathcal{E} - \mathcal{V}$ is ‘‘physics’’. So \mathcal{P} has to be holomorphic invariant. \mathcal{P} will turn out to be a linear combination of the invariants listed in (4.9) to (4.11) below. By means of (4.1) one derives

$$j = (\partial H)H^{-1} \rightarrow (\partial h)h^{-1} + hjh^{-1} \quad \text{but} \quad \bar{\partial}j \rightarrow h(\bar{\partial}j)h^{-1} \quad . \quad (4.2)$$

For convenience we might leave the above $N \times N$ -matrix-language and ask for the holomorphic transformation of j^a and j^{ab} . Let us also shorten the notation a bit by

$$\bar{\partial}j =: \bar{j} \quad , \quad \bar{\partial}j^a =: \bar{j}^a \quad , \quad \bar{\partial}j^{ab} =: \bar{j}^{ab} \quad , \quad \bar{\partial}^2 j =: \bar{\bar{j}} \quad , \quad \text{etc.} \quad (4.3)$$

The last equation in (4.2) now reads $\bar{j} \rightarrow h\bar{j}h^{-1}$. From (1.2), which now reads $\bar{j}^a = 2 \text{Tr}(T^a \bar{j})$, one derives that

$$\bar{j}^a \rightarrow 2 \text{Tr}(T^a h\bar{j}h^{-1}) = 2 \text{Tr}(h^{-1}T^a h T^b) 2 \text{Tr}(T^b \bar{j}) = h^{ab} \bar{j}^b \quad , \quad (4.4)$$

where $h^{ab} = 2 \text{Tr}(T^a h T^b h^{-1}) = 2 \text{Tr}(T^b h^{-1} T^a h) = (h^{-1})^{ba}$. Hence $\bar{j}^a \bar{j}^a$ is invariant, and so is $\mathcal{V} = \int \bar{j}^a \bar{j}^a$. From $j^{ab} = ((\partial H)H^{-1})^{ab}$ and (4.2) one obtains

$$\begin{aligned} j^{ab} &\rightarrow ((\partial h H h^\dagger)h^{\dagger-1}H^{-1}h^{-1})^{ab} = ((\partial h)h^{-1} + hjh^{-1})^{ab} \quad \text{but} \\ [\partial - j]^{ab} &\rightarrow (\partial - (\partial h)h^{-1} - hjh^{-1})^{ab} = (h [\partial - j] h^{-1})^{ab} \quad . \end{aligned} \quad (4.5)$$

The above details allow to recognize the following set

$$Q_n = \int \bar{j}^a \left([[\partial - j] \bar{\partial}]^n \right)^{ab} \bar{j}^b \quad , \quad n = 0, 1, 2, \dots \quad , \quad (4.6)$$

as holomorphic invariants. Note that $Q_0 = \mathcal{V}$.

The invariants are conveniently written as adjoint traces. (4.9) below shows this trace for Q_n . It contains the operator

$$\mathcal{D} := \partial - [j, \quad] \quad \text{where} \quad [j, \quad] \text{ any} := j \text{ any} - \text{any} j \quad . \quad (4.7)$$

Starting from (4.6) the version (4.9) can be derived.⁴

⁴ Consider $Q_n = \int \bar{j}^a \left([[\partial - j] \bar{\partial}]^{n-1} \right)^{ac} [\partial - j]^{cb} \bar{j}^b$ and insert $j^{cb} = -i j^d f^{dcb}$ and $\bar{j}^b = i f^{buv} \bar{j}^{uv}/N$ at the right end. Now the Jacobi identity $f^{dcb} f^{buv} = -f^{udb} f^{bcv} - f^{cub} f^{bdv}$ leads to $[[\partial - j] \bar{\partial}]^{cb} \bar{j}^b = i f^{cbv} [\mathcal{D} \bar{\partial} \bar{j}]^{bv}/N$ – and so on.

As is seen in section 5 (and latest in section 6) *more* holomorphic invariants are to be included. They will all be of the form

$$\frac{1}{N} \int (any)^{aa} \quad \text{with} \quad any \rightarrow h any h^{-1} \quad . \quad (4.8)$$

Our attempts to solve $\mathcal{TP} = \mathcal{E} - \mathcal{V}$ (up to some maximal ϵ -power) involve the following invariants

$$Q_n = \frac{1}{N} \int (\bar{j} [\mathcal{D} \bar{\partial}]^n \bar{j})^{aa} \quad , \quad (4.9)$$

$$R_{20} = \frac{1}{N} \int (\bar{j} \bar{j} \bar{j} \bar{j})^{aa} \quad , \quad R_{21} = \frac{1}{N} \int (\bar{j} \bar{j} \mathcal{D} \bar{j})^{aa} \quad , \quad (4.10)$$

$$R_{31}^{(1)} = \frac{1}{N} \int (\bar{j} \bar{j} \bar{j} \mathcal{D} \bar{j})^{aa} \quad , \quad R_{31}^{(2)} = \frac{1}{N} \int (\bar{j} \bar{j} \bar{j} \mathcal{D} \bar{j})^{aa} \quad , \quad R_{32} = \frac{1}{N} \int (\bar{j} \bar{j} \mathcal{D} \bar{\partial} \mathcal{D} \bar{j})^{aa} \quad . \quad (4.11)$$

The first index on R refers to the n of that Q_n , which under application of \mathcal{T} produces the R in addition. The second index just denotes the number of \mathcal{D} 's contained. $R_{n\dots}$ has the same mass dimension as Q_n , namely m^{2n+2} . In passing, ∂ , $\bar{\partial}$, j , \mathcal{D} and δ_j have mass dimension m^1 , ϵ and \int have m^{-2} and \mathcal{T} is dimensionless.

4.2 Technicalities in applying \mathcal{T} to invariants

If the \mathcal{D} 's are made explicit, an invariant is a linear combination of products. To begin with \mathcal{T}_1 or \mathcal{T}_2 , let the $\delta_{j_r^a}$ in there act on one factor of such a product, for example on $\bar{\partial} \bar{j}$:

$$\begin{aligned} \int \omega_r^a \delta_{j_r^a} \frac{1}{N} \int'' (\bullet \bullet \bullet_{r''} \partial'' \bar{j}_{r''} \circ \circ \circ_{r''})^{cc} &= \frac{-i}{N} \int \omega_r^a \delta_{j_r^a} f^{cbd} \int'' (\partial'' \bar{j}_{r''}^d) (\circ \circ \circ \bullet \bullet \bullet_{r''})^{bc} \\ &= \frac{-i}{N} \int \omega_r^a f^{cba} \int'' (\partial'' \bar{\partial}''^2 \delta(\vec{r} - \vec{r}'')) (\circ \circ \circ \bullet \bullet \bullet_{r''})^{bc} \\ &\quad \partial'' \rightarrow -\partial \text{ (on } \delta) \rightarrow +\partial \text{ (by partial int.) , and similar with } \bar{\partial} : \\ &= \frac{-i}{N} \int (\partial \bar{\partial}^2 \omega_r^a) f^{cba} (\circ \circ \circ \bullet \bullet \bullet_r)^{bc} = \frac{1}{N} \int ((\partial \bar{\partial}^2 \omega) \circ \circ \circ \bullet \bullet \bullet_r)^{cc} \quad , \quad (4.12) \end{aligned}$$

where $-i f^{cba} \omega^a =: \omega^{cb}$ was defined. To check (4.12): if ω were j , the operator $\int j^a \delta_{j^a}$ would just count the number of j 's in the product, indeed. Remember that ω^a is the sum $\omega^a = \omega^{\heartsuit a} + \omega^{\diamond a}$ with $\omega^{\heartsuit a}$ given by (3.8). But $\omega^{\diamond a}$ is rather taken from (4.18) below. Since $\omega^a = \omega^{\diamond a} + O(\epsilon)$ and $L_{00}^{uv} = j^{uv} + O(\epsilon)$ we have $\omega^a = j^a + O(\epsilon)$.

To study the action of \mathcal{T}_3 on an invariant, two factors in a product might be made explicit, for example \bar{j} and $\partial \bar{j}$:

$$\iint' \Omega_{r'r'}^{ab} \delta_{j_r^a} \delta_{j_{r'}^b} \frac{1}{N} \int'' (\bullet \bullet \bullet_{r''} \bar{j}_{r''} \circ \circ \circ_{r''} \partial'' \bar{j}_{r''} \diamond \diamond \diamond_{r''})^{cc} \quad . \quad (4.13)$$

Here $\Omega_{r'r'}^{ab}$ may be replaced by $\Omega_{r'r'}^{ab} + \Omega_{r'r'}^{ba} =: \mathcal{K}_{r'r'}^{ab}$, if the two δ 's are applied ordered, i.e. interchanging them is forbidden. Assume $\bar{j}_{r''}$ carries indices^{uv} at its position in $(\)^{cc}$.

Then $\delta_{j_r^a} \bar{J}_{r''} = -i f^{uva} \bar{\partial}'' \delta(\vec{r} - \vec{r}'') = i f^{uav} \dots$. We may relax the notation by $f^{uav} =: f^a$ since the indices u, v are fixed by the position of f^a in $(\)^{cc}$. Hence

$$\begin{aligned}
(4.13) &= \frac{-1}{N} \iint \iint'' \mathcal{K}_{rr'}^{ab} \left(\bar{\partial}'' \delta(\vec{r} - \vec{r}'') \right) \left(\partial'' \bar{\partial}'' \delta(\vec{r}' - \vec{r}'') \right) (\bullet \bullet \bullet f^a \circ \circ \circ f^b \diamond \diamond \diamond)_{r''}^{cc} \\
&\stackrel{\text{---}}{=} \frac{-1}{N} \iint \iint' \left(\bar{\partial} \partial' \bar{\partial}' \mathcal{K}_{rr'}^{ab} \right) \delta(\vec{r} - \vec{r}') (\bullet \bullet \bullet f^a \circ \circ \circ f^b \diamond \diamond \diamond)_r^{cc} \\
&\stackrel{\text{---}}{=} \frac{-1}{N} \int \left\{ \bar{\partial} \partial' \bar{\partial}' \mathcal{K}_{rr'}^{ab} \right\} (\bullet \bullet \bullet f^a \circ \circ \circ f^b \diamond \diamond \diamond)_r^{cc} \quad . \quad (4.14)
\end{aligned}$$

The pair of curly brackets stands for the coincidence limit $\{any_{rr'}\} := \lim_{\vec{r}' \rightarrow \vec{r}} any_{rr'}$.

Imagine a definite invariant with its linear-combined products, each booked down with all combinations of f^a and f^b (the latter to the right of the first). Let each of the various terms be given the form (4.14). So far things are done by hand on paper. But now it is convenient to put the result (for \mathcal{T}_3 Invariant) in a MAPLE-file and continue with it by keyboard and screen. There the following 5 steps are done.

I. Reintroducing Ω by $\mathcal{K}_{rr'}^{ab} = \Omega_{rr'}^{ab} + \Omega_{r'r}^{ba}$ means that each $\Omega_{rr'}^{ab}$ becomes a partner with interchanged indices and variables (hence ∂ 's become primed and ∂' 's lose their primes).

II. With reason explained in step **V**, convert all unprimed ∂ 's inside $\{ \}$ into primed ones by using

$$\{ \partial \dots \} = \partial \{ \dots \} - \{ \partial' \dots \} \quad \text{and} \quad \{ \bar{\partial} \dots \} = \bar{\partial} \{ \dots \} - \{ \bar{\partial}' \dots \} \quad . \quad (4.15)$$

To understand (4.15) note that $\int \{ \partial \dots \} any_r = \iint' (\partial \dots) \delta(\vec{r} - \vec{r}') any_r$. Now partial integrate under \int , use $\partial \delta = -\partial' \delta$ and partial integrate with ∂' under \int' . Hence $\int \{ \partial \dots \} any_r = -\int \{ \partial' \dots \} any_r - \int \{ \dots \} \partial any_r$, which leads to (4.15) by a last partial integration.

III. Partial integrate the unprimed ∂ 's in front of $\{ \}$. Thereby some of the $(\)^{cc}$'s are differentiated, but they remain $(\)^{cc}$. Note the $1/N$ -prefactor of each invariant. Hence a typical term has the form

$$\frac{1}{N} \int (\dots f^a \dots f^b \dots)^{cc} \{ \text{primed } \partial \text{'s } \Omega \}^{ab} = \frac{1}{N^2} \int (\dots f^a \dots f^b \dots)^{cc} \{ \text{primed } \partial \text{'s } (\partial - j) \Theta \}^{ab} \quad , \quad (4.16)$$

where (2.10) was inserted.

IV. Use (4.15) for the last ∂ in (4.16) and partial integrate if it appears in front of $\{ \}$.

V. It will be shown in the sequel that Θ can be written as

$$\Theta_{rr'}^{ab} = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^s}{s!} \frac{\bar{\sigma}^t}{t!} L_{st}^{ab} \quad , \quad (4.17)$$

with the coefficients L depending on only the unprimed z and \bar{z} . According to (3.13) primed variables occur in $\Theta_{rr'}$, (3.14), only through $\sigma = z' - z$ and $\bar{\sigma} = \bar{z}' - \bar{z}$. This was

the reason, to favour primed differentiations in point **II**. A coincidence limit makes σ and $\bar{\sigma}$ vanish. So, it depends on the ∂' 's and $\bar{\partial}'$'s in front of Θ which coefficient L survives. Replace $\{\text{primed } \partial' \text{ s } \Theta\}$ by this coefficient. End of the 5 MAPLE-steps.

An example for the result is shown in Appendix C. Copy the result for L 's to a separate MAPLE-file. Here \mathcal{T}_1 *Invariant* and \mathcal{T}_2 *Invariant* might be included, and the ready list (see Appendix D) of L 's can be inserted there.

By (4.17) the “second ω ” becomes explicit. (2.10) simply turns into

$$\omega_r^{\diamond a} = \frac{i}{N} f^{auv} \Theta_{rr}^{uv} = \frac{i}{N} f^{auv} L_{00}^{uv} \quad (4.18)$$

4.3 Derivation of the coefficients L

(3.21) suggests to distinguish two parts: $\Theta = {}^1\Theta + {}^2\Theta$ with ${}^1\Theta$ the first term of (3.21) and ${}^2\Theta$ the remaining two terms in (3.21) containing \mathcal{A} . Correspondingly L splits into 1L and 2L . To start with

$$\begin{aligned} {}^1\Theta &= \frac{1}{\sigma} \left(1 - H_r \varepsilon^{\sigma(-\bar{\sigma}/\epsilon + \partial)} H_{z\bar{z}}^{-1} \right) = -\frac{1}{\sigma} H_r \sum_{s=1}^{\infty} \frac{1}{s!} \sigma^s (-\bar{\sigma}/\epsilon + \partial)^s H_r^{-1} \\ &= -\sum_{s=1}^{\infty} \sum_{t=0}^s \frac{1}{t!(s-t)!} \sigma^{s-1} (-\bar{\sigma}/\epsilon)^t H_r \partial^{s-t} H_r^{-1} \quad , \quad s \rightarrow s+1 : \\ &= \sum_{s=0}^{\infty} \sum_{t=0}^{s+1} \frac{\sigma^s \bar{\sigma}^t}{s! t!} \frac{s!}{(s+1-t)!} \frac{(-1)^{1+t}}{\epsilon^t} B(s+1-t) \quad . \end{aligned} \quad (4.19)$$

Defining $\theta_{t \leq s+1}$ to be 1 for $t \leq s+1$ and zero otherwise 1L is obtained as

$${}^1L_{st} = \theta_{t \leq s+1} \frac{s! (-1)^{1+t} B(s+1-t)}{(s+1-t)! \epsilon^t} \quad . \quad (4.20)$$

Towards 2L let us assume that \mathcal{A} can be written as

$$\mathcal{A}^{ab} = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^s \bar{\sigma}^t}{s! t!} C_{st}^{ab} \quad , \quad (4.21)$$

in general where the coefficients C do not depend on z', \bar{z}' . If (4.21) can be reached, 2L can be traced back to C as follows:

$${}^2\Theta = \partial' \mathcal{A} + \mathcal{A} j_r = \sum_{s=1}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^{s-1}}{(s-1)!} \frac{\bar{\sigma}^t}{t!} C_{st}^{ab} + \mathcal{A} \sum_{u=0}^{\infty} \frac{\sigma^u}{u!} \sum_{v=0}^{\infty} \frac{\bar{\sigma}^v}{v!} \partial^u \bar{\partial}^v j_r \quad . \quad (4.22)$$

Combining (4.22) with (4.21) one has to manipulate

$$\sum_{s=0}^{\infty} \sum_{u=0}^{\infty} \frac{\sigma^{s+u}}{s! u!} = \sum_{u=0}^{\infty} \sum_{s=u}^{\infty} \frac{\sigma^s}{(s-u)! u!} = \sum_{s=0}^{\infty} \frac{\sigma^s}{s!} \sum_{u=0}^s \frac{s!}{u!(s-u)!} \quad . \quad (4.23)$$

Doing the same with the sums over t and v one can read off that

$${}^2L_{st} = C_{s+1} t + \sum_{u=0}^s \binom{s}{u} \sum_{v=0}^t \binom{t}{v} C_{s-u} t-v \partial^u \bar{\partial}^v j_r \quad . \quad (4.24)$$

It remains to determine C_{st} as defined by (4.21). Clearly \mathcal{A} from (3.24) needs a further rearrangement such that C_{st} can be read off. The two lines of (3.24) have something in common, namely the two sums over p and q followed by $\frac{1}{p!q}$ and explicit powers p and q . In the following formula $\alpha = -\sigma \dot{\partial}'$, $\beta = -\frac{\epsilon}{\sigma} \bar{\partial}'_{\uparrow}$ apply to the first line of (3.24) while $\alpha = -\sigma \dot{\partial}' + \sigma \bar{\sigma}/\epsilon$ and $\beta = -\frac{\epsilon}{2\sigma} \bar{\partial}'_{\uparrow} + \frac{1}{2}$ apply to the second:

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{1}{p!} \alpha^p \sum_{q=1}^p \frac{1}{q} \beta^q &= \int_0^1 d\tau \sum_{p=1}^{\infty} \frac{1}{p!} \alpha^p \beta \frac{1 - (\tau\beta)^p}{1 - \tau\beta} = \int_0^1 d\tau \frac{\beta (e^{(\tau\beta-1)\alpha} - 1)}{\tau\beta - 1} e^{\alpha} \\ &= \sum_{c=1}^{\infty} \frac{1}{c!} \frac{1}{c} \left((\alpha\beta - \alpha)^c - (-\alpha)^c \right) e^{\alpha} \quad . \end{aligned} \quad (4.25)$$

The last expression was obtained by expanding the exponential (sum over c) and performing the τ integration.

The explicit exponential e^{α} at the right end of (4.25) has welcome effects. It removes dot-subscripts by $e^{-\sigma \dot{\partial}'} H_{z'\bar{z}'} = H_{z\bar{z}'}$ and $e^{-\sigma \dot{\partial}'} H_{z'\bar{z}}^{-1} H_{z'\bar{z}'} H_{z'\bar{z}'}^{-1}$ in the first and second line of (3.24), respectively. Moreover the second term in the second-line- α compensates the prefactor $e^{-\sigma \bar{\sigma}/\epsilon}$ in (3.24).

The corresponding intermediate result is

$$\begin{aligned} \mathcal{A} &= \sum_{c=1}^{\infty} \frac{1}{c!} \frac{1}{c} \left\{ \left([\sigma + \epsilon \bar{\partial}']^c - \sigma^c \right) \partial^c H_{z\bar{z}'} H_r^{-1} \right. \\ &\quad \left. - H_r \frac{1}{2^c} \left([\sigma + \epsilon \bar{\partial}'_{\uparrow}]^c - (2\sigma)^c \right) \left(\partial - \frac{\bar{\sigma}}{\epsilon} \right)^c H_r^{-1} H_{z\bar{z}'} H_{z'\bar{z}'}^{-1} \right\} \quad . \end{aligned} \quad (4.26)$$

where the arrow-subscript prevents $\bar{\partial}'$ to act on the explicit $\bar{\sigma}$.

There is still some cumbersome analysis left. We first separate powers of σ and $\bar{\sigma}$ binomially:

$$\begin{aligned} \mathcal{A} &= \sum_{c=1}^{\infty} \sum_{k=1}^c \frac{\epsilon^k \sigma^{c-k}}{c k! (c-k)!} \mathcal{B}_1 - \sum_{c=1}^{\infty} \sum_{k=1}^c \sum_{\ell=0}^c \frac{\epsilon^{k-\ell} \sigma^{c-k} c! (-\bar{\sigma})^{\ell}}{c k! (c-k)! \ell! (c-\ell)! 2^c} \mathcal{B}_2 \\ &\quad + \sum_{c=1}^{\infty} \sum_{\ell=0}^c \frac{\sigma^c (-\bar{\sigma})^{\ell} (1 - 2^{-c})}{c \ell! (c-\ell)! \epsilon^{\ell}} \mathcal{B}_3 \quad \text{with} \end{aligned} \quad (4.27)$$

$$\mathcal{B}_1 = \bar{\partial}'^k \partial^c H_{z\bar{z}'} H_r^{-1} \quad , \quad \mathcal{B}_2 = H_r \bar{\partial}'^k \partial^{c-\ell} H_{z\bar{z}}^{-1} H_{z\bar{z}'} H_r^{-1} \quad , \quad \mathcal{B}_3 = \mathcal{B}_2|_{k=0} \quad . \quad (4.28)$$

Now remember that primed indices z' and \bar{z}' must not occur in C_{st} . With view to $z' = z + \sigma$, $\bar{z}' = \bar{z} + \bar{\sigma}$ they can be removed by Taylor expansions:

$$\begin{aligned} \mathcal{B}_1 &= \frac{\partial^k}{\partial \uparrow} (\partial^c H_{z \bar{z} + \bar{\sigma}}) H_{z + \sigma \bar{z} + \bar{\sigma}} = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^s \bar{\sigma}^t}{s! t!} \partial^{t+k} (\partial^c H_{z \bar{z}}) \partial^s H_{z \bar{z}}^{-1} \\ &= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^s \bar{\sigma}^t}{s! t!} M(0, t+k, c, s) \quad . \end{aligned} \quad (4.29)$$

Remember $M(a, b, c, d)$ from (3.9). Furthermore

$$\begin{aligned} \mathcal{B}_2 &= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} H_r \partial^{c-\ell} H_{z \bar{z}}^{-1} \frac{\bar{\sigma}^t}{t!} \partial^{t+k} H_{z \bar{z}} \frac{\sigma^s}{s!} \partial^s H_{z \bar{z}}^{-1} \quad , \\ &= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^s \bar{\sigma}^t}{s! t!} \sum_{a=0}^{c-\ell} \frac{(c-\ell)!}{a! (c-\ell-a)!} H (\partial^{c-\ell-a} H^{-1}) \partial^{t+k} (\partial^a H) \partial^s H^{-1} \\ &= \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \frac{\sigma^s \bar{\sigma}^t}{s! t!} \sum_{a=0}^{c-\ell} \frac{(c-\ell)!}{a! (c-\ell-a)!} M(c-\ell-a, t+k, a, s) \quad . \end{aligned} \quad (4.30)$$

Inserting (4.29), (4.30) and $\mathcal{B}_3 = \mathcal{B}_2|_{k=0}$ into (4.27) we are forced to commute a few double sums to reach the structure (4.21). The final result is

$$\begin{aligned} C_{st} &= \sum_{c=0}^s \frac{s!}{(s-c)!} \sum_{k=1}^{\infty} \left(\frac{\epsilon^k M(0, t+k, c+k, s-c)}{c! k! (c+k)} \right. \\ &\quad \left. - \sum_{\ell=0}^{\min(t, c+k)} \frac{t!}{(t-\ell)!} \sum_{a=0}^{c+k-\ell} \frac{\epsilon^{k-\ell} (-1)^\ell (c+k-1)! M(c+k-\ell-a, t+k-\ell, a, s-c)}{k! c! \ell! a! (c+k-\ell-a)! 2^{c+k}} \right) \\ &\quad + \theta_{s \geq 1} \sum_{c=1}^s \frac{s!}{(s-c)!} \sum_{\ell=0}^{\min(t, c)} \sum_{a=0}^{c-\ell} \frac{t!}{(t-\ell)!} \frac{(1-2^{-c}) (-1)^\ell M(c-\ell-a, t-\ell, a, s-c)}{\epsilon^\ell c! \ell! a! (c-\ell-a)!} \quad . \end{aligned} \quad (4.31)$$

We shall work with (4.31) neither by pen nor on screen. But it is an easy task to write (4.31) into a file. MAPLE performs the sums for index pairs st of interest. It also processes ${}^2L_{st}$, (4.24), and adds ${}^1L_{st}$, (4.18). The coefficients $M(a, b, c, d)$ appearing in L can be made explicit by hand on screen and read in. Hence all L_{st} of interest are known and expressed by the currents j . A list of some L 's is shown in Appendix D.

We are quite sure that every detail of this section 4.3 is correct. This rests on various successful tests. \mathcal{T} applied to a holomorphic invariant must result in a linear combination of holomorphic invariants. Any erraneous little detail in e.g. (4.31) (and that had happened) would lead to the desaster of non-holomorphic remnants in the result (and that had led to find the error).

The above section is the last one which avoids approximations or truncations. The kinetic energy operator \mathcal{T} – see (2.8), (3.8), (4.18), (4.17) and all about C 's and L 's – contains the ϵ -powers to all orders. It remained general. The sad counterpart is that the whole apparatus developed above will by far not be exhausted in what follows.

————— end of the general analysis that keeps ϵ arbitrary —————

5 Results of \mathcal{T} -application

$$\begin{aligned} \mathcal{T} Q_0 &= -\frac{\aleph}{2\epsilon^2} + \frac{13}{8} Q_0 + \epsilon \frac{5}{4} Q_1 + \epsilon^2 \left(\frac{19}{384} Q_2 + \frac{1}{48} R_{21} - \frac{5}{32} R_{20} \right) \\ &\quad + \epsilon^3 \left(\frac{7}{576} Q_3 + \frac{35}{288} R_{32} - \frac{77}{144} R_{31}^{(1)} - \frac{49}{144} R_{31}^{(2)} \right) + \mathcal{O}(\epsilon^4) \quad . \quad (5.1) \end{aligned}$$

$$\begin{aligned} \mathcal{T} Q_1 &= +\frac{\aleph}{2\epsilon^3} + \frac{69}{32} Q_1 + \epsilon \left(\frac{149}{128} Q_2 - \frac{11}{8} R_{21} + \frac{3}{32} R_{20} \right) \\ &\quad + \epsilon^2 \left(\frac{33}{512} Q_3 - \frac{27}{256} R_{32} + \frac{165}{128} R_{31}^{(1)} + \frac{105}{128} R_{31}^{(2)} \right) + \mathcal{O}(\epsilon^3) \quad . \quad (5.2) \end{aligned}$$

$$\begin{aligned} \mathcal{T} Q_2 &= -\frac{3\aleph}{4\epsilon^4} + \frac{1}{8\epsilon^2} Q_0 + \frac{3}{2\epsilon} Q_1 + \frac{1943}{768} Q_2 - \frac{25}{96} R_{21} + \frac{7}{64} R_{20} \\ &\quad + \epsilon \left(\frac{35}{32} Q_3 - \frac{43}{16} R_{32} - \frac{13}{8} R_{31}^{(1)} - \frac{11}{8} R_{31}^{(2)} \right) + \mathcal{O}(\epsilon^2) \quad . \quad (5.3) \end{aligned}$$

$$\begin{aligned} \mathcal{T} Q_3 &= +\frac{3\aleph}{2\epsilon^5} + \frac{3}{8\epsilon^3} Q_0 - \frac{103}{32\epsilon^2} Q_1 + \frac{1}{\epsilon} \left(\frac{259}{64} Q_2 + \frac{43}{32} R_{21} \right) \\ &\quad + \frac{2749}{1024} Q_3 - \frac{1333}{1536} R_{32} + \frac{1403}{768} R_{31}^{(1)} + \frac{1111}{768} R_{31}^{(2)} + \mathcal{O}(\epsilon) \quad . \quad (5.4) \end{aligned}$$

$$\mathcal{T} R_{20} = -\frac{5}{2\epsilon^2} Q_0 + \frac{13}{4} R_{20} + \epsilon \left(-\frac{25}{8} R_{31}^{(1)} - \frac{7}{4} R_{31}^{(2)} + \frac{9}{4} S \right) + \mathcal{O}(\epsilon^2) \quad . \quad (5.5)$$

$$\begin{aligned} \mathcal{T} R_{21} &= +\frac{1}{8\epsilon^2} Q_0 + \frac{3}{16\epsilon} Q_1 - \frac{3}{64} Q_2 + \frac{45}{16} R_{21} - \frac{13}{32} R_{20} \\ &\quad + \epsilon \left(\frac{29}{32} R_{31}^{(1)} - \frac{13}{16} R_{31}^{(2)} \right) + \mathcal{O}(\epsilon^2) \quad . \quad (5.6) \end{aligned}$$

$$\begin{aligned} \mathcal{T} R_{31}^{(1)} &= -\frac{1}{2\epsilon^3} Q_0 + \frac{1}{2\epsilon^2} Q_1 - \frac{1}{8\epsilon} R_{21} \\ &\quad + \frac{1}{32} R_{32} + \frac{133}{32} R_{31}^{(1)} + \frac{17}{64} R_{31}^{(2)} - \frac{3}{4} S + \mathcal{O}(\epsilon) \quad . \quad (5.7) \end{aligned}$$

$$\begin{aligned} \mathcal{T} R_{31}^{(2)} &= -\frac{1}{4\epsilon^3} Q_0 + \frac{1}{4\epsilon^2} Q_1 - \frac{1}{4\epsilon} R_{21} \\ &\quad - \frac{1}{16} R_{32} - \frac{1}{8} R_{31}^{(1)} + \frac{7}{2} R_{31}^{(2)} + \frac{1}{4} S + \mathcal{O}(\epsilon) \quad . \quad (5.8) \end{aligned}$$

$$\begin{aligned} \mathcal{T} R_{32} &= +\frac{1}{4\epsilon^2} Q_1 + \frac{1}{8\epsilon} Q_2 + \frac{5}{8\epsilon} R_{21} \\ &\quad - \frac{7}{384} Q_3 + \frac{637}{192} R_{32} + \frac{125}{96} R_{31}^{(1)} + \frac{35}{48} R_{31}^{(2)} + \mathcal{O}(\epsilon) \quad . \quad (5.9) \end{aligned}$$

Obviously, on the left hand sides \mathcal{T} is applied to all the invariants named in (4.9) to (4.11). The necessity to include ϵ -powers diminishes with increasing index on Q or left on R . On the right hand sides there are two unknown quantities, which have to be invariants too. The most simple holomorphic invariant of the form (4.8) is obtained by choosing

“any” = 1 :

$$\aleph = \frac{1}{N} \int (1)^{aa} = \frac{N^2 - 1}{N} F \quad (5.10)$$

with F the divergent area of the plane. The equation $\mathcal{T}\mathcal{P} = \mathcal{E} - \mathcal{V}$ to be solved shows, that \aleph -terms can be absorbed in \mathcal{E} . The quantity S

$$S := \frac{1}{N^2} \int (f^a \bar{j} \bar{j} f^a \bar{j} \mathcal{D} \bar{j})^{cc} \quad , \quad (5.11)$$

is also a holomorphic invariant (sum over a). To check this invariance, use $h^{ab} = 2 \text{Tr}(T^a h T^b h^{-1})$, $f^{eas} T^s = -i [T^e, T^a]$ and $2 \text{Tr}(\dots T^e) 2 \text{Tr}(T^e \dots) = 2 \text{Tr}(\dots)$ to get $(h^{-1} f^a h)^{cr} = f^{cdr} h^{ad}$. Hence the integrand $(\quad)^{cc} =: S_{\text{int}}$ of S transforms into

$$\begin{aligned} S_{\text{int}} &\rightarrow (h^{-1} f^a h)^{cr} (\bar{j} \bar{j})^{rs} (h^{-1} f^a h)^{st} (\bar{j} \mathcal{D} \bar{j})^{tc} \\ &= f^{cdr} (h^{-1})^{da} (\bar{j} \bar{j})^{rs} f^{sgt} h^{ag} (\bar{j} \mathcal{D} \bar{j})^{tc} = (f^g \bar{j} \bar{j} f^g \bar{j} \mathcal{D} \bar{j})^{cc} = S_{\text{int}} \quad , \quad \text{q.e.d.} \quad (5.12) \end{aligned}$$

All invariants occurring on the right hand sides of (5.1) to (5.9) might also appear under \mathcal{T} -operation on the left. The next section will explain why. $\mathcal{T}\aleph = 0$ is no problem. However through the evaluation of $\mathcal{T}S$ some unsurmountable difficulties will become obvious in Section 6.3.

6 The first steps in solving $\mathcal{T}\mathcal{P} = \mathcal{E} - \mathcal{V}$

The solution \mathcal{P} to this equation is some function of ϵ . Only after this function is obtained the limit $\epsilon \rightarrow +0$ can be performed. This requirement is violated from the outset if we truncate the *ansatz* for \mathcal{P} at some highest power n of ϵ (“step n ”). Let us nevertheless do so – with thoughts on truncated perturbative expansions.

In the most simple \mathcal{P} -*ansatz* let no positive ϵ -power be allowed at all: **step zero**. Remember that \mathcal{V} equals Q_0 (combine (1.7) with (1.1)). With the

$$\text{ansatz} \quad \mathcal{P} = c_0 Q_0 \quad . \quad (6.1)$$

The equation reads $c_0 \mathcal{T}Q_0 = \mathcal{E} - Q_0$. Omitting the positive ϵ powers in (5.1) one obtains

$$c_0 = -\frac{8}{13} \quad \text{and} \quad \mathcal{E} = \frac{4}{13 \epsilon^2} \aleph \quad . \quad (6.2)$$

The result $\mathcal{P} = -\frac{8}{13} Q_0$ differs from (1.8) i.e. from $-\frac{1}{2} Q_0$, – the well known misfortune.

6.1 Step one : including ϵ^1

Now \mathcal{P} must contain ϵ^0 and ϵ^1 . The mass dimension m^2 of all terms in $\mathcal{T}\mathcal{P} = \mathcal{E} - Q_0$ is dictated by the inhomogeneity Q_0 (ϵ has m^{-2} , Q_1 has m^4). So we are lead to the

$$\text{ansatz} \quad \mathcal{P} = c_0 Q_0 + c_1 \epsilon Q_1 \quad (6.3)$$

and expect two equations for the coefficients by comparison. With (5.1) and (5.2) one obtains

$$(-c_0 + c_1) \frac{\aleph}{2\epsilon^2} + c_0 \frac{13}{8} Q_0 + \epsilon \left(c_0 \frac{5}{4} + c_1 \frac{69}{32} \right) Q_1 = \mathcal{E} - Q_0 \quad , \quad (6.4)$$

and hence

$$c_0 = -\frac{8}{13} \quad , \quad c_1 = \frac{320}{13 * 69} \quad \text{and} \quad \mathcal{E} = \frac{\aleph}{2\epsilon^2} \left(\frac{320}{69 * 13} + \frac{8}{13} \right) \quad . \quad (6.5)$$

The limit $\epsilon \rightarrow +0$ “afterwards” reduces \mathcal{P} to $-\frac{8}{13} Q_0$ again. No change.

6.2 Step two : including ϵ^2

\mathcal{P} must contain ϵ^0 , ϵ^1 and ϵ^2 . Already (5.1) shows that now the invariants R_{21} and R_{20} come into play. It will be seen that their inclusion has influence on the Q_0 -prefactor in question. Inspecting (5.5) and (5.6) one may expect a closed set of equation. However five coefficients are to be distinguished :

$$\text{ansatz} \quad \mathcal{P} = c_0 Q_0 + c_1 \epsilon Q_1 + c_2 \epsilon^2 Q_2 + c_3 \epsilon^2 R_{21} + c_4 \epsilon^2 R_{20} \quad . \quad (6.6)$$

Now the \mathcal{T} -applications (5.1) to (5.3) and (5.5), (5.6) are in use. We compare the coefficients of Q_0 , ϵQ_1 , $\epsilon^2 Q_2$, $\epsilon^2 R_{21}$ and $\epsilon^2 R_{20}$. The resulting 5 equations can be read off from the following MAPLE program :

$$\begin{aligned} \text{resu} &:= \text{solve}(\{ 13 * c_0 + c_2 + c_3 - 20 * c_4 = -8, \\ &\quad 40 * c_0 + 69 * c_1 + 48 * c_2 + 6 * c_3 = 0, \\ &\quad 38 * c_0 + 894 * c_1 + 1943 * c_2 - 36 * c_4 = 0, \\ &\quad 25 * c_0 - 132 * c_1 - 25 * c_2 + 270 * c_3 = 0, \\ &\quad -10 * c_0 + 6 * c_1 + 7 * c_2 - 26 * c_3 + 208 * c_4 = 0 \}, \{ c_0, c_1, c_2, c_3, c_4 \}); \quad . \quad (6.7) \end{aligned}$$

Again the limit $\epsilon \rightarrow +0$ “afterwards” reduces \mathcal{P} to the first term, i.e. only c_0 remains of interest for \mathcal{P} :

$$\begin{aligned} \text{request} &: c_0 = -0.5000 \\ \text{step 1} &: c_0 = -0.6154 = -8/13 \\ \text{step 2} &: c_0 = -0.6330 \\ (\text{no R's}) &: c_0 = -0.5986 \end{aligned} \quad (6.8)$$

Aside, the energy is $\mathcal{E} = (-2c_0 + 2c_1 - 3c_3) \aleph / (4\epsilon^2)$. Obviously, step 2 has led further away from the “request”. But one should not expect too much from the present lowest non-trivial order. The important outcome is merely **that** c_0 **does change** under the inclusion of higher ϵ powers.

In the last line of (6.8) we were curious what happens when R_{21} and R_{20} were deleted from the *ansatz* and from (5.1) to (5.3). Then (6.7) would reduce to the first three lines with $c_3 = c_4 = 0$. So, the R 's *do* influence the value of c_0 . Stimulated by the above we might study one more step at least.

6.3 The problems with step three : including ϵ^3

Now \mathcal{P} must include ϵ^0 to ϵ^3 , and (at least) all \mathcal{T} -applications (5.1) to (5.9) become relevant. In addition $\mathcal{T}S$, see (5.11)), has to be evaluated, but this makes trouble:.

$$\mathcal{T}S = \frac{1}{2\epsilon^3} Q_0 - \frac{1}{2\epsilon^2} Q_1 + \frac{1}{16\epsilon} R_{21} - \frac{1}{2\epsilon} S^{(2)} + \mathcal{O}(\epsilon^0) \quad . \quad (6.9)$$

In (6.9) one more object appears. $S^{(2)}$ is a holomorphic invariant too and reads

$$S^{(2)} := \frac{1}{N^3} \int (f^a \bar{j} f^b f^a \bar{j} f^b \mathcal{D}\bar{j})^{cc} \quad (\text{sum over } a \text{ and } b)^5 \quad . \quad (6.10)$$

It has mass dimension m^6 (see end of § 4.1). S and all terms in (6.9) have mass dimension m^8 . In (6.9) the term $\mathcal{O}(\epsilon^0)$ is not made explicit (but should be). It is again an invariant and has the structure of $S^{(2)}$ with one more \mathcal{D} in. Clearly, now \mathcal{T} has to be applied to both of these new invariants. It may be supposed that thereby even more holomorphic invariants come into play. And so on, and so on – a disaster. As a warning, further steps in the desert become quite time consuming.

Is step three at its very end? There is a last chance. We abandon S from the *ansatz*:

$$\mathcal{P} = c_0 Q_0 + c_1 \epsilon^2 Q_1 + c_2 \epsilon^3 Q_3 + c_4 \epsilon^2 R_{21} + c_5 \epsilon^2 R_{20} + c_6 \epsilon^3 R_{32} + c_7 \epsilon^3 R_{31}^{(1)} + c_8 \epsilon^3 R_{31}^{(2)} \quad . \quad (6.11)$$

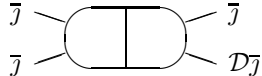
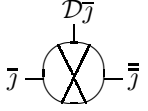
Then via (5.5), (5.8), (5.9) a term $\Delta := (9c_5 - 3c_7 + c_8)\epsilon^3 S/4$ remains in \mathcal{TP} . May we hope for the miracle that this term vanishes? MAPLE solves the 9 equations for c_0 to c_8 with ease. It produces $c_0 = -0.5562$ – and $\Delta = -0.275 \epsilon^3 S \neq 0$: no miracle.

7 Conclusions

The kinetic energy operator \mathcal{T} is worked out as a power series in the regularization parameter ϵ . \mathcal{T} contains all powers of ϵ and their prefactors are made explicit. \mathcal{T} is applied to various holomorphic invariants and leads to linear combination of such invariants. Admittedly the detailed results include only the third power of ϵ at highest.

To obtain the ground state $\psi = e^P$ (with P restricted to its leading nontrivial term towards $m \rightarrow \infty$) the Hamilton-Jacobi type equation $\mathcal{TP} = \mathcal{E} - \mathcal{V}$ needs to be solved for \mathcal{P} . But actually only ϵ^1 and ϵ^2 could have been included in an *ansatz* for \mathcal{P} .

So far tested, the point splitting regularization works well and is thus hopefully freed from possible doubts. Due to the difficulties with breaking apart the ϵ power series, one might think about to sum it up. This could be possible, but then: who has the strength to guess a suitable *ansatz* for \mathcal{P} ?

⁵ S and $S^{(2)}$ can be depicted as  and , respectively.

Appendix A : How (2.5) and (2.8), (2.9), (2.10) come about

————— (2.5) :

When restricting \mathcal{T} to act in the space $\psi [j_r^a]$ and since $p_{r^m}^v$ is the operator at the right end of (2.1) the first detail to be studied is $p_{r^m}^v \psi [j_r^a]$. With (2.3) this reads

$$p_{r^m}^v \psi = \partial'' (M_{r^m}^{-1})^{vc} \delta_{Ac(\vec{r}'')} \psi = \partial'' M_{r^m}^{cv} \int'''' \left(\delta_{Ac(\vec{r}'')} j_{r^m}^e \right) \delta_{j_{r^m}^e} \psi \quad . \quad (\text{A.1})$$

For the above round bracket one may combine $j^e = 2 \text{Tr} (T^e (\partial H) H^{-1})$ with $H = M^\dagger M$ to get $j^e = 2 \text{Tr} (T^e (\partial M^\dagger) M^{\dagger-1} + T^e M^\dagger (\partial M) M^{-1} M^{\dagger-1})$. Now $(\partial M) M^{-1} = iT^b A^b$ (see e.g. p.22 of [5]) and the A -independence of M^\dagger lead to

$$\delta_{Ac(\vec{r}'')} j_{r^m}^e = i (M_{r^m}^\dagger)^{ec} \delta(\vec{r}'' - \vec{r}''') \quad (\text{A.2})$$

Using (A.2) in (A.1) one arrives at (2.5) in the main text.

————— (2.8) :

As explained below (2.7) the kinetic energy operator \mathcal{T} splits into three parts. The first is $\mathcal{T}_1 = \int \omega_r^{\heartsuit a} \delta_{j_r^a}$ and requires evaluation of

$$\overline{p}_{r^m}^u \mathcal{G}_{rr^m}^{bv} \stackrel{=}{=} -\overline{\partial}' (M_{r^m}^\dagger)^{ud} \delta_{Ad^*(\vec{r}')}' G_{rr^m} \left(\delta^{bv} - e_{rr^m} (H_{z''\overline{z}''}^{-1} H_{z''\overline{z}''})^{bv} \right) \quad , \quad (\text{A.3})$$

where e_{rr^m} is shorthand for $\exp(-(\vec{r} - \vec{r}'')^2/\epsilon)$,

$$\stackrel{=}{=} \overline{\partial}' (M_{r^m}^\dagger)^{ud} G_{rr^m} e_{rr^m} \left[(\mathcal{R} \delta_{Ad^*(\vec{r}')}' H_{r^m}^{cb}) H_{r^m}^{cv} + (\mathcal{R} H_{r^m}^{cb}) \delta_{Ad^*(\vec{r}')}' H_{r^m}^{cv} \right] \quad , \quad (\text{A.4})$$

where the operator \mathcal{R} replaces \overline{z}'' by \overline{z} within its round bracket .

To proceed with (A.4) we need

$$\delta_{Ad^*(\vec{r}')}' H_{r^m}^{cb} = \overline{G}_{r^m r^m} (M_{r^m}^\dagger)^{gd} f^{cgh} H_{r^m}^{hb} \quad (\text{A.5})$$

and the same with v in place of b . To obtain (A.5) derive from $M^{\dagger-1} \overline{\partial} M^\dagger = -iT^a A^{a*}$ a differential equation for $\delta_{Ad^*(\vec{r}')}' M_{r^m}^\dagger$ and solve it to get

$$\delta_{Ad^*(\vec{r}')}' M_{r^m}^\dagger = -i M_{r^m}^\dagger T^d M_{r^m}^{\dagger-1} \overline{G}_{r^m r^m} M_{r^m}^\dagger \quad \text{or} \quad \delta_{Ad^*(\vec{r}')}' H_{r^m} = -i \overline{G}_{r^m r^m} M_{r^m}^\dagger T^d M_{r^m}^{\dagger-1} H_{r^m} \quad , \quad (\text{A.6})$$

since in $H = M^\dagger M$ only M^\dagger depends on A^* . Now using this and $H_{r^m}^{cb} = 2 \text{Tr} (T^c H_{r^m} T^b H_{r^m}^{-1})$ on the lhs of (A.5) one obtains

$$\delta_{Ad^*(\vec{r}')}' H_{r^m}^{cb} = -i \overline{G}_{r^m r^m} 2 \text{Tr} (T^c D B - T^c B D) = -i \overline{G}_{r^m r^m} 2 \text{Tr} ([T^h, T^c] D) 2 \text{Tr} (T^h B) \quad (\text{A.7})$$

with the temporary abbreviations $D = M_{r^m}^\dagger T^d M_{r^m}^{\dagger-1}$ and $B = H_{r^m} T^b H_{r^m}^{-1}$. From (A.7) one derives the rhs of (A.5) indeed.

Using (A.5) in (A.4) we arrive at

$$\overline{p}_{r^m}^u \mathcal{G}_{rr^m}^{bv} = G_{rr^m} e_{rr^m} f^{uhc} H_{r^m}^{hv} H_{z''\overline{z}''}^{cb} \left(\overline{\partial}' \overline{G}_{r^m r^m} - \mathcal{R} \overline{\partial}' \overline{G}_{r^m r^m} \right) \quad . \quad (\text{A.8})$$

Note that $\bar{\partial}'$ and \mathcal{R} do commute. The round bracket in (A.8) restricts their action. \bar{G} is ε -regulated, see (2.4). Due $\varepsilon \rightarrow 0$ the first term may be replaced by $-\delta(\vec{r}'' - \vec{r}')$ but the second requires more care:

$$-\mathcal{R} \bar{\partial}' \bar{G}_{r''r'} = \frac{\varepsilon^2/\pi}{[(z'' - z')(\bar{z} - \bar{z}') + \varepsilon^2]^2} = e^{(\bar{z}'' - \bar{z}')\bar{\partial}'} \frac{\varepsilon^2/\pi}{[(z'' - z')(\bar{z}'' - \bar{z}') + \varepsilon^2]^2} , \quad (\text{A.9})$$

where the exponential is a translation operator ($\bar{z}' \rightarrow \bar{z}' + \bar{z}'' - \bar{z}$).

The couple (A.8) (including (A.9)) is now well prepared for insertion into (2.6) at the right inner position. Thereby slightly rearranged the result \mathcal{T}_1 reads

$$\begin{aligned} \mathcal{T}_1 &= \frac{i\pi}{N} \int \int' \int'' G_{r''r''} e_{r''r''} f^{uhc} H_r^{ab} H_{z''\bar{z}''}^{cb} \left(-\bar{\mathcal{G}}_{r''r''}^{au} \delta(\vec{r}'' - \vec{r}') \right. \\ &\quad \left. + \frac{\varepsilon^2/\pi}{[(z'' - z')(\bar{z}'' - \bar{z}') + \varepsilon^2]^2} e^{(\bar{z} - \bar{z}')\bar{\partial}'} \bar{\mathcal{G}}_{r''r''}^{au} \right) H_{r''}^{hv} \partial'' H_{r''}^{ev} \delta_{j_{r''}^e} . \end{aligned} \quad (\text{A.10})$$

Obviously, under way from (A.9) to (A.10), the translation operator has changed its place. This was justified by partial integration under \int' . The right end of (A.10) that follows the round bracket can be dealt with as $[\partial'' + j_{r''}]^{eh} \delta_{j_{r''}^e}$.

We now encounter a surprizing problem in forming the limit $\varepsilon \rightarrow 0$ (it is the ‘‘harmless’’ ε). In (A.10) the second term in the round bracket consists of three factors (counting $\bar{\mathcal{G}}_{r''r''}^{au}$ as two, see (2.2), and including only $\bar{G}_{r''r''}$ next). The product of the first two reads

$$\mathcal{P} := \frac{\varepsilon^2/\pi}{[(z'' - z')(\bar{z}'' - \bar{z}') + \varepsilon^2]^2} \cdot \frac{(\bar{z}'' - \bar{z}')/\pi}{(z - z')(\bar{z}'' - \bar{z}') + \varepsilon^2} . \quad (\text{A.11})$$

If (A) the limit is first performed in the first fraction, giving $\delta(\vec{r}'' - \vec{r}')$, $\mathcal{P} = 0$ is obtained. But if (B) the limit is first done in the second fraction one ends up with $\mathcal{P} = \delta(\vec{r}'' - \vec{r}') \cdot \frac{1}{\pi(z - z')}$. Who wins – (A) or (B) or none of? To resolve this puzzle we integrate \mathcal{P} over all \vec{r}'' and ignore the less important dependences on \vec{r}'' outside \mathcal{P} :

$$\mathcal{Q} := \int d^2 r'' \mathcal{P} = \int d^2 r'' \frac{\varepsilon^2/\pi}{[r''^2 + \varepsilon^2]^2} \cdot \frac{\bar{z}''/\pi}{(z - z')\bar{z}'' + \varepsilon^2} . \quad (\text{A.12})$$

With $(z - z') = \omega e^{i\beta}$ (ω real), $\bar{z}'' = \rho e^{i(\varphi - \beta)}$ (hence $z'' = \rho e^{-i(\varphi - \beta)}$) and with the notation $\tilde{\varepsilon} := \varepsilon/(\omega\rho)$ the integral turns into

$$\mathcal{Q} = \frac{1}{\pi^2 \omega e^{i\beta}} \int_0^\infty d\rho \frac{\rho}{[\rho^2 + 1]^2} \mathcal{J} \quad \text{with} \quad \mathcal{J} = \int_{-\pi}^\pi d\varphi \frac{1}{1 + \tilde{\varepsilon} e^{-i\varphi}} = \dots = 2\pi\theta(\omega\rho - \varepsilon) \quad (\text{A.13})$$

and θ the step function. In the final result for \mathcal{Q} the limit $\varepsilon \rightarrow 0$ is unambiguous:

$$\mathcal{Q} = \frac{1/\pi}{z - z'} \int_{\varepsilon/\omega}^\infty d\rho \frac{2\rho}{[\rho^2 + 1]^2} \longrightarrow \frac{1}{\pi(z - z')} . \quad (\text{A.14})$$

Version (B) wins. We conclude that \mathcal{P} can be replaced by $\frac{1}{\pi(z - z')} \delta(\vec{r}'' - \vec{r}')$.

The Delta function is now common to both terms in the round bracket of (A.10) and helps performing \int'' . The intermediate result for \mathcal{T}_1 so far reached is

$$\begin{aligned} \mathcal{T}_1 &= \frac{i}{N} \int \int' \frac{e_{rr'}}{\pi(\vec{r} - \vec{r}')^2} f^{cuh} . \\ &\quad \left(e_{rr'} H_r^{ab} H_{z'\bar{z}'}^{cb} H_{z\bar{z}}^{ad} H_{r'}^{ud} - H_r^{ab} H_{z'\bar{z}'}^{cb} H_r^{ad} H_{z'\bar{z}'}^{ud} \right) [\partial' + j_{r'}]^{eh} \delta_{j_{r'}^e} . \end{aligned} \quad (\text{A.15})$$

Note that the second term in the round bracket reduces to δ^{cu} . So it vanishes in contact with f^{cuh} . By partial integration the square bracket can be placed as $[-\partial' + j_{r'}]^{eh}$ at the beginning of the integrand. Then of course $-\partial'$ does no more act on $\delta_{j_{r'}^e}$. Finally we interchange primed and unprimed variables. Remembering $\mathcal{T}_1 = \int \omega_r^{\diamond a} \delta_{j_r^a}$ the object $\omega_r^{\diamond a}$ can now be read off to be (2.8) in the main text.

————— (2.9) :

To learn about $\mathcal{T}_2 = \int \omega_r^{\diamond a} \delta_{j_r^a}$ the functional derivative \bar{p}_r^u in (2.7) has to be placed in front of $H_{r''}^{ev}$. Using (2.3) and (A.5) we have

$$\begin{aligned} \bar{p}_{r''}^u H_{r''}^{ev} &= -\bar{\partial}' (M_{r'}^\dagger)^{ud} \delta_{A^{d*}(\vec{r}')} H_{r''}^{ev} \\ &= -\bar{\partial}' (M_{r'}^\dagger)^{ud} \bar{\mathcal{G}}_{r''r'} (M_{r'}^\dagger)^{gd} f^{egh} H_{r''}^{hv} = -\delta(\vec{r}'' - \vec{r}') f^{euh} H_{r''}^{hv} \end{aligned} \quad (\text{A.16})$$

and consequently

$$\begin{aligned} \mathcal{T}_2 &= -\frac{\pi}{N} \int \int' \int'' H_r^{ab} \bar{\mathcal{G}}_{r'r'}^{au} \mathcal{G}_{r''r'}^{bv} i\partial'' \delta(\vec{r}'' - \vec{r}') f^{euh} H_{r''}^{hv} \delta_{j_{r''}^e} \\ &= \frac{i\pi}{N} \int \int' H_r^{ab} \bar{\mathcal{G}}_{r'r'}^{au} (\partial' \mathcal{G}_{r'r'}^{bv}) f^{euh} H_{r''}^{hv} \delta_{j_{r''}^e} \text{ or by just changing notation} \\ &\quad \vec{r} \rightarrow \vec{r}'', \vec{r}' \rightarrow \vec{r}, v \rightarrow c, h \rightarrow v, a \rightarrow d, e \rightarrow a : \\ &= \int \left[\frac{i}{N} f^{auv} (-\pi) \int'' \bar{\mathcal{G}}_{r''r'}^{du} H_{r''}^{db} (\partial \mathcal{G}_{r''r'}^{bc}) (H_r^{-1})^{cv} \right] \delta_{j_r^a} . \end{aligned} \quad (\text{A.17})$$

The square bracket is $\omega_r^{\diamond a}$ indeed, see (2.9).

————— (2.10) :

According to the text below (2.7) for the third part $\mathcal{T}_3 = \iint' \Omega_{r'r'}^{ab} \delta_{j_r^a} \delta_{j_{r'}^b}$, we have to place \bar{p}_r^u in front of $\delta_{j_{r''}^e} \psi[j]$ ($=: \chi[j]$ for brevity).

$$\bar{p}_r^u \chi = -\bar{\partial}' (M_{r'}^\dagger)^{ud} \delta_{A^{d*}(\vec{r}')} \chi = -\bar{\partial}' (M_{r'}^\dagger)^{ud} \int'' \left(\delta_{A^{d*}(\vec{r}')} j_{r''}^c \right) \delta_{j_{r''}^c} \chi \quad (\text{A.18})$$

The round bracket is evaluated via $j_{r''}^c = 2 \text{Tr}(T^c (\partial'' H_{r''}) H_{r''}^{-1})$ and the right half of (A.6). Again abbreviating $M_{r'}^\dagger T^d M_{r'}^{\dagger-1}$ by D it follows that

$$\begin{aligned} \delta_{A^{d*}(\vec{r}')} j_{r''}^c &= -i(\partial'' \bar{\mathcal{G}}_{r''r'}) 2 \text{Tr}(T^c D) - i\bar{\mathcal{G}}_{r''r'} 2 \text{Tr}([T^c, D] j_{r''}) \\ &= -i(\partial'' \bar{\mathcal{G}}_{r''r'}) (M_{r'}^\dagger)^{cd} - i\bar{\mathcal{G}}_{r''r'} 2 \text{Tr}([T^c, D] T^p) 2 \text{Tr}(T^p j_{r''}) \\ &= -i(M_{r'}^\dagger)^{cd} \partial'' \bar{\mathcal{G}}_{r''r'} + (M_{r'}^\dagger)^{qd} \bar{\mathcal{G}}_{r''r'} f^{cqp} j_{r''}^p . \end{aligned} \quad (\text{A.19})$$

Using (A.19) in (A.18) we obtain

$$\begin{aligned}\bar{p}_r^u \chi &= \int'' (-i\delta^{uc} \partial'' \delta(\vec{r}'' - \vec{r}') + \delta(\vec{r}'' - \vec{r}') f^{cup} j_{r''}^p) \delta_{j_{r''}^c} \chi \\ &= i(\partial' - j_{r'})^{uc} \delta_{j_{r'}^c} \chi\end{aligned}\quad (\text{A.20})$$

to be inserted in (2.7):

$$\begin{aligned}\mathcal{T}_3 &= \frac{\pi}{N} \int \int' \int'' H_r^{ab} \bar{\mathcal{G}}_{r'r}^{au} \mathcal{G}_{r'r''}^{bv} i\partial'' H_{r''}^{ev} i(\partial' - j_{r'})^{uc} \delta_{j_{r''}^c} \delta_{j_{r''}^e} \ , \\ &\quad \text{partial integrations and } \vec{r}, \vec{r}', \vec{r}'' \rightarrow \vec{r}'', \vec{r}, \vec{r}' : \\ &= \int \int' \frac{1}{N} (\partial - j_r)^{cu} (-\pi) \int'' \bar{\mathcal{G}}_{r''r}^{au} H_{r''}^{ab} (\partial' \mathcal{G}_{r''r'}^{bv}) (H_{r'}^{-1})^{ve} \delta_{j_{r''}^c} \delta_{j_{r''}^e} \ .\end{aligned}\quad (\text{A.21})$$

Now (2.10) can be read off from (A.21). End of [1]–recapitulations.

Appendix B : Evaluation of the integral $I_{p,q}(\epsilon, \sigma)$, (3.15)

At first we drop the double primes in (3.15) and set $z = re^{-i\varphi}$, $\bar{z} = re^{i\varphi}$:

$$I_{p,q}(\epsilon, \sigma) = \frac{1}{\pi} \int_0^\infty dr r e^{-r^2/\epsilon} r^{p+q-1} \int d\varphi \frac{(e^{i\varphi})^{q-p-1}}{re^{-i\varphi} + \sigma} \ .\quad (\text{B.1})$$

Next we write $\sigma = |\sigma|e^{-i\chi}$ and

$$I_{p,q}(\epsilon, \sigma) = \frac{1}{\pi} \int_0^\infty dr r e^{-r^2/\epsilon} r^{p+q-1} \sigma^{p-q} E_{q-p} \quad \text{with}\quad (\text{B.2})$$

$$E_n := \int d\varphi \frac{|\sigma|^n (e^{-i\chi})^n (e^{i\varphi})^{n-1}}{re^{-i\varphi} + |\sigma|e^{-i\chi}} = \int d\varphi \frac{(|\sigma|e^{i\varphi})^n}{r + |\sigma|e^{i\varphi}} \ ,\quad (\text{B.3})$$

where the last step rests on the shift $\varphi \rightarrow \varphi + \chi$. Writing the numerator as $(|\sigma|e^{i\varphi})^{n-1} (|\sigma|e^{i\varphi} + r - r)$ one easily derives the recurrence relation

$$E_n = \delta_{n,1} 2\pi - r E_{n-1} \quad , \quad E_0 = \int d\varphi \frac{1}{r + |\sigma|e^{i\varphi}} = \frac{2\pi}{r} \theta(r - |\sigma|) \ .\quad (\text{B.4})$$

θ is the step function, cf. (A.13). The solution to the problem (B.4) is

$$E_n = \begin{cases} (-r)^{n-1} 2\pi \theta(|\sigma| - r) & \text{for } n \geq 1 \\ -(-r)^{n-1} 2\pi \theta(r - |\sigma|) & \text{for } n \leq 0 \end{cases} \ .\quad (\text{B.5})$$

With (B.5) at $n = q - p$, substituting $t = r^2$ and using $|\sigma|^2 = \sigma\bar{\sigma}$ from (3.13), (B.2) turns into

$$I_{q \leq p} = (-\sigma)^{p-q} \int_{\sigma\bar{\sigma}}^\infty dt t^{q-1} e^{-t/\epsilon} \quad \text{and} \quad I_{q > p} = -(-\sigma)^{p-q} \int_0^{\sigma\bar{\sigma}} dt t^{q-1} e^{-t/\epsilon} \ .\quad (\text{B.6})$$

Through $\int_0^{\sigma\bar{\sigma}} = \int_0^\infty - \int_{\sigma\bar{\sigma}}^\infty$ one arrives at (3.17) in the main text.

Appendix C : Example of a MAPLE–file

In evaluating $\mathcal{T}_3 Q_1$ the five steps described below (4.14) lead to

$\begin{aligned} & \text{aa9} := \\ & +2*\text{ab}*L23 \\ & +4*\text{abj}*L13 \\ & +6*\text{abtj}*L12 \\ & -6*\text{abttj}*L11 \\ & -4*\text{abtttj}*L10 \\ & +4*\text{abdj}*L03 \\ & +6*\text{abdtj}*L02 \\ & -6*\text{abdttj}*L01 \\ & -4*\text{abdtttj}*L00 \\ & +2*\text{ab}*jL13 \\ & +4*\text{abj}*jL03 \\ & +6*\text{abtj}*jL02 \\ & -6*\text{abttj}*jL01 \\ & -4*\text{abtttj}*jL00 : \end{aligned}$	<p>Of course a stands for f^a and b for f^b. Read the factor containing them as an adjoint trace, hence especially $\text{ab} = (\text{ab})^{cc} = f^{cad} f^{dbc} = -N\delta^{ab}$. But the factors containing L carry indices^{ab}. Hence e.g. $jL02 = (j * L_{02})^{ab}$.</p> <p>dttj is $\partial \bar{\partial}^2 j$, hence $\text{abdttj} = f^{cad} f^{dbe} \partial \bar{\partial}^2 j^{ec}$. Using $f^{cad} f^{dbe} f^{euc} = -(N/2) f^{abu}$ this turns into $(N/2) \partial \bar{\partial}^2 j^{ab}$.</p> <p>By the cyclic invariance of trace a can always be moved to the left end inside $()^{cc}$. By inversion of the content of $()^{cc}$ (the factors are anti-symmetric) also b can be placed near to a. But to have always ab at the left end is a speciality of the example $\mathcal{T}Q_1$ choosen.</p>
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Appendix D : List of the L's which occur in Appendix C

How L -coefficients look like. Here they include only the first power of ϵ (to limit the length of this paper). Each term carries indices^{ab}. ...j*something means ...j*something.

$$L00 = +j + \epsilon * \left(+3/8 * dtj + 3/8 * tjj - 3/8 * jtj \right) \quad (D.1)$$

$$\begin{aligned} L01 &= +1/2 * \frac{1}{\epsilon} + 1/4 * tj \\ &+ \epsilon * \left(+7/16 * dttj + 7/16 * ttjj - 7/16 * jttj + 1/8 * tjtj \right) \end{aligned} \quad (D.2)$$

$$\begin{aligned} L02 &= +\epsilon * \left(+15/32 * dtttj + 15/32 * tttjj - 15/32 * jtttj \right. \\ &\quad \left. +17/32 * ttjtj - 7/32 * tjttj \right) \end{aligned} \quad (D.3)$$

$$\begin{aligned} L03 &= -11/48 * tttj + \epsilon * \left(+31/64 * dttttj + 31/64 * ttttjj \right. \\ &\quad \left. +1 * tttjtj + 3/16 * ttjttj - 5/8 * tjtttj - 31/64 * jttttj \right) \end{aligned} \quad (D.4)$$

$$\begin{aligned} L10 &= +1/2 * dj - 1/2 * jj + \epsilon * \left(+7/24 * ddtj + 5/24 * dtjj - 7/24 * djtj \right. \\ &\quad \left. +7/24 * tjdj - 7/12 * jdtj - 1/12 * tjjj - 5/24 * jtjj + 7/24 * jjtj \right) \end{aligned} \quad (D.5)$$

$$\begin{aligned} L11 &= -1/2 * \frac{j}{\epsilon} - 1/4 * tjj + \epsilon * \left(+31/96 * ddtjtj + 5/24 * dttjj \right. \\ &\quad -1/12 * dtjtj - 31/96 * djttj + 31/96 * ttjdj - 5/48 * tjdtj \\ &\quad \left. -31/48 * jdttj - 11/96 * ttjjj - 5/24 * jttjj + 31/96 * jjttj \right) \end{aligned}$$

$$-11/48 * tjtjj + 1/48 * tjjtj + 1/12 * jtjtj) \quad (D.6)$$

$$\begin{aligned} L12 = & -1/4 * \frac{1}{\epsilon^2} - 1/4 * \frac{tj}{\epsilon} - 11/32 * dtjtj \\ & -11/32 * ttjj + 11/32 * jttj - 3/16 * tjtj \\ & + \epsilon * \left(+1/3 * ddtttj + 19/96 * dtttjj + 1/16 * dtjtj - 7/16 * dtjttj - 1/3 * djtttj \right. \\ & +1/3 * tttj dj + 1/8 * ttj dtj - 5/8 * tjdttj - 2/3 * jdtttj - 13/96 * tttjjj \\ & -19/96 * jtttjj + 1/3 * jjtttj - 13/32 * ttjtjj - 1/16 * ttjttj - 13/32 * tjtjjj \\ & \left. -1/16 * jttjtj + 3/16 * tjjttj + 7/16 * jtjttj - 1/8 * tjtjtj \right) \quad (D.7) \end{aligned}$$

$$\begin{aligned} L13 = & -1/16 * \frac{ttj}{\epsilon} - 61/96 * dtttj - 13/32 * tttjj + 61/96 * jtttj - 17/32 * ttjtj \\ & + 7/32 * tjttj \\ & + \epsilon * \left(+43/128 * ddttttj + 3/16 * dttttjj + 7/32 * dtttjtj - 15/32 * dtjtjj \\ & -19/16 * dtjtttj - 43/64 * djttttj - 43/64 * jdttttj - 39/32 * tjdtttj \\ & -45/64 * ttjdtj + 13/32 * tttj dtj + 37/128 * tttj dj - 19/128 * ttttjjj \\ & -3/16 * jtttjj + 43/128 * jjtttj - 19/32 * ttjtjj - 3/16 * tttjttj \\ & -19/32 * tjtttj - 7/32 * jtttjtj + 13/32 * tjjtttj + 13/16 * jtjtttj \\ & -57/64 * ttjttjj + 15/64 * ttjttj + 15/32 * jttjttj - 9/16 * ttjtjtj \\ & \left. -9/16 * tjttjtj + 15/32 * tjtjttj \right) \quad (D.8) \end{aligned}$$

$$\begin{aligned} L23 = & +1/4 * \frac{1}{\epsilon^3} + 3/8 * \frac{tj}{\epsilon^2} + 3/8 * \frac{dtjtj}{\epsilon} + 1/2 * \frac{ttjj}{\epsilon} - 3/8 * \frac{jttj}{\epsilon} + 3/8 * \frac{tjtj}{\epsilon} \\ & -79/96 * ddtttj - 3/8 * dtttjj - 1/4 * dtjtj + 25/32 * dtjttj + 79/96 * djtttj \\ & +79/48 * jdtttj + 35/32 * tjdttj - 13/32 * ttj dtj - 19/32 * tttj dj + 7/32 * tttjjj \\ & +3/8 * jtttjj - 79/96 * jjtttj + 21/32 * ttjtjj + 5/32 * ttjttj + 1/4 * jttjtj \\ & +21/32 * tjttjj - 5/16 * tjjttj + 25/32 * jtjttj + 5/16 * tjtjtj \\ & + \epsilon * \left(+129/512 * ddttttj + 43/512 * dttttjj - 1/128 * ddtttjtj - 139/256 * ddtjtjj \\ & -275/384 * dtjtttj - 129/512 * ddjtttj - 387/512 * jdttttj - 199/128 * tjddtttj \\ & -309/256 * ttjddttj + 9/128 * tttjddtj + 129/512 * tttjddj + 139/512 * dtttj dj \\ & +3/128 * dtttj dtj - 399/256 * dtjtttj - 261/128 * dtjdtttj - 387/512 * djdttttj \\ & -53/512 * dttttjjj - 53/128 * dtttjtjj - 5/128 * dtttjjtj + 121/256 * dtttjttj \\ & -159/256 * dtjtttjj - 15/128 * dtjtjtj - 53/128 * dtjtttjj + 233/384 * dtjjtttj \\ & -15/128 * dtjttjtj + 121/128 * dtjtjttj - 43/512 * djttttjj + 1/128 * djtttjtj \\ & +139/256 * djttjttj + 275/384 * djtjtttj + 129/512 * djjttttj - 43/256 * jdttttjj \\ & +1/64 * jdtttjtj + 139/128 * jdttjttj + 275/192 * jdtjtttj + 129/256 * jdjttttj \\ & -43/64 * tjdtttjj - 15/64 * tjdtjtj + 85/64 * tjdtjttj + 161/192 * tjdjtttj \\ & -129/128 * ttjdtjtj - 15/64 * ttjdtjtj + 85/128 * ttjdjttj - 43/64 * tttj dtj \\ & -5/64 * tttj dtj - 43/256 * tttj dj + 387/512 * jjdttttj + 137/128 * tjjdtttj \end{aligned}$$

$$\begin{aligned}
& +261/128 * jtjdtttj + 219/256 * ttjjdttj + 219/128 * tjtjdtj + 399/256 * jttjdtj \\
& -15/128 * tttjjdtj - 45/128 * ttjtjdtj - 45/128 * tjttjdtj - 3/128 * jtttjdtj \\
& -119/512 * ttttjjdj - 119/128 * tttjtj dj - 357/256 * ttjttjdj - 119/128 * tjtttjdj \\
& -139/512 * jttttjdj + 33/512 * ttttjjjj + 53/512 * jttttjjj + 43/512 * jjttttjj \\
& -129/512 * jjjttttj + 33/128 * tttjtjjj + 33/128 * tttjtjjj + 5/128 * tttjjjtj \\
& +5/128 * jtttjjtj + 53/128 * jtttjtjj + 33/128 * tjtttjjj + 33/128 * tjjtttjj \\
& +53/128 * jtjtttjj - 1/128 * jjtttjtj - 89/384 * tjjjtttj - 233/384 * jtjtttj \\
& -275/384 * jjtjtttj + 99/256 * ttjttjjj + 99/256 * ttjtttjj - 49/256 * ttjjjttj \\
& +159/256 * jttjtttj - 121/256 * jttjtttj - 139/256 * jjttjtttj + 99/128 * ttjtjtjj \\
& +15/128 * ttjtjtjj + 15/128 * ttjttjtj + 99/128 * tjttjtjj + 15/128 * tjttjtjj \\
& +15/128 * jttjtjtj + 99/128 * tjtjttjj + 15/128 * tjttjtjj + 15/128 * jtjttjtj \\
& -49/128 * tjtjttj - 49/128 * tjttjttj \\
& -121/128 * jtjtjttj + 15/64 * tjttjtjj \Big) \tag{D.9}
\end{aligned}$$

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