Lax Representations for Matrix Short Pulse Equations

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Abstract

The Lax representation for different matrix generalizations of Short Pulse Equations (SPE) is considered. The four-dimensional Lax representations of four-component Matsuno, Feng and Dimakis-Müller-Hoissen-Matsuno equations is obtained. The four-component Feng system is defined by generalization of the two-dimensional Lax representation to the four-component case. This system reduces to the original Feng equation or to the two-component Matsuno equation or to the Yao-Zang equation. The three component version of Feng equation is presented. The four-component version of Matsuno equation with its Lax representation is given . This equation reduces the new two-component Feng system. The two-component Dimakis-Müller-Hoissen-Matsuno equations are generalized to the four parameter family of the four-component SPE. The bi-Hamiltonian structure of this generalization, for special values of parameters, is defined. This four-component SPE in special case reduces to the new two-component SPE.

1 Introduction

The short pulse equation (SPE)

$$u_{x,t} = u + \frac{1}{6}(u^3)_{xx}, \tag{1}$$

derived by Schäfer and Wayne [1] as a model of ultra-short optical pulses in nonlinear media has attracted considerable attention. For the first time the SPE has appeared in an attempt to construct integrable differential equations associated with pseudo spherical surfaces [2]. The associated linear scattering problem appeared for the first time in differential geometry [2, 3].

The integrability of the SPE have been studied from various points of view [4, 5, 6]. More specifically it was shown that SPE equation admits a Lax pair and possess a bi-Hamiltonian formulation.

The problem of mathematical description of the propagation ultra-short pulses has been considered from different points of view. For example, the SPE was generalized in different manners to the 2-component systems. The first generalization have been considered by Pietrzyk Kanattsikov and Bandelow in the form [7].

$$u_{xt} = u + \frac{1}{6}(u^3 + 3uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6}(v^3 + 3u^2v)_{xx}, \tag{2}$$

$$u_{xt} = u + \frac{1}{6}(u^3 - 3uv^2)_{xx}, \quad v_{xt} = v - \frac{1}{6}(v^3 - 3u^2v)_{xx},$$

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad v_{xt} = v + \frac{1}{2}(u^2v)_{xx}.$$

Sakovich [8] presented another 2-component generalization

$$u_{xt} = u + \frac{1}{6}(u^3 + uv^2)_{xx}, \quad v_{xt} = v + \frac{1}{6}(v^3 + u^2v)_{xx}, \tag{3}$$

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \quad v_{xt} = v + \frac{1}{6}(u^2v)_{xx}.$$

As was shown, these generalizations are integrable and possess Lax representation and bi-Hamiltonian formulation.

Recently Matsuno [9, 10] generalized, in two different manners, the SPE to the n-component case as

$$u_{i,xt} = u_i + \frac{1}{2}(Fu_{i,x})_x, \quad F = \frac{1}{2}\sum_{1 \le j < k \le n} c_{j,k}u_ju_k, \tag{4}$$

$$u_{i,xt} = u_i + (Fu_{i,x})_x - \frac{1}{2}Gu_i, \quad G = \sum_{1 \le j,k \le n} c_{j,k}u_{j,x}u_{k,x}. \tag{5}$$

where $c_{j,k}$ are arbitrary constants such as $c_{j,k} = c_{k,j}$.

For these equations Matsuno applied the bilinear method and presented multi-solitons solutions in the parametric form. Moreover, Matsuno found the Lax representation, local and nonlocal conservation laws for the equation 5, for special case $n = 2, u_1 = u, u_2 = v$ and $c_{11} = c_{22} = 0, c_{12} = 1$

$$u_{x,t} = u + \frac{1}{2}v(u^2)_{xx}, \quad v_{x,t} = v + \frac{1}{2}u(v^2)_{xx}. \tag{6}$$

Matsuno defined the zero curvature condition for the equation 6 as $\mathbf{X}_t - \mathbf{Z}_x + [\mathbf{X}, \mathbf{Z}] = 0$ where

$$\mathbf{X} = \lambda \begin{pmatrix} 1 - u_xv_x & 2u_x \\ 2v_x & -1 + u_xv_x \end{pmatrix} \tag{7}$$

$$\mathbf{Z} = \begin{pmatrix} \lambda uv(1 - u_xv_x) + \frac{1}{4\lambda} & 2\lambda u_x uv - u \\ 2\lambda u_x uv + v & -\lambda uv(1 - u_xv_x) - \frac{1}{4\lambda} \end{pmatrix}.$$

Dimakis and Müller-Hoissen [11] studied matrix generalization of the SPE in the form

$$U_{xt} = U + \frac{1}{2}(U^2 U_x)_x, \tag{8}$$

where U is matrix valued function. Assuming that U^2 has to be a scalar times the identity matrix, Dimakis and Müller-Hoissen were able to construct the Lax pair. When $U^2 = (u^2 + v^2)I$, where I is an identity matrix, the equation (8) reduces, after the transformations of variables $(u, v) \to ((u+v)/2, (u-v)/2i)$, to

$$u_t = \partial^{-1}u + \frac{1}{2}uvu_x, \qquad v_t = \partial^{-1}v + \frac{1}{2}uvv_x. \tag{9}$$

This equation, as it was shown by Matsuno [9], is a special case of the system (4) when $n = 2, c_{1,1} = c_{2,2} = 0, c_{1,2} = 1, u_1 = u, u_2 = v$.

For this equation, Brunelli and Sakovich found [12] the bi-Hamiltonian formulation

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = K \begin{pmatrix} H_{1,u} \\ H_{1,v} \end{pmatrix} = J \begin{pmatrix} H_{0,u} \\ H_{0,v} \end{pmatrix}, \tag{10}$$

where

$$H_1 = \int dx[v(\partial^{-1}u) + \frac{1}{8}u^2(v^2)_x], \qquad H_0 = \frac{1}{2}\int dx\ uv, \tag{11}$$

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} u_x\partial^{-1}u_x & 2\partial^{-1} + u_x\partial^{-1}v_x \\ 2\partial^{-1} + v_x\partial^{-1}u_x & v_x\partial^{-1}v_x \end{pmatrix},$$

and Matsuno [9] discovered the Lax representation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \lambda \begin{pmatrix} 1 & u_x \\ v_x & -1 \end{pmatrix}, \qquad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} \frac{\lambda}{2}uv + \frac{1}{4\lambda} & \frac{\lambda}{2}uvu_x - \frac{1}{2}u \\ \frac{\lambda}{2}uvv_x + \frac{1}{2}v & -\frac{\lambda}{2}uv - \frac{1}{4\lambda} \end{pmatrix}. \tag{12}$$

Quite different generalization of two-component SPE was proposed by Feng [13]

$$u_{x,t} = u + uu_x^2 + \frac{1}{2}(u^2 + v^2)u_{xx}, \qquad v_{x,t} = v + vv_x^2 + \frac{1}{2}(u^2 + v^2)v_{xx}. \tag{13}$$

Brunelli and Sakovich in [12] obtained the zero curvature condition $X_t - T_t + [X, T] = 0$ and the bi-Hamiltonian structure for this equation, where

$$X = \lambda \begin{pmatrix} 1 + u_xv_x & u_x - v_x \\ u_x - v_x & -1 - u_xv_x \end{pmatrix} \tag{14}$$

$$T = \begin{pmatrix} \frac{\lambda}{2}(u^2 + v^2)(1 + u_xv_x) + \frac{1}{4\lambda} & \frac{\lambda}{2}(u^2 + v^2)(u_x - v_x) - \frac{1}{2}(u - v) \\ \frac{\lambda}{2}(u^2 + v^2)(u_x - v_x) + \frac{1}{2}(u - v) & -\frac{\lambda}{2}(u^2 + v^2)(1 + u_xv_x) - \frac{1}{\lambda} \end{pmatrix}.$$

The last two-component generalizations of SPE were given by Yao-Zeng [14]

$$u_{x,t} = u + \frac{1}{6}(u^3)_{xx}, \qquad v_{x,t} = v + \frac{1}{2}(u^2v_x)_x \tag{15}$$

for which Brunelli and Sakovich found, using the four-dimensional matrices, the Lax representation in [12].

In this paper we would like to study several different generalizations of the two-component SPE to the matrix and then to the four-component case. Therefore we investigate the equations obtained from the matrix generalization of the Lax representations eqs. (7,12,14).

The paper is organized as follows. In the following two chapters we study the matrix version of the Lax representation of Matsuno and Feng equations. The four-component Matsuno equation is discussed in the first chapter too, and its reduction to the new version of the two-component Feng equation is given. In the third section, we study the four-component version of Feng equation and its different reductions. This generalization describes the interaction between the Feng, Matsuno and Yao-Zeng equations and reduces to the original Feng equation or to the two-component Matsuno equation or to the Yao-Zeng system. Also, the three-component version of Feng equation is obtained as a result of reduction. In the last section we study the matrix Lax representation of the two-component Dimakis-Müller-Hoissen - Matsuno equation. This representation in the case of four-dimensional matrices produces the four parameters family of four-component SPE. For special values of free parameters we defined the bi-Hamiltonian structure for this equation. As a result of reduction of this four parameter equations, we obtained a new two-component version of SPE. The last section is a conclusion. The paper contains two appendixes.

2 Lax Representation of Matrix Matsuno Equation

Let us consider the Lax representation for matrix Matsuno equation

$$\Pi_x = \mathbf{X}\Pi, \qquad \Pi_t = \mathbf{T}\Pi, \qquad \Pi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad (16)$$

$$\mathbf{X} = \lambda \begin{pmatrix} I - U_x V_x & 2U_x \\ 2V_x & -I + U_x V_x \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} \lambda(I - U_x V_x)UV + I\frac{1}{4\lambda} & 2\lambda UVU_x - U \\ 2\lambda UVV_x + V & -\lambda(I - U_x V_x)UV - I\frac{1}{4\lambda} \end{pmatrix},$$

where I is identity matrix, U and V are arbitrary dimensional matrices such that UV and $U_x V_x$ are scalar functions multiplied by identity matrix.

From the integrability condition $\Pi_{x,t} = \Pi_{t,x}$ we obtained the following matrix equations

$$U_{t,x} = U + V(UU_x)_x + V_x(UU_x - U_x U), \qquad (17)$$
$$V_{t,x} = V + V(UV_x)_x + V_x(UV_x - U_x V).$$

These equations reduce to the Matsuno eq.(6) when U, V are scalar functions.

The four-component version of the Matsuno equation are concluded from eq.(17) assuming that

$$U = \frac{1}{2}\big((v_0 + v_1)I - i(v_2 + v_3)\sigma_1 - (v_2 - v_3)\sigma_2 - (v_1 - v_0)\sigma_3\big), \qquad (18)$$

$$V = \frac{1}{2}\big((v_0 + v_1)I + i(v_2 + v_3)\sigma_1 + (v_2 - v_3)\sigma_2 + (v_1 - v_0)\sigma_3\big).$$

where σ_i are Pauli matrices and I is identity matrix.

$$v_{0,t,x} = v_0 + \frac{1}{2}v_1(v_0^2)_{xx} + v_2v_3v_{0,xx} + v_{0,x}(v_2v_3)_x - v_0v_{2,x}v_{3,x}, \quad (19)$$

$$v_{1,t,x} = v_1 + \frac{1}{2}v_0(v_1^2)_{xx} + v_2v_3v_{1,xx} + v_{1,x}(v_2v_3)_x - v_1v_{2,x}v_{3,x},$$

$$v_{2,t,x} = v_2 + \frac{1}{2}v_3(v_2^2)_{xx} + v_0v_1v_{2,xx} + v_{2,x}(v_0v_1)_x - v_2v_{0,x}v_{1,x},$$

$$v_{3,t,x} = v_3 + \frac{1}{2}v_2(v_3^2)_{xx} + v_0v_1v_{3,xx} + v_{3,x}(v_0v_1)_x - v_3v_{0,x}v_{1,x}.$$

The system eq.(19) allows interesting reduction to the two-component case when $v_1 = v_0 = \frac{1}{2}(f+g), \quad v_3 = v_2 = \frac{1}{2}(f-g)$

$$f_{t,x} = f + f_x^2 f + \frac{1}{2}(f^2+g^2)f_{xx} - \frac{1}{2}(f_x^2+g_x^2)f + f_xg_xg, \quad (20)$$

$$g_{t,x} = g + g_x^2 g + \frac{1}{2}(f^2+g^2)g_{xx} - \frac{1}{2}(f_x^2+g_x^2)g + f_xg_xf,$$

and describes the Feng fields plus additional interaction.

3 Lax Representation of Matrix Feng Equation

We postulate the Lax representation for the matrix Feng equation as

$$\Pi_x = \mathbf{X}\Pi, \qquad \Pi_t = \mathbf{T}\Pi, \qquad \Pi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (21)$$

$$\mathbf{X} = \lambda \begin{pmatrix} I + U_xV_x & U_x - V_x \\ U_x - V_x & -I - U_xV_x \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} \frac{1}{2}\lambda(U^2+V^2)(I+U_xV_x) + I\frac{1}{4\lambda} & \frac{\lambda}{2}(U^2+V^2)(U_x-V_x) + \frac{1}{2}(V-U) \\ \frac{\lambda}{2}(U^2+V^2)(U_x-V_x) + \frac{1}{2}(U-V) & -\frac{1}{2}\lambda(U^2+V^2)(I+U_xV_x) - I\frac{1}{4\lambda} \end{pmatrix},$$

where I is identity matrix, U and V are arbitrary matrices such that UV and U_xV_x are scalar functions multiplied by the identity matrix.

The matrix Feng equation is obtained from the integrability condition $\Pi_{x,t} = \Pi_{t,x}$ and reads

$$U_{t,x} = U + \frac{1}{2}\big(((U^2+V^2)U_x)_x + (UU_x - U_xU - U_xV - VU_x)V_x\big), \quad (22)$$

$$V_{t,x} = V + \frac{1}{2}\big(V_{xx}(U^2+V^2) + (V-U)V_x^2 + V_x(V-U)V_x\big).$$

When U and V are scalar function, then eq.(22) reduces to the Feng equation (13).

In the next section we show that this system, in the case where we consider the two dimensional matrices U and V, contains the Feng, Matsuno and Yao-Zheng equations.

4   Unification of Feng, Matsuno and Yao - Zeng equations.

The four-component Feng equation is concluded from eq.(22) assuming that

$$U = \frac{1}{2}\big((v_0 + v_1)I - (v_2 + v_3)\sigma_1 + i(v_2 - v_3)\sigma_2 - (v_1 - v_0)\sigma_3\big), \qquad (23)$$

$$V = \frac{1}{2}\big((v_0 + v_1)I + (v_2 + v_3)\sigma_1 - i(v_2 - v_3)\sigma_2 + (v_1 - v_0)\sigma_3\big),$$

where σ_i are Pauli matrices and I is identity matrix.

$$
\begin{aligned}
v_{0,t,x} &= v_0 + v_{0,x}^2 v_0 + \frac{1}{2}(v_0^2 + v_1^2)v_{0,xx} && + (v_{0,x}v_2 v_3)_x + v_1 v_{2,x} v_{3,x}, && (24)\\
v_{1,t,x} &= v_1 + v_{1,x}^2 v_1 + \frac{1}{2}(v_0^2 + v_1^2)v_{1,xx} && + (v_{1,x}v_2 v_3)_x + v_0 v_{2,x} v_{3,x}, \\
v_{2,t,x} &= v_2 + v_3(v_2^2)_{xx} && + \frac{1}{2}\big(v_{2,x}(v_0^2 + v_1^2)\big)_x + v_{0,x}v_{1,x}v_2, \\
v_{3,t,x} &= v_3 + v_2(v_3^2)_{xx} && + \frac{1}{2}\big(v_{3,x}(v_0^2 + v_1^2)\big)_x + v_{0,x}v_{1,x}v_3,
\end{aligned}
$$

We see that this system of equations describes the interaction between Feng fields $\{v_0, v_1\}$ and Matsuno fields $\{v_2, v_3\}$. For this reasons the system of equations could be considered as the interacting system of Feng - Matsuno type.

The Matrices $\mathbf{X}$ and $\mathbf{T}$ in the Lax representation eq.(21) are rewritten in terms of the 16 dimensional Lie algebra, spanned by the generators $\{e_1, e_2, \ldots, e_{16}\}$ as

$$
\begin{aligned}
\mathbf{X} &= \lambda(1 + v_{0,x}v_{1,x} - v_{2,x}v_{3,x})(e_1 + e_2) + && (25)\\
&\quad \lambda(v_{0,x} - v_{1,x})(e_3 + e_4) + 2v_{2,x}e_5 + 2v_{3,x}e_6,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T} &= \lambda\gamma(1 + v_{0,x}v_{1,x} - v_{2,x}v_{3,x})(e_1 + e_2) + \frac{1}{4\lambda}(e_1 + e_2) + \lambda\gamma(v_{0,x} - v_{1,x})(e_3 + e_4) + \\
&\quad 2\lambda\gamma v_{2,x}e_5 + 2\lambda\gamma v_{3,x}e_6 + \frac{1}{2}(v_0 - v_1)(e_7 + e_8) + v_2 e_9 + v_3 e_{10},
\end{aligned}
$$

where $\gamma = \frac{1}{2}(v_0^2 + v_1^2 + 2v_2 v_3)$,

The explicit representation of the Lie algebra $e_e, e_2, \ldots e_{16}$ is given in the appendix A.

The system of equations (24) allows three different reductions to the two-component case.

When $v_2 = v_0 = \frac{1}{2}(a + b)$, $v_3 = v_1 = \frac{1}{2}(a - b)$ then eq.(24) reduces to the Yao-Zeng eq.(15) in the variables a, b.

For this type of reduction the matrices $\mathbf{X}, \mathbf{T}$ are

$$
\begin{aligned}
\mathbf{X} &= \lambda(e_1 + e_2) + \lambda a_x(e_5 + e_6) + \lambda b_x(e_3 + e_4 + e_5 - e_6), && (26)\\
\mathbf{T} &= \frac{1}{4\lambda}(2a^2 + 1)(e_1 + e_2) + \frac{\lambda}{2}a^2 a_x(e_5 + e_6) + \frac{1}{2}a(e_9 + e_{10}) \\
&\quad -\frac{\lambda}{2}a^2 b_x(e_3 + e_4 + e_5 - e_6) - \frac{1}{2}b(e_7 + e_8 + e_9 - e_{10}).
\end{aligned}
$$

This representation is different than the one given in [12].

When $v_1 = v_0 = \frac{1}{2}(a+b), v_3 = v_2 = \frac{1}{2}(a-b)$ then eq.(24) reduces to the Feng eq.(13).

When $v_3 = v_0 = \frac{1}{2}(a+b), v_2 = v_1 = \frac{1}{2}(a-b)$ then eq.(24) reduces to the Yao-Zeng eq.(15) again but now we have different representations of the matrices $\mathbf{X}, \mathbf{T}$

$$
\begin{aligned}
\mathbf{X} &= \lambda(e_1 + e_2) + \lambda a_x(e_5 + e_6) + \lambda b_x(e_3 + e_4 - e_5 + e_6), \quad (27)\\
\mathbf{T} &= \frac{\lambda}{2}a^2(e_1 + e_2) + \frac{1}{4\lambda}(e_1 + e_2) + \frac{\lambda}{2}a^2 a_x(e_5 + e_6) + \frac{1}{2}a(e_{10} + e_9)\\
&\quad \frac{\lambda}{2}a^2 b_x(e_3 + e_4 - e_5 + e_6) + \frac{1}{2}b(e_7 + e_8 - e_9 + e_{10}).
\end{aligned}
$$

The system (24) allows the reduction to the three component case as well. For example, assuming that

$$
\begin{aligned}
v_0 &= a + b + c, \quad v_1 = a - b + c \quad (28)\\
v_2 &= a + b - c, \quad v_3 = v_0 - v_1 - v_2
\end{aligned}
$$

then equations (24) reads as

$$
\begin{aligned}
a_{t,x} &= a + 2a_{xx}(2ac + b^2) + 2c(a_x^2 + b_x^2 - c_x^2) + 4a_x b_x b + 4a_x c_x a, \quad (29)\\
b_{t,x} &= b + 2b_{xx}(2ac + b^2) + 2b(a_x^2 + b_x^2 + c_x^2) + 4a_x b_x c + 4b_x c_x a,\\
c_{t,x} &= c + 2c_{xx}(2ac + b^2) + 2a(b_x^2 + c_x^2 - a_x^2) + 4a_x c_x c + 4b_x c_x b.
\end{aligned}
$$

For this type of reduction the matrices $\mathbf{X}, \mathbf{T}$ are

$$
\begin{aligned}
\mathbf{X} &= \lambda\big(1 + 2(a_x^2 - b_x^2 + c_x^2)\big)(e_1 + e_2) + 2\lambda(a_x - c_x)(e_5 - e_6) \quad (30)\\
&\quad + 2\lambda b_x(e_3 + e_4 + e_5 + e_6),\\
\mathbf{T} &= 2\lambda(2ac + b^2)\big(1 + 2(a_x^2 - b_x^2 + c_x^2)\big)(e_1 + e_2) + \frac{1}{4\lambda}(e_1 + e_2)\\
&\quad (c - a)(e_{10} - e_9) + b(e_7 + e_8 + e_9 + e_{10}) + 4\lambda(2ac + b^2)b_x(e_3 + e_4 + e_5 + e_6)\\
&\quad + 4\lambda(c_x - a_x)(2ac + b^2)(e_6 - e_5).
\end{aligned}
$$

5 Lax representation of Matrix Dimakis-Müller-Hoissen-Matsuno Equation

Now let us consider the matrix generalization of Lax pair representation eq.(12)

$$
\Psi_x = \Omega\Psi = \xi \begin{pmatrix} I & U_x \\ V_x & -I \end{pmatrix}\Psi, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (31)
$$

$$
\Psi_t = P\Psi = \begin{pmatrix} \frac{\xi}{2}UV + I\frac{1}{4\xi} & -\frac{1}{2}U + \frac{\xi}{4}(U_x VU + UVU_x) \\ \frac{1}{2}V + \frac{\xi}{4}(V_x UV + VUV_x) & -\frac{1}{2}VU - I\frac{1}{4\xi} \end{pmatrix}\Psi.
$$

where I is an N dimensional identity matrix and $U(x,t), V(x,t)$ are at the moment arbitrary N and dimensional matrices.

The zero-curvature condition $\Omega_t - P_x + [\Omega, P] = 0$ produces the following equations on U, V

$$
\begin{aligned}
U_{tx} &= U + \frac{1}{4}(U_x V U + U V U_x)_x, \\
V_{tx} &= V + \frac{1}{4}(V_x U V + V U V_x)_x,
\end{aligned}
\tag{32}
$$

with the constraints on U, V

$$
[U_x V_x, UV] = 0, \qquad [V_x U_x, VU] = 0
\tag{33}
$$

The equations (32) with the constraints 33 are our matrix generalization of two-component Dimakis-Müller-Hoissen-Matsuno equation different than the one proposed in [11].

Now we consider the case where U and V are two-dimensional matrices. We parametrize U by two functions u_1, u_2 and V by two functions u_3, u_4 as

$$
U = \begin{pmatrix} u_1 & u_2 \\ s_1 u_1 + s_2 u_2 & s_3 u_1 + s_4 u_2 \end{pmatrix}, \quad V = \begin{pmatrix} u_3 & u_4 \\ z_1 u_3 + z_2 u_4 & z_3 u_3 + z_4 u_4 \end{pmatrix}
\tag{34}
$$

where $s_i, z_i, i = 1, 2, 3, 4$ are an arbitrary constants. [1]

Substituting U, V defined by eq.(34) to the equations (33), we find the connection between z_i and s_i

$$
z_1 = -\frac{s_1}{s_3}, \quad z_2 = \frac{s_2 s_3 - s_1 s_4}{s_3}, \quad z_3 = \frac{1}{s_3}, \quad z_4 = \frac{s_4}{s_3}
\tag{35}
$$

Now the Lax representation generates four parameters family of equations

$$
\begin{aligned}
u_{1,t} &= (\partial^{-1} u_1) + \frac{1}{4s_3}\Big(u_3(s_3 u_1^2 + s_2 u_2^2)_x + u_4(s_1 s_3 u_1^2 + s_2 s_4 u_2^2 + 2s_2 s_3 u_1 u_2)_x\Big), \\
u_{2,t} &= (\partial^{-1} u_2) + \frac{1}{4s_3}\Big(u_4\big(s_3^2 u_1^2 + (s_2 s_3 + s_4^2 - s_1 s_4)u_2^2 + 2s_3 s_4 u_1 u_2\big)_x \\
&\qquad\qquad + u_3\big((s_4 - s_1)u_2^2 + 2s_3 u_1 u_2\big)_x\Big), \\
u_{3,t} &= (\partial^{-1} u_3) + \frac{1}{4}\Big(u_1\big(u_3^2 + (s_2 s_3 - s_1 s_4)u_4^2\big)_x\Big) \\
&\qquad + \frac{1}{4s_3}\Big(u_2\big(-s_1 u_3^2 + (s_2 s_3 - s_1 s_4)(s_4 u_4^2 + 2u_3 u_4)\big)_x\Big), \\
u_{4,t} &= (\partial^{-1} u_4) + \frac{1}{4}\Big(u_1\big((s_1 + s_4)u_4^2 + 2u_3 u_4\big)_x + \frac{1}{s_3}u_2\big(u_3^2 + (s_2 s_3 + s_4^2)u_4^2 + 2s_4 u_3 u_4\big)_x\Big).
\end{aligned}
\tag{36}
$$

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[1] In general it is possible to parametrize the two dimensional matrix by two arbitrary functions in six different manners. However, all these parametrizations could be obtained from eq.(34) using the linear transformations of these functions.

These equations are integrable in the sense that they possess the Lax representation and allow several different reduction to the one or two-component SPE equation case.

When $u_4 = u_3 = u_2 = u_1 = u, s_1 = -s_2, s_4 = -s_3 - 1$ the equations (36) reduce to the original SPE equation (1).

For $u_1 = u_2 = u, u_3 = u_4 = v, s_1 = 1 - s_2, s_4 = s_3 - 1$ the equations (36) reduce to the two-component Dimakis-Müller-Hoissen-Matsuno (9).

When $u_1 = u_3 = u, u_2 = u_4 = v, s_1 = s_4 = 0, s_3 = 1$ and $s_2 = \pm 1, 0$ we obtained three generalizations of the SPE equation considered by Pietrzyk at.al.

On the other hand for $u_1 = u_3 = u, u_2 = u_4 = v, s_1 = 0, s_2 = 1, s_3 = 1, s_4 = -2$ we obtained new two-component generalization of the SPE equation

$$u_{t,x} = u + \frac{1}{2}\left(u_x(u^2 + v^2) - (v^2)_x(u - v)\right) \tag{37}$$

$$v_{t,x} = v + u_x(vu - v^2) + \frac{1}{2}v_x(u^2 + 5v^2 - 4uv)$$

This equation was missing in the classification of Pietrzyk at.all [7]. For this equation the Lax representation is exactly as in the paper [7]

$$\mathbf{X} = \lambda \begin{pmatrix} I & \mathbf{U}_x \\ \mathbf{U}_x & -I \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} \frac{\lambda}{2}\mathbf{U}^2 + \frac{1}{4\lambda}I & \frac{\lambda}{6}(\mathbf{U}^3)_x - \frac{1}{2}\mathbf{U} \\ \frac{\lambda}{6}(\mathbf{U}^3)_x + \frac{1}{2}\mathbf{U} & -\frac{\lambda}{2}\mathbf{U}^2 - \frac{1}{4\lambda}I \end{pmatrix}, \tag{38}$$

but now

$$\mathbf{U} = \begin{pmatrix} u & v \\ v & u - 2v \end{pmatrix}. \tag{39}$$

We found the bi-Hamiltonian formulations for the equations (36) for $s_1 = 0, s_2 = 1$.

$$u_{1,t} = (\partial^{-1}u_1) + \frac{1}{4s_3}\left(u_3(s_3u_1^2 + u_2^2)_x + u_4(s_4u_2^2 + 2s_3u_1u_2)_x\right), \tag{40}$$

$$u_{2,t} = (\partial^{-1}u_2) + \frac{1}{4s_3}\left(u_4\left(s_3^2u_1^2 + (s_3 + s_4^2)u_2^2 + 2s_3s_4u_1u_2\right)_x + u_3\left(s_4u_2^2 + 2s_3u_1u_2\right)_x\right),$$

$$u_{3,t} = (\partial^{-1}u_3) + \frac{1}{4}(u_1\left(u_3^2 + s_3u_4^2\right)_x + u_2(s_4u_4^2 + 2u_3u_4)_x),$$

$$u_{4,t} = (\partial^{-1}u_4) + \frac{1}{4}\left(u_1\left(s_4u_4^2 + 2u_3u_4\right)_x + \frac{1}{s_3}u_2\left(u_3^2 + (s_3 + s_4^2)u_4^2 + 2s_4u_3u_4\right)_x\right).$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = J_0 \begin{pmatrix} H_{1,u_1} \\ H_{1,u_2} \\ H_{1,u_3} \\ H_{1,u_4} \end{pmatrix} = J_1 \begin{pmatrix} H_{0,u_1} \\ H_{0,u_2} \\ H_{0,u_3} \\ H_{0,u_4} \end{pmatrix} \tag{41}$$

$$H_1 = -\int dx \left[u_1(\partial^{-1}u_3) + u_2(\partial^{-1}u_4)\right] + \frac{1}{8s_3}\Big((u_2^2 + s_3u_1^2)(u_3^2)_x + \tag{42}$$

$$(u_2^2(s_3 + s_4^2) + s_3u_1^2 + 2s_3s_4u_1u_2)(u_4^2)_x + (2s_4u_2^2 + 4s_3u_1u_2)(u_3u_4)_x\Big)$$

$$H_0 = \int dx \; (u_1u_3 + u_2u_4)$$

$$J_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{43}$$

$$J_1 = \begin{pmatrix} 0 & 0 & \partial^{-1} & 0 \\ 0 & 0 & 0 & \partial^{-1} \\ \partial^{-1} & 0 & 0 & 0 \\ 0 & \partial^{-1} & 0 & 0 \end{pmatrix} + \tag{44}$$

$$\frac{s_4}{2s_3}\begin{pmatrix} 0 & u_{2,x}\partial^{-1}u_{2,x} & 0 & u_{2,x}\partial^{-1}u_{2,x} \\[1em] u_{2,x}\partial^{-1}u_{2,x} & \begin{matrix} u_{2,x}\partial^{-1}(s_4u_{2,x} + s_3u_{1,x}) \\ +s_3u_{1,x}\partial^{-1}u_{2,x} \end{matrix} & s_3u_{2,x}\partial^{-1}u_{4,x} & \begin{matrix} u_{2,x}\partial^{-1}(s_4u_{4,x} + u_{3,x}) \\ +s_3u_{1,x}\partial^{-1}u_{4,x} \end{matrix} \\[1em] 0 & s_3u_{4,x}\partial^{-1}u_{2,x} & 0 & s_3u_{4,x}\partial^{-1}u_{4,x} \\[1em] u_{4,x}\partial^{-1}u_{2,x} & \begin{matrix} u_{4,x}\partial^{-1}(s_4u_{2,x} + s_3u_{1,x}) \\ +u_{3,x}\partial^{-1}u_{2,x} \end{matrix} & s_3u_{4,x}\partial^{-1}u_{4,x} & \begin{matrix} u_{4,x}\partial^{-1}(s_4u_{4,x} + u_{3,x}) \\ +u_{3,x}\partial^{-1}u_{3,x} \end{matrix} \end{pmatrix} +$$

$$\frac{1}{2s_3}\begin{pmatrix} \begin{matrix} u_{2,x}\partial^{-1}u_{2,x} \\ +s_3u_{1,x}\partial^{-1}u_{1,x} \end{matrix} & \begin{matrix} s_3u_{2,x}\partial^{-1}u_{1,x} \\ +s_3u_{1,x}\partial^{-1}u_{1,x} \end{matrix} & \begin{matrix} s_3u_{2,x}\partial^{-1}u_{4,x} \\ +s_3u_{1,x}\partial^{-1}u_{2,x} \end{matrix} & \begin{matrix} u_{2,x}\partial^{-1}u_{2,x} \\ +s_3u_{1,x}\partial^{-1}u_{4,x} \end{matrix} \\[1em] \begin{matrix} s_3u_{1,x}\partial^{-1}u_{2,x} \\ +s_3u_{1,x}\partial^{-1}u_{2,x} \end{matrix} & \begin{matrix} s_3u_{2,x}\partial^{-1}u_{2,x} \\ +s_3^2u_{1,x}\partial^{-1}u + 1, x \end{matrix} & \begin{matrix} s_3u_{2,x}\partial^{-1}u_{3,x} \\ +u_{1,x}\partial^{-1}u_{4,x} \end{matrix} & \begin{matrix} s_3u_{2,x}\partial^{-1}u_{4,x} \\ +s_3u_{1,x}\partial^{-1}u_{3,x} \end{matrix} \\[1em] \begin{matrix} s_3u_{4,x}\partial^{-1}u_{2,x} \\ +s_3u_{3,x}\partial^{-1}u_{1,x} \end{matrix} & \begin{matrix} s_3u_{3,x}\partial^{-1}u_{2,x} \\ +s_3u_{4,x}\partial^{-1}u_{1,x} \end{matrix} & \begin{matrix} s_3u_{3,x}\partial^{-1}u_{3,x} \\ +s_3u_{4,x}\partial^{-1}u_{4,x} \end{matrix} & \begin{matrix} s_3u_{4,x}\partial^{-1}u_{3,x} \\ +s_3u_{3,x}\partial^{-1}u_{4,x} \end{matrix} \\[1em] \begin{matrix} s_3u_{4,x}\partial^{-1}u_{1,x} \\ +s_3u_{3,x}\partial^{-1}u_{2,x} \end{matrix} & \begin{matrix} s_3u_{4,x}\partial^{-1}u_{2,x} \\ +s_3u_{3,x}\partial^{-1}u_{1,x} \end{matrix} & \begin{matrix} s_3u_{4,x}\partial^{-1}u_{3,x} \\ +s_3u_{3,x}\partial^{-1}u_{2,x} \end{matrix} & \begin{matrix} s_3u_{4,x}\partial^{-1}u_{3,x} \\ +s_3u_{3,x}\partial^{-1}u_{3,x} \end{matrix} \end{pmatrix}$$

The proof that this operator satisfies the Jacobi identity is postponed to the appendix A.

Moreover we showed that the Hamiltonians operators J_0 and J_1 are compatible. It means that $\mu J_0 + J_1$ is also the Hamiltonian operator, where μ is an arbitrary constant.

$$U = \begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 \end{pmatrix} \tag{45}$$

The equation (46) for $s_4 = -2, s_3 = 1$ possesses very nice bi-Hamiltonian structure

$$u_{1,t} = (\partial^{-1}u_1) + \frac{1}{4}u_3(u_1^2 + u_2^2)_x + \frac{1}{2}u_4(-u_2^2 + u_1u_2)_x,$$

$$u_{2,t} = (\partial^{-1}u_2) + \frac{1}{4}u_4(u_1^2 + 5u_2^2 - 4u_1u_2)_x + \frac{1}{2}u_3(-u_2^2 + u_1u_2)_x,$$

$$u_{3,t} = (\partial^{-1}u_3) + \frac{1}{4}u_1(u_3^2 + u_4^2)_x + \frac{1}{2}u_2(-u_4^2 + u_3u_4)_x,$$

$$u_{4,t} = (\partial^{-1}u_4) + \frac{1}{4}u_2(u_3^2 + 5u_4^2 - 4u_3u_4)_x + \frac{1}{2}u_1(-u_4^2 + u_3u_4)_x.$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_{1,u_1} \\ H_{1,u_2} \\ H_{1,u_3} \\ H_{1,u_4} \end{pmatrix} = J \begin{pmatrix} H_{0,u_1} \\ H_{0,u_2} \\ H_{0,u_3} \\ H_{0,u_4} \end{pmatrix}, \tag{46}$$

$$H_1 = -\int dx \left[u_3(\partial^{-1}u_1) + u_4(\partial^{-1}u_2) \right] + \frac{1}{8}\left((u_1^2)_x(u_3^2 + u_4^2) + \right. \tag{47}$$

$$\left. (u_2^2)_x(u_3^2 + 5u_4^2 - 4u_3u_4) + (u_1u_2)_x(u_3u_4 - u_4^2) \right)$$

$$H_0 = \int dx \, (u_1u_3 + u_2u_4), \tag{48}$$

$$J = \begin{pmatrix} U\partial^{-1}U & \partial^{-1}I + U\partial^{-1}V \\ \partial^{-1}I + V\partial^{-1}U & V\partial^{-1}V \end{pmatrix} \tag{49}$$

,

$$U = \begin{pmatrix} u_1 & u_2 \\ u_2 & u_1 - 2u_2 \end{pmatrix}, \quad V = \begin{pmatrix} u_3 & u_4 \\ u_4 & u_3 - 2u_4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

6   Conclusion

In this paper we studied several different generalizations of the two-component SPE to the matrix case and then to the four-component case. In particular we studied four-component version of Feng, Matsuno and Dimakis-Müller-Hoissen-Matsuno equations with its different reductions. The four-component Feng equation is very reticular, because it contains the Feng, Matsuno and Yao-Zeng equations and thus unifies these equations.

The four parameters of four-component Dimakis-Müller-Hoissen-Matsuno equations were discussed. For special values of free parameters, we obtained the bi-Hamiltonian structure and presented new two-component SPE (37).

We obtained several new integrable equations in the sense that these possess the Lax representation and this open perspective for further investigations. For example the problem of finding the bi-Hamiltonian structures for other generalized SPE is still an open issue, similarly as the problem of presentation of their solutions.

7   Appendix A. Representation of $e_1, e_2, \ldots e_{16}$

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad e_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad e_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_7 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad e_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$e_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad e_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad e_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad e_{15} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$e_{16} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

8   Appendix B. Hamiltonian operators and verification of the Jacobi identity

.

We assume the most general form on the second Hamiltonian operator as $J_1 = J_0 + \hat{J}$ where

$$\hat{J}_{j,s} = \sum_{k,l=1}^{4} c_{j,s,k,l} u_{k,x} \partial^{-1} u_{l,x}, \qquad j \leq s \tag{50}$$

$$\hat{J}_{s,j} = -(\hat{J}_{j,s})^{\star} \qquad \qquad j > s,$$

$$J_0 = \partial^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{51}$$

where $c_{j,k,s,l}$ are arbitrary constants $j, k, s, l = 1 \ldots 4$ for $j \neq k$ and $c_{j,j,s,k} = c_{j,j,k,s}$.

Now we will investigate the equations obtained from the second Hamiltonian structure

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = J \begin{pmatrix} H_{0,u_1} \\ H_{0,u_2} \\ H_{0,u_3} \\ H_{0,u_4} \end{pmatrix} \tag{52}$$

and compare the solutions with the solutions obtained from Lax representation eq. (31). This fixes all coefficients $c_{i,j,k,l}$ and moreover restricts $s_1 = 0, s_2 = 1$.

We use traditional manner to verify the Jacobi identity [15]. In order to prove that the operator J_1, defined in (44), satisfies the Jacobi identity, we utilize the standard form of the Jacobi identity

$$Jacobi = \int dx A J^{\star}_{1_{J_1(B)}} C + cyclic(A, B, C) = 0, \tag{53}$$

where A, B and C are the test vector functions, for example $A = (a_1, a_2, a_3.a_4)$ while $\star$ denotes the Gato derivative along the vector $\mathcal{L}(B)$. We check this identity utilizing the computer algebra Reduce and package Susy2 [16]. We will, briefly explain our procedures applied for the verification of the Jacobi identity (53).

In the first stage, we remove the derivatives from the test functions $a_{i,x}, b_{i,x}$ and $c_{i,x}$ in the Jacobi identity, using the rule

$$f_x = \overleftarrow{\partial} f - f \overrightarrow{\partial} \tag{54}$$

where f is an arbitrary function.

Because of this, the Jacobi identity can be split into three segments. The first and second segments contain terms in which the integral operator ∂^{-1} appears twice and once respectively. The last segment does not contain any integral operators. We consider each segment separately.

The first segment is a combination of the following expressions:

$$\int dx \, n_c \partial^{-1} n_a \partial^{-1} n_b + \int dx \, \tilde{n}_a \partial^{-1} \tilde{n}_c \partial^{-1} \tilde{n}_b + cyclic(a, b, c)$$

where n_a denotes $n_j a_i$ or $n_{j,x} a_i$ or $n_{j,xx} a_i$, $\quad i, j = 1, 2, 3, 4$ and similarly for n_b, n_c, $\tilde{n}_a, \tilde{n}_b, \tilde{n}_c$.

Here ∂^{-1} is an integral operator, and, therefore, each ingredient could be rewritten as, for example,

$$\int dx \, n_c \partial^{-1} n_a \partial^{-1} n_b = -\int dx \, n_a (\partial^{-1} n_c)(\partial^{-1} n_b), \tag{55}$$

Now, we replace n_a in the last formula by

$$n_a = \overleftarrow{\partial}(\partial^{-1} n_a) - (\partial^{-1} n_a)\overrightarrow{\partial}.$$

Hence, the expression (55) transforms to

$$\int dx \, n_c \partial^{-1} n_a \partial^{-1} n_b = \int dx, \, n_c(\partial^{-1} n_a)(\partial^{-1} n_b) + \int dx \, n_b(\partial^{-1} n_a)(\partial^{-1} n_c)$$

Now repeating this procedure for n_a and $\tilde{n}_a$ in the first segment, it appears that this segment reduces to zero.

The second segment is constructed from the combinations of the following terms:

$$\int dx\ \Lambda_a \Lambda_c \partial^{-1} \Lambda_b + \int dx\ \tilde{\Lambda}_b \partial^{-1} \tilde{\Lambda}_a \tilde{\Lambda}_c + cyclic(a,b,c).$$

where Λ_a takes values in $\{n_j a_i,\ n_{j,x} a_i,\ n_{j,xx} a_i,\ n_j a_{i,x},\ n_j a_{i,xx}\}, i, j = 1, 2, 3, 4$. In a similar manner the $\Lambda_b, \Lambda_c, \tilde{\Lambda}_a, \tilde{\Lambda}_b, \tilde{\Lambda}_c$ are defined. These terms are rewritten as

$$\int dx\ \Lambda_a \Lambda_c (\partial^{-1} \Lambda_b) - \int dx\ \tilde{\Lambda}_a \tilde{\Lambda}_c (\partial^{-1} \tilde{\Lambda}_b) + cyclic(a,b,c).$$

Next, using rule (54), we replace once more the derivatives in $a_{k,x}$ and $b_{k,x}$ in the second segment. After this replacement, it appears that the second segment contains no term with an the integral operator. Therefore, we add this segment to the third segment.

Now, it is easy to check that this last segment vanishes. Indeed, it is enough to use the rule (54) in order to remove the derivatives from $a_{k,x}$ in the last segment.

This finishes the proof.

References

[1] T. Schäfer and C. E. Wayne, *propagation of ultra-short optical pulses in cubic nonlinear media*, Physics **D 196** (2004) 90-105.

[2] M.L.Rabelo, *On equations which describe pseudospherical surfaces* Stud. Appl.Math. **81** 221-248 (1989).

[3] R. Beals, M.L.Rabelo and K. Tenenblat, *Bäcklund transformation and inverse scattering solutions for some pseudospherical surfaces equations*, Stud. Appl.Math. **81** 125- 151 (1989).

[4] A. Sakovich and S. Sakovich, *The short pulse equations is integrable*, J. Phys. Soc. Jpn. **74** 230-241 (2005).

[5] J. C. Brunelli *The short pulse hierarchy*, J. Math. Phys. **46** 123507 (2005).

[6] J. C. Brunelli *The bi-Hamiltonian structure of the short pulse equation*, Phys. Letta Z **353** 475-478 (2006).

[7] M. Pietrzyk, I. Kanatsikov and U. Bandelow *On the propagation of vector ultra-short pulses*, J. Nonlinear Math. Phys. **15**, 162 (2008).

[8] S. Sakovich *On integrability of the vector short pulse equation* J. Phys. Soc. Jpn. **77** 123001 (2008).

[9] Y. Matsuno *A novel multi-component generalization of the short pulse equation and its multisoliton solutions*, J. Math. Phys. **52** 123702 (20111).

[10] Y. Matsuno *Integrable multi-com-ponent generalization of a modified short pules equations*, J. Math. Phys. **57** 111507 (20116).

[11] A. Dimakis and F. Müller-Hoissen *Bidifferentional Calculus Approach to AKNS Hirarchies and Their Solutions* Symmetry, Intgrability and Geometry: Methods and Applications **6** 055 (2010).

[12] J.C. Brunelli and S. Sakovich *Hamiltonian Integrability of Two-Component Short Pulse Equations* J. Math. Phys. **54** (2013) 012701.

[13] B.F. Feng *An integrable coupled short pulse equation* J. Phys.A **45** 085202 (2012).

[14] Y. Yao and Y.Zeng *Coupled Short Pulse Hierarchy and Its Hamiltonian Structure* J. Phys.Soc.Jpn. 80, 064004 (2011).

[15] M. Blaszak *Multi-Hamiltonian Theory of Dynamical Systems* Springer 1998.

[16] Z. Popowicz *SUSY 2*, Computer Physics Communications 100, 277-296 (1997).