

# Some Three-body force cancellations in Chiral Lagrangians <sup>\*</sup>

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## Abstract

The cancellation between off-shell two body forces and three body forces implies a tremendous simplification in the study of three body resonances in two meson-one baryon systems. While this can be done by means of Faddeev equations we provide an alternative and simpler derivation using just the chiral Lagrangean and the field reparameterization invariance.

## 1 Introduction

The main and stunning difference between point particles and waves is that the latter contain an effective size corresponding to their wavelength. This provides a natural resolution scale,  $\lambda$ , *below* which details cannot be resolved. For instance, the boundaries of macroscopic matter objects do not show up their roughness below the visible light wavelength of the order of  $\lambda_\gamma \sim 400 - 800nm$  and hence appear as locally smooth. In non-relativistic Quantum Mechanics, due to the wave-particle duality, the de Broglie wavelength sets up the scale  $\lambda_{dB} = \hbar/Mv$  and becomes large for slow enough particles. In Relativistic Quantum Field Theory and because of microcausality, interactions are generated by particle exchange and the corresponding Compton wavelength  $\lambda_C = \hbar/mc$  provides the range of the interaction, exemplified by the Yukawa-like potential  $V(r) \sim e^{-r/\lambda_C}/r$ .

Thus, if we think of  $N$  interacting composite quantum particles with rest masses  $M_i$  and typical momenta  $p_i$  and interacting through exchanges or particles with masses  $m_j$  we have that for  $p_{\max} \equiv \max |p_i - p_j| \ll m_{\min} \equiv \min m_i$  we expect not to resolve the precise form of the interaction and the few-body problem may be analyzed within a systematic expansion in  $p_{\max}/m_{\min}$ . The coefficients of such an expansion are called low energy constants (LEC's). These

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<sup>\*</sup>Dedicated to the memory of Juan Antonio Morente Chiquero (1955-2012).

LEC's implement shape independence, short-distance insensitivity and effective elementarity and summarize all effects not explicitly taken into account. Nonetheless, they also depend on the resolution scale  $\lambda$  and a too coarse resolution may not allow to distinguish between N-body correlations from the difficult N-body forces due to (N-1) irreducible particle exchange (with range  $\sim \lambda_C/N$ ). A concise form of summarising all these features is by using an Effective Field Theory (EFT) [1] where the renormalization scale  $\mu = \hbar c/\lambda$  is used instead. A more tangible approach uses the concept of coarse grained interactions (see e.g. [2] for a Nuclear Physics setup).

While this discussion is quite general we will focus here on the application to hadronic and relativistic systems described by quantum fields and show how further simplifications can arise in modern chiral Lagrangians which effectively describe scattering and bound states involving one baryon and two mesons, such as e.g. the  $\pi\pi N$  system.

## 2 Hadronic Interpolating Fields

Quantum Chromo Dynamics (QCD) is expected to describe all known composite hadronic systems from the pion to finite nuclei in terms of  $(u, d, s, \dots)$ , quarks (or anti-quarks) and gluons. One can construct local *hadronic* interpolating and composite fields out of quark fields  $q(x)$  and gluon fields  $A_\mu(x)$  in terms of covariant derivatives  $D_\mu q = (\partial_\mu + iA_\mu)q$  and field strength tensor  $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$  carrying the same quantum numbers. This of course generates the problem of operator mixing which resembles the freedom of choice of a basis in the standard quantum mechanical variational method. In EFT this is usually reorganized in a dimensional expansion with growing energy dimensions. For instance, for a scalar-isoscalar particle with  $J^{PC} = 0^{++}$  we have the composite field expansion in the resolution wavelength  $\lambda$

$$\sigma(x) = Z_2 \lambda^2 \bar{q}(x)q(x) + Z_3 \lambda^3 G^2(x) + \lambda^4 Z_4 \bar{q}(x)D^2 q(x) + Z_5 \lambda^5 [\bar{q}(x)q(x)]^2 + \dots \quad (1)$$

where  $G^2(x) = \text{tr}_c G_{\mu\nu}(x)G^{\mu\nu}(x)$  and  $Z_n(\lambda)$  are dimensionless constants. Physically, the expansion corresponds to a Fock space decomposition of composite particles which are treated as elementary with constituents placed at the same point  $x$ . The effective elementarity occurs when  $\partial_x \sigma \ll \sigma/\lambda$ . For similar reasons, interactions in an effective Lagrangian can be written in a dimensional expansion where 1) classical equations of motion are used and 2) fields may be reparameterized by any local transformation of the field  $\sigma(x)$ . An important issue is that if quantum corrections to the effective Lagrangian are also suppressed in the resolution scale  $\lambda$  a fully consistent EFT may be built.

However, a direct calculation of Green functions in EFT does not necessarily guarantee off-shell finiteness from on shell renormalization conditions (see e.g. Ref. [3]) and suitable field redefinitions may be requested to ensure off-shell renormalizability. The on-shell scheme of Georgi [4] for EFT's allows to consider on-shell vertices and the problem is circumvented from the start,

since off-shellness cannot be measured [5] (see also [6]). However, this does not mean that *any* off-shellness can be removed as it was pointed out for chiral two-pion exchange NN interactions [7].

### 3 Chiral Lagrangians

Chiral Lagrangians in non-linear realizations [8] (for a review see e.g. [9] and references therein) capture many known relevant features of low energy hadronic physics in a systematic expansion in  $1/f$  ( $f \sim 88\text{MeV}$  is the pion weak decay constant for massless quarks). At lowest order it contain kinetic and mass baryon pieces and meson-baryon interaction terms and is given by [9]

$$\mathcal{L}_1 = \text{Tr}\{\bar{B}(i\nabla - M_B)B\} + \frac{1}{2} \mathcal{D} \text{Tr}\{\bar{B}\gamma^\mu\gamma_5\{u_\mu, B\}\} + \frac{1}{2} \mathcal{F} \text{Tr}\{\bar{B}\gamma^\mu\gamma_5[u_\mu, B]\}, \quad (2)$$

The meson kinetic and mass pieces and the baryon mass chiral corrections are second order and read

$$\mathcal{L}_2 = \frac{f^2}{4} \text{Tr}\{u_\mu^\dagger u^\mu + (U^\dagger \chi + \chi^\dagger U)\} - b_0 \text{Tr}(\chi_+) \text{Tr}(\bar{B}B) - b_1 \text{Tr}(\bar{B}\chi_+ B) - b_2 \text{Tr}(\bar{B}B\chi_+) \quad (3)$$

where “Tr” stands for the trace in  $SU(3)$ . In addition,

$$\begin{aligned} \nabla_\mu B &= \partial_\mu B + [\Gamma_\mu, B], & \Gamma_\mu &= \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger), \\ U = u^2 &= e^{i\sqrt{2}\Phi/f}, & u_\mu &= iu^\dagger \partial_\mu U u^\dagger \\ \chi_+ &= u^\dagger \chi u^\dagger + u \chi^\dagger u, & \chi &= 2B_0 \mathcal{M}. \end{aligned} \quad (4)$$

$M_B$  is the common mass of the baryon octet, due to spontaneous chiral symmetry breaking for massless quarks. The  $SU(3)$  coupling constants which are determined by semileptonic decays of hyperons are  $\mathcal{F} \sim 0.46$ ,  $\mathcal{D} \sim 0.79$  ( $\mathcal{F} + \mathcal{D} = g_A = 1.25$ ). The constants  $B_0$  and  $f$  are not determined by the symmetry. The current quark mass matrix is  $\mathcal{M} = \text{Diag}(m_u, m_d, m_s)$ . The parameters  $b_0$ ,  $b_1$  and  $b_2$  are coupling constants with dimension of an inverse mass. The values of  $b_1$  and  $b_2$  can be determined from baryon mass splittings, whereas  $b_0$  gives an overall contribution to the octet baryon mass  $M_B$ <sup>1</sup>. See e.g. [10, 11] for applications and [12] for extensions to  $SU(6)$  in hadronic reactions.

The Chiral Lagrangian preserves the Baryon current

$$\partial_\mu \text{tr}(\bar{B}\gamma_\mu B) = 0. \quad (6)$$

<sup>1</sup>Using  $SU(3)$  flavour symmetry for the meson and the baryon octet are written in terms of the meson  $\Phi$  and baryon  $B$  spinor fields respectively and are given by

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}. \quad (5)$$

respectively. These interpolating fields are not unique as they contain contributions from off-shell states, but they fulfill free Klein-Gordon and Dirac equations respectively.

Therefore, for any scalar and  $SU(2)$  invariant field  $\mathcal{M}$  we have

$$\text{tr}(\bar{B}\partial\mathcal{M}B) = \partial_\mu[\mathcal{M}\text{tr}(\bar{B}\gamma_\mu B)] , \quad (7)$$

which yields a vanishing contribution to the action since it is a total derivative.

## 4 Reparameterization invariance

The unitarity condition  $u^\dagger u = 1$  implies that in general we may use the polar decomposition,

$$u = e^{iH} = 1 + iH + \dots , \quad (8)$$

where  $H = H^\dagger$  is a hermitian matrix. In the case of  $SU(2)$  one usually writes

$$H = \vec{\tau} \cdot \vec{\phi} , \quad (9)$$

with  $\vec{\tau}$  the Pauli matrices which imply that  $\text{Tr}H = 0$ . This standard choice is not unique, and in particular one may take

$$H = \vec{\tau} \cdot \vec{\phi} + \xi \vec{\tau} \cdot \vec{\phi}(\vec{\phi} \cdot \vec{\phi}) + O(\phi^5) , \quad (10)$$

which complies equally well the unitarity of  $u$ . The arbitrariness of  $\xi$  will be exploited below. While the non-linear character of  $u(\phi)$  generally implies the appearance of many body forces, they turn out to be chirally suppressed by the pion weak decay constant. On the other hand, fields appearing in an EFT are not unique since one has the freedom to make a change of variables or field redefinition  $\phi \rightarrow \phi' = F(\phi)$  with the same quantum numbers with no consequences on the physical S-matrix. This is the so-called equivalence theorem.

The lack of reparameterization invariance shows up in final results only if incomplete calculations are carried out. For instance, the description of hadronic resonances makes the use of unitarity mandatory. Any unitarization procedure corresponds to an infinite but partial sum of Feynman diagrams and the reparameterization invariance is violated *after* unitarization (see e.e.g [6] for the case of  $\pi\pi$  scattering within a Bethe-Salpeter framework.).

## 5 Cancellation of three body forces in $SU(2)$

The problem of existence of three-body forces in the Baryon-Meson system relies on whether or not there is a particular choice of the arbitrary variable  $\xi$  where the terms in the Lagrangian with four pion fields  $\vec{\phi}$  vanish. Actually any value of  $\xi$  corresponds to a specific perturbative choice of coordinates on the  $SU(2)$  group around the origin. Expanding in powers of  $\phi$  we get in the effective Lagrangian Eq. (2) with Eq. (4)

$$\Gamma^\mu = \Gamma_2^\mu + \Gamma_4^\mu + O(H^6) , \quad (11)$$

where

$$\Gamma_2^\mu = \frac{1}{2} [H, \partial^\mu H], \quad (12)$$

$$\Gamma_4^\mu = \frac{1}{6} H^3 \partial^\mu H + \frac{1}{4} H^2 \partial H^2 - \frac{1}{6} H \partial^\mu H^3 + \frac{1}{24} \partial^\mu H^4. \quad (13)$$

Now, to the desired order in  $SU(2)$  we get (sandwiching with B-fields and partial integration understood)

$$\Gamma_2^\mu = i(\varphi \wedge \partial^\mu \varphi) \cdot \tau [1 + 2\xi \varphi^2] + O(\varphi^6), \quad (14)$$

$$\begin{aligned} \Gamma_4^\mu &= \frac{1}{6} \varphi^2 [(\tau \cdot \varphi), \partial^\mu (\tau \cdot \varphi)] + \frac{1}{6} \varphi^2 (\tau \cdot \varphi) \partial^\mu (\tau \cdot \varphi) + \frac{1}{4} \varphi^2 \partial^\mu (\varphi^2) \\ &\quad - \frac{1}{6} (\tau \cdot \varphi) \partial^\mu [(\tau \cdot \varphi) \varphi^2] + O(\varphi^6). \end{aligned} \quad (15)$$

Finally, discarding total derivatives we get after some algebra

$$\Gamma^\mu = i(\varphi \wedge \partial^\mu \varphi) \cdot \tau \left[ 1 + \left( 2\xi + \frac{1}{6} \right) \varphi^2 \right] + O(\varphi^6). \quad (16)$$

If we choose  $\xi = -1/12$  the terms with four pions cancel. Note that in our case the cancellation does not make use of the equations of motion and hence holds off-shell. As a consequence, any sub-diagram containing these contributions will cancel. Thus, for this field coordinates there are no three body  $\pi\pi N$  forces at order  $1/f^4$  in the chiral Lagrangian.

In a series of works [13, 14] the study of three hadron resonances with baryon number  $B = 1$  was vigorously started within a unitary approach based on the Faddeev equations. The kind of cancellation found above complies with the result found in Ref. [13, 14] where a direct analysis of the Faddeev equation using the standard polar field coordinates (corresponding to  $\xi = 0$ ) and using the on-shell conditions corresponding to the equations of motion. This simplification is crucial as it reduces the complicated analysis of the three body problem to a more feasible linear algebraic value problem.

## 6 Discussion and outlook

The suppression of three body forces was a major original motivations to introduce EFT approaches based on the chiral symmetry of QCD in Nuclear Physics [15]. Actually, the possibility of computing Pion-Deuteron scattering in a model independent leads to the absence of three-body corrections at threshold at order  $O(1/f^4)$  since “the sum of all corrections vanishes for a variety of reasons, among them the threshold kinematics and the isoscalar character of the deuteron” after an intricate diagrammatic analysis [16] which might be simplified using a suitable field reparameterizations as done here.

As a final remark we note that *local* field redefinitions in EFT are innocuous in dimensional regularization where the functional Jacobian vanishes. Unfortunately, the application of this regularization is subtle beyond perturbation

theory. Actually, finite cut-offs may jeopardize the reparameterization invariance. This is a potential drawback inherent to the framework, and relevant when implementing exact unitarity (see e.g. [17] as applied to ultracold atomic systems and the interplay with van der Waals forces) which needs clarification.

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