Geometric phase of neutrinos: differences between Dirac and Majorana neutrinos

A. Capolupo,[1] S.M. Giampaolo,[2] B. C. Hiesmayr,[3] and G. Vitiello[1]

[1]*Dipartimento di Fisica "E.R. Caianiello" Universitá di Salerno, and INFN - Gruppo Collegato di Salerno, Italy*
[2]*International Institute of Physics, Universidade Federal do Rio Grande do Norte, 59078-400 Natal-RN, Brazil*
[3]*Fakultat fur Physik, Universitat Wien, Boltzmanngasse 5, 1090 Vienna, Austria.*
(Dated: November 21, 2021)

We analize the non-cyclic geometric phase for neutrinos. We find that the geometric phase and the total phase associated to the mixing phenomenon provide a tool to distinguish between Dirac and Majorana neutrinos. Our results hold for neutrinos propagating in vacuum and through the matter. Future experiments, based on interferometry, could reveal the nature of neutrinos.

–Introduction. The mixing and oscillation of neutrinos represent one of the most intriguing phenomena in particle physics. The main issues of neutrino physics are the neutrino mass and the nature of the Dirac vs Majorana neutrino. Many experiments [1]-[6] have confirmed neutrino oscillations. Such a phenomenon implies that the neutrino has mass. Since neutrinos are neutral particles, they can be Majorana (with neutrinos and antineutrinos being the same), or Dirac particles (with neutrinos and antineutrinos being different objects). The Majorana neutrinos allow processes in which the total lepton number is not conserved, as the neutrinoless double β decay. Such processes are inhibited for Dirac neutrinos. Experiments based on the detection of the double beta decay may contribute to determine the neutrino nature [7], however, at the moment the nature of the neutrino remains an unsolved question.

Here, we show that the study of the total and the geometric phases for neutrino propagation in vacuum and through matter could open a new scenario in the investigation of the neutrino properties.

In the case of neutrinos travelling through a dense medium, as for example, the Sun or the Earth, the neutrino propagation can be affected by interactions with the particles in the medium. Therefore, the oscillation probabilities can be considerably different than the ones due to vacuum propagation. Such an effect is called Mikhaev-Smirnov-Wolfenstein (MSW) effect [8, 9]. The phenomenon originates from the fact that electron–neutrinos (and antineutrinos) have different interactions with matter compared to the two other flavour neutrinos. This means that in the effective Hamiltonian H, which governs the propagation of neutrinos in matter, there is an extra-potential term for the electron neutrinos. In the ordinary matter, due to the coherent forward scattering on electrons such an extra-potential term is $A_e = \pm\sqrt{2}G_F n_e$, with n_e the electron density in the matter. G_F is the Fermi constant, and the positive (negative) sign applies to electron-neutrino (antineutrinos).

On the other hand, the interferometry and the study of geometric phases [10]-[23] which appear in the evolution of many physical systems [24–26] have attracted the attention in recent years. A geometric phase arises in the evolution of any state $|\psi\rangle$, describing a quantum system characterized by a Hamiltonian defined on a parameter space (in the present case the relevant parameters are Δm^2 and the mixing angles). The geometric phase results from the geometrical properties of such a parameter space and can be used to study the properties of the system [25]. Berry and Berry-like phases have been shown in experimental observations to characterize physical properties of specific quantum systems. In particular, geometric phases turn out to be by themselves physical observables [20]-[25]. Typical examples are graphene systems [24], superconducting nanocircuits and devices, semiconductors [25], etc.

Berry-like phases and noncyclic invariants associated to particle oscillations (see, for example, Ref.[27] and references therein) have also been studied extensively.

In the present paper we consider the two flavor neutrino mixing case (the discussion can be generalized to the three flavor mixing case). We show that different choices of the Majorana phase ϕ, violating the charge-parity (CP) symmetry, in the mixing matrix lead to different values of the total and of the geometric phases. The oscillation formulas, on the contrary, are independent from the Majorana phase and consequently cannot be used to distinguish between Majorana and Dirac neutrinos [28, 29].

Therefore, in order to derive the oscillation formulas for Majorana neutrinos, one can use indifferently any of the mixing matrices, containing the Majorana phase ϕ, obtained by the rephasing the lepton charge fields in the charged current weak-interaction Lagrangian, (for details see Ref.[28]). For example, one can consider the following mixing matrix

$$U_1 = \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ -\sin\theta & \cos\theta e^{i\phi} \end{pmatrix}, \qquad (1)$$

or

$$U_2 = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ -\sin\theta e^{i\phi} & \cos\theta \end{pmatrix}, \qquad (2)$$

and obtain the same oscillation formulas. The probabili-

ties of neutrino transitions are indeed invariant under the rephasing $U_{\alpha k} \to e^{i\phi_k} U_{\alpha k}$ ($\alpha = e, \mu; k = 1, 2$).

The amplitude of transitions between different flavors are instead depending on the choice of the matrix U; see Eq.(22) of [28], here reported for reader's convenience: $\langle \mu^+, \nu_e | \psi(t) \rangle = A_\mu \left(U_{e1} U_{\mu 1}^* e^{-iE_1 t} + U_{e2} U_{\mu 2}^* e^{-iE_2 t} \right)$. Obviously, when squaring the amplitudes, the phases disappear and the resulting oscillation formulas do not depend on the phase.

As we will show below, the total phase (the dynamical and the geometric one) depends on the transition amplitudes and, in the case of transition between different flavors, it depends on the choice of the matrix U. In these transitions, different choices of U lead thus to different values of the total phases.

In our computations we consider the matrices corresponding to U_1 and U_2, for the case of neutrinos travelling through a dense medium.

We show that, by using the matrix corresponding to U_2 in Eq.(2), the geometric and the total phase due to the transition between different flavors are different for Majorana and for Dirac neutrinos. On the other hand, by using the matrix corresponding to U_1, all the phases are independent from ϕ and Majorana neutrinos cannot be distinguished from the Dirac ones.

A measurement of the total and geometric phase may thus determine the choice of the mixing matrix U. Similar results are obtain in the case of propagation in the vacuum.

–*Majorana and Dirac neutrino.* A Majorana field has a physical phase ϕ which violates the CP symmetry. In this case, assuming for simplicity only two flavour eigenstates, $|\nu_e\rangle$, $|\nu_\mu\rangle$, by considering the mixing matrix U_2, one can write

$$
\begin{aligned}
|\nu_e\rangle &= \cos\theta |\nu_1\rangle + e^{-i\phi} \sin\theta |\nu_2\rangle \\
|\nu_\mu\rangle &= -e^{i\phi} \sin\theta |\nu_1\rangle + \cos\theta |\nu_2\rangle ,
\end{aligned} \tag{3}
$$

where θ is the mixing angle and $|\nu_1\rangle$, $|\nu_2\rangle$ are the eigenstates of the free Hamiltonian. The phase ϕ cannot be eliminated since the Lagrangian of Majorana neutrinos is not invariant under $U(1)$ global transformation and the rephasing is not possible (see below). In this case, only left-handed components of the neutrino fields $\nu_{eL} = \frac{1+\gamma_5}{2}\nu_e$, $\nu_{\mu L} = \frac{1+\gamma_5}{2}\nu_\mu$ and right-handed components of the antineutrino fields $\nu_{eR}^C = \frac{1-\gamma_5}{2}\nu_e^C = (\nu_{eL})^C$, $\nu_{\mu R}^C = \frac{1-\gamma_5}{2}\nu_\mu^C = (\nu_{\mu L})^C$ appear in the Hamiltonian. Here $\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$, with γ^j Dirac matrices [30], $\nu_{e,\mu}^C = C\bar{\nu}_{e,\mu}$ is the charge conjugated spinor and the matrix C satisfies the relations: $C^\dagger C = 1, C\gamma_\alpha^T C^{-1} = -\gamma_\alpha$, $C^T = -C$. For Majorana neutrinos the interaction Hamiltonian, which does not conserve the lepton numbers, has the form

$$
H = m_{\bar{e}e}\bar{\nu}_{eR}^C\nu_{eL} + m_{\bar{\mu}\mu}\bar{\nu}_{\mu R}^C\nu_{\mu L} + m_{\bar{\mu}e}\left(\bar{\nu}_{\mu R}^C\nu_{eL} + \bar{\nu}_{eR}^C\nu_{\mu L}\right) + h.c.
$$

where the parameters $m_{\bar{e}e}$, $m_{\bar{\mu}\mu}$, $m_{\bar{\mu}e}$ have the dimensions of a mass.

On the contrary, for Dirac neutrino, the Lagrangian is invariant under $U(1)$ global transformation and the phase ϕ can be removed. The Hamiltonian interaction in this case is

$$
H = m_{\bar{e}e}\bar{\nu}_e\nu_e + m_{\bar{\mu}\mu}\bar{\nu}_\mu\nu_\mu + m_{\bar{\mu}e}\left(\bar{\nu}_\mu\nu_e + \bar{\nu}_e\nu_\mu\right) ,
$$

where $\bar{\nu}_\sigma$, with $\sigma = e, \mu$ are the antineutrino fields.

The phase ϕ does not affect the oscillation formulas for neutrino propagating in the vacuum and through matter. Therefore, neglecting the dissipation [31], the oscillation formulas cannot reveal the nature of neutrinos [32].

–*Neutrinos propagating through the matter.* In the case of neutrinos propagating through the matter, the evolution equation in the flavour basis is described by the Schrödinger equation

$$
i\frac{d}{dt}\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = H_f \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} ,
$$

where the Wolfenstein effective Hamiltonian [9] for flavored neutrinos can be written as (for a formal derivation see also Ref.[33])

$$
H_f = \left(p + \frac{m_1^2 + m_2^2}{4E} + \frac{\sqrt{2}}{2}G_F(n_e - n_n) \right)\mathbf{I} + H_I. \tag{4}
$$

Here, p is the neutrino momentum, m_i are the masses of the free fields ν_i ($i = 1, 2$) (for relativistic neutrinos $E_i \approx p + \frac{m_i^2}{2E}$), n_e is the electron density, n_n is the neutron number density, G_F the Fermi weak coupling constant and H_I for Majorana neutrinos is [34]

$$
H_I = \begin{pmatrix} \frac{G_F n_e}{\sqrt{2}} - \frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}e^{-i\phi}\sin 2\theta \\ \frac{\Delta m^2}{4E}e^{i\phi}\sin 2\theta & -\frac{G_F n_e}{\sqrt{2}} + \frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix}. \tag{5}
$$

$\mathbf{I}$ is the 2×2 identity matrix, ϕ the CP violating phase (which can be put equal to zero in the Dirac neutrino case), E the neutrino energy and $\Delta m^2 = m_2^2 - m_1^2$. The first term of H (the one proportional to the identity matrix) is responsible only of an overall phase factor in the neutrino evolution, therefore we will neglected it in the computation of the geometric phase and focus on H_I.

By defining $\Delta m_m^2 = \Delta m^2 R$, $\sin 2\theta_m = \sin 2\theta / R$, with

$$
R = \sqrt{\left(\cos 2\theta - \frac{2\sqrt{2}G_F n_e E}{\Delta m^2} \right)^2 + \sin^2 2\theta} , \tag{6}
$$

H_I in Eq. (5) can be written as

$$
H_I = \frac{\Delta m_m^2}{4E}\begin{pmatrix} -\cos 2\theta_m & e^{-i\phi}\sin 2\theta_m \\ e^{i\phi}\sin 2\theta_m & \cos 2\theta_m \end{pmatrix} . \tag{7}
$$

The eigenvalues of H_I are $\lambda_\pm = \pm\frac{\Delta m_m^2}{4E}$. Moreover, H_I can be diagonalized by means of the matrix U_m,

$$
U_m = \begin{pmatrix} \cos\theta_m & e^{-i\phi}\sin\theta_m \\ -e^{i\phi}\sin\theta_m & \cos\theta_m \end{pmatrix} , \tag{8}
$$

so that $H_I = U_m H_0 U_m^{-1}$, with $H_0 = diag(\lambda_+, \lambda_-)$. The matrix U_m relates the flavor states $|\nu_e(z)\rangle$ and $|\nu_\mu(z)\rangle$ with the energy eigenstates $|\nu_1(z)\rangle$ and $|\nu_2(z)\rangle$. Here we consider, in natural units, the approximation, $z \approx t$, where t is the neutrino propagation time and z the distance traveled by neutrinos. Explicitly one has

$$|\nu_e(z)\rangle = \cos\theta_m e^{i\frac{\Delta m_m^2}{4E}z}|\nu_1\rangle + e^{-i\phi}\sin\theta_m e^{-i\frac{\Delta m_m^2}{4E}z}|\nu_2\rangle,$$

$$|\nu_\mu(z)\rangle = -e^{i\phi}\sin\theta_m e^{i\frac{\Delta m_m^2}{4E}z}|\nu_1\rangle + \cos\theta_m e^{-i\frac{\Delta m_m^2}{4E}z}|\nu_2\rangle,$$

$$(9)$$

with $\langle\nu_e(z)|\nu_e(z)\rangle = \langle\nu_\mu(z)|\nu_\mu(z)\rangle = 1$ and $\langle\nu_e(z)|\nu_\mu(z)\rangle = 0$. In the case of the propagation in vacuum, $R = 1$, then $\Delta m_m^2 \rightarrow \Delta m^2$, $\theta_m \rightarrow \theta$ and U_m in Eq.(8) coincides with U_2 of Eq.(2).

– *Geometric phases for neutrinos in matter.* In the following we consider the states given by Eq. (9). To compute the geometric and the total phases generated by the states (9), we use the definition of Mukunda-Simon geometric phase [16], which generalizes the Berry phase to the non-adiabatic and non-cyclic case. Such a phase, derived within a kinematical approach, is associated to any open curve of unit vectors in Hilbert space. For a quantum system whose state vector $|\psi(s)\rangle$ belongs to a curve Γ, with the real parameter s such that $s \in [s_1, s_2]$, the Mukunda–Simon phase is defined as the difference between the total and the dynamic phase:

$$\Phi^g(\Gamma) = \Phi_\psi^{tot}(s) - \Phi_\psi^{dyn}(s)$$

$$= \arg\langle\psi(s_1)|\psi(s_2)\rangle - \Im\int_{s_1}^{s_2}\langle\psi(s)|\dot\psi(s)\rangle ds. (10)$$

Here the dot denotes the derivative with respect to the real parameter s. In Eq.(10), $\arg\langle\psi(s_1)|\psi(s_2)\rangle$ represents the total phase, and $\Im\int_{s_1}^{s_2}\langle\psi(s)|\dot\psi(s)\rangle ds$ is the dynamical one.

In the specific case of neutrinos, the geometric phase of electron neutrino, for an initial state $|\nu_e\rangle$, is

$$\Phi_{\nu_e}^g(z) = \Phi_{\nu_e}^{tot}(z) - \Phi_{\nu_e}^{dyn}(z) \qquad (11)$$

$$= \arg[\langle\nu_e(0)|\nu_e(z)\rangle] - \Im\int_0^z\langle\nu_e(z')|\dot\nu_e(z')\rangle dz',$$

and explicitly one obtains

$$\Phi_{\nu_e}^g(z) = \arg\left[\cos\left(\frac{\Delta m_m^2 z}{4E}\right) + i\cos 2\theta_m \sin\left(\frac{\Delta m_m^2 z}{4E}\right)\right]$$

$$- \frac{\Delta m_m^2 z}{4E}\cos 2\theta_m. \qquad (12)$$

In a similar way, the geometric phase for the muon neutrino, for an initial state $|\nu_\mu\rangle$, is given by $\Phi_{\nu_\mu}^g(z) = \arg[\langle\nu_\mu(0)|\nu_\mu(z)\rangle] - \Im\int_0^z\langle\nu_\mu(z')|\dot\nu_\mu(z')\rangle dz'$. One has $\Phi_{\nu_\mu}^g(z) = -\Phi_{\nu_e}^g(z)$. Eq.(12) does not depends on the CP violating phase, thus, it holds both for Majorana and for Dirac neutrinos.
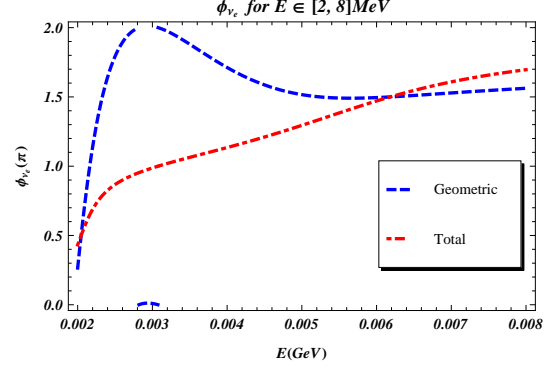

Figure 1: *(Color online) Plots of the total and the geometric phases associated to the evolution of ν_e, as a function of the neutrino energy E, for a distance length $z = 100km$. - The red dot dashed line is the total phase; - the blue dashed line is the geometric phase.*

However, in addition to the phases $\Phi_{\nu_e}^g(z)$ and $\Phi_{\nu_\mu}^g(z)$, one also has the following phases due to the neutrino transitions between different flavors,

$$\Phi_{\nu_e\rightarrow\nu_\mu}^g(z) = \arg[\langle\nu_e(0)|\nu_\mu(z)\rangle] - \Im\int_0^z\langle\nu_e(z')|\dot\nu_\mu(z')\rangle dz',$$

$$(13)$$

$$\Phi_{\nu_\mu\rightarrow\nu_e}^g(z) = \arg[\langle\nu_\mu(0)|\nu_e(z)\rangle] - \Im\int_0^z\langle\nu_\mu(z')|\dot\nu_e(z')\rangle dz'.$$

$$(14)$$

In this case, by using the states in Eq.(9) for Majorana neutrinos, since ϕ cannot be removed, one obtains

$$\Phi_{\nu_e\rightarrow\nu_\mu}^g(z) = \arg\left[\sin 2\theta_m \sin\left(\frac{\Delta m_m^2 z}{4E}\right)(\sin\phi - i\cos\phi)\right]$$

$$+ \left(\frac{\Delta m_m^2}{4E}\sin 2\theta_m \cos\phi\right)z, \qquad (15)$$

which holds provided there is mixing ($\Delta m^2 \neq 0$ and $\theta \neq 0$). Thus, we have

$$\Phi_{\nu_e\rightarrow\nu_\mu}^g(z) = \frac{3\pi}{2} + \phi + \left(\frac{\Delta m_m^2}{4E}\sin 2\theta_m \cos\phi\right)z, (16)$$

and in similar way

$$\Phi_{\nu_\mu\rightarrow\nu_e}^g(z) = \frac{3\pi}{2} - \phi + \left(\frac{\Delta m_m^2}{4E}\sin 2\theta_m \cos\phi\right)z. (17)$$

Then, $\Phi_{\nu_e\rightarrow\nu_\mu} \neq \Phi_{\nu_\mu\rightarrow\nu_e}$, which reveals an asymmetry between the transitions $\nu_e \rightarrow \nu_\mu$ and $\nu_\mu \rightarrow \nu_e$ due to the presence of ϕ.

On the contrary, in the case of Dirac neutrinos, the phase ϕ can be removed by means of $U(1)$ global transformation, then, the phases of Eqs.(13) and (14) (or equivalently Eqs. (16) and (17)) become equivalent and reduce

to

$$\begin{aligned}
\Phi^g_{\nu_e \to \nu_\mu}(z) &= \Phi^g_{\nu_\mu \to \nu_e}(z) \\
&= \frac{3\pi}{2} + \left(\frac{\Delta m_m^2}{4E} \sin 2\theta_m \right) z. \quad (18)
\end{aligned}$$

Let us now observe that the result of Eq.(12) is independent on the choice of the mixing matrix. The same result is indeed obtained by using, for example, the mixing matrix corresponding to U_1 which diagonalize the interaction Hamiltonian H_I^D, derived by H_I in Eq.(7) by setting $\phi = 0$.

On the contrary, the phases defined in Eqs.(13) and (14) are dependent on the choice of the mixing matrix. Indeed, the result of Eq.(18) is obtained also for Majorana neutrinos when one considers the mixing matrix corresponding to U_1. We therefore conclude that the phases $\Phi^g_{\nu_e \to \nu_\mu}$ and $\Phi^g_{\nu_\mu \to \nu_e}$ discriminate between the two matrices U_1 and U_2.

Notice that all the above phases vanish in the limit of zero neutrino masses where the mixing is absent and Dirac and Majorana neutrinos are equivalent.

–Numerical analysis. In our analysis, for the total and geometric phases associated with the evolution of ν_e, we consider the energy of neutrinos produced in nuclear reactors $E \in [2-8] MeV$, the earth density $n_e = 10^{24} cm^{-3}$, $\Delta m^2 = 7.6 \times 10^{-3} eV^2$ and a distance $z = 100 km$. We obtain the results reported in Fig.1 which could be detected in experiments like RENO [2]. Similar results can be found considering energies of few GeV, which are characteristic of neutrino beams produced at particle accelerators, and distances of several hundred of km, as in long base line experiments. For the analysis of geometric phases due to the transition between different flavors (Eqs.(13) and (14)), we consider energies $E \sim 1 GeV$ and a distance $z = 300 km$, which are values similar to the ones in $T2K$ experiment. Moreover we consider $\phi = 0.3$, and the values of n_e and Δm^2 considered above. The plots of the phases in Eqs.(16), (17) and (18) are reported in Fig.2.

–Conclusions. We have shown that the total and geometric phase, due to a transition between different neutrino flavors, take different values depending on the representation of the mixing matrix and on the nature of neutrinos. Measurement of different values for $\Phi^g_{\nu_e \to \nu_\mu}$ and $\Phi^g_{\nu_\mu \to \nu_e}$ could allow the detection of the Majorana phase ϕ. Such a measurement implies that the mixing matrix to be used is of type U_2, which then removes the ambiguity in the use of U_1 and U_2. On the other hand, if the measurement of the geometric phase leads to $\Phi^g_{\nu_e \to \nu_\mu} = \Phi^g_{\nu_\mu \to \nu_e}$, then the ambiguity between U_1 and U_2 remains and nothing can be said on the nature (Dirac or Majorana) of neutrinos. The phases above analyzed are in principle detectable, and next long baseline neutrino oscillation experiment like $T2K$ [35], or short base line experiments like RENO [2], could reveal the correct
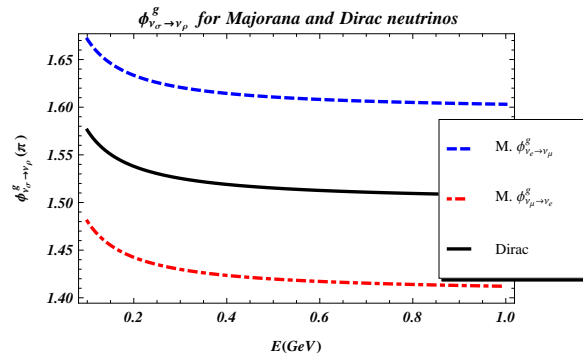

Figure 2: *(Color online) Plot of the geometric phases $\Phi^g_{\nu_e \to \nu_\mu}$ (the blue dashed line) and $\Phi^g_{\nu_\mu \to \nu_e}$ (the red dot dashed line) for Majorana neutrinos as a function of the neutrino energy E, for a distance length $z = 300 km$. The geometric phases $\Phi^g_{\nu_e \to \nu_\mu} = \Phi^g_{\nu_\mu \to \nu_e}$ for Dirac neutrinos is represented by the black solid line.*

mixing matrix for Majorana neutrinos and the nature of the neutrinos by means of interferometric analysis.

In a forthcoming paper we will consider the case of Dirac and Majorana neutrinos propagating through a magnetic field.

The theoretical aspects of particle mixing have been studied extensively in the contexts of quantum mechanics (QM) [36]-[39] and of quantum field theory (QFT) [40]-[42] where corrections to the QM oscillation formulas in vacuum have been derived (see also Refs. [43]-[44]). In the discussion here presented these corrections can be safely neglected [41].

[1] F. P. An et al. [Daya-Bay Collaboration], Phys. Rev. Lett. 108, 171803 (2012).

[2] J. K. Ahn et al. [RENO Collaboration], Experiment, Phys. Rev. Lett. 108, 191802 (2012).

[3] 3 Y. Abe et al. [Double Chooz Collaboration], Phys. Rev. Lett. 108, 131801 (2012).

[4] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 107, 041801 (2011).

[5] P. Adamson et al. [MINOS Collaboration], Phys. Rev. Lett. 107, 181802 (2011).

[6] K. Nakamura and S.T. Petcov, Phys. Rev. D 86, 010001 (2012).

[7] A. Giuliani, A. Poves, Advances in High Energy Physics, Volume 2012, 1(2012).

[8] S. P. Mikheev, A. Yu. Smirnov, Sov. J. Nuc. Phys. **42** (6): 913917, (1985).

[9] L. Wolfenstein Phys. Rev. D **17** (9): 2369, (1978).

[10] M. V. Berry, Proc. Roy. Soc. Lond. A **392**, 45 (1984); Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987); J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339 (1988).

[11] S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 1225 (1956).

[12] A. Shapere and F. Wilczek, Geometric Phases in Physics, *World Scientific*, Singapore, 1989.

[13] J. C. Garrison and E. M. Wright, Phys. Lett. A **128**, 177 (1988).

[14] A. K. Pati, J. Phys. A **28**, 2087 (1995).

[15] A. K. Pati, Phys. Rev. A **52**, 2576 (1995).

[16] N. Mukunda and R. Simon, Ann. Phys.(N.Y) **228**, 205 (1993).

[17] A. Mostafazadeh, J. Phys. A **32**, 8157 (1999).

[18] J. Anandan, Phys. Lett. A **133**, 171 (1988).

[19] A. Tomita, R. Y. Chiao, Phys. Rev. Lett. **57**, 937 (1986).

[20] J. A. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature **403**, 869 (2000).

[21] P. J. Leek, et. al, Science **318**, 1889 (2007).

[22] M. Neeley, et al., Science **325**, 722 (2009).

[23] M. Pechal, et al., Phys. Rev. Lett. **108**, 170401 (2012).

[24] Y. Zhang, Y.-W. Tan, H. L. Stormer, P. Kim, Nature **438**, 201-204 (2005).

[25] G. Falci, et al. Nature **407**, 355-358 (2000); M. Mottonen, J. J. Vartiainen, J. P. Pekola, Phys. Rev. Lett. **100**, 177201 (2008); H. Murakawa et al. Science, **342**, Issue 6165, 1490-1493 (2013).

[26] D. Xiao et al., Rev. Mod. Phys. 82, 1959 (2010); F. Wilczek, A. Shapere, Geometric Phases in Physics, World Scientific, (1989); A. Capolupo and G. Vitiello, Adv. High Energy Phys. **2013**, 850395 (2013); Phys. Rev. D **88**, 024027 (2013); Adv. High Energy Phys. **2015**, 878043, (2015); A. Bruno, A. Capolupo, S. Kak, G. Raimondo and G. Vitiello, Mod. Phys. Lett. B **25**, 1661 (2011); J. Hu and H. Yu Phys. Rev. A **85**, 032105 (2012).

[27] A. Capolupo, Phys. Rev. D **84**, 116002 (2011); M. Blasone, A. Capolupo, E. Celeghini, G. Vitiello, Phys. Lett. B **674**, 73 (2009); S. Joshi, S. R. Jain, Phys. Lett. B **754**, 135 (2016); V.A. Naumov, Phys. Lett. B **323**, 351 (1994); N. Nakagawa, Ann. Phys. **179**, 145 (1987); L. Johns and G. M. Fuller. Phys. Rev. D **95**, 043003 (2017) A. Capolupo, G. Lambiase and G. Vitiello, Adv. High Energy Phys. **2015**, 826051 (2015).

[28] C. Giunti, Phys. Lett. B **686**, 41 (2010).

[29] S. M. Bilenky, J. Hosek and S. T. Petcov, Phys. Lett. B **94**, 495 (1980); M. Doi, T. Kotani, H. Nishiura, K. Okuda and E. Takasugi, Phys. Lett. B **102**, 323 (1981); J. Schechter and J. W. F. Valle, Phys. Rev. D **23**, 1666 (1981); P. Langacker, S. T. Petcov, G. Steigman and S. Toshev, Nucl. Phys. B **282**, 589 (1987).

[30] M. Peskin, D. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, 1995).

[31] F. Benatti, R. Floreanini, JHEP **02**, 32, (2000).

[32] Note that dissipative effects in neutrino oscillation in matter can be mostly neglected since they are of the order of E^2/M_P, with E, neutrino energy and M_P Plank Mass. For more details see Ref. [31].

[33] S.M. Bilenky, C. Giunti, W. Grimus, Prog. Part. Nucl. Phys. **43**, 1 (1999).

[34] J.W.F. Valle, . J.C. Romao, Neutrinos in high energy and astroparticle physics, Wiley-vch, ISBN: 978-3-527-41197-9 448 pages, (2015).

[35] T2K Collaboration; K. Abe, et al. Phys. Rev. Lett. **107** (4): 041801 (2011).

[36] P. K. Kabir, *The CP Puzzle*, Academic Press, London (1968); O. Nachtmann, *Elementary Particle Physics: Concepts and Phenomena*, Springer, Berlin (1990).

[37] S.M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978).

[38] A. Lenz and U. Nierste, J. High Energy Phys. **06**, 072 (2007).

[39] M. Fidecaro and H. J. Gerber, Rept. Prog. Phys. **69**, 1713, (2006), [(2006) Erratum-ibid. **69**, 2841] and references therein.

[40] M. Blasone, A. Capolupo, G. Vitiello, Phys. Rev. D **66**, 025033 (2002) and references therein; M. Blasone, A. Capolupo, O. Romei and G. Vitiello, Phys. Rev. D **63**, 125015 (2001).

[41] A. Capolupo, C. R. Ji , Y. Mishchenko and G. Vitiello, Phys. Lett. B **594**, 135 (2004).

[42] M. Blasone, A. Capolupo, F. Terranova, G. Vitiello, Phys. Rev. D **72**, 013003 (2005); M. Blasone, A. Capolupo, C. -R. Ji, G. Vitiello, Int. J. Mod. Phys. A **25**, 4179 (2010).

[43] A. Capolupo, Adv. High Energy Phys. **2016**, 8089142 (2016); A. Capolupo, S. Capozziello and G. Vitiello, Phys. Lett. A **373**, 601 (2009); Phys. Lett. A **363**, 53 (2007); Int. J. Mod. Phys. A **23**, 4979 (2008); M. Blasone, A. Capolupo, S. Capozziello, G. Vitiello, Nucl. Instrum. Meth. A **588**, 272 (2008); M. Blasone, A. Capolupo, G. Vitiello, Prog. Part. Nucl. Phys. **64**, 451 (2010); M. Blasone, A. Capolupo, S. Capozziello, S. Carloni and G. Vitiello, Phys. Lett. A **323**, 182 (2004).

[44] A. Capolupo, M. Di Mauro, A. Iorio, Phys. Lett. A **375**, 3415 (2011); A. Capolupo and M. Di Mauro, Acta Phys. Polon. B **44**, 81 (2013).