Universitat de València

Facultat de Física

Master Thesis

Neutrino-Pair Exchange Long-Range Force Between Aggregate Matter

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Abstract

We study the long-range force arising between two neutral—of electric charge—aggregates of matter due to a neutrino-pair exchange, in the limit of zero neutrino mass. The conceptual basis for the construction of the effective potential comes from the coherent scattering amplitude at low values of t. This amplitude is obtained using the methodology of an unsubtracted dispersion relation in t at threshold for s, where $(s,\, t)$ are the Lorentz invariant scattering variables. The ultraviolet behavior is irrelevant for the long-range force. In turn, the absorptive part in the t-dependence is given by the corresponding unitarity relation. We show that the potential describing this force decreases as r^{-5} at large separation distance r. This interaction is described in terms of its own charge, which we call the weak flavor charge of the interacting systems, that depends on the flavor of the neutrino as $Q_W^e = 2Z - N$, $Q_W^\mu = Q_W^\tau = -N$. The flavor dependence of the potential factorizes in the product of the weak charges of the interacting systems, so that the resulting force is always repulsive. Furthermore, this charge is proportional to the number of constituent particles, which differs from the global mass, so this interaction could be disentangled from gravitation through deviations from the Equivalence Principle.

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1. Introduction

It's been 85 years since Wolfgang Pauli postulated the existence of the neutrino in order to explain the continuous spectrum in β-decays, and 59 years since Reines and Cowan discovered it. In those years, we've learnt many properties about this particle, such as the fact that it only interacts through weak interactions—all of its charges but weak isospin are zero. In fact, in the framework of the Standard Model [1], there are only left-handed neutrinos, so Standard Model neutrinos are massless—we can't generate a neutrino mass through a Yukawa-type coupling with a Higgs doublet.

Other interesting phenomena related to this particle are neutrino oscillations [2], which have been well established experimentally since 1998. This process is understood as the fact that there is a mismatch between mass eigenstates and flavor eigenstates, so that flavors get mixed along free propagation. Indeed, the observation of neutrino oscillations is a direct measurement of the mass difference between the three states, proving that neutrinos are massive particles, which is a first signal of Physics beyond the Standard Model.

Therefore, the study of the origin of neutrino mass is one of the directions in which we can expect finding new Physics, even though its small value ($m_\nu \lesssim 1$ eV [3]) makes it hard to observe experimentally. As well as determining the absolute mass of the neutrino, there's still a more fundamental question about their nature unanswered: since neutrinos can be neutral of all charges, their finite mass could be explained through a Dirac mass term (implying that neutrinos and antineutrinos are different particles, described by $4-$component Dirac spinors) or through a Majorana one (implying that neutrinos are self-conjugate of all charges, described by 2 independent degrees of freedom).

In any case, the fact that their masses are very low stands, and we discuss here another property of neutrinos as mediators of a new force. As is well known, the processes represented in Quantum Field Theory by the exchange of a massless particle give raise to long-range interactions. An easy example is the scattering of two particles mediated by a photon, which—at tree level—describes Coulomb scattering. Our objective in this work is the application of these ideas to a process mediated by neutrinos. According to the Electroweak Lagrangian, the lowest-order process is a neutrino-pair exchange, which—since neutrinos are nearly massless—describes an interaction of long range.

With this idea in mind, we review in Section 2 the relation between the Feynman amplitude in Born approximation and an effective potential, which is a Fourier Transform.

The amplitude at low t, associated to the long-range behavior, is obtained by means of an unsubtracted dispersion relation. Its ultraviolet dependence is of no relevance. In order to simplify the calculation of the potential, in Section 3 we exploit the untitarity of the S matrix, writing the absorptive part of the $1-$loop scattering amplitude with the amplitude of the tree-level scattering process.

In Section 4, we study the low-energy limit of the Electroweak Lagrangian in terms of a contact interaction, establishing the framework for the calculation of the scattering amplitude including both neutral current and charged current vertices. We compute in detail this amplitude in Section 5, where it's natural to introduce the concept of a weak flavor charge of matter. In terms of this amplitude, obtaining the interaction potential is straightforward, and we find in Section 6 that it leads to a repulsive force which decreases as r^{-6}.

We conclude this work analyzing the possibility of an experimental measurement of this interaction, which is relevant between nanometers and microns, where there are also residual electromagnetic interactions—such as Van der Waals or Casimir-Polder forces—and gravitation. The measurement of this weak interaction is very compelling, since it could give information about properties of the neutrino such as its absolute mass, which is still unknown, or it could even help us to answer the most fundamental question regarding neutrinos, whether they are Dirac or Majorana particles. These points are considered in Sections 7 and 8.

2. From a Quantum Field Theory to an Effective Potential

We are interested in calculating the interaction potential resulting from a neutrino-pair exchange between aggregates of matter, which is an interaction described in the framework of a Quantum Field Theory. Therefore, we will begin this work relating the concepts of interaction potential and Feynman amplitude.

2.1. The Coulomb potential

It is known that the interaction between two electrically charged particles, say A and B, is described by the Coulomb potential,

$$V_C(r) = \frac{e^2}{4\pi} \frac{Q_A Q_B}{r} \,, \tag{2.1}$$

where e is the charge of the proton, Q_J the charge of the particle J in units of e and r the distance between the two particles. Throughout this work, we'll use the Natural System of Units and the Heaviside electric system—all conventions are stated in Appendix A.

We are interested in calculating this potential using the Quantum Electrodynamics (QED), which is described by the interaction Lagrangian

$$\mathscr{L}_{\text{QED}} = -eQ\,\bar{\psi}\gamma^\mu\psi\, A_\mu \,. \tag{2.2}$$

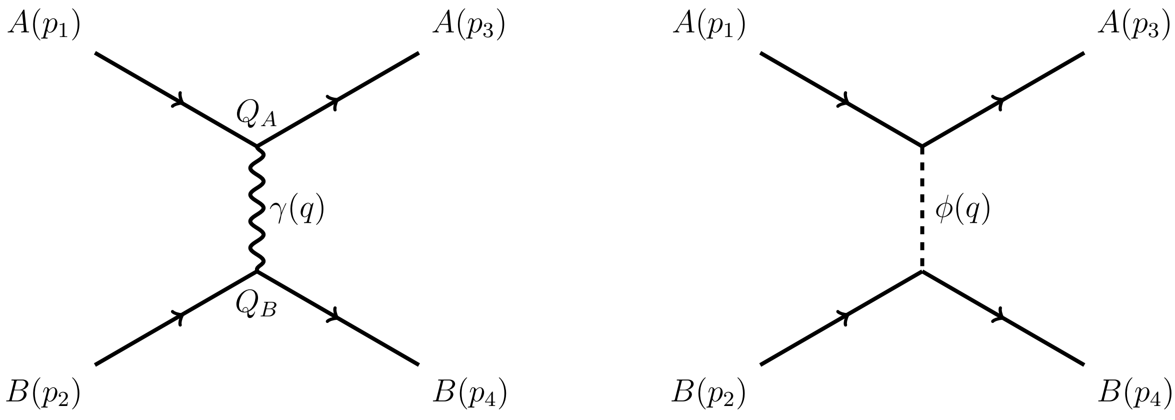
In this framework, the $AB \to AB$ elastic scattering is described—at leading order—by the Feynman graph from Fig.1a. Using the QED Feynman rules [1], the amplitude of the process is

$$\mathcal{M} = e^2 Q_A Q_B \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \frac{1}{q^2} \left[\bar{u}(p_4)\gamma_\mu u(p_2)\right] . \tag{2.3}$$

Since we are looking for a long-range coherent interaction, we can simplify

$$\mathcal{M} \approx e^2 Q_A Q_B \left[\bar{u}(p_3)\gamma^0 u(p_1)\right] \frac{1}{q^2} \left[\bar{u}(p_4)\gamma_0 u(p_2)\right] \tag{2.4}$$

taking into account the fact that γ^0 is related to the electric charge, which is coherent, while $\boldsymbol{\gamma}$ is related to the electromagnetic current, which is not a coherent quantity.



(a) (b)

Figure 1: Lowest-order Feynman diagrams for $AB \to AB$ elastic scattering. (a) QED interaction, mediated by a photon, where A and B are particles of electric charge Q_A and Q_B. (b) Yukawa interaction, mediated by a scalar ϕ of mass μ.

Using $\left(\gamma^0\right)^2 = 1$ and dropping external-line factors, we get

$$M(q^2) = e^2 Q_A Q_B \, \frac{1}{q^2} \, , \tag{2.5}$$

where we defined $M(q^2)$ as $\mathcal{M}(q^2) \equiv \bar{u}^{(A)}(p_3)\bar{u}^{(B)}(p_4)M(q^2)u^{(B)}(p_2)u^{(A)}(p_1)$.

Since this is a scattering process, we can work in the Breit reference frame (defined by $q^0 = 0$), which describes the non-relativistic limit (low energy transfer), where

$$M(q^2) = -e^2 Q_A Q_B \, \frac{1}{\boldsymbol{q}^{\,2}} \, . \tag{2.6}$$

We can compute the 3-dimensional Fourier Transform of this quantity (see Appendix B.2.1), and we find

$$\mathcal{F}\left\{M\right\}(r) \equiv \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \, e^{i\boldsymbol{q}\,\boldsymbol{r}} \, M(q^2) = -\frac{e^2}{4\pi} \, \frac{Q_A Q_B}{r} = -V_C(r) \, . \tag{2.7}$$

This expression shows the relation between the Quantum Field Theory Feynman amplitude and the interaction potential used in a potential description of the system dynamics. Before considering a more general case, let's look at another simple one: the Yukawa interaction.

2.2. The Yukawa interaction

Another well-known potential is Yukawa's, which describes an effective central strong nuclear force acting between nucleons,

$$V_Y(r) = -\frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}\,. \tag{2.8}$$

From a Quantum Field Theory point of view, this interaction is described by the Lagrangian

$$\mathscr{L}_Y = -g\phi\bar{\psi}\psi\,, \tag{2.9}$$

where ϕ is a scalar field and ψ is a fermionic field. Such a scalar can be physically associated to the σ meson for the interacting π-π mediation. The $AB \to AB$ scattering amplitude described by this Lagrangian is the one represented in Fig.1b, so it is

$$\mathcal{M} = -g^2\left[\bar{u}(p_3)u(p_1)\right]\frac{1}{q^2-\mu^2}\left[\bar{u}(p_4)u(p_2)\right]\,, \tag{2.10}$$

where μ is mass of the scalar, and

$$M(q^2) = \frac{-g^2}{q^2-\mu^2}\,. \tag{2.11}$$

Again, we can work in the Breit reference frame, so that

$$M(q^2) = \frac{g^2}{\boldsymbol{q}^{\,2}+\mu^2}\,. \tag{2.12}$$

The potential must be related to the Fourier Transform of this $M(q^2)$, which is also calculated in Appendix B.2.1,

$$\mathcal{F}\left\{M\right\}(r) = \frac{g^2}{4\pi}\frac{e^{-\mu r}}{r} = -V_Y(r)\,, \tag{2.13}$$

which is the same relation between $M(q^2)$ and $V(r)$ that we obtained in the Coulomb case.

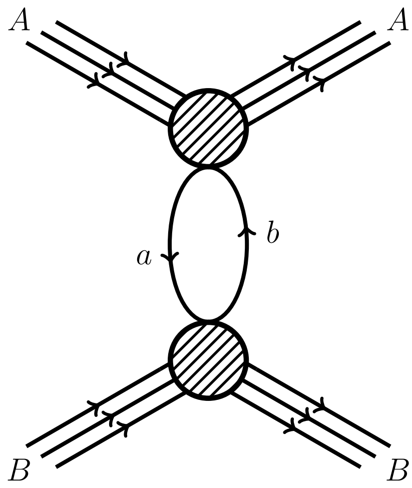


Figure 2: Feynman diagram for $AB \to AB$ elastic scattering mediated by a, b exchange.

2.3. A more general case: particle-pair exchange

As we have just seen, the interaction potential between particles A and B is the Fourier Transform

$$V(r) = -\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, e^{i\boldsymbol{q}\,\boldsymbol{r}}\, M(q^2)\,, \tag{2.14}$$

where M is the lowest order Feynman amplitude for the process $AB \to AB$, with both A and B on-shell, but without external-leg factors, as is discussed in [4][1]. In the case of a pair exchange, this process will be the one represented in Fig.2.

In order to compute integral (2.14), we rewrite the amplitude as a dispersion relation following the steps mentioned in [5]. We can extend t to the complex plane and expand the amplitude using Cauchy's Formula [6],

$$f(z) = \frac{1}{2\pi i}\int_C \mathrm{d}z'\, \frac{f(z')}{z'-z}\,, \tag{2.15}$$

which is valid whenever $f(z)$ is analytic inside C.

The physical region of the t variable of elastic scattering processes has $t < 0$, so we want the $\mathbb{R}^-$ axis inside C. Also, the $t-$channel amplitude will have a branching point at $t = (m_a + m_b)^2 \equiv t_0 \geqslant 0$, so we can use Cauchy's Formula with the integration path shown in Fig.3. In fact, the physical region is $-s \leqslant t \leqslant t_0$, but we are only interested in the long-range interaction, which is associated to low values of $|t|$. Since $|t| \sim s \sim (M_A + M_B)^2$ describes interactions of much shorter range than the nuclear size whenever A and B are

[1]Beware a minus sign between their convention for the Feynman amplitude and ours.

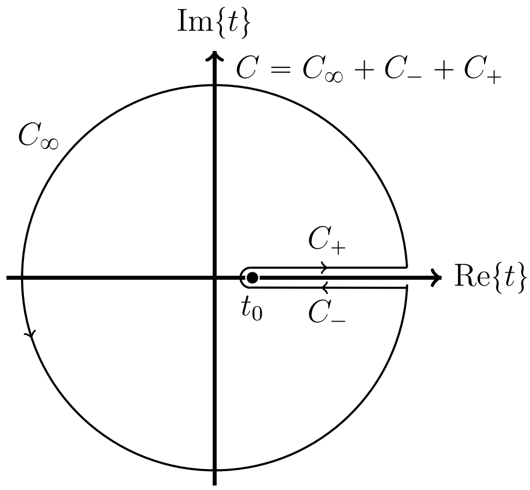


Figure 3: Integration path (in the complex plane of the t Mandelstam variable) used in the dispersion relation decomposition of the Feynman amplitude of the process, as discussed in the text.

aggregates of matter, we can take $s \to \infty$ without affecting the long-range amplitude, as we have done in considering the path in Fig.3.

If the amplitude vanishes along the C_∞ circumference, as $|t| \to \infty$, the only contribution is the one coming from the integral on both sides of the cut along the real t axis,

$$\begin{aligned} M(t) &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{\infty}^{t_0} \mathrm{d}t' \, \frac{M(t' - i\epsilon)}{t' - t} + \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{t_0}^{\infty} \mathrm{d}t' \, \frac{M(t' + i\epsilon)}{t' - t} = \\ &= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{t_0}^{\infty} \mathrm{d}t' \, \frac{M(t' + i\epsilon) - M(t - i\epsilon)}{t' - t} \, . \end{aligned} \tag{2.16}$$

If not vanishing at C_∞, we'd have to either rewrite the dispersion relation for the subtracted amplitude or include the contribution of C_∞. We continue with the formulation without subtractions, because the contribution along C_∞ is of short range. We then understand Eq.(2.16) for the long-range amplitude.

In order to compute the analytically extended amplitude both above and below the unitarity cut, we can relate them using Schwarz Reflexion Principle [6],

$$M(t - i\epsilon) = M^*(t + i\epsilon) \, . \tag{2.17}$$

Using this relation, we can easily write

$$M(t)=\frac{1}{\pi}\int_{t_0}^{\infty}\mathrm{d}t'\,\frac{\mathrm{Im}\left\{M(t')\right\}}{t'-t}\,, \tag{2.18}$$

which is the so-called $t-$channel dispersion relation of the Feynman amplitude. Putting this expression into (2.14) and rewriting $(t'-t)^{-1}$ as (B.20) states, we get

$$\begin{aligned} V(r) &= \frac{-1}{4\pi^2}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,e^{i\boldsymbol{q}\,\boldsymbol{r}}\int_{t_0}^{\infty}\mathrm{d}t'\,\mathrm{Im}\left\{M(t')\right\}\int\mathrm{d}^3r'\,e^{-i\boldsymbol{q}\,\boldsymbol{r}\,'}\,\frac{e^{-\sqrt{t'}r'}}{r'} = \\ &= \frac{-1}{4\pi^2}\int_{t_0}^{\infty}\mathrm{d}t'\,\mathrm{Im}\left\{M(t')\right\}\int\mathrm{d}^3r'\,\frac{e^{-\sqrt{t'}r'}}{r'}\,\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}\,') = \\ &= \frac{-1}{4\pi^2}\int_{t_0}^{\infty}\mathrm{d}t'\,\mathrm{Im}\left\{M(t')\right\}\,\frac{e^{-\sqrt{t'}r}}{r}\,. \end{aligned} \tag{2.19}$$

Therefore, the non-relativistic potential

$$\boxed{V(r)=\frac{-1}{4\pi^2 r}\int_{t_0}^{\infty}\mathrm{d}t'\,\mathrm{Im}\left\{M(t')\right\}\,e^{-\sqrt{t'}r}} \tag{2.20}$$

is determined by the absorptive part of the Feynman amplitude. Since we are not interested in the whole $M(t)$, but only in the $\mathrm{Im}\{M(t)\}$, we can make a profit from the unitarity of the S matrix to simplify our calculations.

3. Unitarity Relation. Absorptive Part

Physical processes are determined by matrix elements of the scattering matrix S. The S matrix relates the orthonormal basis of initial states with the final states' one, so it has to be a unitary operator,

$$S^\dagger S = 1. \tag{3.1}$$

We define the reduced scattering matrix T as $S \equiv 1 + i\,T$, which describes processes where there really is an interaction—initial and final states are not the same ones. In terms of this operator, the unitarity relation (3.1) becomes

$$1 = S^\dagger S = (1 - i\,T^\dagger)(1 + i\,T) = 1 - i\,T^\dagger + i\,T + T^\dagger T,$$

so

$$-\,i(T - T^\dagger) = T^\dagger T\,. \tag{3.2}$$

In order to describe a physical process, we have to consider the matrix element $\langle f|\, S - 1\,|i\rangle = i\,\langle f|\, T\,|i\rangle \equiv i\,(2\pi)^4\delta^{(4)}(p_f - p_i)\mathcal{M}(i \to f)$, where $|i\rangle$ is the initial state and $|f\rangle$ is the final one. Therefore, we need to sandwich the previous relation between those states—we begin computing the left-hand side (LHS),

$$\begin{aligned}\langle f|\, \mathrm{LHS}\,|i\rangle &= -i\,\langle f|\, T - T^\dagger\,|i\rangle = \\ &= -i\left[\langle f|\, T\,|i\rangle - \langle i|\, T\,|f\rangle^*\right] = \\ &= -i\,\times 2i\,\mathrm{Im}\left\{\langle f|\, T\,|i\rangle\right\}\,,\end{aligned} \tag{3.3}$$

where we assumed that time reversal is a good symmetry to write

$$T(i \to f) - T(f \to i)^* = 2\,\mathrm{Im}\left\{T(i \to f)\right\}\,. \tag{3.4}$$

On the other hand,

$$\begin{aligned}\langle f|\, \mathrm{RHS}\,|i\rangle &= \langle f|\, T^\dagger T\,|i\rangle = \\ &= \langle f|\, T^\dagger \left[\sum_n \int \prod_{j=1}^{n} \frac{\mathrm{d}^3 q_j}{(2\pi)^3 2E_{q_j}}\, |q_n\rangle\langle q_n|\right] T\,|i\rangle = \\ &= \sum_n \int \prod_{j=1}^{n} \frac{\mathrm{d}^3 q_j}{(2\pi)^3 2E_{q_j}}\, \langle f|\, T^\dagger\,|q_n\rangle\langle q_n|\, T\,|i\rangle\,,\end{aligned} \tag{3.5}$$

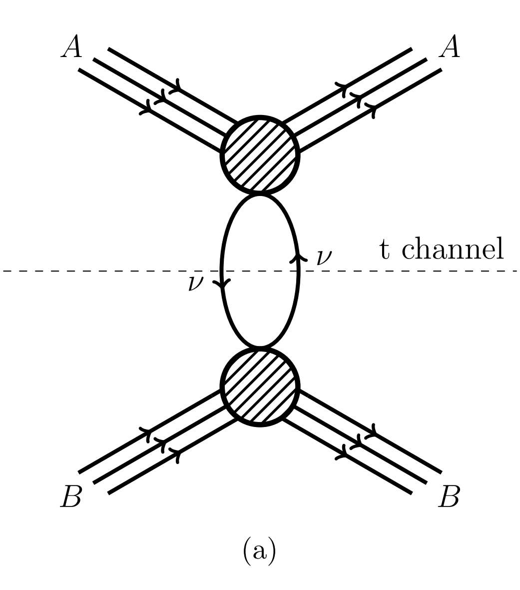


Figure 4: Feynman diagrams for the neutrino-pair mediated (a) $AB \to AB$ scattering and (b) $A\bar{A} \to B\bar{B}$ scattering. The labels in the figures denote the fields which describe the particles in the process.

where in the second line we have inserted an identity—a sum over all possible states, with $|q_n\rangle$ representing a state of n particles with 4-momenta $q_1, q_2 ... q_n$.

Now we can write the unitarity relation $\langle f|\,\mathrm{LHS}\,|i\rangle = \langle f|\,\mathrm{RHS}\,|i\rangle$ as

$$\boxed{\mathrm{Im}\left\{\langle f|\,T\,|i\rangle\right\} = \frac{1}{2}\sum_n \int \mathrm{d}Q_n \,\langle q_n|\,T\,|f\rangle^* \langle q_n|\,T\,|i\rangle}\,. \tag{3.6}$$

Let's apply this relation to our process. We are interested in calculating the absorptive part of the $AB \to AB$ amplitude mediated by a neutrino-pair, so we need to do a t-channel unitarity cut of the diagram in Fig.4a. Therefore, we should write Eq.(3.6) for the crossed process $A\bar{A} \to B\bar{B}$, Fig.4b, with a $\nu\nu$ intermediate state[2],

$$\mathrm{Im}\left\{\langle B\bar{B}|\,T\,|A\bar{A}\rangle\right\} = \frac{1}{2}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{k_1}}\,\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{k_1}}\,\langle \nu(k_1)\bar{\nu}(k_2)|\,T\,|B\bar{B}\rangle^* \langle \nu(k_1)\bar{\nu}(k_2)|\,T\,|A\bar{A}\rangle\,. \tag{3.7}$$

[2]Since the intermediate state is a fermionic one, there should be a spin sum. However, only left-handed neutrinos exist, so in this case it is not necessary.

Dropping the $(2\pi)^4\delta^{(4)}(p_f - p_i)$ global factor from both sides, this equation becomes

$$\begin{aligned}&\mathrm{Im}\left\{\mathcal{M}(A\bar{A}\to B\bar{B})\right\} = \\ &\quad = \frac{1}{2}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_{k_1}}\,\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_{k_2}}\,(2\pi)^4\delta^{(4)}(k_1+k_2-p_i)\mathcal{M}(B\bar{B}\to\nu\bar{\nu})^*\mathcal{M}(A\bar{A}\to\nu\bar{\nu})\,.\end{aligned} \tag{3.8}$$

Finally, we can write this expression in an explicitly Lorentz invariant manner,

$$\boxed{\begin{aligned}&\mathrm{Im}\left\{\mathcal{M}(A\bar{A}\to B\bar{B})\right\} = \\ &\quad = \frac{1}{2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^3}\,\delta(k_1^2)\,\frac{\mathrm{d}^4k_2}{(2\pi)^3}\,\delta(k_2^2)\,(2\pi)^4\delta^{(4)}(k_1+k_2-p_i)\mathcal{M}(B\bar{B}\to\nu\bar{\nu})^*\mathcal{M}(A\bar{A}\to\nu\bar{\nu})\end{aligned}} \tag{3.9}$$

4. Low-Energy Contact Interaction

The weak interactions of fermions, charged and neutral currents, are described by the Lagrangian densities [1, 7]

$$\mathscr{L}_{\mathrm{CC}} = -\frac{e}{2\sqrt{2}\sin\theta_W}\left\{W_\mu^\dagger\left[\bar{u}_i\gamma^\mu\left(1-\gamma_5\right)V_{ij}\,d_j + \bar{\nu}_i\gamma^\mu\left(1-\gamma_5\right)e_i\right] + \text{h.c.}\right\}, \tag{4.1a}$$

$$(i, j = 1^{\text{st}},\ 2^{\text{nd}},\ 3^{\text{rd}}\ \text{gen.})$$

$$\begin{aligned}\mathscr{L}_{\mathrm{NC}} &= -eA_\mu\,Q_j\bar{\psi}_j\,\gamma^\mu\,\psi_j - \frac{e}{4\sin\theta_W\cos\theta_W}\,Z_\mu\,\bar{\psi}_j\,\gamma^\mu\,\left(g_{V_j} - g_{A_j}\gamma_5\right)\psi_j \\ &\equiv \mathscr{L}_{\mathrm{QED}} + \mathscr{L}_{\mathrm{Z}}\,, \qquad\qquad (\psi_j = u, d, \nu_e, e...)\end{aligned} \tag{4.1b}$$

where θ_W is the weak mixing angle.

For any elementary particle, the weak neutral couplings are given by

$$g_V = 2T_3 - 4Q\sin^2\theta_W, \qquad\qquad g_A = 2T_3\,, \tag{4.2}$$

where T_3 is the third component of weak isospin and Q is the electric charge. The electroweak charges of the SM fermions are written in Table 1.

We are interested in calculating the potential associated to a process at low energy, where the limit $|q^2| \ll M_W^2, M_Z^2$ is valid, so now we'll focus in obtaining the low-energy effective interactions from the above Lagrangians.

4.1. Effective charged current couplings

We are describing neutrino scattering against an aggregate of matter, so only the ν_e-e charged current contributes to the scattering. Therefore, the only two terms of the

Table 1: Electroweak charges of the Standard Model fermions. The index $i = 1, 2, 3$ labels the three generations, so that $u_1 = u$, $u_2 = c$, $u_3 = t$...

Particle	Q	g_V	g_A
u_i	2/3	$1 - \frac{8}{3}\sin^2\theta_W$	1
d_i	-1/3	$-1 + \frac{4}{3}\sin^2\theta_W$	-1
ν_i	0	1	1
e_i	-1	$-1 + 4\sin^2\theta_W$	-1

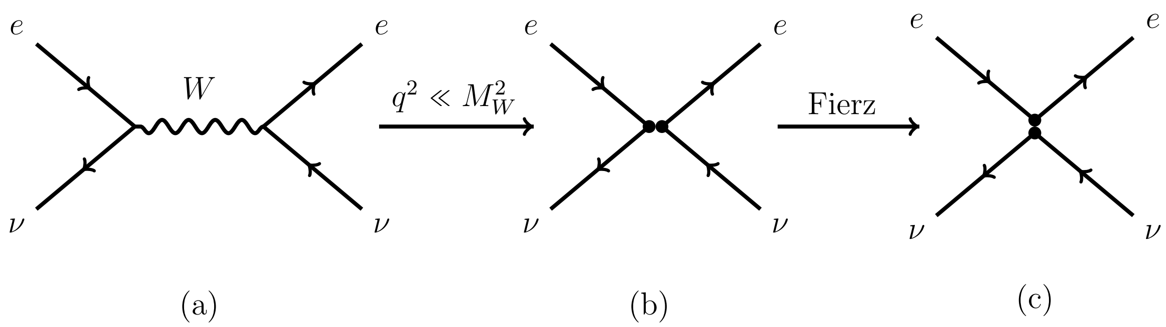


Figure 5: Tree-level Feynman diagrams for the $\bar{\nu}e \to \bar{\nu}e$ scattering corresponding to (a) the Standard Model charged current Lagrangian, (b) the effective low-energy Lagrangian obtained integrating out the W degrees of freedom and (c) this last Lagrangian after Fierz reordering the fields and writing the interaction currents as flavor-diagonal.

interaction Lagrangian which are interesting to our process are

$$\mathscr{L}_{\mathrm{CC}} = M_W^2 W_\mu^\dagger W^\mu + W_\mu^\dagger \, \bar{\nu}_e \, \Gamma^\mu \, e + W_\mu \, \bar{e} \, \Gamma^\mu \, \nu_e \, , \tag{4.3}$$

where

$$\Gamma^\mu \equiv -\frac{e}{2\sqrt{2}\sin\theta_W}\gamma^\mu(1-\gamma_5)$$

and we also wrote the kinetic term of the W_μ field.

In order to calculate the effective Lagrangian, we integrate the W_μ degrees of freedom out of the Lagrangian using its equations of motion,

$$0 = \frac{\partial \mathscr{L}_{\mathrm{CC}}}{\partial W_\mu^\dagger} = M_W^2 W^\mu + \bar{\nu}_e \, \Gamma^\mu \, e \, , \tag{4.4}$$

so

$$W_\mu = -\frac{1}{M_W^2} \, \bar{\nu}_e \, \Gamma_\mu \, e = \frac{e}{2\sqrt{2} M_W^2 \sin\theta_W} \, \bar{\nu}_e \, \gamma_\mu \, (1-\gamma_5) \, e \, . \tag{4.5}$$

Putting this relation into Eq.(4.3) one easily gets

$$\mathscr{L}_{\mathrm{CC}}^{\mathrm{eff}} = -\frac{G_F}{\sqrt{2}} \left[\bar{\nu}_e \, \gamma^\mu \, (1-\gamma_5) \, e\right] \left[\bar{e} \, \gamma_\mu \, (1-\gamma_5) \, \nu_e\right] \, , \tag{4.6}$$

where the Fermi constant is given by $\frac{G_F}{\sqrt{2}} = \frac{e^2}{8 M_W^2 \sin^2\theta_W}$.

It is convenient to write this Lagrangian as flavour diagonal—as shown in Fig.5—, so that we can add both CC and NC Lagrangians. In order to do so, we use the Fierz

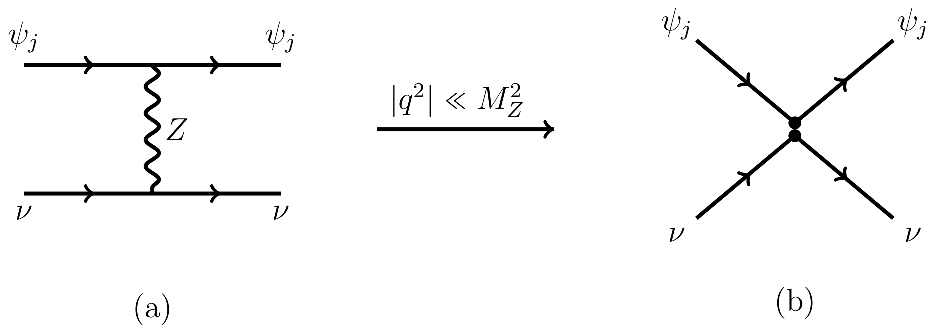


Figure 6: Tree-level Feynman diagrams for the $\psi\nu \to \psi\nu$ scattering corresponding to (a) the Standard Model neutral current Lagrangian and (b) the effective low-energy Lagrangian obtained integrating out the Z degrees of freedom.

identity (B.8) and the relation $\gamma_\mu\gamma_\nu\gamma^\mu = -2\gamma_\nu$ to write

$$\mathscr{L}_{\mathrm{CC}}^{\mathrm{eff}} = -\frac{G_F}{\sqrt{2}} \left[\bar{\nu}_e \, \gamma^\mu \, (1-\gamma_5) \, \nu_e\right] \left[\bar{e} \, \gamma_\mu \, (1-\gamma_5) \, e\right] . \tag{4.7}$$

4.2. Effective neutral current couplings

In this case, the interesting Lagrangian to our process is

$$\mathscr{L}_{\mathrm{NC}} = \frac{1}{2} M_Z^2 Z_\mu Z^\mu - \frac{e}{4\sin\theta_W \cos\theta_W} Z_\mu \, \bar{\psi}_j \, \gamma^\mu \, \left(g_{V_j} - g_{A_j}\gamma_5\right) \psi_j \, , \tag{4.8}$$

where $j = u, d, e, \nu_e, \nu_\mu, \nu_\tau$.

As in the previous section, we integrate out the Z degrees of freedom using its equations of motion,

$$0 = \frac{\partial \mathscr{L}_{\mathrm{NC}}}{\partial Z_\mu} = M_Z^2 Z^\mu - \frac{e}{4\sin\theta_W \cos\theta_W} \, \bar{\psi}_j \, \gamma^\mu \, \left(g_{V_j} - g_{A_j}\gamma_5\right) \psi_j \, , \tag{4.9}$$

so

$$Z_\mu = \frac{e}{4\sin\theta_W \cos\theta_W M_Z^2} \, \bar{\psi}_j \, \gamma_\mu \, \left(g_{V_j} - g_{A_j}\gamma_5\right) \psi_j \, . \tag{4.10}$$

Putting this relation in Eq.(4.8) we get

$$\mathscr{L}_{\mathrm{NC}}^{\mathrm{eff}} = -\frac{G_F}{2\sqrt{2}} \left[\bar{\nu} \, \gamma^\mu \, (1-\gamma_5) \, \nu\right] \left[\bar{\psi}_j \, \gamma_\mu \, \left(g_{V_j} - g_{A_j}\gamma_5\right) \psi_j\right] , \tag{4.11}$$

as is represented in Fig.6.

4.3. Low-energy effective Lagrangian for matter particles

Let us now consider some aggregate of matter A. In the scattering process $A\nu \to A\nu$ at low energy, the neutrino can interact with the three "elementary" particles which matter is formed with—electrons, protons and neutrons.

We can consider that nucleons are point-like Dirac particles because the scattering happens at low energy—i.e. the neutrino is like a large scale probe, so it cannot resolve the structure of nucleons. The vector current is conserved, so both the electric charge Q and the weak vector charge g_V of the nucleon are the sum of its valence quarks' charges,

$$\begin{aligned} Q_p &= 1\,, & g_V^p &= 1 - 4\sin^2\theta_W\,, \\ Q_n &= 0\,, & g_V^n &= -1\,. \end{aligned} \tag{4.12}$$

On the other hand, the axial current is not conserved, so this argument does not apply to the weak axial charge of the nucleon. In fact, Eq.(4.2) shows that the axial coupling is independent of the electric charge—it only depends on the weak isospin coupled to the W_μ^3 boson. Therefore, it can be expected due to weak isospin[3] symmetry that the weak neutral axial coupling at low momentum transfer, $q^2 \to 0$, is the same as the coupling to the $W_\mu^\pm$ mediated charge current responsible of the $n \to p$ process, $g_A = 1.2723 \pm 0.0023$ [3].

Taking all of this into account, the Lagrangian describing the $A\nu \to A\nu$ interaction has three terms, related to the processes

$$\begin{aligned} \nu_e + e &\longrightarrow \nu_e + e\,, & & \\ \nu_i + e &\longrightarrow \nu_i + e\,, & &(i = \mu, \tau) \\ \nu_j + N &\longrightarrow \nu_j + N\,. & &(j = e, \mu, \tau) \end{aligned}$$

The first one is mediated by both charged and neutral currents. Therefore, we have to add the Lagrangians (4.7) and (4.11), so we get

$$\begin{aligned} \mathscr{L}_1 = -\frac{G_F}{\sqrt{2}}\left[\bar{\nu}_e\,\gamma^\mu\,(1-\gamma_5)\,\nu_e\right]\left[\bar{e}\,\gamma_\mu\,(1-\gamma_5)\,e\right] - \\ -\frac{G_F}{2\sqrt{2}}\left[\bar{\nu}_e\,\gamma^\mu\,(1-\gamma_5)\,\nu_e\right]\left[\bar{e}\,\gamma_\mu\,(g_V^e - g_A^e\gamma_5)\,e\right] = \\ = -\frac{G_F}{2\sqrt{2}}\left[\bar{\nu}_e\,\gamma^\mu\,(1-\gamma_5)\,\nu_e\right]\left[\bar{e}\,\gamma_\mu\,(\tilde{g}_V^e - \tilde{g}_A^e\gamma_5)\,e\right]\,, \end{aligned} \tag{4.13}$$

[3] In fact, for the first generation of quarks, weak and strong isospin coincide.

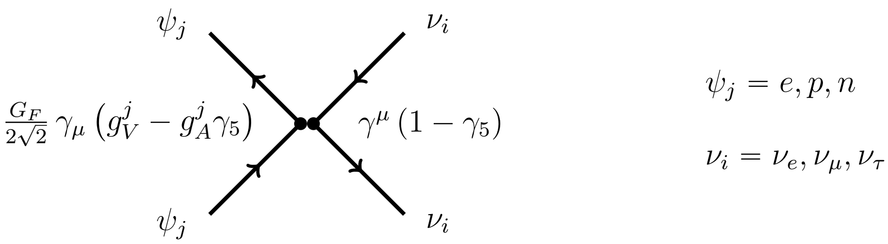


Figure 7: Fundamental vertex of the effective low-energy Lagrangian (4.17). The couplings g_V, g_A depend on both the neutrino flavor and which is the charged fermion, as discussed in the text.

where we have defined

$$\begin{aligned}\tilde{g}_V^e &= 2 + g_V^e = 1 + 4\sin^2\theta_W\,, \\ \tilde{g}_A^e &= 2 + g_A^e = 1\,. \end{aligned} \tag{4.14}$$

The second and third ones are only mediated by neutral currents, so they are described by the Lagrangian (4.11),

$$\mathscr{L}_2 = -\frac{G_F}{2\sqrt{2}}\left[\bar{\nu}_i\,\gamma^\mu\left(1-\gamma_5\right)\nu_i\right]\left[\bar{e}\,\gamma_\mu\,\left(g_V^e - g_A^e\gamma_5\right)e\right]\,, \qquad (i = \mu, \tau) \tag{4.15}$$

$$\mathscr{L}_3 = -\frac{G_F}{2\sqrt{2}}\left[\bar{\nu}_j\,\gamma^\mu\left(1-\gamma_5\right)\nu_j\right]\left[\bar{N}\,\gamma_\mu\,\left(g_V^N - g_A^N\gamma_5\right)N\right]\,. \qquad \begin{array}{l}(j = e, \mu, \tau)\\ (N = p, n)\end{array} \tag{4.16}$$

With all this information, we finally have our whole interaction Lagrangian $\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3$, which is

$$\boxed{\begin{aligned}\mathscr{L} = -\frac{G_F}{2\sqrt{2}}\Big\{&\left[\bar{\nu}_e\,\gamma^\mu\left(1-\gamma_5\right)\nu_e\right]\left[\bar{e}\,\gamma_\mu\,\left(\tilde{g}_V^e - \tilde{g}_A^e\gamma_5\right)e\right] + \\ &+\left[\bar{\nu}_i\,\gamma^\mu\left(1-\gamma_5\right)\nu_i\right]\left[\bar{e}\,\gamma_\mu\,\left(g_V^e - g_A^e\gamma_5\right)e\right] + && (i = \mu, \tau) \\ &+\left[\bar{\nu}_j\,\gamma^\mu\left(1-\gamma_5\right)\nu_j\right]\left[\bar{N}\,\gamma_\mu\,\left(g_V^N - g_A^N\gamma_5\right)N\right]\Big\} && \begin{array}{l}(j = e, \mu, \tau)\\ (N = p, n)\end{array}\end{aligned}} \tag{4.17}$$

All fundamental vertices of this Lagrangian have the same structure, as is represented in Fig.7. As a check, the couplings we obtained here are (indeed) the same ones stated in [7].

5. Neutrino-Matter Scattering

Once the interaction Lagrangian is written, we can focus on calculating the scattering amplitude between an aggregate of matter and a neutrino, $\mathcal{M}(A\nu \to A\nu)$. For simplicity, A can be understood as a molecule, composed of Z_A protons and electrons and N_A neutrons—we'd better study electrically neutral systems, with the same number of protons and electrons, because any net-charge electric interaction would be much stronger than the weak interaction we're looking for.

As shown in Fig.8, the process $A\nu \to A\nu$ is described by an elementary vertex of the interaction Lagrangian (4.17). The different terms of this Lagrangian show explicitly that the coupling neutrino-matter must depend on the flavor of the neutrino, so we will consider them separately.

5.1. Electron neutrino

In the $A(p_1)\,\nu_e(k_1) \to A(p_2)\,\nu_e(k_2)$ case, the amplitude is determined by

$$i\,T_{\nu_e} = \langle A\nu_e|\, i \int \mathrm{d}^4x\, [\mathscr{L}_1(x) + \mathscr{L}_2(x) + \mathscr{L}_3(x)]\, |A\nu_e\rangle \equiv i\,T^{(1)}_{\nu_e} + i\,T^{(2)}_{\nu_e} + i\,T^{(3)}_{\nu_e}\,, \tag{5.1}$$

which we can calculate separately. The contribution of the first term is

$$\begin{aligned} T^{(1)}_{\nu_e} &= \langle A\nu_e| \int \mathrm{d}^4x\, \frac{-G_F}{2\sqrt{2}}\, [\bar{\nu}_e\, \gamma^\mu\, (1-\gamma_5)\, \nu_e]\, [\bar{e}\, \gamma_\mu\, (\tilde{g}^e_V - \tilde{g}^e_A\gamma_5)\, e]\, |A\nu_e\rangle = \\ &= -\frac{G_F}{2\sqrt{2}} \int \mathrm{d}^4x\, \langle \nu_e|\, \bar{\nu}_e\, \gamma^\mu\, (1-\gamma_5)\, \nu_e\, |\nu_e\rangle\, \langle A|\, \bar{e}\, \gamma_\mu\, (\tilde{g}^e_V - \tilde{g}^e_A\gamma_5)\, e\, |A\rangle \equiv \\ &\equiv -\frac{G_F}{2\sqrt{2}} \int \mathrm{d}^4x\, j^\mu_{\nu_e}(x) J^{(1)}_\mu(x)\,. \end{aligned} \tag{5.2}$$

Since neutrinos are elementary particles, the leptonic current is the usual

$$\begin{aligned} j^\mu(x) &= \langle \nu_e(k_2)|\, \bar{\nu}_e(x)\, \gamma^\mu\, (1-\gamma_5)\, \nu_e(x)\, |\nu_e(k_1)\rangle = \\ &= [\bar{u}(k_2)\, \gamma^\mu\, (1-\gamma_5)\, u(k_1)]\; e^{-i(k_1-k_2)x}\,. \end{aligned} \tag{5.3}$$

On the other hand, the molecular current matrix element will be

$$J^{(1)}_\mu(x) = e^{-i(p_1-p_2)x}\, J^{(1)}_\mu = e^{-i(p_1-p_2)x}\, \langle A(p_2)|\, \bar{e}(0)\gamma_\mu\, (\tilde{g}^e_V - \tilde{g}^e_A\gamma_5)\, e(0)\, |A(p_1)\rangle\,. \tag{5.4}$$

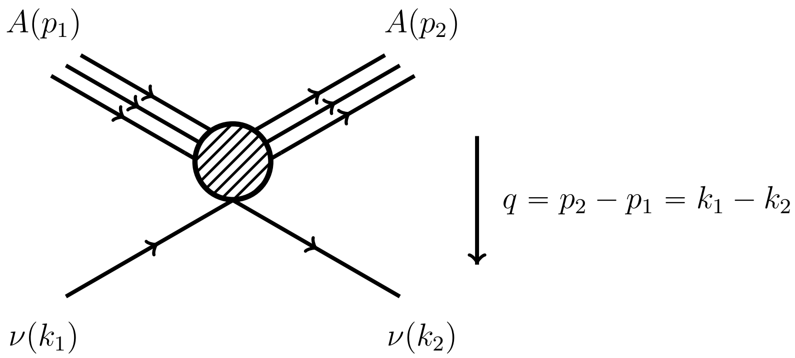


Figure 8: Lowest order Feynman diagram for the $A\nu \to A\nu$ scattering in the low-energy effective weak theory.

Since we're looking for a low-energy coherent interaction, it's interesting to analyze separately the different terms in J_μ:

- γ^0 is a scalar quantity, related to the matrix element of $e^\dagger e$, which is the number operator, so its contribution is coherent.

- $\gamma^0\gamma_5$ is a pseudo-scalar quantity, so its matrix element is related to $\boldsymbol{\sigma q}/M$, where $\boldsymbol{\sigma}$ is the spin of A, M its mass and $\boldsymbol{q} = \boldsymbol{p}_1 - \boldsymbol{p}_2$. Since this contribution depends on $\boldsymbol{\sigma}$, it's not coherent. Also, any contribution of the form $\boldsymbol{q}/M$ gives a relativistic correction to the potential, so this is another reason why we can ignore this term.

- $\boldsymbol{\gamma}$ is a polar vector, so its matrix element must be proportional to $\boldsymbol{q}/M$. Again, this is a relativistic correction we won't consider.

- $\boldsymbol{\gamma}\gamma_5$ is an axial vector, directly related to the spin of the particle, so this contribution is not coherent.

Therefore, the coherent contribution to the molecular current is given by

$$\begin{aligned}
J_0^{(1)}(x) &= \langle A(p_2)|\, \bar{e}(x)\, \tilde{g}_V^e \gamma^0 e(x)\, |A(p_1)\rangle = \tilde{g}_V^e \,\langle A(p_2)|\, e^\dagger(x)\, e(x)\, |A(p_1)\rangle = \\
&= \tilde{g}_V^e \,\langle A(p_2)| \left[\int \mathrm{d}^4 y\, |y\rangle\langle y|\right] e^\dagger(x)\, e(x) \left[\int \mathrm{d}^4 z\, |z\rangle\langle z|\right] |A(p_1)\rangle = \\
&= \tilde{g}_V^e \int \mathrm{d}^4 y\, \mathrm{d}^4 z\, e^{ip_2 y}\, e^{-ip_1 z}\, \langle A(y)|\, e^\dagger(x)\, e(x)\, |A(z)\rangle = \\
&= \tilde{g}_V^e \int \mathrm{d}^4 y\, \mathrm{d}^4 z\, e^{ip_2 y}\, e^{-ip_1 z}\, \delta^{(4)}(x-y)\, \delta^{(4)}(x-z) Z_A = \\
&= Z_A\, \tilde{g}_V^e\, e^{-i(p_1-p_2)x}\,,
\end{aligned} \tag{5.5}$$

where we have inserted two Closure Relations,

$$I = \int \mathrm{d}^4 x \, |x\rangle\langle x| \ , \tag{5.6}$$

and we have taken into account the fact that $e^\dagger(x)e(x)$ is the electron number operator at x.

Even though J_0 is the only relevant component of J_μ, it is convenient to keep a relativistic framework—later we'll consider the non-relativistic limit. Therefore, the T matrix element (5.2) is

$$T^{(1)}_{\nu_e} = (2\pi)^4 \delta^{(4)}(q + k_1 - k_2) \times \frac{-G_F}{2\sqrt{2}} \, J^{(1)}_\mu \left[\bar{u}(k_2) \, \gamma^\mu \, (1 - \gamma_5) \, u(k_1)\right] \ , \tag{5.7}$$

where $q \equiv p_1 - p_2$ and $J^{(1)}_0 = Z_A \tilde{g}^e_V$—this last equality, and the following giving J_0 values, must be understood as the coherent contribution to J_0 given by the number operator of the particle constituents.

Analogously,

$$T^{(2)}_{\nu_e} = 0 \, , \tag{5.8}$$

$$T^{(3)}_{\nu_e} = (2\pi)^4 \delta^{(4)}(q + k_1 - k_2) \times \frac{-G_F}{2\sqrt{2}} \, J^{(3)}_\mu \left[\bar{u}(k_2) \, \gamma^\mu \, (1 - \gamma_5) \, u(k_1)\right] \ . \tag{5.9}$$

where $J^{(3)}_0 = Z_A g^p_V + N_A g^n_V$.

Adding all contributions and dropping the $(2\pi)^4 \delta(p_i - p_f)$ factor, we get

$$\mathcal{M}(A\nu_e \to A\nu_e) = -\frac{G_F}{2\sqrt{2}} \, J^e_{A,\mu} \left[\bar{u}(k_2) \, \gamma^\mu \, (1 - \gamma_5) \, u(k_1)\right] \ , \tag{5.10}$$

where $J^e_{A,\mu}$ is the molecular current in the scattering with an electron neutrino. Using the weak charges from (4.12) and (4.14),

$$\begin{aligned} g^p_V &= 1 - 4\sin^2\theta_W \ , \qquad & g^e_V &= -1 + 4\sin^2\theta_W = -g^p_V \ , \\ g^n_V &= -1 \, , & \tilde{g}^e_V &= 2 + g^e_V = 2 - g^p_V \ , \end{aligned}$$

we find $J^e_{A,0} = 2Z_A - N_A$. At this level, we remind the reader that the first term comes from charged current interaction with electrons, while the second one comes from neutral currents with neutrons—neutral currents with protons and electrons cancel out.

5.2. Muon and tau neutrino

The muon and tau flavors have the same contribution to the Effective Lagrangian, so the scattering amplitudes for the processes $A\nu_\mu \to A\nu_\mu$ and $A\nu_\tau \to A\nu_\tau$ must be the same. Therefore, we can consider both of them simultaneously and calculate

$$i\,T_{\nu_j} = \langle A\nu_j|\, i\int \mathrm{d}^4x\, [\mathscr{L}_1(x) + \mathscr{L}_2(x) + \mathscr{L}_3(x)]\, |A\nu_j\rangle \equiv i\,T^{(2)}_{\nu_j} + i\,T^{(3)}_{\nu_j}\,, \tag{5.11}$$

where $j = \mu, \tau$ and $\mathscr{L}_1$ does not contribute because it only has electron neutrinos. Following the same steps than in the previous section, we get

$$T^{(2)}_{\nu_j} = (2\pi)^4\delta^{(4)}(q + k_1 - k_2) \times \frac{-G_F}{2\sqrt{2}}\, J^{(2)}_{A,\mu}\, [\bar{u}(k_2)\,\gamma^\mu\,(1-\gamma_5)\,u(k_1)]\ , \tag{5.12}$$

$$T^{(3)}_{\nu_j} = (2\pi)^4\delta^{(4)}(q + k_1 - k_2) \times \frac{-G_F}{2\sqrt{2}}\, J^{(3)}_{A,\mu}\, [\bar{u}(k_2)\,\gamma^\mu\,(1-\gamma_5)\,u(k_1)]\ , \tag{5.13}$$

where $J^{(2)}_{A,0} = Z_A g_V^e$ and $J^{(3)}_{A,0} = Z_A g_V^p + N_A g_V^n$.

Finally, dropping the $(2\pi)^4\delta(p_i - p_f)$ factor, we can write

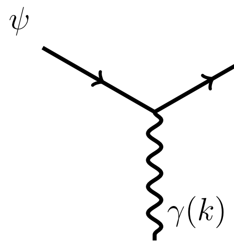
$$\mathcal{M}(A\nu_j \to A\nu_j) = -\frac{G_F}{2\sqrt{2}}\, J^j_{A,\mu}\, [\bar{u}(k_2)\,\gamma^\mu\,(1-\gamma_5)\,u(k_1)]\ , \qquad (j = \mu, \tau)\ , \tag{5.14}$$

where $J^j_{A,0} = -N_A$. As before, the neutral current interactions for electrons and protons cancel each other in neutral (of electric charge) matter.

5.3. The weak flavor charge of aggregate matter

Up to this point, we have calculated the amplitudes of the processes described by all fundamental vertices of our Lagrangian, so it's convenient to sum up our results and analyze them. In order to do that, it's useful to compare with well-known theories.

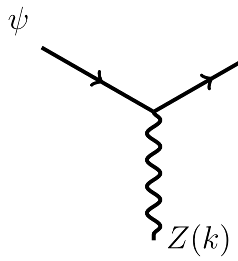
Let's consider QED. In this theory, a process described by the fundamental vertex would involve two fermions and a photon. If we take the photon on-shell and drop external-leg fermion factors, the iM would be



$$= -ieQ\gamma^{\mu}\epsilon_{\mu}(k)\,, \tag{5.15}$$

where Q is the electric charge of the fermion field. Due to vector current conservation, this vertex does also apply to non-fundamental particles, which have an electric charge equal to the sum of its constituents' charges—and the amplitude would be this charge times the coupling e.

This same behavior appears in weak interactions. The only flavor-diagonal interaction is the one mediated by neutral currents, with the fundamental vertex

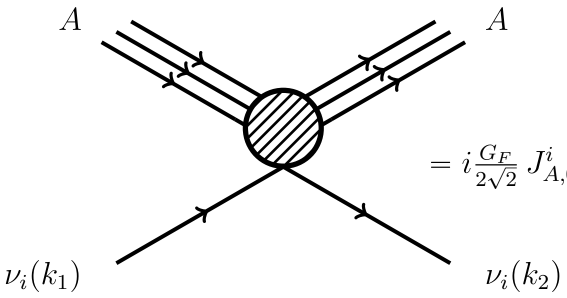


$$= -i\tfrac{e}{4\sin\theta_W\cos\theta_W}\left\{\left(2T_3 - 4Q\sin^2\theta_W\right)\gamma^{\mu} - 2T_3\gamma^{\mu}\gamma_5\right\}\epsilon_{\mu}(k)\,. \tag{5.16}$$

As is thoroughly discussed in [8], the vector current conservation allows us to talk about a weak charge of the fermion field ψ, which is $Q_W = 2T_3 - 4Q\sin^2\theta_W$, such that the weak charge of a composed particle is the sum of its constituents', as happens with electric charge. However, we can't talk about the axial coupling as a charge, since the axial current is not conserved.

Once the concept of a weak charge has been introduced, we see that the vector part of this amplitude has the same structure as the QED one, a coupling times a charge times a mediating-particle external-leg factor. According to this idea, we can expect our amplitudes to have this structure too.

Indeed, both Eqs.(5.10) and (5.14) can be written (in the non-relativistic limit) as



$$= i\frac{G_F}{2\sqrt{2}}\, J^i_{A,0}\, \left[\bar{u}(k_2)\gamma^0(1-\gamma_5)u(k_1)\right] , \tag{5.17}$$

where $Q^i_{W,A} \equiv J^i_{A,0}$ is the weak charge of the aggregate of matter A. It depends on the flavor of the neutrino, so we can speak of three weak flavor charges of aggregate matter, which are given by

$$\begin{aligned} Q^e_{W,A} &= 2Z_A - N_A \,, \\ Q^\mu_{W,A} = Q^\tau_{W,A} &= -N_A \,. \end{aligned} \tag{5.18}$$

Eqs.(5.18) state the fact that, whereas aggregate matter is neutral of electric charge, it is not neutral of weak charges!

It's interesting to analyze the value of those charges for "normal" matter. In order to do that, we'll look at stable nuclei. According to the semi-empirical mass formula [9], the (Z, N) values of stable nuclei are related by

$$Z \approx \frac{A}{2 + 0.0157A^{2/3}} \,, \tag{5.19}$$

where $A \equiv Z + N$, as is represented in Fig.9. Using those pairs of values, the weak charges of each element (neutral atom) are represented in Fig.10, where we see that the electron neutrino weak charge is always positive, while the muon and tau neutrino charges are always negative. The weak charge of aggregate matter is obtained from Fig.10 by multiplying by the number of the constituent atoms.

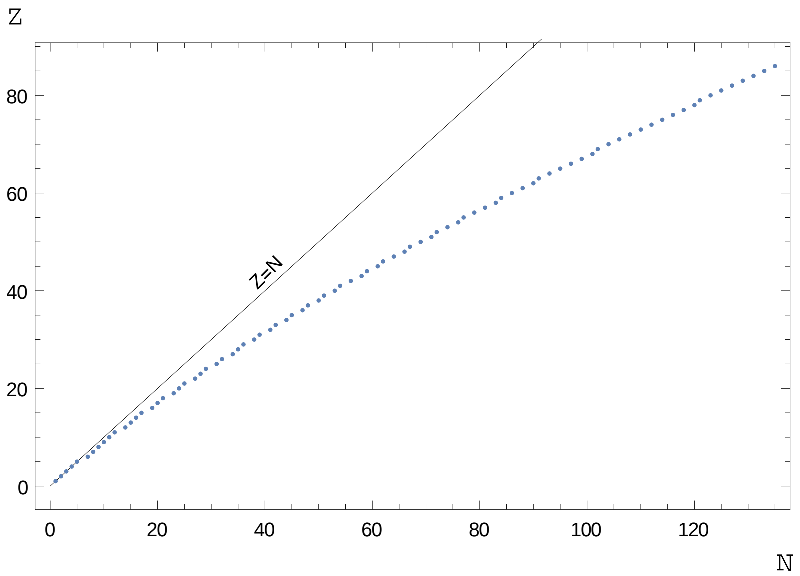


Figure 9: The valley of stability, composed of the pairs of (Z, N) for all elements, according to the semi-empirical mass formula, Eq.(5.19).

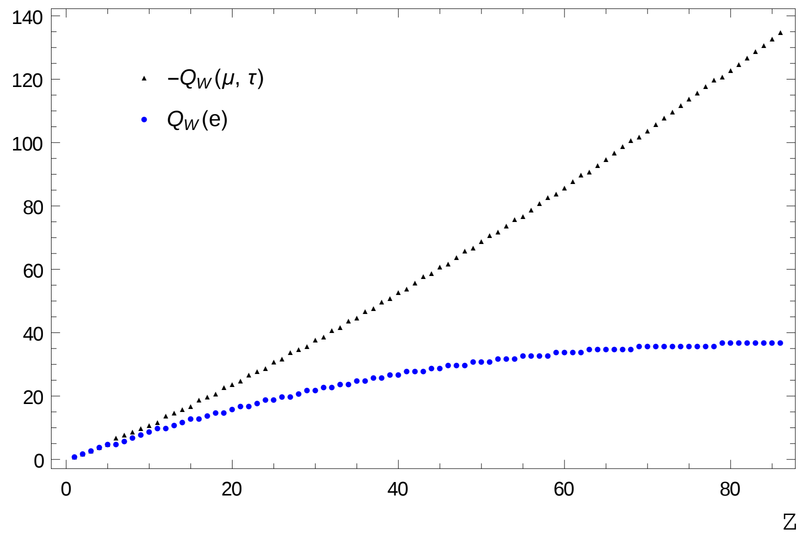


Figure 10: Weak flavor charges of the elements with (Z, N) in the valley of stability, Fig.9. Beware a minus sign in the μ, τ flavor charges.

6. Long-Range Weak Interaction Potential

After analyzing the $A\nu \to A\nu$ scattering amplitude, we can focus on calculating the interaction potential. We'll begin using Eq.(3.9) to determine the absorptive part of the $AB \to AB$ amplitude. After that, we'll use Eq.(2.20) to obtain the potential.

6.1. Absorptive part of $AB \to AB$ at low t

In order to get the absorptive part of the scattering amplitude, we need to compute the crossed quantity $\mathrm{Im}\{\mathcal{M}(A\bar{A} \to B\bar{B})\}$, as written in Eq.(3.9). Therefore, we need to cross the amplitude we calculated in the previous section, from

$$\mathcal{M}(A\nu_i \to A\nu_i) = -\frac{G_F}{2\sqrt{2}}\, J^i_{A,\mu}\, [\bar{u}(k_2)\, \gamma^\mu\, (1-\gamma_5)\, u(k_1)]$$

to

$$\mathcal{M}(A\bar{A} \to \nu_i\bar{\nu}_i) = -\frac{G_F}{2\sqrt{2}}\, \tilde{J}^i_{A,\mu}\, [\bar{u}(k_2)\, \gamma^\mu\, (1-\gamma_5)\, v(k_1)]\ , \tag{6.1}$$

where $\tilde{J}_\mu$ is the crossed molecular current, which still satisfies $\tilde{J}^i_0 = Q^i_W$ in the non-relativistic limit.

According to Eq.(3.9), the absorptive part of $\mathcal{M}(A\bar{A} \to B\bar{B})$ is determined by the quantity $\mathcal{M}(A\bar{A} \to \nu_i\bar{\nu}_i)\mathcal{M}^*(B\bar{B} \to \nu_i\bar{\nu}_i)$, which we can now evaluate. From now on we'll work in a simplified case, assuming that neutrinos are massless[4], so

$$\begin{aligned}
&\mathcal{M}(A\bar{A} \to \nu_i\bar{\nu}_i)\mathcal{M}^*(B\bar{B} \to \nu_i\bar{\nu}_i) = \\
&\qquad = \frac{G_F^2}{8}\, \tilde{J}^i_{A,\mu}\, \tilde{J}^i_{B,\nu}\, [\bar{u}(k_2)\, \gamma^\mu\, (1-\gamma_5)\, v(k_1)]\, [\bar{v}(k_1)\, \gamma^\nu\, (1-\gamma_5)\, u(k_2)] = \\
&\qquad = \frac{G_F^2}{8}\, \tilde{Z}^i_{\mu\nu}\, \mathrm{Tr}\, [\not{k}_1\gamma^\nu(1-\gamma_5)\not{k}_2\gamma^\mu(1-\gamma_5)] = \\
&\qquad = \frac{G_F^2}{4}\, \tilde{Z}^i_{\mu\nu}\, \mathrm{Tr}\, [\not{k}_1\gamma^\nu\not{k}_2\gamma^\mu(1-\gamma_5)] = \\
&\qquad = G_F^2\, \tilde{Z}^i_{\mu\nu}\, [k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - g^{\mu\nu}(k_1k_2) + a^{\mu\nu}]\ ,
\end{aligned} \tag{6.2}$$

where we defined $\tilde{Z}^i_{\mu\nu} \equiv \tilde{J}^i_{A,\mu}\, \tilde{J}^i_{B,\nu}$ and $a^{\mu\nu}$ is some antisymmetric tensor which we will no longer consider because it vanishes in the non-relativistic limit, where the only relevant component is $\mu = \nu = 0$.

[4] The non-vanishing mass of neutrinos will affect the behavior of the potential at the longest range—its implications will be announced in Section 8. Prospects.

Considering the contributions of the three neutrino flavors, the absorptive part is

$$\begin{aligned}
&\operatorname{Im}\left\{\mathcal{M}\left(A\bar{A}\to B\bar{B}\right)\right\} = \\
&\quad = \frac{G_F^2}{8\pi^2}\left(\sum_f \tilde{Z}^f_{\mu\nu}\right)\int \mathrm{d}^4k_1\,\delta(k_1^2)\,\delta(k_2^2)\left[k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - \frac{1}{2}sg^{\mu\nu}\right] = \\
&\quad = \frac{G_F^2}{8\pi^2}\left(\sum_f \tilde{Z}^f_{\mu\nu}\right)\int \mathrm{d}^4k_1\,\delta(k_1^2)\,\delta(k_2^2)\left[-2k_1^\mu k_1^\nu + (k_1^\mu q^\nu + k_1^\nu q^\mu) - \frac{1}{2}sg^{\mu\nu}\right] = \\
&\quad = \frac{G_F^2}{8\pi^2}\left(\sum_f \tilde{Z}^f_{\mu\nu}\right)\frac{\pi}{2}\left[-\frac{2}{3}\left(q^\mu q^\nu - \frac{1}{4}tg^{\mu\nu}\right) + \frac{1}{2}\left(q^\mu q^\nu + q^\nu q^\mu\right) - \frac{1}{2}sg^{\mu\nu}\right] = \\
&\quad = \frac{G_F^2}{24\pi}\left(\sum_f \tilde{Z}^f_{\mu\nu}\right)\left[\,q^\mu q^\nu - sg^{\mu\nu}\,\right],
\end{aligned} \tag{6.3}$$

where we used $k_2 = q - k_1$ in the second line and all integrals needed in the third line are stated in [10]—we demonstrate them in Appendix B.3.

As seen, the tensor structure of Eq.(6.3) is transverse, a requirement which any quantity built from conserved currents must satisfy. We can cross this result back to the $t-$channel applying $s \to t$, so that

$$\operatorname{Im}\left\{\mathcal{M}(AB\to AB)\right\} = \frac{G_F^2}{24\pi}\left(\sum_f Z^i_{\mu\nu}\right)\left[q^\mu q^\nu - tg^{\mu\nu}\right]. \tag{6.4}$$

Now it's easy to evaluate the non-relativistic limit. As discussed before, the only relevant component of the molecular current for coherent interactions is the scalar contribution to J_0, so we can take

$$\operatorname{Im}\left\{M(AB\to AB)\right\} = \frac{G_F^2}{24\pi}\left(\sum_f Q^f_{W,A}\,Q^f_{W,B}\right)\left[\left(q^0\right)^2 - t\right]. \tag{6.5}$$

Besides, we are looking for a long-range interaction, so $q^0 \approx 0$ and

$$\operatorname{Im}\left\{M(AB\to AB)\right\} = -\frac{G_F^2}{24\pi}\left(\sum_f Q^f_{W,A}\,Q^f_{W,B}\right)t\,, \tag{6.6}$$

where

$$\sum_f Q^f_{W,A}\,Q^f_{W,B} = (2Z_A - N_A)(2Z_B - N_B) + 2N_AN_B\,. \tag{6.7}$$

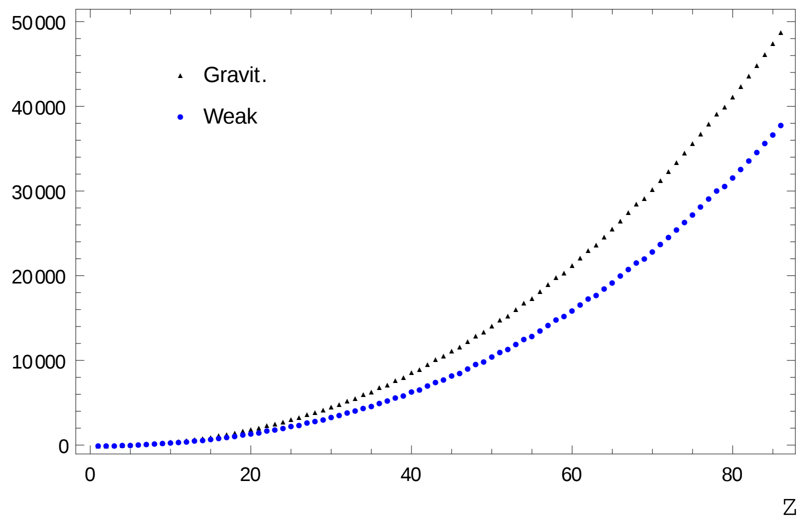


Figure 11: Weak coupling $\sum Q_W^f Q_W^f$, which is written in Eq.(6.7), for the elements of the valley of stability (Fig.9), each one interacting with itself. The gravitational coupling $M^2/m_p^2 \approx (Z+N)^2$, neglecting binding energies, is also represented.

As Eq.(6.6) shows, all the flavor dependence of the absorptive part—and, therefore, of the potential—in the limit of massless neutrinos is factorized in the weak charges.

As we saw in Fig.10, all the stable elements have the same sign for the weak charges, $Q_W^e > 0$ and $Q_W^{\mu,\tau} < 0$. This implies that, for any pair of elements—and therefore for any pair of molecules—this coupling has a positive sign, so the resulting force will have the same character—whether repulsive or attractive—for any pair of aggregates of matter. We'll find out which of those two cases is the right one in the next section.

Let's see the behavior of this quantity for some cases. As we did in the previous section, we'll consider only stable nuclei. Since this is an interaction, we have to choose sets of two elements—we'll consider the interaction of each element with itself. In Fig.11 we show the quantity $\sum Q_{W,A}^f Q_{W,A}^f$ for the element A as a function of the atomic number Z_A for the stable nuclei.

In the same Figure we compare the weak coupling with the gravitational one. It's seen that both of them increase with the number of particles of the systems interacting, but they scale differently, even when the binding energy is neglected. That means that our coherent weak interaction could introduce a deviation from the Equivalence Principle, as was announced in [2].

6.2. Neutrino-pair exchange potential

After obtaining this result, the only remaining step in the calculation of the interaction potential is computing the integral (2.20), with the branching point at $t_0 = 0$ (for massless neutrinos). Using the $\mathrm{Im}\{M\}$ obtained in Eq.(6.6),

$$V(r) = \frac{G_F^2}{24\pi}\left(\sum_f Q^f_{W,A}\, Q^f_{W,B}\right)\frac{1}{4\pi^2 r}\int_0^\infty \mathrm{d}t\, t\, e^{-\sqrt{t}\, r}\,, \tag{6.8}$$

so we have to evaluate this integral. A primitive $P(t)$ is calculated in Appendix B.4, so we obtain the integral (say I) using Barrow's Rule, $I = P(t\to\infty) - P(t\to 0)$. If we assume $r \neq 0$ when calculating the limits—which is valid, since we are looking for a long-range interaction—this contribution to the potential is

$$I = \frac{12}{r^4}\,. \tag{6.9}$$

With this result, we can finally write

$$\boxed{V(r) = \frac{G_F^2}{8\pi^3}\left(\sum_f Q^f_{W,A}\, Q^f_{W,B}\right)\frac{1}{r^5}}\,, \tag{6.10}$$

which has an associated force given by

$$\boldsymbol{F}(r) = -\boldsymbol{\nabla} V(r) = \frac{5G_F^2}{8\pi^3}\left(\sum_f Q^f_{W,A}\, Q^f_{W,B}\right)\frac{\hat{\boldsymbol{r}}}{r^6}\,, \tag{6.11}$$

where $\hat{r}$ is the radial unit vector.

We have obtained a long-range interaction which is repulsive for ordinary matter, since the weak coupling is always positive—as we showed in Fig.11. This is a difference with the gravitational force that we can use to distinguish them—and we can also look for the deviations from the Equivalence Principle we discussed in the previous section.

The other interactions that appear in electrically neutral systems are the residual electromagnetic Van der Waals ($\sim r^{-7}$) or Casimir-Polder ($\sim r^{-8}$) forces, which have a lower range (larger inverse power law) than our weak force, even though they're stronger. It would be interesting to find systems with low electromagnetic momenta, so that this interactions became weaker, as the ones described in [11].

7. Conclusions

We began this work reviewing the relation between the description of an interaction process in the framework of a Quantum Field Theory and in terms of an interaction potential. As shown in Section 2, the Feynman amplitude of an elastic scattering process and the effective potential describing this interaction are related, in Born approximation, by a Fourier Transform. This is the result one would expect, taking into account that $M(q^2)$ describes the interaction process in momentum space, while $V(r)$ describes the interaction in position space.

Another interesting detail that stems from the fact that we are calculating a long-range interaction is that we needn't calculate the whole amplitude of the process in order to determine the potential. This is due to the fact that $M(q^2)$ at $|q^2| \to \infty$ gives short-range contributions, so the potential is determined by $\mathrm{Im}\{M(q^2)\}$ through an unsubtracted dispersion relation. Therefore, we could use the unitarity relation from Section 3 to avoid the calculation of $M(q^2)$ (a $1-$loop quantity) and compute a tree-level process instead. Besides, the fact that we were interested in the low-energy limit allowed us to work in the framework of an effective theory where Charged Currents and Neutral Currents could be written in the form of a contact interaction, so we worked with only one interaction vertex.

The results from Sections 2-4 made it clear that the effective potential was determined by the amplitude of the process $A\nu \to A\nu$, which we calculated in Section 5. Although this was a quite straightforward calculation, it gave rise to a very interesting concept—the weak flavor charge of aggregate matter. Indeed, the coupling of bulk matter to a neutrino is proportional to G_F with a charge that depends on the flavor of the neutrino,

$$Q_W^e = 2Z - N \,, \qquad\qquad Q_W^\mu = Q_W^\tau = -N \,.$$

These are the weak flavor charges for electrically neutral matter—the case we are interested in, since any non-zero electric charge would produce an electromagnetic interaction much stronger than our weak interaction.

This amplitude was the last ingredient needed to compute the effective potential, which gives raise to the repulsive force

$$\boldsymbol{F} = \frac{5G_F^2}{8\pi^3}\left(\sum_f Q_{W,A}^f Q_{W,B}^f\right)\frac{\hat{\boldsymbol{r}}}{r^6} \,,$$

where all flavor dependence is in the weak charges. This is the coherent contribution to the force, which we obtained from the vector charge J^0, proportional to γ^0. The first correction to this result would come from the spin dependent contribution to $\boldsymbol{J}$, that comes from $\boldsymbol{\gamma}\gamma_5$. The other two contributions to the current, proportional to $\gamma^0\gamma_5$ and $\boldsymbol{\gamma}$, give relativistic corrections $\sim \frac{1}{M}$.

In the long-range regime we are looking at, there are two other important interactions: residual electromagnetic interactions and gravitation. For ordinary molecules, Van der Waals forces are much stronger than our weak interaction at short distances, so it would be interesting to look for a system where the first electromagnetic moments are zero. In the case of gravitation, there are two traits of this weak interaction that can help to distinguish between them in an experiment: this force is repulsive—while gravitation is attractive—and its charge is proportional to the number of particles but not to their mass, so it would produce a signal that deviates from the Equivalence Principle.

In any case, joining the previous ideas with the recent development of atomic traps [12]—not ionic traps—can be the key to observe this interaction in an experiment.

8. Prospects

The long-range potential obtained in this work, Eq.(6.10), is valid and of interest for distances between nanometers and microns. The short-distance limit comes from the requirement of having neutral (of electric charge) systems of aggregate matter, while the long-distance limit is imposed by a non-vanishing value of the absolute mass of the neutrino—indeed, the range of this interaction for neutrinos of $m \sim 0.1$ eV is of the order of

$$R \sim \frac{\hbar c}{mc^2} = \frac{197 \text{ MeV fm}}{0.1 \text{ eV}} \sim 10^9 \text{ fm} = 1\,\mu\text{m}\,.$$

In this region, the effective potential will become of Yukawa type instead of the inverse power law. We can get a first idea on the dependence of the potential with m changing slightly this work's result. If we had integrated Eq.(6.8) from a branching point at $t_0 = 4m^2$, the potential would have been

$$V(r) = \frac{G_F^2}{8\pi^3}\left(\sum_f Q_{W,A}^f\, Q_{W,B}^f\right)\left(\frac{1}{r^5} + \frac{2m}{r^4} + \frac{2m^2}{r^3} + \frac{4m^3}{3r^2}\right) e^{-2mr}\,,$$

which depends on m not only in the Yukawa exponential, but also in the preceding inverse power terms[5].

The neutrino mass dependence of the effective potential in the long-range behavior opens novel directions in the study of the most interesting pending questions on neutrino properties: absolute neutrino mass (from the range), flavor dependence and mixing (from the weak charges in the interaction) and, hopefully, with two neutrino exchange, the exploration of the most crucial open problem in neutrino physics: whether neutrinos are Dirac or Majorana particles.

The study of these problems will be the subject of my immediate future research work. On the one hand, it's necessary to calculate the form of this interaction with a finite mass for the neutrino. In fact, two calculations are needed: for Dirac neutrinos and for Majorana neutrinos. On the other hand, the collaboration with the experimental groups involved in neutral traps will be initiated this summer during my stay at CERN, in order to find out whether the low electromagnetic interacting systems mentioned in our Conclusions could be implemented.

[5] Of course, the computation of the potential at finite m is not so trivial—the mass has to be included in the absorptive part of the amplitude—, but it serves for illustrating the kind of changes that will occur.

Appendices

A. Notations and Conventions

A.1. Units

This work is written using the Natural System of Units, where $\hbar = c = k_B = 1$. We describe electromagnetic quantities with the Heavyside system, $\epsilon_0 = \mu_0 = 1$, so that the fine-structure constant is given by $\alpha = e^2/4\pi \approx 1/137$.

A.2. Relativity end Tensors

We define the Minkowsi metric tensor with signature $(+,-,-,-)$, as

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.1}$$

so that any 4-vector can be written as $x^\mu = (x^0, \boldsymbol{x})$. We also use the Einstein Summation Convention, so scalar products can be written as $x \cdot p = x^\mu p_\mu = g_{\mu\nu} x^\mu p^\nu = x^0 p^0 - \boldsymbol{x} \cdot \boldsymbol{p}$. We can also write $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\boldsymbol{x})$, and the derivative operator is $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \boldsymbol{\nabla})$.

We use the totally antisymmetric tensor with the convention $\epsilon^{0123} = +1 = -\epsilon_{0123}$.

A.3. Fourier Transforms

We define Fourier Transforms so that all 2π factors are included in the momentum integration,

<u>4−dimensional FT:</u>

$$f(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, e^{-ikx} \, \tilde{f}(k) \,, \qquad \tilde{f}(k) = \int \mathrm{d}^4 x \, e^{ikx} \, f(x) \,,$$

<u>3−dimensional FT:</u>

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, e^{i\boldsymbol{k}\boldsymbol{x}} \, \tilde{f}(\boldsymbol{k}) \,, \qquad \tilde{f}(\boldsymbol{k}) = \int \mathrm{d}^3 x \, e^{-i\boldsymbol{k}\boldsymbol{x}} \, f(\boldsymbol{x}) \,, \tag{A.2}$$

Other 2π factors come from the following expression for the Dirac delta,

$$\int \mathrm{d}^4x\, e^{ikx} = (2\pi)^4\delta^{(4)}(k)\,. \tag{A.3}$$

A.4. Diracology

As is well known, Dirac gamma matrices are required to satisfy the relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \qquad [\gamma^\mu, \gamma^\nu] = -2i\sigma^{\mu\nu}. \tag{A.4}$$

Also,

$$(\gamma^0)^2 = -(\gamma^i)^2 = 1, \qquad \gamma^\dagger_\mu = \gamma^0\gamma_\mu\gamma^0. \tag{A.5}$$

We define the fifth gamma matrix as $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\alpha\beta\gamma\delta}\,\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$, which satisfies $(\gamma_5)^2 = -1,\ \gamma_5^\dagger = \gamma_5$. Therefore, the quirality projectors can be written as

$$P_R = \frac{1+\gamma_5}{2}, \qquad P_L = \frac{1-\gamma_5}{2}. \tag{A.6}$$

Some useful contractions are

$$\gamma^\mu\gamma_\mu = 4\,, \tag{A.7a}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \tag{A.7b}$$
$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4g^{\alpha\beta}, \tag{A.7c}$$
$$\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma_\mu = -2\gamma^\beta\gamma^\alpha\gamma^\nu. \tag{A.7d}$$

Some useful trace identities are

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = 4g^{\mu\nu}, \tag{A.8a}$$
$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma_5\right] = 0\,, \tag{A.8b}$$
$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right), \tag{A.8c}$$
$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma_5\right] = -4i\,\epsilon^{\mu\nu\alpha\beta}, \tag{A.8d}$$
$$\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2k+1}}\right] = 0\,. \tag{A.8e}$$

For any 4−vector a^μ, we define $\not{a} \equiv \gamma_\mu a^\mu$.

B. Useful Relations

B.1. Fierz Identity

Let us consider the Dirac-scalar quantity

$$[\bar{u}_1 A P_L u_2]\,[\bar{u}_3 P_R B u_4]\ , \tag{B.1}$$

where A and B are arbitrary matrices in Dirac space, $P_{L,R}$ are the quirality projectors from Eq.(A.6) and the four u_i are Dirac spinors[6].

The set of matrices $\Gamma_i = \{1, \gamma_5, \gamma^\mu P_L, \gamma^\mu P_R, \sigma^{\mu\nu}\}$ are a basis of Dirac space, so we can expand

$$u_2\bar{u}_3 = \sum_i \alpha_i \Gamma_i = \alpha_1 1 + \alpha_5\gamma_5 + \alpha_L^\mu \gamma_\mu P_L + \alpha_R^\mu \gamma_\mu P_R + \alpha_S^{\mu\nu}\sigma_{\mu\nu}\,. \tag{B.2}$$

Since we have this expansion between quiarilty projectors, we can simplify

$$P_L u_2 \bar{u}_3 P_R = P_L \left(\sum_i \alpha_i \Gamma_i\right) P_R = \alpha_R^\mu \gamma_\mu P_R\,, \tag{B.3}$$

where we have used the relations

$$P_{L,R}^2 = P_{L,R}\,, \qquad P_{L,R}P_{R,L} = 0\,, \qquad P_{L,R}\gamma_\mu = \gamma_\mu P_{R,L}\ , \qquad P_{L,R}\gamma_5 = \gamma_5 P_{L,R}$$

to show that all other terms are zero.

We need to calculate the α_R^μ coefficient, so we evaluate the quantity

$$\mathrm{Tr}[\gamma^\nu P_L u_2 \bar{u}_3] = \mathrm{Tr}\left[\gamma^\nu P_L \left(\sum_i \alpha_i \Gamma_i\right)\right] = \alpha_R^\mu \mathrm{Tr}[\gamma^\nu P_L \gamma_\mu P_R] = 2\alpha_R^\mu\,, \tag{B.4}$$

which means that we can get the coefficient by computing

$$\alpha_R^\mu = \frac{1}{2}\mathrm{Tr}[\gamma^\mu P_L u_2 \bar{u}_3] = \frac{1}{4}\bar{u}_3\gamma^\mu(1-\gamma_5)u_2 = \frac{1}{2}\bar{u}_3\gamma^\mu P_L u_2\,. \tag{B.5}$$

[6] If any of these spinors were a v spinor, nothing in this section would change—the Identity would still hold.

Putting this relation into Eq.(B.3) one gets

$$P_L u_2 \bar{u}_3 P_R = \frac{1}{2}[\bar{u}_3 \gamma^\mu P_L u_2]\gamma_\mu P_R \,, \tag{B.6}$$

so that Eq.(B.1) can be rewritten as

$$[\bar{u}_1 A P_L u_2]\,[\bar{u}_3 P_R B u_4] = \frac{1}{2}[\bar{u}_1 A \gamma_\mu P_R B u_4][\bar{u}_3 \gamma^\mu P_L u_2] \,, \tag{B.7}$$

which is the identity we wanted to prove. Notice that, unlike spinors, fermionic fields anticommute, so their version of the Fierz Identity is

$$\left[\bar{\psi}_1 A P_L \psi_2\right]\left[\bar{\psi}_3 P_R B \psi_4\right] = -\frac{1}{2}[\bar{\psi}_1 A \gamma_\mu P_R B \psi_4][\bar{\psi}_3 \gamma^\mu P_L \psi_2] \,, \tag{B.8}$$

with an extra minus sign.

B.2. Fourier Transforms

B.2.1. Yukawa/Coulomb propagator

Let's compute the Fourier Transform of the Yukawa propagator,

$$I \equiv \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, e^{i\boldsymbol{q}\boldsymbol{r}}\, \frac{1}{q^2+\mu^2} \,, \tag{B.9}$$

where $q \equiv |\boldsymbol{q}\,|$, which will also give the Fourier Transform of the Coulomb propagator taking the limit $\mu \to 0$. In spherical coordinates, $\mathrm{d}^3 q = q^2 \mathrm{d}q\, \mathrm{d}\cos\theta\, \mathrm{d}\phi$, we get

$$I = \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}q\, \frac{q^2}{q^2+\mu^2} \int_{-1}^{1} \mathrm{d}\cos\theta e^{iqr\cos\theta} \,. \tag{B.10}$$

Integrating over $\cos\theta$ we get

$$\begin{aligned} I &= \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}q\, \frac{q^2}{q^2+\mu^2}\, \frac{2\sin qr}{qr} = \frac{1}{(2\pi)^2}\, \mathrm{Im}\left\{ \int_{-\infty}^{\infty} \mathrm{d}q\, \frac{q^2}{q^2+\mu^2}\, \frac{e^{iqr}}{qr} \right\} = \\ &= \frac{1}{(2\pi)^2 r}\, \mathrm{Im}\left\{ \int_{-\infty}^{\infty} \mathrm{d}y\, \frac{y}{y^2+\mu^2 r^2}\, e^{iy} \right\} \equiv \frac{1}{(2\pi)^2 r}\, \mathrm{Im}\left\{ \int_{-\infty}^{\infty} \mathrm{d}y\, f(y) \right\} \,. \end{aligned} \tag{B.11}$$

The integration of $f(y)$ along the circumference arc in Fig.12 is zero, since it goes

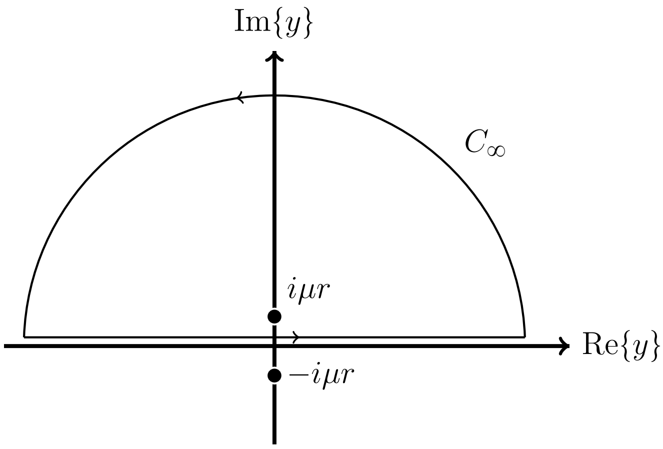


Figure 12: Integration path (in the complex plane of the y variable) used in the Residue Theorem for the integral in expression (B.12).

as $e^{-|y|}/|y|$ when $y \to i\infty$. Therefore, using the Residue Theorem [6],

$$\int_{-\infty}^{\infty} \mathrm{d}y\, f(y) = \int_{C_\infty} \mathrm{d}y f(y) - 2\pi i\ \mathrm{Res}\,[f(y), y = i\mu r] = -2\pi i\ \mathrm{Res}\,[f(y), y = i\mu r]\ . \tag{B.12}$$

The pole of $f(y)$ in $y = i\mu r$ is simple, so we can compute the residue as

$$\mathrm{Res}\,[f(y), y = i\mu r] = \lim_{y\to i\mu r} (y - i\mu r) f(y) = \lim_{y\to i\mu r} \frac{y}{(y + i\mu r)}\, e^{iy} = \frac{1}{2}\, e^{-\mu r}\, . \tag{B.13}$$

Using this result, we can trivially write

$$I = \frac{1}{4\pi}\, \frac{e^{-\mu r}}{r}\, . \tag{B.14}$$

B.2.2. A spherical wave

Let's compute the integral

$$I \equiv \frac{1}{4\pi} \int \mathrm{d}^3 r\, e^{-i\boldsymbol{q}\,\boldsymbol{r}}\, \frac{e^{-\sqrt{t'}r}}{r}\, , \tag{B.15}$$

where the 4π factor has been introduced for convenience.

In spherical coordinates $\mathrm{d}^3 r = r^2 \mathrm{d}r\, \mathrm{d}\Omega = r^2 \mathrm{d}r\, \mathrm{d}\cos\theta\, \mathrm{d}\phi$,

$$I = \frac{1}{2} \int_0^\infty \mathrm{d}r\, r\, e^{-\sqrt{t'}r} \int_{-1}^{1} \mathrm{d}\cos\theta\, e^{-iqr\cos\theta}\, , \tag{B.16}$$

where the $\int \mathrm{d}\phi$ has been trivially computed and $q \equiv |\boldsymbol{q}\,|$. The remaining angular integral is also easy to calculate, so we can write

$$I = \frac{1}{q}\int_0^\infty \mathrm{d}r\, e^{-\sqrt{t'}r}\,\sin(qr)\,. \tag{B.17}$$

This integral can be computed integrating by parts:

$$\begin{aligned} I &= \frac{1}{q}\int_0^\infty \mathrm{d}r\,\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{-1}{\sqrt{t'}}e^{-\sqrt{t'}r}\,\sin(qr)\right] - \left[\frac{-1}{\sqrt{t'}}e^{-\sqrt{t'}r}\,\frac{\mathrm{d}}{\mathrm{d}r}\sin(qr)\right]\right\} = \\ &= 0 + \frac{1}{\sqrt{t'}}\int_0^\infty \mathrm{d}r\, e^{-\sqrt{t'}r}\,\cos(qr) = \\ &= \frac{1}{\sqrt{t'}}\int_0^\infty \mathrm{d}r\,\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{-1}{\sqrt{t'}}e^{-\sqrt{t'}r}\,\cos(qr)\right] - \left[\frac{-1}{\sqrt{t'}}e^{-\sqrt{t'}r}\,\frac{\mathrm{d}}{\mathrm{d}r}\cos(qr)\right]\right\} = \\ &= \frac{1}{t'}\left\{1 - q\int_0^\infty \mathrm{d}r\, e^{-\sqrt{t'}r}\,\sin(qr)\right\} \end{aligned} \tag{B.18}$$

Taking into account the fact that $q^2 \equiv \boldsymbol{q}^{\,2} = -t$, this last relation can be written as

$$I = \frac{1}{t'}(1 + tI) \qquad \longrightarrow \qquad (t' - t)I = 1\,. \tag{B.19}$$

Therefore, we have proved the relation

$$\frac{1}{t' - t} = \frac{1}{4\pi}\int \mathrm{d}^3r\, e^{-i\boldsymbol{q}\,\boldsymbol{r}}\,\frac{e^{-\sqrt{t'}r}}{r}\,. \tag{B.20}$$

B.3. Integrals for the absorptive part

In this appendix we are going to calculate the integrals

$$I \equiv \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2) = \frac{\pi}{2}\,, \tag{B.21a}$$

$$I^\mu \equiv \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,k^\mu = \frac{\pi}{4}\,q^\mu\,, \tag{B.21b}$$

$$I^{\mu\nu} \equiv \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,k^\mu k^\nu = \frac{\pi}{6}\left(q^\mu q^\nu - \frac{1}{4}\,s\,g^{\mu\nu}\right)\,, \tag{B.21c}$$

where $\bar{k} = q - k$ and $q^2 = s$.

Let's consider the first one. Using $k^2 = E^2 - \boldsymbol{k}^{\,2}$ in the first delta function, we compute the $E \equiv k^0$ integral,

$$I = \int \frac{\mathrm{d}^3k}{2E}\,\delta(\bar{k}^2) = \int \frac{\mathrm{d}^3k}{2E}\,\delta\left[q^2 - 2(kq)\right] . \tag{B.22}$$

We can evaluate the integral in the CM reference frame, where $q^\mu = (\sqrt{s}, \mathbf{0}\,)$, so that

$$I = \frac{1}{2}\int \mathrm{d}\Omega\,\mathrm{d}E\,E\,\delta\left(s - 2E\sqrt{s}\right) = \frac{1}{2}\int \mathrm{d}\omega\,\frac{E}{2\sqrt{s}}\bigg|_{E=\sqrt{s}/2} = \frac{1}{8}\int \mathrm{d}\Omega = \frac{\pi}{2} , \tag{B.23}$$

as we wanted to prove.

Due to Lorentz covariance, the I^μ integral must be of the form

$$I^\mu \equiv \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,k^\mu = A\,q^\mu . \tag{B.24}$$

Multiplying this relation by q_μ, we get

$$A\,q^2 = \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,(k\bar{k}) = \frac{1}{2}\,q^2\,I = \frac{\pi}{4}\,q^2 , \tag{B.25}$$

so

$$I^\mu = \frac{\pi}{4}\,q^\mu . \tag{B.26}$$

Finally, we can also use Lorentz covariance to write

$$I^{\mu\nu} \equiv \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,k^\mu k^\nu = Ag^{\mu\nu} + Bq^\mu q^\nu . \tag{B.27}$$

We need to multiply by $g_{\mu\nu}$ and $q_\mu q_\nu$ to determine A and B,

$$g_{\mu\nu}I^{\mu\nu} = 4A + q^2B = \int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,k^2 = 0 , \tag{B.28a}$$

$$\frac{q_\mu q_\nu}{q^2}I^{\mu\nu} = A + q^2B = \frac{1}{q^2}\int \mathrm{d}^4k\,\delta(k^2)\delta(\bar{k}^2)\,(k\bar{k})^2 = \frac{1}{4}\,{}^2\,I = \frac{\pi}{8}q^2 , \tag{B.28b}$$

so we just need to solve the algebraic system of equations

$$\begin{aligned} 4A + q^2B &= 0 , \\ A + q^2B &= \frac{\pi}{8}\,q^2 , \end{aligned} \tag{B.29}$$

which gives

$$A = -\frac{\pi}{24}\, q^2\,, \qquad\qquad B = \frac{\pi}{6}\,. \tag{B.30}$$

Therefore,

$$I^{\mu\nu} = \frac{\pi}{6}\left(q^\mu q^\nu - \frac{1}{4}\, q^2\, g^{\mu\nu}\right)\,. \tag{B.31}$$

B.4. Integral for the interaction potential

We are interested in calculating a primitive $P(t)$ of

$$\int \mathrm{d}t\, t\, e^{-\sqrt{t}\, r}\,, \tag{B.32}$$

which can be obtained quite straightforwardly using $q \equiv \sqrt{t}$,

$$\begin{aligned}
P(t) = \int \mathrm{d}t\, t\, e^{-\sqrt{t}\, r} &= 2\int \mathrm{d}q\, q^3\, e^{-qr} = \\
&= -2\,\frac{\mathrm{d}^3}{\mathrm{d}r^3}\int \mathrm{d}q\, e^{-qr} = \\
&= 2\,\frac{\mathrm{d}^3}{\mathrm{d}r^3}\left[\frac{e^{-qr}}{r}\right] = \\
&= 2\,\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[\left(-\frac{1}{r^2} - \frac{q}{r}\right) e^{-qr}\right] = \\
&= 2\,\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{2}{r^3} + \frac{2q}{r^2} + \frac{q^2}{r}\right) e^{-qr}\right] = \\
&= -\left(\frac{12}{r^4} + \frac{12q}{r^3} + \frac{6q^2}{r^2} + \frac{2q^3}{r}\right) e^{-qr} = \\
&= -\left(\frac{12}{r^4} + \frac{12\sqrt{t}}{r^3} + \frac{6t}{r^2} + \frac{2\sqrt{t^3}}{r}\right) e^{-\sqrt{t}\, r}\,.
\end{aligned} \tag{B.33}$$

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