

Exact solutions for the biadjoint scalar field

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Abstract

Biadjoint scalar theories are novel field theories that arise in the study of non-abelian gauge and gravity amplitudes. In this short paper, we present exact nonperturbative solutions of the field equations, and compare their properties with monopole-like solutions in non-abelian gauge theory. Our results may pave the way for nonperturbative studies of the double copy.

1 Introduction

There has recently been much attention on the relationship between different types of field theory. A notable example is the *double copy* of refs. [1–3], which relates perturbative scattering amplitudes in non-abelian gauge and gravity theories. At tree-level this has a string theoretic explanation from the well-known KLT relations [4]. However, the double copy remains a conjecture at loop level, where it has been tested up to four loops [2, 5–26], to all orders in certain kinematic limits [8, 27–31], and in the self-dual sector [32]. A related programme of work relating amplitudes in different theories has arisen from the *CHY equations* of refs. [33–36]. These express tree-level amplitudes in gauge and gravity theories in terms of an abstract integral representation involving punctures on the Riemann sphere, which is itself reminiscent of string theory: indeed, the CHY equations can be obtained from ambitwistor string theory [37–39], which framework has recently been used to extend the former to one-loop level [40, 41].

All of the above work focuses on perturbative properties of the respective field theories. If the double copy is truly correct, it must also be possible to relate exact classical solutions of gauge theory and gravity, where these are known. First steps in this direction were undertaken in ref. [42], which found an infinite class of classical solutions in General Relativity (namely *stationary Kerr-Schild metrics*), possessing well-defined counterparts in a gauge theory. Examples include the Schwarzschild and Kerr black holes, which give rise to pointlike and rotating distributions of charge in the gauge

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theory respectively. One may generalise the former case to a pointlike dyon in the gauge theory, which double copies to the Taub-NUT solution in GR [43]. More recently, ref. [44] examined an arbitrarily accelerating (and radiating) particle in gauge theory and gravity, and further work has examined whether the source terms in both theories are physical [45].

Both the double copy and the CHY equations relate solutions in well-known gauge theories and gravity. This is not the whole story, however, in that one also finds more exotic theories whose amplitudes are related to gauge theory. In this paper we focus on *biadjoint scalar theories*, containing a field $\Phi = \Phi^{aa'} \mathbf{T}^a \tilde{\mathbf{T}}^{a'}$, where $\Phi^{aa'} \in \mathbb{R}$, and $\{\mathbf{T}^a\}$ and $\{\tilde{\mathbf{T}}^a\}$ are generators of two (possibly different) Lie algebras:

$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \mathbf{T}^c, \quad [\tilde{\mathbf{T}}^a, \tilde{\mathbf{T}}^b] = i \tilde{f}^{abc} \tilde{\mathbf{T}}^c. \quad (1)$$

We write the Lagrangian defining the theory as

$$\mathcal{L} = \frac{1}{2} \partial^\mu \Phi^{aa'} \partial_\mu \Phi^{aa'} + \frac{y}{3} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'}, \quad (2)$$

which gives rise to the equation of motion

$$\partial^2 \Phi^{aa'} - y f^{abc} \tilde{f}^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0. \quad (3)$$

This is a novel theory, which we may think of as describing a scalar field with two types of charge. Although such theories seem not to be directly applicable to nature, they nevertheless have an importance of their own. Firstly, they underlie the structure of gauge and gravity theories, in both the double copy and CHY approaches. For example, ref. [46] explains how to use biadjoint theories as building blocks for constructing amplitude numerators in gauge theory, in such a way as to make the double copy manifest. Secondly, the field equation of eq. (3) was crucial in ref. [42] for arguing that the classical double copy considered there was related to the BCJ story for perturbative amplitudes. Finally, scalar theories are often simpler than gauge or gravity theories. Thus, one may potentially use exact solutions of the former to investigate solutions of the latter. This is essentially the spirit of the classical double copy of refs. [42–45]. However, the solutions considered there were very special in that both the Yang-Mills and Einstein equations linearised. Correspondingly, the cubic term in the Lagrangian of eq. (2) vanished.

The aim of this paper is to derive exact solutions of eq. (3) in which the interaction term is nonzero. We will focus on static solutions, and furthermore those that are fully nonperturbative, in that they involve inverse powers of the coupling constant y in eqs. (2, 3). To the best of our knowledge, this has not been previously carried out. We will study a number of nontrivial solutions, and contrast these with nonperturbative solutions in non-abelian gauge theory. Whilst a full understanding of any double-copy like property is beyond the scope of this paper, we will see intriguing hints that nonperturbative solutions in biadjoint theory and gauge theory are related. Thus, our results constitute an important step in being able to probe nonperturbative aspects of the double copy and / or CHY frameworks.

The structure of the paper is as follows. In section 2 we consider a number of simple ansätze, and a first non-perturbative solution, applicable when the two Lie algebras coincide with each other. In section 3 we focus specifically on the case in which both Lie algebras are $SU(2)$, finding a more general form for the scalar field. Throughout, we compare our solutions with similar solutions in non-abelian gauge theory. Finally, we discuss our results and conclude in section 4.

2 A first nonperturbative solution

As a first attempt at solving eq. (3), one may try the form

$$\Phi^{aa'}(x) = \chi^a(x)\xi^{a'}(x), \quad (4)$$

in which each adjoint index a, a' has a nontrivial kinematic dependence associated with it, but the two types of charge do not talk to each other. Substituting this into eq. (3), one immediately finds (from the antisymmetry of the structure constants) that the interaction term vanishes, leaving

$$\partial^2(\chi^a(x)\xi^{a'}(x)) = 0. \quad (5)$$

One may further classify general solutions of this form but, being solutions of the free theory, they are not interesting for our present purpose, which is to find nonperturbative solutions of the full field equation. However, it is already interesting that, in order to find a nonperturbative solution, the two types of charge in the biadjoint theory must be inextricably linked.

Let us now restrict ourselves to the case where both sets of structure constants in eq. (3) come from the same Lie algebra. One may then write the ansatz

$$\Phi^{aa'} = \delta^{aa'} S(r), \quad (6)$$

where we assume spherical symmetry, and r is the radial coordinate. Substituting eq. (6) into eq. (3) yields ²

$$\delta^{aa'} \nabla^2 S(r) + y f^{abc} f^{a'bc} S^2(r) = 0. \quad (7)$$

In the second term, we may use the colour algebra

$$\begin{aligned} f^{abc} f^{a'bc} &= \text{Tr} \left[\mathbf{T}^a \mathbf{T}^{a'} \right] \\ &= \delta^{aa'} T_A \end{aligned} \quad (8)$$

where \mathbf{T}^a is a colour generator in the adjoint representation, and T_A the relevant normalisation constant. Then eq. (7) becomes

$$\delta^{aa'} \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dS(r)}{dr} \right) + y T_A S^2(r) \right] = 0, \quad (9)$$

where we have used the usual form of the Laplacian in spherical polars. At this point one may scale

$$S(r) = \frac{\bar{S}(r)}{y T_A} \Rightarrow \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\bar{S}(r)}{dr} \right) + \bar{S}^2(r) = 0, \quad (10)$$

and look for a solution of the form $\bar{S}(r) = K r^\alpha$, which finally leads to the nontrivial field

$$\Phi^{aa'} = -\frac{2\delta^{aa'}}{y T_A r^2}. \quad (11)$$

As already mentioned in the introduction, this is a fully nonperturbative solution, in that it contains an inverse factor of the coupling constant. It falls off rapidly as $r \rightarrow \infty$, and is singular at $r = 0$.

²Note that we use the spacetime metric (+, -, -, -) throughout.

Thus, it represents a point-like excitation localised at the origin. To further physically interpret this solution, we may study its energy. From eq. (2), the Hamiltonian density is given by

$$\begin{aligned}\mathcal{H} &= \frac{1}{2} \left[(\dot{\Phi}^{aa'})^2 + \nabla \Phi^{aa'} \cdot \nabla \Phi^{aa'} \right] - \frac{y}{3} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'} \\ &= \frac{32}{3} \frac{\mathcal{N}}{y^2 T_A^2} \frac{1}{r^6},\end{aligned}\tag{12}$$

where in the second line we have substituted eq. (11), and defined \mathcal{N} to be the dimension of the (common) Lie group. The energy can be obtained by imposing a short-distance radial cutoff, to get

$$\begin{aligned}E &= \int d^3 \mathbf{x} \mathcal{H} \\ &= \frac{128\pi}{3} \frac{\mathcal{N}}{y^2 T_A^2} \int_{r_0}^{\infty} \frac{dr}{r^4} \\ &= \frac{128\pi}{9} \frac{\mathcal{N}}{y^2 T_A^2} \frac{1}{r_0^3}.\end{aligned}\tag{13}$$

This is divergent at small distances, analogous to the case of a point-like charge in gauge theory, such as the singular solutions of ref. [47]. However, at large distances the energy is bounded, leading to a well-defined result upon assuming a finite charge radius.

In this section we have seen an exact, non-perturbative solution to the biadjoint field equation, valid when both Lie groups are the same as each other. A more interesting solution is possible if this common group is taken to be $SU(2)$, as we explore in the following section.

3 Further solutions for $SU(2) \otimes SU(2)$

In this section we choose both sets of structure constants in eqs. (2, 3) to correspond to $SU(2)$, so that the field equation becomes

$$\nabla^2 \Phi^{aa'} + y \epsilon^{abc} \epsilon^{a'b'c'} \Phi^{bb'} \Phi^{cc'} = 0,\tag{14}$$

where ϵ^{abc} is the Levi-Cevita symbol, and we have again focused on static solutions. It is then possible to write the following ansatz for the field:

$$\Phi^{aa'} = A(r) \delta^{aa'} + B(r) x^a x^{a'} + C(r) \epsilon^{aa'd} x^d.\tag{15}$$

Note that this ansatz contains mixing between spacetime and colour indices, analogous to nonperturbative solutions in non-abelian gauge theory [48–52]. This requirement motivates our choice of $SU(2)$ as the common Lie group. Substituting eq. (15) into eq. (14), one obtains

$$\begin{aligned}\delta^{aa'} \left[A'' + \frac{2A'}{r} + 2B + 2yA(A + Br^2) \right] + x^a x^{a'} \left[B'' + \frac{6B'}{r} + 2y(C^2 - AB) \right] \\ + \epsilon^{aa'd} x^d \left[C'' + \frac{4C'}{r} + 2yC(A + Br^2) \right] = 0.\end{aligned}\tag{16}$$

Linear independence of each term then results in the three coupled nonlinear equations

$$A'' + \frac{2A'}{r} + 2B + 2yA(A + Br^2) = 0; \quad (17)$$

$$B'' + \frac{6B'}{r} + 2y(C^2 - AB) = 0; \quad (18)$$

$$C'' + \frac{4C'}{r} + 2yC(A + Br^2) = 0. \quad (19)$$

As in the previous section, one may scale out the coupling y by defining $\bar{A} = A/y$ etc., so that eqs. (17–19) become

$$\bar{A}'' + \frac{2\bar{A}'}{r} + 2\bar{B} + 2\bar{A}(\bar{A} + \bar{B}r^2) = 0; \quad (20)$$

$$\bar{B}'' + \frac{6\bar{B}'}{r} + 2(\bar{C}^2 - \bar{A}\bar{B}) = 0; \quad (21)$$

$$\bar{C}'' + \frac{4\bar{C}'}{r} + 2\bar{C}(\bar{A} + \bar{B}r^2) = 0. \quad (22)$$

One may next try a power-like solution of the form

$$\bar{A} = ar^\alpha, \quad \bar{B} = br^\beta, \quad \bar{C} = cr^\gamma. \quad (23)$$

Upon substituting into eqs. (20–22), one finds

$$\alpha = -2, \quad \beta = -4, \quad \gamma = -3, \quad (24)$$

together with the coupled equations

$$(a + 1)(a + b) = -4b + 2c^2 - 2ab = c(a + b) = 0. \quad (25)$$

A trivial solution is $a = -1$, $b = c = 0$, which reproduces eq. (11) for the special case of SU(2) (note that $T_A = 2$ in that case). In addition, there is the one-parameter family of solutions

$$b = -a = k, \quad c = \pm\sqrt{2k - k^2}, \quad (26)$$

where $0 \leq k \leq 2$ if $\Phi^{aa'} \in \mathbb{R}$. The complete field is then given by

$$\Phi^{aa'} = \frac{1}{yr^2} \left[-k \left(\delta^{aa'} - \frac{x^a x^{a'}}{r^2} \right) \pm \sqrt{2k - k^2} \frac{\epsilon^{aa'd} x^d}{r} \right]. \quad (27)$$

Using the Hamiltonian of eq. (12), the energy of this solution is given by

$$E = \frac{16\pi k}{y^2 r_0^3}, \quad (28)$$

where again r_0 is a small-distance cutoff. As for the solution of eq. (11), this represents a pointlike object at the origin.

The two solutions of eqs. (11, 27) have different boundary conditions, namely that of spherical and cylindrical symmetry respectively. We may then ask whether eq. (23) is the most general solution satisfying cylindrical symmetry, by instead looking for a solution of the form of eq. (15), but imposing only the requirement

$$B(r) = -\frac{A(r)}{r^2} \quad \Rightarrow \quad \bar{B}(r) = -\frac{\bar{A}(r)}{r^2}. \quad (29)$$

A similar ansatz was used to find cylindrically symmetric multi-instanton solution of the Yang-Mills equations in ref. [53], as well as the extended Yang-Mills solutions of refs. [48,49,54]. In the present case, substituting eq. (29) into eq. (20–22) yields

$$\bar{A}'' + 2\frac{\bar{A}'}{r} - \frac{2\bar{A}}{r^2} = 0; \quad (30)$$

$$-\frac{\bar{A}''}{r^2} - \frac{2\bar{A}'}{r^3} + \frac{6\bar{A}}{r^4} + 2\bar{C}^2 + \frac{2\bar{A}^2}{r^2} = 0; \quad (31)$$

$$\bar{C}'' + \frac{4\bar{C}'}{r} = 0. \quad (32)$$

The first and third equations are now linear, and have general solutions

$$\bar{A} = a_1 r + \frac{a_2}{r^2}; \quad (33)$$

$$\bar{C} = \frac{c_1}{r^3} + c_2. \quad (34)$$

One may also substitute eq. (30) into eq. (31), such that the latter simplifies to

$$\frac{4\bar{A}}{r^4} + \frac{2\bar{A}^2}{r^2} + 2\bar{C}^2 = 0. \quad (35)$$

This constrains the general constants appearing in eqs. (33, 34), and reproduces precisely the solution of eq. (27). It seems then that, in contrast to pure Yang-Mills theory (e.g. ref. [54]), cylindrically symmetric extended solutions are not possible.

All of the solutions presented here are singular as $r \rightarrow 0$, which is not surprising, given that non-singular solutions are prohibited by Derrick's theorem [55]. There may be solutions, however, that are regular at the origin, but singular elsewhere. Such solutions exist in the spherically symmetric case, for example, as can be seen from the fact that eq. (10) admits the power series solution

$$S(r) = c - \frac{c^2 r^2}{6} + \frac{c^3 r^4}{60} - \frac{11c^4 r^6}{7560} + \mathcal{O}(c^5 r^8), \quad (36)$$

with c an arbitrary constant. For large enough r , however, one may neglect the second term in eq. (10) so that

$$S''(r) + S^2(r) \simeq 0. \quad (37)$$

The solution to this is a Weierstrass elliptic function, which will have a double pole at some finite value of r . Similar solutions (regular at the origin, singular on a spherical shell at finite r) have been constructed in Yang-Mills theory by Singleton [54], where they were compared with the

Schwarzschild solution in gravity.

The above analysis suggests that the spectrum of nonperturbative solutions of biadjoint scalar theory is, perhaps unsurprisingly, simpler than that of nonabelian gauge theory. This is also true for scattering amplitudes in both theories, and indeed biadjoint scalar amplitudes may be used as building blocks for gauge theory amplitudes [46], in order to ensure that the latter are BCJ dual [1] (meaning that the double copy to gravity is made manifest). If the double and zeroth copies relating biadjoint, gauge and gravity theories extend beyond perturbation theory, it is presumably true that nonperturbative solutions in biadjoint theory can also be used to obtain such solutions in gauge theory or gravity. To this end, it is useful to compare our solutions in eqs. (11, 27) to what appears to be the closest equivalent in non-abelian gauge theory, namely the Wu-Yang monopole [56]. In a gauge in which $A_0^a = 0$, the spatial components of the gauge field are given by [47]

$$A_i^a = -\frac{\epsilon_{iak}x^k}{er^2}. \quad (38)$$

This has cylindrical rather than spherical symmetry and, like eq. (27), involves an inverse power of the coupling constant. It is also a pointlike disturbance at the origin, leading to a divergent field energy there. However, it falls off $\sim r^{-1}$ rather than $\sim r^{-2}$, which behaviour can be traced to the dimensionality of the coupling constants in the two theories: both scalar and gauge fields have mass dimension 1, whereas the scalar coupling y and electromagnetic coupling e have mass dimensions 1 and 0 respectively. Thus, there must be an additional power of r^{-1} in eq. (27) relative to eq. (38) on dimensional grounds. Despite this minor difference, the forms of eqs. (27) and eq. (38) are very similar, hinting at a possible zeroth copy-like relationship that may be underlying them. This is particularly true for the case $k = 2$ in eq. (27), when it may then be written as

$$\Phi^{aa'} = -\frac{2}{yr^2}\epsilon^{abc}\epsilon^{a'b'c}\frac{x^bx^{b'}}{r^2} \sim A_i^a A_i^{a'}. \quad (39)$$

That is, the biadjoint solution looks like a product of Wu-Yang gauge fields, where one traces over the space indices. This may simply be a coincidence, but in any case the issue of whether there is a zeroth copy relationship between nonperturbative solutions deserves further investigation. It is not immediately clear how to find this, given that in both the amplitude and classical double copies, solutions of the linearised theory play a crucial role.

4 Conclusion

Biadjoint scalar theories have arisen in studies of scattering amplitudes in gauge and gravity theories, such as the CHY equations [33–36], and the double copy [1–3]. They also play an important role in the recently discovered classical double copy [42–45], which is closely related to the amplitude story. It is widely hoped that there is a nonperturbative explanation of these relationships, and to this end it is useful to study nonperturbative solutions of the biadjoint field equations. The hope is that such solutions could be used as building blocks for nonperturbative solutions in gauge and gravity theories, mirroring the role of biadjoint scalar amplitudes in the perturbative sector.

In this paper, we have presented some nontrivial solutions of the biadjoint field equations, with spherical (eq. (11)) and cylindrical (eq. (27)) symmetry. These correspond to pointlike disturbances

localised at the origin. Extended solutions, with a non-power like form, and also possessing such symmetry, seem not to be possible. The most closely related gauge theory solution to those presented here appears to be the Wu-Yang monopole, and the issue of whether a true zeroth copy-like relationship exists deserves further investigation. If such a connection can be made, it opens up a way to understand the nonperturbative significance of both the zeroth copy, and the double copy of gauge theories to gravity.

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