Theorem: A Static Magnetic *N*-pole Becomes an Oscillating Electric *N*-pole in a Cosmic Axion Field *

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We show for the classical Maxwell equations, including the axion electromagnetic anomaly source term, that a cosmic axion field induces an *oscillating electric N-moment* for any static magnetic *N*-moment. This is a straightforward result, accessible to anyone who has taken a first year graduate course in electrodynamics.

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The action for electrodynamics in the presence of an axion field, $\theta = a(x)/f_a$, is [1]:

$$S = \int d^4x \, \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} g \theta(x) F_{\mu\nu} \widetilde{F}^{\mu\nu} \right) \quad (1)$$

where $\widetilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. g is the anomaly coefficient, which is model dependent but typically of order $\sim 10^{-3}$. Eq.(1) leads to Maxwell's equations, $\partial_{\mu}F^{\mu\nu} = g\partial_{\mu}\theta\widetilde{F}^{\mu\nu}$.

We specialize to a cosmic axion field in its rest-frame, where it is a pure oscillator plus a constant (or slowly varying function), $\overline{\theta}(x,t)$, and m_a is the axion mass:

$$\theta(t) = \theta_0 \sin(m_a(t - t_0)) + \overline{\theta}.$$
 (2)

In this frame we assume electromagnetic fields of the form: $\vec{E} = \vec{E}_r$ and $\vec{B} = \vec{B}_0 + \vec{B}_r$. \vec{B}_0 is a large static, applied magnetic field, and \vec{E}_r and \vec{B}_r are small oscillating "response fields."¹. Maxwell's Equations in these fields, to first order in $g\theta_0$, become:

$$\overrightarrow{\nabla} \times \overrightarrow{B}_r - \partial_t \overrightarrow{E}_r = -g \overrightarrow{B}_0 \partial_t \theta$$

$$\overrightarrow{\nabla} \times \overrightarrow{E}_r + \partial_t \overrightarrow{B}_r = 0$$
(3)

and $\overrightarrow{\nabla} \cdot \overrightarrow{B}_r = \overrightarrow{\nabla} \cdot \overrightarrow{E}_r = 0$. These are standard and lead to, *e.g.*, the RF-cavity solutions [2]

Consider an applied magnetic field, \overline{B}_0 , arising from a static, classical, magnetic dipole, $\overline{\vec{m}}$, with a pointike magnetization $\overline{\vec{m}}\delta^3(\vec{r})$. This produces the familiar result,

$$\vec{B}_0 = -\frac{1}{4\pi} \left(\frac{1}{r^3} \right) \left(\vec{m} - \frac{3\vec{r} (\vec{r} \cdot \vec{m})}{r^2} \right) + \frac{2}{3}\vec{m}\delta^3(\vec{r}) \quad (4)$$

and $\overrightarrow{\nabla} \cdot \overrightarrow{B}_0 = 0$. This expression is well-known, such as in eq.(5.64) of Jackson [3]. Observe, however, that eq.(4) can be rewritten in an equivalent form:

$$\overrightarrow{B}_{0} = \overrightarrow{m}\delta^{3}(\overrightarrow{r}) + \frac{1}{4\pi}\overrightarrow{\nabla}\left(\overrightarrow{m}\cdot\overrightarrow{\nabla}\frac{1}{r}\right)$$
(5)

Eq.(5) yields eq.(4) upon computing the gradient term by using the identity:

$$\nabla_i \frac{r_j}{r^3} = \delta_{ij} \frac{1}{r^3} - 3\frac{r_i r_j}{r^5} + \frac{4\pi}{3}\delta_{ij}\delta^3(\vec{r}) \tag{6}$$

Note that, if we contract (ij), then eq.(6) becomes $\overrightarrow{\nabla} \cdot \frac{\overrightarrow{r}}{r^3} = 4\pi \delta^3(\overrightarrow{r})$, which is just Gauss' law (we emphasize that the singularities can be replaced by smooth, localized Gaussians). The Maxwell equations thus become:

$$\vec{\nabla} \times \vec{B}_r - \partial_t \vec{E}_r = -g \partial_t \theta \left(\vec{m} \delta^3(\vec{r}) + \frac{1}{4\pi} \vec{\nabla} \left(\vec{m} \cdot \vec{\nabla} \frac{1}{r} \right) \right)$$
$$\vec{\nabla} \times \vec{E}_r + \partial_t \vec{B}_r = 0 \tag{7}$$

We now make a redefinition of the electric field by shifting away the gradient term, where we assume $\partial_t \overline{\theta}(x,t) \approx 0$:

$$\overrightarrow{E}_{r} = \overrightarrow{E}_{r}^{\prime} + \frac{1}{4\pi} g \widetilde{\theta} \,\overrightarrow{\nabla} \left(\overrightarrow{m} \cdot \overrightarrow{\nabla} \frac{1}{r} \right) \tag{8}$$

Here $\tilde{\theta}(t) = \int_{t_0}^t d\tau \partial_t \theta(t) = \theta_0 \sin(m_a(t-t_0))$ and has the property that $\tilde{\theta}(t) \to 0$ as $m_a \to 0$.² The shift is a purely longitudinal (gradient) term and cannot affect the radiation field. Since $\vec{\nabla} \times \vec{\nabla}(X) = 0$, we have $\vec{\nabla} \times$

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¹ Note that \vec{E}_r and \vec{B}_r are of order $g\theta_0 << 1$ where $\theta_0 \sim 3 \times 10^{-19}$ when one matches to the local galactic halo density of $\sim 300 \text{ MeV/cm}^3$ [2].

² Note that as $m_a \to 0$ then $\partial_t \theta \to 0$ and $\tilde{\theta}(t) \to 0$. The constant $\overline{\theta}$ has disappeared from the physics and the theory has the "shift symmetry" $\overline{\theta} \to \overline{\theta} + k$ for any constant k. This is called "axion decoupling," and it is somewhat more involved with general $\theta(x,t)$ [2]. We note that in an RF-cavity, such as ADMX, an oscillating electric field develops along the cavity axis driven by the axion and the same eqs.(3) with $\overrightarrow{E}_T \propto \widetilde{\theta}(t)$ [2]. Maxwell's equations act as a "high-pass filter" for $\theta(t) \to \widetilde{\theta}(t)$.

 $\overrightarrow{E}_r = \overrightarrow{\nabla} \times \overrightarrow{E}'_r$, and the second Maxwell equation, (7), is unaffected by the shift. Hence:

$$\vec{\nabla} \times \vec{B}_r - \partial_t \vec{E}'_r = -g \partial_t \theta \vec{m} \delta^3(\vec{r})
\vec{\nabla} \times \vec{E}'_r + \partial_t \vec{B}_r = 0$$
(9)

The Maxwell equations, in terms of \vec{B}_r and \vec{E}'_r , describe the radiation field produced by the anomaly and magnetic dipole source. Note that the extended magnetic dipole field has completely disappeared, leaving only the pointlike source.

Equations (9) are identical to those of an oscillating electric dipole moment $\overrightarrow{p}(t)$ (the "Hertzian" dipole):

$$\vec{\nabla} \times \vec{B}_r - \partial_t \vec{E}'_r = -\partial_t \vec{p}(t) \vec{\nabla} \times \vec{E}'_r + \partial_t \vec{B}_r = 0$$
(10)

where we have the correspondence:

$$\overrightarrow{p}(t) = g\widetilde{\theta}(t)\overrightarrow{m}\delta^{3}(\overrightarrow{r}) \tag{11}$$

One may have noticed that, upon performing the shift, eq.(8), the $\overrightarrow{\nabla} \cdot \overrightarrow{E}'_r$ equation is now modified,

$$\vec{\nabla} \cdot \vec{E}'_r = -\frac{1}{4\pi} g \widetilde{\theta} \vec{\nabla}^2 \vec{m} \cdot \vec{\nabla} \frac{1}{r} = g \widetilde{\theta}(t) \vec{m} \cdot \vec{\nabla} \delta^3(\vec{r}).$$
(12)

In the second term, using eq.(11) we have:

$$\overrightarrow{\nabla} \cdot \overrightarrow{E}'_r = \overrightarrow{\nabla} \cdot \overrightarrow{p} \tag{13}$$

However, eq.(13) together with eq.(10) are precisely the Maxwell equations obtained from an action containing an electric dipole term,

$$S = \int d^4x \,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overrightarrow{p}(t)\cdot\overrightarrow{E}\right) \qquad (14)$$

The axion causes the magnetic dipole to radiate. The radiation from a static magnetic moment in an axion field is identical to the Hertzian electric dipole radiation (see *e.g.*, Jackson, [3] eqs.(9.18, 9.19)). In the near-zone the two terms on the *rhs* of eq.(8) cancel as $m_a \to 0$, causing \vec{E}_r to vanish. Hence there is no persistent electric field in the $\tilde{\theta} \to \text{constant limit!}$ In the far-zone we have:

$$\vec{E}_{r}(x,t) = gm_{a}^{2}\widetilde{\theta}(t-r)\left(\frac{\vec{m}}{r} - \frac{\vec{r}}{r^{2}}\frac{\vec{m}\cdot\vec{r}}{r}\right)$$
$$\vec{B}_{r}(x,t) = -gm_{a}^{2}\widetilde{\theta}(t-r)\left(\vec{m}\times\frac{\vec{r}}{r^{2}}\right)$$
(15)

Note the CP-violation in the alignment of the polarization of the vector \overrightarrow{E}_r with the axial vector \overrightarrow{m} , which arises from the background axion field. From eq.(15) we obtain the total emitted power,

$$P = g^2 m_a^4 \theta_0^2 |\vec{m}|^2 / 12\pi.$$
 (16)

This is equivalent to the quantum result for a spinup to spin-up electron with $|\vec{m}|^2 \rightarrow \mu_{Bohr}^2$. Coherent assemblages of many electrons can produce potentially observable effects. The physical applications of this are discussed in [2]. In the near-zone there is a cancellation of \vec{E}'_r with the shifted piece of eq.(8), so there is no persistent constant electric dipole field in \vec{E}_r in the $\tilde{\theta} \rightarrow$ constant limit (see eq.(56) in [2]).

Alternatively, we can see this result directly from the action, eq.(1), without performing the shift. We decompose \overrightarrow{E}_r into transverse and longitudinal (gradient) components: $\overrightarrow{E}_r = \overrightarrow{E}_{rT} + \overrightarrow{E}_{rL}$ where (i): $\overrightarrow{\nabla} \cdot \overrightarrow{E}_{rT} = 0$, and (ii): $\overrightarrow{E}_{rL} = \overrightarrow{\nabla}[(1/\overrightarrow{\nabla}^2)\overrightarrow{\nabla} \cdot \overrightarrow{E}_r]$. The anomaly term in the action of eq.(1), $\int g\theta \overrightarrow{E}_r \cdot \overrightarrow{B}_0$, becomes, upon integrating by parts and using (i), (ii): $\int g\theta \overrightarrow{E}_{rT} \cdot \overrightarrow{m}\delta^3(\overrightarrow{r})$. Thus, only the *transverse electric field* couples to the axion-induced OEDM. The axionic OEDM radiates transverse radiation, and it will interact in the usual way, as in eq.(14) with an applied transverse $\overrightarrow{E}_T(t)$, *e.g.*, such as a cavity mode or light. Thus, we can write a complete action by replacing $\overrightarrow{p}(t)$ by $\overrightarrow{p}'(t)$ in eq.(14) where $\overrightarrow{p}'(t) = \overrightarrow{p}(t) - \overrightarrow{\nabla}(1/\overrightarrow{\nabla}^2)\overrightarrow{\nabla} \cdot \overrightarrow{p}(t)$, as in [2, 4].

We can build up a quadrupole source from a pair of dipoles, and an octupole source from a pair of quadrupoles, etc. Thus, by superposition, the axion causes any static magnetic N-pole to become an oscillating electric N-pole.

We've seen that the extended magnetic dipole field is irrelevant to the radiation, only the point-like singularity matters, a result that translates to the *N*-pole case. We thus call this an *effective OEDM*, in the sense of *e.g.*, Fermi's effective weak interaction. It is "effective" as a leading order result in a tiny $g\theta_0$ coupling constant.

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