

Residual Z_2 symmetries and leptonic mixing patterns from finite discrete subgroups of $U(3)$

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ABSTRACT: We study embedding of non-commuting Z_2 and Z_m , $m \geq 3$ symmetries in discrete subgroups (DSG) of $U(3)$ and analytically work out the mixing patterns implied by the assumption that Z_2 and Z_m describe the residual symmetries of the neutrino and the charged lepton mass matrices respectively. Both Z_2 and Z_m are assumed to be subgroups of a larger discrete symmetry group G_f possessing three dimensional faithful irreducible representation. The residual symmetries predict the magnitude of a column of the leptonic mixing matrix U_{PMNS} which are studied here assuming G_f as the DSG of $SU(3)$ designated as type C and D and large number of DSG of $U(3)$ which are not in $SU(3)$. These include the known group series $\Sigma(3n^3)$, $T_n(m)$, $\Delta(3n^2, m)$, $\Delta(6n^2, m)$ and $\Delta'(6n^2, j, k)$. It is shown that the predictions for a column of $|U_{\text{PMNS}}|$ in these group series and the C and D types of groups are all contained in the predictions of the $\Delta(6N^2)$ groups for some integer N . The $\Delta(6N^2)$ groups therefore represent a sufficient set of G_f to obtain predictions of the residual symmetries Z_2 and Z_m .

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1 Introduction

Discrete subgroups (DSG) of $SU(3)$ and $U(3)$ have been extensively used as flavour symmetries with a view to understand the observed leptonic mixing patterns, see [1–5] for reviews. Many of these approaches are based on two observations: (a) the mass matrix M_ν for the non-degenerate Majorana neutrinos and a Hermitian combination $M_l M_l^\dagger$ of the charged lepton mass matrix are always invariant under residual symmetries $G_\nu \in Z_2 \times Z_2$ and $G_l \in Z_n \times Z_m \times Z_p$ respectively and (b) the unitary matrices U_{G_ν} , U_{G_l} diagonalizing G_ν , G_l respectively also diagonalize the corresponding mass matrices. Thus the forms of the symmetry generators contained in G_ν , G_l are sufficient to determine the leptonic mixing matrix [6–10]. The former can be determined group theoretically by requiring that G_ν and G_l are subgroups of a larger group G_f .

The predictions of the above mentioned approaches have been extensively studied [11–13]. They have been numerically worked out [14–17] for a large number of DSG of $U(3)$ and the group series $\Delta(6n^2)$ [18] under the assumption of $G_\nu = Z_2 \times Z_2$ and $G_l = Z_m$, $m \geq 3$. General classification of all the possible mixing patterns under the above assumptions is also presented in [19] from a different point of view. The virtue of the above approach is its predictivity since the matrices diagonalizing the generators of $Z_2 \times Z_2$ and Z_m , $m \geq 3$ (with

unequal eigenvalues) are completely fixed apart from over all diagonal phase matrices. A less conservative approach is to assume that only a Z_2 subgroup of the full neutrino residual symmetry $Z_2 \times Z_2$ resides in the flavour group G_f . This allows additional phases and a mixing angle which do not get determined from group theoretical considerations. This approach is desirable since it is found that only a handful of G_f containing the full $Z_2 \times Z_2$ are actually able to explain all the three leptonic mixing angles at the leading order [20, 21]. Moreover, there exist a large number of DSG of $SU(3)$ (those with order not divisible by 4 but divisible by 2) which contain Z_2 but not the full Klein group $Z_2 \times Z_2$. Predicted mixing patterns in this approach are also studied numerically in [9, 20, 22–24] for many groups and analytically for some groups in [25, 26].

The main aim of this work is to present a simple and widely applicable general analytic discussion of the assumption $G_\nu = Z_2$, $G_l = Z_m$ for the type C and D subgroup of $SU(3)$ ¹ as classified in [27–30] and for all the known [31] group series, namely $\Sigma(3n^3)$, $T_n(m)$, $\Delta(3n^2, m)$, $\Delta(6n^2, m)$ and $\Delta(6n^2, j, k)$ of groups which are contained in $U(3)$ but not in $SU(3)$ ². The above stated assumption determines modulus of the elements in one of the columns of the leptonic mixing matrix. The analytic mixing formula derived here are used to show that all possible predictions of this column in all the above groups can always be derived using some member of the the group $\Delta(6n^2)$ making this group series as optimal choice for the flavour symmetry G_f . This analytical discussion is supplemented with numerical analysis performed by scanning large number of known DSG of $U(3)$ and their viability is discussed in the context of the latest global fit to neutrino oscillation data. Results for some of the groups which are not described by the above series are discussed numerically.

We discuss general formalism of fixing the leptonic mixing matrix using the residual symmetries of neutrinos and charged leptons in the next section. Detailed analytic treatments of various groups are presented in section 3 which is then followed by numerical investigations in section 4. We discuss in section 5 additional symmetries fixing the unknown observables which are not determined from $G_\nu = Z_2$ and summarize our results in section 6.

2 Formalism

The leptonic mixing matrix can be determined from the given residual symmetries of neutrinos and charged leptons [6–10]. Consider three generations of massive Majorana neutrinos and the charged leptons both transforming as faithful irreducible representations of a discrete flavour group G_f . The 3×3 matrices S_ν and T_l are elements of G_f such that $S_\nu^2 = T_l^m = \mathbf{1}$. Assume that the neutrino mass matrix M_ν and the charged lepton mass

¹Earlier work [21] in this direction presented some analytic discussion of the possible mixing patterns in the type C, D and the group series $\Sigma(3n^3)$.

² Such groups were earlier used to embedd the residual symmetries of a massless neutrino [32–34].

matrix M_l are invariant under the action of symmetry such that

$$S_\nu^T M_\nu S_\nu = M_\nu \quad \text{and} \quad T_l^\dagger M_l M_l^\dagger T_l = M_l M_l^\dagger . \quad (2.1)$$

It is assumed that S_ν and T_l do not commute in order to induce non-trivial mixing among the leptons. We further assume that all the eigenvalues of T_l are distinct. The eigenvalues of S_ν are ± 1 . We do not consider a trivial case corresponding to $S_\nu = -\mathbf{1}$ as it does not provide any constraint on M_ν . Therefore, two of the eigenvalues in S_ν are always degenerate and the other is unique. Let V_ν and V_l be unitary matrices which diagonalizes S_ν and T_l

$$V_\nu^\dagger S_\nu V_\nu = d_\nu , \quad V_l^\dagger T_l V_l = d_l . \quad (2.2)$$

Here d_ν and d_l are diagonal matrices. Note that V_ν and V_l are arbitrary upto a multiplication by a diagonal phase matrix from right hand side. Further, degeneracy of eigenvalues in d_ν imply that V_ν can also be multiplied by a unitary rotation in the space of the degenerate eigenvalues. The matrices U_ν and U_l diagonalizing the leptonic mass matrices of eq. (2.1) can therefore be written as

$$U_\nu = V_\nu P_\nu U_{ij}(\theta, \beta) , \quad U_l = V_l P_l , \quad (2.3)$$

where $P_{l,\nu}$ are diagonal phase matrices. This leads to the leptonic mixing matrix U_{PMNS}

$$U_{\text{PMNS}} = U_l^\dagger U_\nu , \quad (2.4)$$

The U_{PMNS} depends on the unknown phases in $P_\nu \equiv \text{diag.}(e^{i\chi_1}, e^{i\chi_2}, e^{i\chi_3})$, the phase β and angle θ contained in U_{ij} where ij refers to indices corresponding to degenerate eigenvalues in d_ν . However modulus of one column of U_{PMNS} is independent of these unknowns. If c_ν is an eigenvector of S_ν corresponding to the unique eigenvalue in d_ν then

$$|c_0| = |V_l^\dagger c_\nu| \quad (2.5)$$

represents the absolute values of the elements in a column of U_{PMNS} which is independent of the unknowns. In a special situation corresponding to $\beta = 0$ in eq. (2.3), one can also define further invariants which are independent of the mixing angle θ . These are given by

$$A_\alpha = \text{Im}((U_{\text{PMNS}})_{\alpha i} (U_{\text{PMNS}})_{\alpha j}^*) , \quad (2.6)$$

where $\alpha = e, \mu, \tau$. These have been used earlier to derive useful conclusions in case of two degenerate neutrinos [35, 36]. We shall use them to obtain θ independent restrictions on the elements of U_{PMNS} . Although the absolute values of elements in c_0 are fixed, one can permute the elements within the column and rearrange them in order to comply with the observed U_{PMNS} . Our aim in this paper is to derive possible predictions for $|c_0|$ using various discrete DSG of $U(3)$ as G_f .

3 An analytic study for fixed column prediction

In this section, we obtain a universal formula for a fixed column $|c_0\rangle$ of U_{PMNS} applicable to DSG of $SU(3)$ categorized as C and D and to a large number of DSG of $U(3)$ to be specified below. Apart from five exceptional groups, C and D types are the only DSG of $SU(3)$ which have a 3 dimensional faithful irreducible representation. These are generated [29] by the following matrices:

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.1)$$

$$F = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon^k & 0 \\ 0 & 0 & \epsilon^{-k-1} \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^{-r} & 0 \\ 0 & 0 & \epsilon^r \end{pmatrix}. \quad (3.2)$$

Here, $\epsilon = e^{2\pi i/m}$, $r = m/n$ with m , n and r are nonzero and positive integers. The k and l_1 are positive integers satisfying

$$1 + k + k^2 = l_1 r$$

with $k = 0, 1, \dots, r-1$. Type C groups are generated from the multiple products of E , F , G with m , n , k satisfying the above restrictions. Type D groups involve one more generator B and also need additional restriction on k

$$2k + 1 = l_2 r$$

with l_2 an arbitrary positive integer.

Our derivation is based on the observation that the the elements of groups generated by the multiple products of the above generators can all be represented by six different textures defined as $\mathcal{S} = \{R, S, T, U, V, W\}$ with elements in the set having the following form:

$$\begin{aligned} W(\eta_1, \eta_2, \eta_3) &= \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}, & V(\eta_1, \eta_2, \eta_3) &= \begin{pmatrix} 0 & \eta_1 & 0 \\ 0 & 0 & \eta_2 \\ \eta_3 & 0 & 0 \end{pmatrix}, \\ S(\eta_1, \eta_2, \eta_3) &= - \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & 0 & \eta_2 \\ 0 & \eta_3 & 0 \end{pmatrix}, & R(\eta_1, \eta_2, \eta_3) &= \begin{pmatrix} 0 & 0 & \eta_1 \\ \eta_2 & 0 & 0 \\ 0 & \eta_3 & 0 \end{pmatrix}, \\ T(\eta_1, \eta_2, \eta_3) &= - \begin{pmatrix} 0 & 0 & \eta_1 \\ 0 & \eta_2 & 0 \\ \eta_3 & 0 & 0 \end{pmatrix}, & U(\eta_1, \eta_2, \eta_3) &= - \begin{pmatrix} 0 & \eta_1 & 0 \\ \eta_2 & 0 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}. \end{aligned} \quad (3.3)$$

These textures have the properties that only one entry in a given column and row is non-zero and it is labeled by η_i which is an arbitrary root of unity. Only the exact expressions

of η_i vary for different groups and for different elements within the groups. The above textures do not just describe the elements of the groups of type C and D but have larger validity and can describe a large number of DSG of $U(3)$ and not just $SU(3)$. This follows from the expressions of generators of the DSG of $U(3)$ given by [31]. These generators have the same textures as above with replacement of η_i by powers of the n^{th} roots of unity. Combinations of these generators with in general different integer n also lead to elements having the same textures as above [33]. All the groups which can be obtained from the above set of generators and having order < 512 are listed in [31]. In addition, elements of the five infinite series of the DSG of $U(3)$ identified in [31] can also be parameterized as in eq. (3.3). Thus the above textures describe a large bulk of DSG of $SU(3)$ and $U(3)$. For example, total 55 out of 59 DSG of $SU(3)$ and 70 out of 75 DSG of $U(3)$ (which are not DSG of $SU(3)$) listed in [31] with order less than 512 belong to this category.

We now derive analytic expressions for the possible patterns of a column in U_{PMNS} if S_ν and T_l have textures specified in \mathcal{S} . Since $S_\nu^2 = \mathbf{1}$, it forms a Z_2 subgroup of a given flavour group G_f which is a DSG of $SU(3)$ or $U(3)$. The elements V and R in \mathcal{S} do not form Z_2 subgroups as their eigenvalues are all different and are given by $\det.(V)(1, \omega, \omega^2)$ and $\det.(R)(1, \omega, \omega^2)$ respectively, with $\omega = e^{2\pi i/3}$. Thus S_ν can be any of W with the eigenvalues $(-1, -1, 1)$ or any of S, T and U with specific values of η_i . For example,

$$S(1, \eta_\nu^*, \eta_\nu) = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \eta_\nu^* \\ 0 & \eta_\nu & 0 \end{pmatrix} \quad (3.4)$$

and equivalent T and U obtained by permuting the above S have eigenvalues $(-1, -1, 1)$. The symmetry of the charged leptons T_l can have any structure within \mathcal{S} with all the eigenvalues being distinct and $[S_\nu, T_l] \neq 0$.

As discussed in the previous section, the lepton mixing matrix depends on the matrices diagonalizing S_ν and T_l . If V_G is a unitary matrix that diagonalizes a given element $G \in \mathcal{S}$ such that $V_G^\dagger G V_G = G_{\text{diag}}$, then different V_G are given by [33]

$$\begin{aligned} V_W &= \mathbf{1}, & V_V &= \text{Diag.}(1, \eta_1^* \Delta, \eta_2^* \eta_1^* \Delta^2) U_\omega, \\ V_S &= \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \eta_2^* \lambda_{23} & -\eta_2^* \lambda_{23} \end{pmatrix}, & V_R &= \text{Diag.}(1, \eta_1^* \eta_3^* \Delta^2, \eta_1^* \Delta) U_\omega, \\ V_T &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ \eta_1^* \lambda_{13} & 0 & -\eta_1^* \lambda_{13} \end{pmatrix}, & V_U &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ \eta_1^* \lambda_{12} & -\eta_1^* \lambda_{12} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \end{aligned} \quad (3.5)$$

Here $\lambda_{ij} = \sqrt{\eta_i \eta_j}$, $\Delta = (\eta_1 \eta_2 \eta_3)^{1/3}$ and U_ω is defined as

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (3.6)$$

It is possible to work out all possible patterns of a fixed column using the above diagonalizing matrices and eq. (2.5). The column c_ν in eq. (2.5) refers to an eigenvector corresponding to a unique eigenvalue in S_ν . The following two possibilities exist for c_ν corresponding to $S_\nu = W(-1, -1, 1)$ or $S_\nu \in \{S, T, U\}$ with the particular structure as specified in eq. (3.4):

$$c_\nu = (0, 0, 1)^T \quad \text{or} \quad c_\nu = \left(0, \frac{1}{\sqrt{2}}, -\frac{\eta_\nu}{\sqrt{2}}\right)^T, \quad (3.7)$$

or with permutations of the elements within a given c_ν . The different possible choices of T_l then lead to the following structures for fixed column $|c_0|$ according to eq. (2.5).

(1) For $S_\nu = W(-1, -1, 1)$ and $T_l \in \{R, V\}$,

$$|c_0|^2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T \quad (3.8)$$

(2) For $S_\nu = W(-1, -1, 1)$ and $T_l \in \{S, T, U\}$ or $S_\nu \in \{S, T, U\}$ and $T_l = W(\eta_1, \eta_2, \eta_3)$,

$$|c_0|^2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T \quad (3.9)$$

or its permutations.

(3) For $S_\nu \in \{S, T, U\}$ and $T_l = V(\eta_1, \eta_2, \eta_3)$,

$$|c_0|^2 = \frac{1}{6} \left(|1 - \eta_\nu \eta_2 \Delta^*|^2, |1 - \omega \eta_\nu \eta_2 \Delta^*|^2, |1 - \omega^2 \eta_\nu \eta_2 \Delta^*|^2\right)^T \quad (3.10)$$

or its permutations. The choice $T_l = R(\eta_1, \eta_2, \eta_3)$ leads to a similar structure with η_2 replaced by η_3 .

(4) For $S_\nu \in \{S, T, U\}$ and $T_l \in \{S, T, U\}$ but with textures different than S_ν ,

$$|c_0|^2 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)^T \quad (3.11)$$

or its permutations.

(5) For $S_\nu \in \{S, T, U\}$ and $T_l \in \{S, T, U\}$ but with textures same as S_ν ,

$$|c_0|^2 = \frac{1}{4} \left(0, |1 + \sqrt{\eta_2^* \eta_3^* \eta_\nu}|^2, |1 - \sqrt{\eta_2^* \eta_3^* \eta_\nu}|^2\right)^T \quad (3.12)$$

or its permutations.

The above five equations exhaust all the predictions for one of the columns of U_{PMNS} for all the groups under considerations. Three of the predictions are universal in the sense that they are independent of the choice of $\eta_{1,2,3}$ and η_ν in all the formula above. Among them eq. (3.8), namely the tri-maximal column, provides a viable prediction for the second

column of U_{PMNS} . The bi-maximal column in eq. (3.8) at best can be identified with the third column of U_{PMNS} with large correction required through breaking of the residual symmetries. The prediction in eq. (3.12) depends on the free parameters but one of the entries of the column is predicted to be zero and thus it also requires large correction. The solution in eq. (3.10) offers several possibilities which can lead to correct leading order predictions for the columns of U_{PMNS} matrix. The complex numbers η_i are some roots of unity. They do not represent the same root nor powers of some specific roots of unity in general. Their exact expressions are specified by the properties of the groups and we discuss various cases in the next subsections. We also show that eq. (3.10) can fit either the first or third column of PMNS matrix at the leading order. Therefore, tri-maximal is the only solution offered by the groups under considerations which can be consistently identified with the second column of the U_{PMNS} matrix.

3.1 Mixing angle predictions for the DSG of $SU(3)$

Among the DSG of $SU(3)$, the groups of type C and D are generated [29] using two or more elements from the textures given in eq. (3.3). We discuss below both these group series.

3.1.1 Type C groups

The type C groups defined earlier are isomorphic to $(Z_m \times Z_n) \rtimes Z_3$ where Z_m, Z_n are respectively generated by F, G in eq. (3.1) and Z_3 by E . This isomorphism can be used to find all elements of the C type groups by using the result that any element $s \in S = G \rtimes H$ can be uniquely written as $s = gh$ in terms of $g \in G$ and $h \in H$. Thus all the elements $\in C$ are written as

$$F^p G^q E^a$$

with $p = 0, 1, \dots, m-1$; $q = 0, 1, \dots, n-1$ and $a = 0, 1, 2$. It is then seen that all the elements fall in three textures labeled as W, R and V in eq. (3.3) with the identification

$$\eta_1 = \epsilon^p, \quad \eta_2 = \epsilon^{kp-rq}, \quad \eta_3 = \epsilon^{-p(k+1)+rq}. \quad (3.13)$$

Since R and V do not form Z_2 subgroups only possible candidates for Z_2 subgroups are the diagonal generators W when m and n are even. The resulting mixing pattern is always tri-maximal as in eq. (3.8). This leads to two predictions:

$$c_{13}^2 s_{12}^2 = \frac{1}{3} \quad \text{and} \quad \cos \delta = \frac{(c_{23}^2 - s_{23}^2)(c_{12}^2 - s_{12}^2 s_{13}^2)}{4c_{12}c_{23}s_{12}s_{13}s_{23}}, \quad (3.14)$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$. The first leads to a solar angle quite close to its global fit value [37]. The second prediction correlates the Dirac CP phase with the values of s_{23}^2 and s_{13}^2 . The sign of $\cos \delta$ is strongly correlated to the quadrant of the atmospheric mixing angle and one gets positive (negative) values for $\cos \delta$ for θ_{23} smaller (greater) than 45° . It is to be noted that T2K results prefer a negative $\cos \delta$ [38] and the global best fit value of θ_{23} lies in the second quadrant according to [37]. Both these results are consistent with eq. (3.14). However either sign of $\cos \delta$ and both quadrants for θ_{23} are allowed at 3σ .

Let us mention possible groups obtained for the lowest values of r . $r = 1$ leads to $\Delta(3n^2)$ series and the lowest member with Z_2 subgroup is A_4 . For $r = 3$, either one gets the series $Z_3 \times \Delta(3n^2)$ or the series $(Z_{9n} \times Z_{3n}) \rtimes Z_3$ and the lowest member of the latter containing Z_2 subgroup is an order 324 group. Other possible group series of type C are listed in [29].

3.1.2 Type D groups

These groups are generated by F, G, E, B of eq. (3.1) and are isomorphic to $(Z_m \times Z_n) \rtimes S_3$. S_3 is now generated by E and B . As before, we can explicitly write the expressions for all the elements of the D type groups, $(Z_m \times Z_n) \rtimes S_3$ in terms of their generators. To do this, we note that S_3 can be written as $Z_3 \rtimes Z_2$. Hence all its six elements can be written as $E^a B^l$, where $a = 0, 1, 2$ and $l = 0, 1$. Then the semi-direct product structure of D implies that all its elements can be uniquely written as

$$F^p G^q E^a B^l. \quad (3.15)$$

One sees from the explicit forms of generators that all these elements are given by the six textures in eq. (3.3). But now $\eta_{1,2,3}$ are given in terms of p, q by eq. (3.13). All possible mixing patterns within the type D groups are generated by substituting these values of η_1, η_2, η_3 in eqs. (3.8-3.12) and varying p, q for a given group, *i.e.* for given values of k, m and n . Due to restrictions on the values of r and k , essentially two type-D group series exist [29]. They are obtained with $k = 0, r = 1$ and $k = 1, r = 3$. These are respectively isomorphic to $\Delta(6n^2) \equiv (Z_n \times Z_n) \rtimes S_3$ and $D_{9n,3n}^1 \equiv (Z_{9n} \times Z_{3n}) \rtimes S_3$. It is possible to write down simple expressions for the predicted mixing for both these series and we discuss them below.

Let us first consider the case of $\Delta(6n^2)$.

- Since $\Delta(3n^2)$ are subgroups of $\Delta(6n^2)$, the universal prediction given by eq. (3.8) is obtained in all $\Delta(6n^2)$ groups with even n as in the case of $\Delta(3n^2)$.
- The Z_2 subgroups generated by B, EBE^{-1} and E^2BE^{-2} exist in all the $\Delta(6n^2)$ groups and consequently the predictions given in eqs. (3.9,3.11) follow in all $\Delta(6n^2)$ groups. In addition, groups with $n > 2$ also allow values of η_i, η_ν other than ± 1 and lead to predictions as given by eq. (3.12). This allows possibility of predicting non-maximal θ_{23} but the θ_{13} remains zero in all these cases.
- The non-universal prediction as given by eq. (3.10) can lead to non-trivial expressions for the columns of U_{PMNS} . It is obtained for the choice $S_\nu = S(1, \eta^{-q'}, \eta^{q'})$ and $T_l = V(\eta_1, \eta_2, \eta_3)$. Here $\eta_{1,2,3}$ are given by eq. (3.13) with $k = 0, r = 1$ and $m = n$. Substituting these in eq. (3.10), we obtain

$$|c_0|^2 = \frac{1}{6} \left(|1 - \epsilon^{q'-q}|^2, |1 - \omega \epsilon^{q'-q}|^2, |1 - \omega^2 \epsilon^{q'-q}|^2 \right), \quad (3.16)$$

where $q, q' = 0, 1, \dots, n-1$.

The mixing patterns implied by the above equation are obtained by varying n and correspondingly q and q' . Not all of these give physically different predictions. Replacement $\epsilon \leftrightarrow \epsilon^*$ permutes the last two entries in eq. (3.16). Whenever n is divisible by 3 then $\epsilon^{q-q'}$, $\omega\epsilon^{q-q'}$, $\omega^2\epsilon^{q-q'}$ are all n^{th} roots of unity. In such case, *i.e.* $n = 3p$, one gets $p/2$ ($(p-1)/2$) number of non-trivial solutions with all non-zero entries for p even (odd). If n is not divisible by 3, then eq. (3.16) leads to $n/2$ ($(n-1)/2$) non-trivial solutions with all non-zero entries for n even (odd). These observations agree with the results of numerical calculations given in the next section. It is to be noted that eq. (3.16) does not contain the tri-maximal solution $|c_0|^2 = (1/3, 1/3, 1/3)$ for any value of $|q - q'|$. It has to come from eq. (3.8) which requires n to be even. It is straightforward to work out all solutions of eq. (3.16) for different n . We show in Fig. 1 the values of n and $|q' - q|$ leading to the predictions for a column of lepton mixing matrix which is consistent with a column of the experimentally obtained leptonic mixing matrix, namely $U_{\text{PMNS}}^{\text{exp}}$. The recent global fit ‘‘NuFIT 3.0’’ of

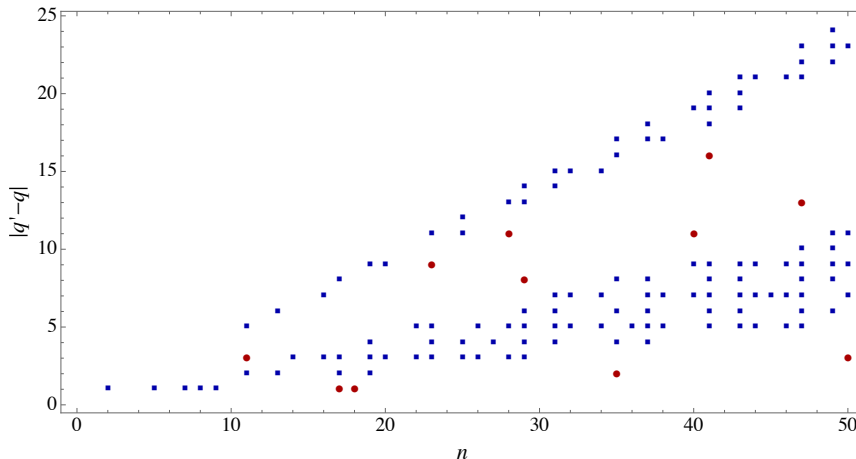


Figure 1. The values of $n \leq 50$ and $|q' - q|$ in eq. (3.16) leading to the viable columns of leptonic mixing matrix. The blue squares (red dots) indicate that the corresponding prediction is consistent with the first (third) column of U_{PMNS} matrix within 3σ . Each point represents a unique solution obtained by the smallest possible values of n and $|q' - q|$.

neutrino oscillation data by NuFIT collaboration [37] (for the latest global fit results, see NuFIT 3.0 (2016), <http://www.nu-fit.org>) allows the following 3σ ranges in the elements of U_{PMNS} .

$$|U_{\text{PMNS}}^{\text{exp}}| = \begin{pmatrix} 0.800 \rightarrow 0.844 & 0.515 \rightarrow 0.581 & 0.139 \rightarrow 0.155 \\ 0.229 \rightarrow 0.516 & 0.438 \rightarrow 0.699 & 0.614 \rightarrow 0.790 \\ 0.249 \rightarrow 0.528 & 0.462 \rightarrow 0.715 & 0.595 \rightarrow 0.776 \end{pmatrix} \quad (3.17)$$

We find that eq. (3.16) can predict either the first or the third column of U_{PMNS} . The simplest example corresponds to $n = 2$. This leads to only one non-trivial solution given by $|c_0|^2 = (2/3, 1/6, 1/6)^T$ corresponding to the first column of the famous tri-bimaximal mixing. The smallest group in $\Delta(6n^2)$ series which reproduces a viable third column of U_{PMNS} corresponds to $n = 11$. It gives a fixed column $|c_0|^2 = (0.0239, 0.3808, 0.5954)^T$

leading to the values $\sin^2 \theta_{13} = 0.0239$ and $\sin^2 \theta_{23} = 0.39$ or 0.61 . We discuss this case in more detail in the next section. It is straightforward to show that eq. (3.16) cannot reproduce the second column of $|U_{\text{PMNS}}^{\text{exp}}|$ for any value of n and $(q' - q)$. The differences between two of the elements of $|c_0|^2$, defined as $\Delta_{ij} = |c_{0i}|^2 - |c_{0j}|^2$, are given by

$$\begin{aligned}\Delta_{12} &= -\frac{1}{2} \left(\cos \xi + \frac{1}{\sqrt{3}} \sin \xi \right), \\ \Delta_{13} &= -\frac{1}{2} \left(\cos \xi - \frac{1}{\sqrt{3}} \sin \xi \right), \\ \Delta_{23} &= \frac{1}{\sqrt{3}} \sin \xi,\end{aligned}\tag{3.18}$$

where $\xi = 2\pi(q' - q)/n$. The 3σ ranges allowed in $|U_{\text{PMNS}}^{\text{exp}}|$ necessarily require all the three $|\Delta_{ij}| < 0.277$ if the predicted column is identified with the second column of U_{PMNS} . These three conditions cannot be simultaneously satisfied by any value of ξ in eq. (3.18). Therefore eq. (3.16), and more generally eq. (3.10), does not lead to any prediction for the second column of U_{PMNS} matrix which is consistent with the current global fit results.

The remaining groups of type D are given by the series $D_{9n', 3n'}^1$ isomorphic to $(Z_{9n'} \times Z_{3n'}) \times S_3$. We now show that mixing angle predictions given in eqs. (3.8-3.12) in these groups coincide with the corresponding ones obtained in the case of $\Delta(6n^2)$ with $n = 9n'$. All the elements of the groups are given by eq. (3.15). The η_i for these groups are given by eq. (3.13) with substitution $k = 1$, $r = 3$ and $m = 9n'$, $n = 3n'$. The symmetry of neutrinos corresponding to eq. (3.12) is $S_\nu = S(1, \eta^{-q'}, \eta^{q'})$. Substituting these values in eq. (3.12), we get

$$|c_0|^2 = \frac{1}{6} \left(|1 - \epsilon^{3(q-q')-p}|^2, |1 - \omega \epsilon^{3(q-q')-p}|^2, |1 - \omega^2 \epsilon^{3(q-q')-p}|^2 \right),\tag{3.19}$$

where $p = 0, 1, \dots, 9n' - 1$ and $q, q' = 0, 1, \dots, 3n' - 1$. It is noted that $3(q - q') - p = 0, 1, \dots, 9n' - 1$. Therefore, the above expression of the predicted column coincides with the prediction eq. (3.16) obtained in the case of $\Delta(6n^2)$ series with $n = 9n'$. A similar argument also holds for the prediction given in eq. (3.12). The smallest group in the series $D_{9n', 3n'}^1$ is order 162 group obtained for $n' = 1$. It predicts the same column $|c_0|^2$ as the group $\Delta(486)$ and by the counting argument given above there is only one nontrivial prediction with all non-zero entries. This is given in the Table 1. The next higher group with $n' = 2$ is an order 648 group and it has three independent non-trivial predictions which can also be obtained from $\Delta(6 \times 18^2)$. As can be seen from Fig. 1, this group will give only a third column consistent with the 3σ range of experimental data. Similar arguments holds for other predictions as well and thus group series $D_{9n', 3n'}^1$ are equivalent to the group series $\Delta(6 \times (9n')^2)$. This is expected since all the D types of groups are shown to be [39] subgroups of $\Delta(6n^2)$ groups for some n .

3.2 Mixing in DSG of $U(3)$

We now analyse the DSG of $U(3)$ which are not subgroups of $SU(3)$. Ludl [31] has identified several group series belonging to this category. Elements in all of these series have textures

given in eq. (3.3). Hence the mixing formulas eq. (3.8-3.12) are applicable to them. Using these, we now argue that none of the groups in these series of groups lead to any new mixing patterns compared to the ones already found with DSG of $SU(3)$. The same result is also found to be numerically true for all the groups of order less than 512 and not just for the groups belonging to the said series as we discuss in the next section. This can be shown in a more general way using the following redefinition of the residual symmetries of the charged leptons and neutrinos³. Let S_ν be a generator of $G_\nu = Z_2$ and T_l a generator of $G_l = Z_m$, $m \geq 3$. Both G_ν and G_l are subgroups of some G_f which is DSG of $U(3)$ and not of $SU(3)$. One can define

$$S'_\nu = (\det.(S_\nu)) S_\nu \quad \text{and} \quad T'_l = (\det.(T_l))^{-\frac{1}{3}} T_l, \quad (3.20)$$

where $\det.(S_\nu)$ is ± 1 . $\det.(S'_\nu) = \det.(T'_l)$ are now +1 and G'_f containing them is a DSG of $SU(3)$. They lead to the same invariance as in eq. (2.1). Both S'_ν and T'_l are respectively diagonalized by the same matrices U_ν and U_l which diagonalize S_ν and T_l resulting in the same prediction for a column of U_{PMNS} as obtained from G_f . Therefore, all the predictions of G_f belonging to $U(3)$ are always obtained from some G'_f within $SU(3)$. Above mentioned argument also remains true if the residual symmetry of the neutrino mass matrix is a Klein group $Z_2 \times Z_2$ generated by $S_{1\nu}$ and $S_{2\nu}$ contained in some DSG of $U(3)$. One can define new generators $S'_{1\nu}, S'_{2\nu}$ and T'_l as in eq. (3.20) having determinant +1. These generators would be contained in some DSG of $SU(3)$ and lead to the same mixing pattern as given by a DSG of $U(3)$ containing $S_{1\nu}, S_{2\nu}, T_l$.

Although the DSG of $U(3)$ do not offer any new prediction than those obtained from the DSG of $SU(3)$ as shown above, it is still interesting from the practical point of view to know which of the five sets of predictions given in eqs. (3.8-3.12) can be obtained from a given DSG of $U(3)$. We continue our analytical treatment in the case of DSG of $U(3)$ in order to explore the connection between the given DSG of $U(3)$ and their $SU(3)$ counterparts which offer the same sets of predictions. Here, we specifically show that

- All the groups in series $\Delta(3n^2, m)$ give the same pattern of column as $\Delta(3n^2)$, namely only tri-maximal for even n .
- The group series $\Delta(6n^2, m)$ does not contain any Z_2 subgroups (except the trivial one corresponding to $S_\nu = -\mathbf{1}$) for odd n . For even n , it gives only the tri-maximal and the bi-maximal columns.
- The series $\Delta'(6n^2, j, k)$ contain $\Delta(6n^2)$ as subgroup only if $k = 1$. In that case, one gets the same mixing pattern as $\Delta(6n^2)$. The groups with $k > 1$, either do not contain any non-trivial Z_2 subgroups or they are diagonal when they exist. In the latter case one gets either the bi-maximal or the tri-maximal mixing pattern.

³We thank referee for pointing out this general result.

3.2.1 The group series $\Delta(3n^2, m)$

These groups are isomorphic to $(Z_n \times Z_n) \rtimes Z_{3^m}$. Here n is not an integer multiple of 3 and m is an integer > 1 . Its generators are

$$\delta_m E, F(n, 0, 1) = W(1, \eta, \eta^*) \quad (3.21)$$

with $\delta_m = e^{2\pi i/3^m}$ and $\eta = e^{2\pi i/n}$. Semi-direct structure implies that all the group elements can be written as

$$X^j Y^k (\delta_m E)^l \quad (3.22)$$

where $X = W(1, \eta, \eta^*)$, $Y = EXE^{-1} = W(\eta^*, \eta, 1)$ and $j, k = 0, 1, \dots, n-1$ while $l = 1, 2, \dots, 3^m$. Using the fact that

$$\begin{aligned} (\delta_m E)^l &= \delta_m^l E \quad \text{for } l = 1, 4, 7, \dots \\ (\delta_m E)^l &= \delta_m^l E^2 \quad \text{for } l = 2, 5, 8, \dots \\ (\delta_m E)^l &= \delta_m^l \quad \text{for } l = 3, 6, 9, \dots \end{aligned}$$

one sees that all the $3^m n^2$ elements of $\Delta(3n^2, m)$ groups have the same texture as in $\Delta(3n^2)$ apart from an over all multiplication by δ_m^l . As a consequence, only Z_2 groups are diagonal if n is even and no non-trivial Z_2 subgroup exists if n is odd. Only mixing pattern in the former case is tri-maximal as was shown in case of $\Delta(3n^2)$ group series.

3.2.2 The group series $\Delta(6n^2, m)$

These groups are isomorphic to $\Delta(3n^2) \rtimes Z_{2^m}$ where n is not an integer multiple of 3 and m is an integer > 1 . The elements of these groups are generated by

$$E, F(n, 0, 1), \beta_m T \quad (3.23)$$

with $\beta_m = e^{2\pi i/2^m}$ and

$$T = T(0, 0, 0) = - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3.24)$$

Because of the semi-direct product structure, the elements can be written as

$$g = X^j Y^k E^l (\beta_m T)^r, \quad (3.25)$$

with X, Y as defined in the previous case, $j, k = 0, 1, \dots, n-1$; $l = 0, 1, 2$ and $r = 1, 1, \dots, 2^m$. It is seen that $(\beta_m T)^r = \beta_m^r T$ if r is odd and $(\beta_m T)^r = \beta_m^r$ if r is even. As a result of this, the elements can be divided into two groups

$$\begin{aligned} g &= \beta_m^{2k} (W, R, V), \\ &= \beta_m^{2k-1} (WB, RB, VB), \end{aligned} \quad (3.26)$$

Here, $k = 1, 2, \dots, 2^{m-1}$ and (W, R, V, WB, RB, VB) are six textures characterizing elements $X^j Y^k E^l T^q$, ($q = 0, 1$) of the $\Delta(6n^2)$ groups. The $\Delta(6n^2)$ group is however not contained

in $\Delta(6n^2, m)$ due to the factor β_m^{2k-1} (which is $\neq \pm 1$ for any k) multiplying the second line of the above equation. As a result the only non-trivial Z_2 subgroups are diagonal and are obtained from the first line in eq. (3.26) corresponding to even r . These are given by $W = W(1, -1, -1)$ and its permutations for $r = 2k = 2^m$ and by $-1, -W(1, -1, -1)$ and permutations for $r = 2k = 2^{m-1}$. The r is even in both the cases since $m > 1$. With any of these as S_ν and $T_l = \beta_m^{2k}(R, V)$ one gets the tri-maximal pattern. If T_l is chosen as any of the elements in the second line in eq. (3.26), one gets bi-maximal pattern. These are thus the only patterns possible with this group series. This is true for the series $S_4(m)$ also which are special cases of $\Delta(6n^2, m)$ with $n = 2$.

3.2.3 The group series $\Delta'(6n^2, j, k)$

These groups are generated by

$$E, X, \beta_{j,k}T \quad (3.27)$$

Here X is as defined earlier, n is a multiple of 3 and $\beta_{j,k} = e^{\frac{2\pi i}{3^j 2^k}}$. T is the standard Z_2 generator defined in eq. (3.24). This group does not have a semi-direct product structure as in the previous cases but still it is possible to write all its elements in the form

$$g = X^m Y^p E^q (\beta_{j,k}T)^r \quad (3.28)$$

with $m = 0, 1, \dots, n-1$; $p = 0, 1, \dots, \frac{n}{3}-1$; $q = 0, 1, 2$ and $r = 0, 1, \dots, 3^j 2^k - 1$. This group therefore has $n^2 3^j 2^k$ elements. This is seen by noting that the multiple products of X and E among themselves generate a set of elements $X^s Y^t E^q$ with s, t running over full range $0, 1, \dots, n-1$. Powers of $\beta_{j,k}T$ occurring anywhere between them can be transferred to the end using $(\beta_{j,k}T)E(\beta_{j,k}T)^{-1} = E^2$ and $(\beta_{j,k}T)X(\beta_{j,k}T)^{-1} = Y$ etc. Furthermore, X and Y are seen to satisfy

$$Y^{\frac{n}{3}} = X^{\frac{n}{3}} \omega^2 \mathbf{1} = X^{\frac{n}{3}} (\beta_{j,k}T)^{3^{(j-1)} 2^{k+1}}$$

Thus Y^a with $a \geq n/3$ can be written in terms of powers of X and $(\beta_{j,k}T)$ with Y^b , $b < n/3$. This allows one to restrict the range of p in eq. (3.28) as given there.

The elements in eq. (3.28) can be divided into two groups as before

$$\begin{aligned} g &= \beta_{j,k}^r (W, R, V) \quad \text{for even } r, \\ &= \beta_{j,k}^r (WT, RT, VT) \sim \beta_{j,k}^r (T, U, S) \quad \text{for odd } r. \end{aligned} \quad (3.29)$$

We have used the fact that the (WT, RT, VT) have the same structure as (T, U, S) respectively. It follows that

- If $k = 1$ then an odd power of $\beta_{j,k}$ namely $\beta_{j,k}^{3^j}$ equals -1 . In that case $\Delta'(6n^2, j, 1)$ contains $(W, R, V, -S, -T, -U)$ which form a $\Delta(6n^2)$ subgroup. This is not the case for $k > 1$ and $\Delta(6n^2)$ is not a subgroup of $\Delta'(6n^2, j, k)$. The group $\Delta(3n^2)$ is always a subgroup of $\Delta'(6n^2, j, k)$ obtained when $r = 0$.

- All the elements in eq. (3.29) have the same six textures as in eq. (3.3) with $\eta_i = \eta^a \beta_{j,k}^r$, η^a being some power of the n^{th} root of unity. Mixing predictions for the group are given by substituting these η_i in eqs. (3.10) and eq. (3.12) when $\Delta(6n^2)$ is a subgroup, *i.e.* for $k = 1$. The overall factor $\beta_{j,k}^r$ in η_i is seen to cancel in these equations and the mixing angle predictions coincide in this case with the ones obtained for $\Delta(6n^2)$ even though one has a larger set of T_l to choose from. This can be explicitly seen from the numerical tables to be presented where one sees that $\Delta(54)$ and $\Delta(54, 3^j, 1)$ give identical mixing pattern namely bi-maximal. A slightly non-trivial example is the group $\Delta(6 \times 6^2, 2, 1)$ which is an order 648 group. All the solutions obtained for this group numerically coincide with the prediction of the $\Delta(6 \times 6^2) = \Delta(216)$ group displayed in the Table 1.
- For $k > 1$ and even n , the Z_2 subgroups are diagonal. The choice $T_l = \beta_{j,k}^r(R, V)$ then lead to the tri-maximal pattern. The choice $T_l = \beta_{j,k}^r(T, U, S)$ lead to the bi-maximal mixing. These two provide the only possible mixing patterns within $\Delta'(6n^2, j, k)$ with $k > 1$.

Continuing similar argument, one can also show that group series $\Sigma(3n^3)$ first studied in [5] can only lead to the tri-maximal mixing pattern. The other remaining series identified by Ludl [31] namely $T_n(m)$, does not contain any Z_2 subgroup and cannot therefore contain residual symmetry S_ν .

4 Numerical analysis

We numerically analyze all the DSG of $U(3)$ having three dimensional IR and order < 512 and derive their prediction for column in leptonic mixing matrix. These groups are listed in [31] along with their generators. Using the given generators, we numerically generate all the elements for a given group G_f . We then identify its Z_2 elements as the symmetry S_ν of neutrinos and find an eigenvector c_ν corresponding to the unique eigenvalue in S_ν . The T_l are chosen such that $T_l^m = 1$, $m \geq 3$, $[T_l, S_\nu] \neq 0$ and all the eigenvalues of T_l are distinct. We then find a unitary matrix V_l which diagonalizes T_l and find a column $|c_0\rangle$ using eq. (2.5). The column $|c_0\rangle$ or its permutation predicts a column in U_{PMNS} matrix.

We first perform a scan over DSG of $SU(3)$. Apart from the groups of type C and D which are already discussed analytically in the previous section, it contains six exceptional groups: A_5 , $PSL(2, 7)$, $\Sigma(36\phi)$, $\Sigma(72\phi)$, $\Sigma(216\phi)$ and $\Sigma(360\phi)$. The last two groups have order > 512 . The results of scan are listed in Table 1 and 2. We list only the results for $SU(3)$ subgroups of type D with order < 512 in Table 1 and six exceptional groups in Table 2. The remaining groups are of type C which give only tri-maximal $|c_0\rangle$. In the Tables, the various groups are represented by $\llbracket O, N \rrbracket$ following the convention used by GAP [40] which is a computer package that contains the groups of small order. Here, O is an order of a group and N indicates identification number of that group in GAP. Some common

predictions for column are denoted as TM (tri-maximal), BM (bi-maximal), TL (tri-large) and BL (bi-large). They correspond to

$$\text{TM} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^T, \quad \text{BM} = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T, \quad \text{TL} = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)^T, \quad \text{BL} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)^T \quad (4.1)$$

A number in the last column in tables indicates n^{th} column in $|U_{\text{PMNS}}^{\text{exp}}|$ with which the corresponding prediction is in the agreement.

We make the following observations from the results given in Table 1 and 2.

- The numerical results obtained in case of groups of type D are consistent with the analytic results derived in terms of eq. (3.16) and Fig. 1. In case of $\Delta(6n^2)$ groups and $n \leq 9$, there exist five different solutions which can reproduce the first column of $U_{\text{PMNS}}^{\text{exp}}$ within 3σ . Using the current best fit values, $\sin^2 \theta_{23} = 0.440$ and $\sin^2 \theta_{13} = 0.02163$, obtained from the global fit [37], one can determine the predictions for solar angle and Dirac CP phase from these columns. They are given as:

$$\begin{aligned} \text{TL} &\Rightarrow \sin^2 \theta_{12} = 0.319, \quad \cos \delta = -0.263 \\ (0.6594, 0.2303, 0.1103)^T &\Rightarrow \sin^2 \theta_{12} = 0.326, \quad \cos \delta = 0.607 \\ (0.6466, 0.2755, 0.078)^T &\Rightarrow \sin^2 \theta_{12} = 0.339, \quad \cos \delta \rightarrow \text{inconsistent} \\ (0.6629, 0.2116, 0.1255)^T &\Rightarrow \sin^2 \theta_{12} = 0.322, \quad \cos \delta = 0.364 \\ (0.6553, 0.2471, 0.0976)^T &\Rightarrow \sin^2 \theta_{12} = 0.330, \quad \cos \delta = 0.817 \end{aligned} \quad (4.2)$$

One column in the above gives $\cos \delta > 1$ which is inconsistent, however these results change when deviations in θ_{13} and θ_{23} from their best fit values are considered. The group $D_{9,3}^1$ gives same results as $\Delta(6 \times 9^2)$. No prediction consistent with the second column of $U_{\text{PMNS}}^{\text{exp}}$ other than TM exists as shown analytically in the previous section.

- The DSG of $SU(3)$ which are not of type C or D lead to a new solution for the second column other than TM. The predicted column $(0.2764, 0.3618, 0.3618)^T$ implies

$$\sin^2 \theta_{12} = 0.283 \quad \text{and} \quad \cos \delta = 0.638 \quad (4.3)$$

given the best fit values of θ_{23} and θ_{13} . This prediction is offered by the group A_5 and the group $\Sigma(360\phi)$ which contains A_5 as a subgroup.

- The group $PSL(2, 7)$ or $\Sigma(168)$ provides two new solutions consistent with the first column of $U_{\text{PMNS}}^{\text{exp}}$. They imply

$$\begin{aligned} (0.664, 0.2045, 0.1315)^T &\Rightarrow \sin^2 \theta_{12} = 0.321, \quad \cos \delta = 0.270 \\ (0.7057, 0.25, 0.0443)^T &\Rightarrow \sin^2 \theta_{12} = 0.279, \quad \cos \delta \rightarrow \text{inconsistent} \end{aligned} \quad (4.4)$$

for the best fit values of θ_{23} and θ_{13} . Predictions for this group were also analyzed in [25] where a solution for the second column leading to correct $\sin^2 \theta_{12}$ was found.

Group	Classification	$ c_0 ^2$	Best fit
[[24,12]]	$\Delta(6 \times 2^2)$	BM, BL	–
		TM	2
		TL	1
[[54,8]]	$\Delta(6 \times 3^2)$	BM	–
[[96,64]]	$\Delta(6 \times 4^2)$	BM, BL	–
		TM	2
		TL	1
		$(0.622, 0.3333, 0.0447)^T, (0.8536, 0.1464, 0)^T$	–
[[150,5]]	$\Delta(6 \times 5^2)$	BM, BL	–
		$(0.6594, 0.2303, 0.1103)^T$	1
		$(0.603, 0.3682, 0.0288)^T, (0.9045, 0.0955, 0)^T$	–
		$(0.6545, 0.3455, 0)^T$	–
[[162,14]]	$D_{9,3}^1$	BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.75, 0.25, 0)^T$	–
[[216,95]]	$\Delta(6 \times 6^2)$	BM, BL	–
		TM	2
		TL	1
		$(0.933, 0.067, 0)^T$	–
[[294,7]]	$\Delta(6 \times 7^2)$	BM, BL	–
		$(0.6629, 0.2116, 0.1255)^T$	1
		$(0.5777, 0.4075, 0.0148)^T, (0.6337, 0.3084, 0.0579)^T$	–
		$(0.9505, 0.0495, 0)^T, (0.6113, 0.3887, 0)^T$	–
		$(0.8117, 0.1883, 0)^T$	–
[[384,568]]	$\Delta(6 \times 8^2)$	BM, BL	–
		TM	2
		TL, $(0.6553, 0.2471, 0.0976)^T$	1
		$(0.622, 0.3333, 0.0447)^T, (0.569, 0.4196, 0.0114)^T$	–
		$(0.9619, 0.0381, 0)^T, (0.6913, 0.3087, 0)^T$	–
		$(0.8536, 0.1464, 0)^T$	–
[[486,61]]	$\Delta(6 \times 9^2)$	BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.9698, 0.0302, 0)^T, (0.75, 0.25, 0)^T$	–
		$(0.5868, 0.4132, 0)^T, (0.883, 0.117, 0)^T$	–

Table 1. Predictions for column c_0 in lepton mixing matrix from DSG of $SU(3)$ of type D and order < 512 . The last column shows the number of a column in $|U_{\text{PMNS}}^{\text{exp}}|$ with which the prediction is in agreement. See the text for more details.

Prediction of the entire column obtained in this case does not however reproduce the entire second column consistent with eq. (3.17). The group $\Sigma(216\phi)$ gives a solution consistent with the first column which can also be obtained from $\Delta(486)$ and hence also a type D group $D_{9,3}^1$.

Group	Classification	$ c_0 ^2$	Best fit
[[60,5]]	$A_5, \Sigma(60)$	BM	–
		TM, $(0.3618, 0.3618, 0.2764)^T$	2
		$(0.8727, 0.0637, 0.0637)^T, (0.4363, 0.4363, 0.1273)^T$	–
		$(0.7236, 0.1382, 0.1382)^T$	–
[[108,15]]	$\Sigma(36\phi)$	BM	–
		$(0.5915, 0.25, 0.1585)^T$	–
[[168,42]]	$PSL(2, 7), \Sigma(168)$	BM, BL	–
		TM	2
		TL	1
		$(0.664, 0.2045, 0.1315)^T, (0.7057, 0.25, 0.0443)^T$	1
		$(0.7985, 0.1667, 0.0348)^T$	–
[[216,88]]	$\Sigma(72\phi)$	BM	–
		$(0.5915, 0.25, 0.1585)^T$	–
[[648,259]]	$\Sigma(216\phi)$	BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.5915, 0.25, 0.1585)^T, (0.75, 0.25, 0)^T$	–
[[1080, 260]]	$\Sigma(360\phi)$	BM, BL	–
		TM, $(0.3618, 0.3618, 0.2764)^T$	2
		TL	1
		$(0.8727, 0.0637, 0.0637)^T, (0.4363, 0.4363, 0.1273)^T$	–
		$(0.5467, 0.3618, 0.0915)^T, (0.7236, 0.1382, 0.1382)^T$	–
		$(0.7992, 0.1382, 0.0626)^T, (0.8026, 0.1019, 0.0955)^T$	–
		$(0.5915, 0.25, 0.1585)^T, (0.6545, 0.3066, 0.0389)^T$	–

Table 2. Predictions for column c_0 in lepton mixing matrix from some exceptional DSG of $SU(3)$ that are not of type C or D. The last column shows the number of a column in $|U_{PMNS}^{\text{exp}}|$ with which the prediction is in agreement. See the text for more details.

- No viable prediction for the third column can be obtained from the DSF of $SU(3)$ with order < 512 .

The results obtained for the groups in Table 2 together with analytic results obtained for the groups of type C and D exhaust all the results that can be obtained from DSG of $SU(3)$ possessing three dimensional IR. As can be seen from Fig. 1, the smallest group reproducing consistent values for the third column is a group $\Delta(6n^2)$ with $n = 11$. Interestingly for $n = 10$, one finds a fixed column $|c_0|^2 = (0.603, 0.3682, 0.0288)^T$ implying $\sin^2 \theta_{13} = 0.0288$ which is larger than the current value allowed at the 3σ by the latest global fit NuFIT 3.0 [37]. The group $\Delta(6 \times 10^2)$ was identified in [21, 41] as the smallest group predicting all the three neutrino mixing angles within their 3σ ranges implied by the global fits of that time. If the results from latest global fit NuFIT 3.0 [37] are considered then this conclusion gets modified and $\Delta(6 \times 11^2)$ becomes the smallest group which reproduces consistent $\sin^2 \theta_{13}$. The predicted value, $\sin^2 \theta_{13} = 0.0239$, however is just below the maximum value allowed

at 3σ by the current fit. The group $\Delta(6 \times 11^2)$ also gives two different viable solutions for the first column of U_{PMNS} . However, it cannot reproduce both the first and third columns simultaneously as the Z_2 symmetries generating these two different solutions do not commute with each others. Since the order of $\Delta(6 \times 11^2)$ is not divisible by 4, it cannot contain two mutually commuting Z_2 groups and hence cannot lead to the correct predictions for any two columns simultaneously. An another arguments is that the group $\Delta(6 \times 11^2)$ corresponding to odd value of n does not contain the tri-maximal solution which is the only viable solution for the second column of U_{PMNS} offered by $\Delta(6n^2)$ groups. Therefore, $\Delta(6n^2)$ with odd n cannot lead to the viable predictions for all the tree mixing angles simultaneously. As seen from Fig. 1, the smallest even n reproducing a consistent solution for the third column is $n = 18$ corresponding to the group of order 1944⁴. We get the following elements in $\Delta(6 \times 18^2)$ which can be identified as the $Z_2 \times Z_2$ symmetry of neutrinos and Z_3 symmetry of the charged leptons.

$$S_{1\nu} \equiv GB = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \epsilon^* \\ 0 & \epsilon & 0 \end{pmatrix}, \quad S_{2\nu} \equiv G^9 = W(1, -1, -1), \quad T_l = E, \quad (4.5)$$

where the generators G , B and E are given in eq. (3.1) with $\epsilon = e^{2\pi i/18}$. The $S_{1\nu,2\nu}$ are diagonalized by appropriate V_S given in eq. (3.5) while T_l is diagonalized by U_ω given in eq. (3.6). Following eq. (2.4) and rearranging columns, they lead to

$$|U_{\text{PMNS}}| = \begin{pmatrix} 0.804 & 0.577 & 0.142 \\ 0.525 & 0.577 & 0.625 \\ 0.279 & 0.577 & 0.767 \end{pmatrix} \quad (4.6)$$

which is consistent with $U_{\text{PMNS}}^{\text{exp}}$ given in eq. (3.17). It predicts

$$\sin^2 \theta_{12} = 0.340, \quad \sin^2 \theta_{23} = 0.399 \text{ or } 0.601, \quad \sin^2 \theta_{13} = 0.0201, \quad \cos \delta = 1. \quad (4.7)$$

The same predictions can also be obtained from the group $D_{9n',3n'}^1$ with $9n' = n = 18$ as shown analytically in the previous section. The group $D_{18,6}^1$ has the rank 648 and therefore is the smallest group predicting the viable values for all the three mixing angles.

Let us now consider DSG of $U(3)$ which are not subgroups of $SU(3)$. There exist 75 such groups with three dimensional IR and order < 512 . As can be seen from Table 5 in [31], most of these group are generated by a texture of type $R(\eta_1, \eta_2, \eta_3)$ given in eq. (3.3). All such groups have elements of type W , R and V . Among these, only W can generate Z_2 subgroups leading to tri-maximal $|c_0|$ and predictions eq. (3.14) and this is the only solution possible for all such groups. We therefore give results for the remaining groups in Table 3 which are generated by the textures of type S , T , U or X_i with $i = 1, 2, \dots, 10$. None

⁴In [21, 41], the group $\Delta(6 \times 16^2)$ was identified as the smallest viable group reproducing all the mixing angles within the 3σ ranges of their global fit values. Again, it predicts $\sin^2 \theta = 0.0254$ which is outside the 3σ range implied by the latest global fit considered by us.

Group	Classification	$ c_0 ^2$	Best fit
[[48,30]]	$S_4(2)$	BM	–
		TM	2
[[96,65]]	$S_4(3)$	BM	–
		TM	2
[[162,10]]		BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.75, 0.25, 0)^T$	0
[[162,12]]		BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.75, 0.25, 0)^T$	0
[[162,44]]	$\Delta(6 \times 3^2, 2, 1)$	BM	–
[[192,186]]	$S_4(4)$	BM	–
		TM	2
[[324,111]]		BM, $(0.5915, 0.25, 0.1585)^T$	–
[[384,571]]	$\Delta(6 \times 4^2, 3)$	BM	–
		TM	2
[[384,581]]	$S_4(5)$	BM	–
		TM	2
[[432,239]]		BM, $(0.5915, 0.25, 0.1585)^T$	–
[[432,260]]	$\Delta(6 \times 4^2, 2)$	BM	–
		TM	2
[[432,273]]		BM, $(0.5915, 0.25, 0.1585)^T$	–
[[486,26]]		BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.75, 0.25, 0)^T$	0
[[486,28]]		BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.75, 0.25, 0)^T$	0
[[486,125]]		BM, BL	–
		$(0.6466, 0.2755, 0.078)^T$	1
		$(0.75, 0.25, 0)^T$	0
[[486,164]]	$\Delta(6 \times 3^2, 3, 1)$	BM	–

Table 3. Predictions for column c_0 in lepton mixing matrix from DSG of $U(3)$ with order < 512 . The last column shows the number of a column in $|U_{\text{PMNS}}^{\text{exp}}|$ with which the prediction is in agreement. See the text for more details.

of these groups generate any new mixing patterns compared to ones already found with the DSG of $SU(3)$. This result is anticipated from our analytic discussion in the previous section for all the groups which do not involve exceptional generators X_i . All the DSG of $U(3)$ of order < 512 lead to a column of type TM or $(0.6466, 0.2755, 0.078)^T$ which can be identified with the second and first column of U_{PMNS} respectively.

Before ending this section, we make a comparison of the results found by us with the previous similar studies. In [9], the DSG of $SU(3)$ with order up to 511 were numerically searched for their column prediction. Our analysis includes *all* the DSG of $SU(3)$ and also DSG of $U(3)$ which are not subgroups of $SU(3)$. An analytical treatment of $SU(3)$ subgroups of type C and D allows us to extend our results to larger groups. As a result of this, we find groups which can predict viable third column for U_{PMNS} unlike [9]⁵. Our results for the $SU(3)$ subgroups of order < 512 matches with their results. Ref. [21] mainly considered Dirac neutrinos but also included a discussion of Majorana neutrinos. Their numerical scan of groups of order < 2000 looked at the Klein groups and identified 10 groups giving consistent patterns. All these predict vanishing Dirac CP phase. Use of the only Z_2 groups allows non-trivial prediction for this phases as is apparent, for example, from eq. (4.2). Our results provide generalization of the existing results in which prediction for one or more columns of PMNS matrix were obtained using residual Z_2 or $Z_2 \times Z_2$ symmetries. Most importantly, we analytically and numerically show that the groups from $\Delta(6n^2)$ series are sufficient and sometime are the smallest groups leading to specific prediction for a column of PMNS matrix which follows from the groups of type C, D and some known group series representing DSG of $U(3)$.

5 Additional symmetries

The U_{PMNS} matrix of eq. (2.3) derived from the previous considerations contain the unknown phases and angle θ which can affect the observables. Some or all of these can be fixed by imposing additional symmetries. These have been discussed analytically [21] for the type C and D groups and also have been numerically explored. We show that essentially the same analysis gets carried over also to the DSG of $U(3)$ being studied here. We discuss here only the case giving non-zero values for all the entries of U_{PMNS} corresponding to S_ν in (3.4) and T_l having the texture of R, V . Other cases are not much of phenomenological interest but can be discussed analogously. The U_{PMNS} in this case can explicitly be written as

$$U_{\text{PMNS}} = \begin{pmatrix} \frac{\eta_1 \Delta^*}{\sqrt{6}}(1 - \eta_\nu \eta_2 \Delta^*) & \frac{1}{\sqrt{3}} & \frac{\eta_1 \Delta^*}{\sqrt{6}}(1 + \eta_\nu \eta_2 \Delta^*) \\ \frac{\eta_1 \omega \Delta^*}{\sqrt{6}}(1 - \omega \eta_\nu \eta_2 \Delta^*) & \frac{1}{\sqrt{3}} & \frac{\eta_1 \omega \Delta^*}{\sqrt{6}}(1 + \omega \eta_\nu \eta_2 \Delta^*) \\ \frac{\eta_1 \omega^2 \Delta^*}{\sqrt{6}}(1 - \omega^2 \eta_\nu \eta_2 \Delta^*) & \frac{1}{\sqrt{3}} & \frac{\eta_1 \omega^2 \Delta^*}{\sqrt{6}}(1 + \omega^2 \eta_\nu \eta_2 \Delta^*) \end{pmatrix} P \begin{pmatrix} 1 & 0 & \\ 0 & \cos \theta & -\sin \theta e^{i\beta} \\ 0 & \sin \theta e^{-i\beta} & \cos \theta \end{pmatrix}. \quad (5.1)$$

The first column of the above matrix is θ independent and already given in eq. (3.10). P is a diagonal phase matrix with $P_{jj} = e^{i\chi_j}$. The θ dependence can be eliminated by imposing another Z_2 symmetry commuting with the first Z_2 defined by $S_{1\nu} \equiv S_\nu$ as given in eq. (3.4). The most general 3×3 matrix having the texture in eq. (3.3) and satisfying

⁵The group $\Delta(6 \times 5^2)$ generates a solution $|c_0|^2 = (0.0289, 0.368, 0.604)^T$ which was identified in [9] as a consistent prediction for the third column of U_{PMNS} . However, it predicts $\sin^2 \theta_{13}$ greater than the highest value allowed by the current global fit at 3σ .

$[S_{1\nu}, S_{2\nu}] = 0$, $S_{2\nu}^2 = \mathbf{1}$ and $\det.(S_{2\nu}) = 1$ is either $W(1, -1, -1) = \text{diag.}(1, -1, -1)$ or

$$S_{2\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \eta_\nu^* \\ 0 & \eta_\nu & 0 \end{pmatrix}. \quad (5.2)$$

The matrices $\mathcal{Z} \equiv \{W(1, -1, -1), S_{1\nu}, S_{2\nu}\}$ together with the identity matrix define $Z_2 \times Z_2$ groups. These and their cyclic permutations define all possible Klein groups having non-diagonal structures in DSG of $SU(3)$ and $U(3)$ being considered here. Since η_ν represents n^{th} root of identity for some n , both $S_{1\nu}$ and $S_{2\nu}$ can be present in the same group only for even n and the residual Klein symmetry can be imposed only in such groups. The set \mathcal{Z} is simultaneously diagonalized by

$$V_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \\ -\eta_\nu & 0 & \eta_\nu \end{pmatrix} \quad (5.3)$$

$$V_\nu^\dagger S_{1\nu} V_\nu = \text{diag.}(1, -1, -1), \quad V_\nu^\dagger S_{2\nu} V_\nu = \text{diag.}(-1, -1, 1).$$

This V_ν with $V_l = V_{R,S}$ defined in eq. (3.5) give the U_{PMNS} as in eq. (5.1) but now θ is fixed to be zero allowing group theoretical prediction for $|U_{\text{PMNS}}|$. Eq. (5.1) with $\theta = 0$ is the most general U_{PMNS} possible in all the groups under consideration when T_l has the structure R or V . An alternative and phenomenologically less interesting possibility for the $Z_2 \times Z_2$ groups is a set of diagonal matrices. This gives democratic structure for $|U_{\text{PMNS}}|$.

- Eq. (5.1) with $\theta = 0$ has been derived earlier for the groups $\Delta(6n^2)$ [18] as well as all type D groups [21]. The derivation here shows that it has more general validity and holds for all the groups having elements with textures as in eq. (3.3) and containing the Klein group. Only change is the occurrence of the factor Δ which is defined as $\Delta^3 = \det.(T_l)$. In particular, the general result that one of the columns of $|U_{\text{PMNS}}|$ necessarily has tri-maximal form if entries in other two columns are required to be non-zero remains true here as well.
- For the $U(3)$ group series considered here the root η_2 differs from a power of the n^{th} root of unity only by an overall factor equal to Δ with the result that eq. (5.1) coincide with the corresponding result for $\Delta(6n^2)$. As already discussed only group series having $\Delta(6n^2)$ as subgroup is $\Delta(6n^2, j, 1)$. In that case the Klein groups in the latter coincide with the ones used in the above derivation and the use of $\Delta(6n^2, j, 1)$ instead of $\Delta(6n^2)$ does not give any new result. The remaining group series of $U(3)$ can contain only diagonal Klein groups and give only democratic mixing.
- Eq. (5.1) leads to vanishing Jarlskog invariant [42] for $\theta = 0$ as can be explicitly verified. Thus the Dirac CP violation is absent in very large class of models containing the Klein group. This result changes if one only requires Z_2 symmetry which does not fix θ . We give an explicit example below.

Eq. (5.1) reduces to the tri-bimaximal form in special cases of $\eta_2\eta_\nu\Delta^* = (-1, -\omega, -\omega^2)$. This pattern thus gets realized in large number of groups (e.g. all the $\Delta(6n^2)$ groups with even n) as the derivation here shows. The required departures from the tri-bimaximal mixing can be generated by using only a single Z_2 symmetry generated by $S_{1\nu}$. In that case the first column has the desired absolute values $(\sqrt{2/3}, 1/\sqrt{3}, 1/\sqrt{3})^T$ but the other two columns are θ dependent. Even in that case, the invariants given in eq. (2.6) are θ independent if $\beta = 0$ is assumed. Using eq. (5.1) one now gets

$$\begin{aligned} A_e &= 0 = c_{13}s_{13}s_{12}\sin(\delta + \beta_2 - \beta_3) , \\ A_\mu &= -\frac{1}{\sqrt{6}} \operatorname{Re} \left(\eta_1 \Delta^* e^{i(\chi_2 - \chi_3)} \right) \\ &= c_{13}c_{12}c_{23}s_{23}\sin(\beta_2 - \beta_3) - c_{13}s_{13}s_{12}s_{23}^2\sin(\delta + \beta_2 - \beta_3) . \end{aligned} \quad (5.4)$$

Here the second entry in each of these equations is obtained from eq. (5.1) while the last entry from the standard parameterization of the PMNS matrix [42] with β_2, β_3 denoting the Majorana phases. From now on, we specialize our discussions to groups $\Delta(6n^2)$ which as argued earlier are sufficient to describe mixing patterns of all other groups. In that case, eq. (5.4) imply

$$\begin{aligned} \beta_2 - \beta_3 + \delta &= k\pi , \\ c_{23}s_{23}\sin\delta &= \pm \frac{1}{2} \operatorname{Re} \left(\eta_1 e^{i(\chi_2 - \chi_3)} \right) . \end{aligned} \quad (5.5)$$

There still remains dependence on the arbitrary phases $\chi_{2,3}$. This can be fixed by invoking CP as a residual symmetry as has been discussed in variety of ways [20, 43–57]⁶. We give a simple example here based on $\Delta(6n^2)$ with CP as residual symmetry of neutrinos. A more detailed analysis of such case is carried out in [55]. Assume that M_ν satisfies eq. (2.1) with $S_\nu = S_{1\nu}$ and transforms as follows under a CP symmetry X_ν

$$X_\nu^T M_\nu X_\nu = M_\nu^* , \quad (5.6)$$

where X_ν denotes a 3×3 matrix corresponding to the μ - τ interchange symmetry. The X_ν and $S_{1\nu}$ satisfy $X_\nu S_{1\nu}^* - S_{1\nu} X_\nu = 0$ as required to define the direct product of Z_2 and CP as a consistent residual symmetry [51]. The most general M_ν satisfying eqs. (2.1,5.6) has the form

$$M_\nu = \begin{pmatrix} x & a\eta_\nu^{\frac{1}{2}} & a\eta_\nu^{-\frac{1}{2}} \\ a\eta_\nu^{\frac{1}{2}} & b\eta_\nu & c \\ a\eta_\nu^{-\frac{1}{2}} & c & b\eta_\nu^* \end{pmatrix} \quad (5.7)$$

where x, a, b, c are real. This M_ν is diagonalized by

$$U_\nu = V_\nu \operatorname{diag.}(\eta_\nu^{-\frac{1}{2}}, 1, \eta_\nu^{-\frac{1}{2}}) R_{23}(\theta)$$

⁶There is an extensive literature on the subject and we refer to [57] for a fairly exhaustive list of references.

with real rotation $R_{23}(\theta)$. This implies $\beta = 0$ and $e^{i(\chi_2 - \chi_3)} = \eta_\nu^{1/2}$. Eq. (5.5) now gives for the $\Delta(6n^2)$ groups

$$c_{23}s_{23} \sin \delta = \pm \frac{1}{2} \cos \left(\frac{2\pi \left(q + \frac{q_\nu}{2} \right)}{n} \right), \quad (5.8)$$

where we have defined $\eta_1 = e^{2\pi i q/n}$, $\eta_\nu = e^{2\pi i q_\nu/n}$. This equation is θ independent but $c_{23}s_{23}$ depends on θ implicitly. One however has the following θ independent correlation

$$s_{23}^2 = \left[\frac{1}{2} \pm \frac{\sqrt{2}s_{13}\sqrt{1-3s_{13}^2}}{c_{13}^2} \text{Im}(\eta_1) \right]^{\frac{1}{2}}.$$

θ_{23} is thus predicted to be nearly maximal and eq. (5.8) is now approximately given by

$$\sin \delta \approx \pm \cos \left(\frac{2\pi \left(q + \frac{q_\nu}{2} \right)}{n} \right). \quad (5.9)$$

It is seen that all the $\Delta(6n^2)$ groups giving identical prediction $(\sqrt{2/3}, 1/\sqrt{3}, 1/\sqrt{3})^T$ for the first column differ in their prediction for the CP phases. The minimal case corresponding to $n = 2$, namely the S_4 group gives $\delta = 0, \pm\frac{\pi}{2}, \pm\pi$, while $n = 4$ corresponding to $\Delta(96)$ predict $\delta \approx (0, \pm 45^\circ, \pm 90^\circ)$. In general, for $n = 2k$, $\delta \approx \pm \frac{\pi l}{2k}$ and $l = 0, 1, \dots, k$.

6 Summary

Discrete subgroups G_f of $SU(3)$ as flavour symmetries of the leptonic world have been extensively discussed in the last decade. The possible mixing patterns implied by these groups have been studied both analytically and numerically. We have given here an analytic discussion of the mixing angle predictions within the DSG of $U(3)$ assuming that neutrino mass matrix is invariant under a single Z_2 group $\in G_f$. The resulting mixing formula eqs. (3.8-3.12) encompasses predictions of large number of groups.

The consequences of assuming a single Z_2 symmetry as a residual symmetry of M_ν have been already discussed in variety of cases. Ref. [25, 26] discussed predictions of the finite and some of the infinite subgroups of the von-Dyck groups. These predictions have also been studied for the DSG of $SU(3)$ of C and D type which emerge here as special cases of our general formula. Variant of these have also been studied numerically in [22, 24].

Most of the above studies concentrated on the DSG of $SU(3)$. We have also discussed here symmetries which are contained in $U(3)$ but not in $SU(3)$. Not all DSG of $U(3)$ are known but a large number of them have elements with textures given in eq. (3.3). The mixing angle predictions do not directly involve basic structure of the groups and use only these textures. This makes them useful to study many groups. One can group theoretically predict absolute magnitude of any one column of the mixing matrix U_{PMNS} under the assumption of single Z_2 symmetry. We have analytically studied this predictions for all the known series of groups which are contained in $U(3)$ using the structures derived here. One of the important conclusion of this work is the realization that the prediction

of a single column of $|U_{\text{PMNS}}|$ in all these group series can be obtained using $\Delta(6n^2)$ as a flavour symmetry for some appropriate n . This conclusion applies to group series of types C, D and various series contained in $U(3)$ [31]. Specifically, (a) the group series $D_{9n',3n'}^1$ lead to the same mixing pattern as the groups $\Delta(486n'^2)$, (b) the series $\Sigma(3n^3)$, $\Delta(3n^2, j)$ for even n lead to tri-maximal pattern as implied by $\Delta(3n^2)$ contained in $\Delta(6n^2)$, (c) The $\Delta(6n^2, m)$ groups lead to the tri-maximal and the bi-maximal patterns, and (d) mixing in case of the $\Delta'(6n^2, j, k)$ groups coincide with that in $\Delta(6n^2)$ for $k = 1$ and with tri-maximal and bi-maximal for other k . All in all, the group series $\Delta(6n^2)$ represent a sufficient set of discrete groups for the study of predictions of a single Z_2 residual symmetry.

The above analytic results are supplemented with numerical discussion using the latest global fit of the neutrino mixing angles. It is found that some of the earlier numerical results get modified in view of the latest result and earlier allowed cases such as $\Delta(150)$, $\Delta(600)$ do not give results consistent with the present 3σ ranges of the mixing angles. The groups $\Delta(6n^2)$ (and consequently the large class of DSG of $U(3)$ as discussed above) lead to only tri-maximal mixing as the viable solution for the second column of leptonic mixing matrix. These groups can predict either first or third column of $|U_{\text{PMNS}}|$ for some values of n . The smallest group which can accommodate the third column of U_{PMNS} matrix within 3σ is found to be $\Delta(6 \times 11^2)$. The group A_5 which does not belong to type C or D provides a viable solution for the second column alternative to the tri-maximal. Selections of two commuting Z_2 symmetries as the symmetry of neutrino mass matrix within a given group G_f is known for fixing $|U_{\text{PMNS}}|$ entirely. The smallest such group in the $\Delta(6n^2)$ series which can simultaneously predict all the three mixing angles within their 3σ ranges is $\Delta(6 \times 18^2)$ of order 1944. The same mixing pattern also follows from the order 648 group $D_{18,6}^1$ which represents the smallest possible group containing residual symmetries leading to all the mixing angles within their 3σ ranges.

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