

A GENERALIZATION OF NEUMANN'S QUESTION

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ABSTRACT. Let G be a group, $m \geq 2$ and $n \geq 1$. We say that G is an $\mathcal{T}(m, n)$ -group if for every m subsets X_1, X_2, \dots, X_m of G of cardinality n , there exists $i \neq j$ and $x_i \in X_i, x_j \in X_j$ such that $x_i x_j = x_j x_i$. In this paper, we give some examples of finite and infinite non-abelian $\mathcal{T}(m, n)$ -groups and we discuss finiteness and commutativity of such groups. We also show solvability length of a solvable $\mathcal{T}(m, n)$ -group is bounded in terms of m and n .

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1. Introduction

Let m, n be positive integers or infinity (denoted ∞) and \mathcal{X} be a class of groups. We say that a group G satisfies the condition $\mathcal{X}(m, n)$ (G is an $\mathcal{X}(m, n)$ -group, or $G \in \mathcal{X}(m, n)$), if for every two subsets M and N of cardinalities m and n , respectively, there exist $x \in M$ and $y \in N$ such that $\langle x, y \rangle \in \mathcal{X}$. Bernhard H. Neumann in 2000 [9], put forward the question: Let G be a finite group of order $|G|$ and assume that however a set M of m elements and a set N of n elements of the group is chosen, at least one element of M commutes with at least one element of N , that is G is an $\mathcal{C}(m, n)$ -group, where \mathcal{C} is the class of abelian groups. What relations between $|G|$, m and n guarantee that G is abelian? Even though the latter question was posed for finite groups, the property introduced therein can be considered for all groups.

Following Neumann's question, authors in [1], showed that infinite groups satisfying the condition $\mathcal{C}(m, n)$ for some m and n are abelian. They obtained an upper bound in terms of m and n for the solvability length of a solvable group G . Also the third author in [13] studied the $\mathcal{N}(m, n)$ -groups, where \mathcal{N} is the class of nilpotent groups. Considering the analogous question for rings, Bell and Zarrin in [3] studied the $\mathcal{C}(m, n)$ -rings and they showed that all infinite $\mathcal{C}(m, n)$ -rings (like infinite $\mathcal{C}(m, n)$ -groups) are commutative and proved several commutativity results.

As a substantial generalization of $\mathcal{C}(m, n)$ -rings, Bell and Zarrin in [4] studied the $\mathcal{T}(m, n)$ -rings. Let $m \geq 2$ and $n \geq 1$. A ring R (or a semigroup) is said to be a $\mathcal{T}(m, n)$ -ring (or is an $\mathcal{T}(m, n)$ -semigroup), if for every m n -subsets A_1, A_2, \dots, A_m of R , there exists $i \neq j$ and $x_i \in A_i, x_j \in A_j$ such that $x_i x_j = x_j x_i$. They showed that torsion-free $\mathcal{T}(m, n)$ -rings are commutative. Note that unlike $\mathcal{C}(m, n)$ -rings there are the vast classes of infinite noncommutative $\mathcal{T}(m, n)$ -rings. Also they discussed finiteness and commutativity of such rings.

In this paper, considering the analogous definition for groups, we prove some results for $\mathcal{T}(m, n)$ -groups and present some examples of such groups and give several

commutativity theorems. Note that infinite $\mathcal{T}(m, n)$ -groups, unlike infinite $\mathcal{C}(m, n)$ -groups, need not be commutative. For instance, we can see $A_5 \times A$ is an infinite non-Abelian $\mathcal{T}(22, 1)$ -group, where A_5 is the alternating group of degree 5 and A is an arbitrary infinite abelian group. However, certain infinite $\mathcal{T}(m, n)$ -groups can be shown to be commutative. Clearly, every finite group is an $\mathcal{T}(m, n)$ -group, for some m and n . It is not necessary that every group is an $\mathcal{T}(m, n)$ -group, for some $m \geq 2, n \geq 1$. For example, if F be a **free** group, then it is not difficult to see that F is not an $\mathcal{T}(m, n)$ -group, for every $m \geq 2, n \geq 1$.

It is easy to see that every $\mathcal{C}(m, n)$ -group is an $\mathcal{C}(\max\{m, n\}, \max\{m, n\})$ -group and every $\mathcal{C}(\max\{m, n\}, \max\{m, n\})$ -group is an $\mathcal{T}(2, \max\{m, n\})$ -group. Therefore every $\mathcal{C}(m, n)$ -group is an $\mathcal{T}(2, r)$ -group for some r . Thus a next step might be to consider $\mathcal{T}(3, r)$ -groups. We show solvability length of a solvable $\mathcal{T}(3, n)$ -group is bounded above in terms of n . Also we give a solvability criterion for $\mathcal{T}(m, n)$ -groups in terms of m and n .

Finally, in view of $\mathcal{T}(m, n)$ -groups and $\mathcal{X}(m, n)$ -groups, we can give a substantial generalization of $\mathcal{X}(m, n)$ -groups. Let m, n be positive integers or infinity (denoted ∞) and \mathcal{X} be a class of groups. We say that a group G satisfies the condition $\mathcal{GX}(m, n)$ (or $G \in \mathcal{GX}(m, n)$), if for every m, n -subsets A_1, A_2, \dots, A_m of G , there exists $i \neq j$ and $x_i \in A_i, x_j \in A_j$ such that $\langle x_i, x_j \rangle \in \mathcal{X}$. Also a set $\{A_1, A_2, \dots, A_m\}$ of n -subsets of a group G is called (m, n) -obstruction if it prevents G from being an $\mathcal{GX}(m, n)$ -group. Therefore, with this definition, $\mathcal{T}(m, n)$ -groups are exactly $\mathcal{GC}(m, n)$ -groups. Obviously, every $\mathcal{X}(m, n)$ -group is an $\mathcal{GX}(m+n, 1)$ -group.

2. Some properties of $\mathcal{T}(m, n)$ -groups

Here, we use the usual notation, for example $A_n, S_n, SL_n(q), PSL_n(q)$ and $Sz(q)$, respectively, denote the alternating group on n letters, the symmetric group on n letters, the special linear group of degree n over the finite field of size q , the projective special linear group of degree n over the finite field of size q and the Suzuki group over the field of size q .

At first, we give some properties of $\mathcal{T}(m, n)$ -groups and then give some examples of such groups. If $2 \leq m_1 \leq m_2, n_1 \leq n_2$, then every $\mathcal{T}(m_1, n_1)$ -group is a $\mathcal{T}(m_2, n_2)$ -group. Therefore every $\mathcal{T}(m, n)$ -group is an $\mathcal{T}(\max\{m, n\}, \max\{m, n\})$ -group.

Lemma 2.1. *If G is an $\mathcal{T}(m, n)$ -group, $H \leq G$ and $N \trianglelefteq G$, then two groups H and $\frac{G}{N}$ are $\mathcal{T}(m, n)$ -group.*

Lemma 2.2. *Let $m+n \leq 4$, then G is an $\mathcal{T}(m, n)$ -group if and only if it is abelian.*

Proof. It is enough to consider only the groups that belongs to $\mathcal{T}(3, 1)$ and $\mathcal{T}(2, 2)$. If G be a non-Abelian $\mathcal{T}(3, 1)$ -group, then there exist elements x and y of G , such that $[x, y] \neq 1$. Therefore $A_1 = \{x\}, A_2 = \{y\}, A_3 = \{xy\}$ is a $(3, 1)$ -obstruction of G , a contrary.

If G is a non-Abelian $\mathcal{T}(2, 2)$ -group and $[x, y] \neq 1$. Then we can see that $A_1 = \{x, y\}, A_2 = \{xy, yx\}$ is a $(2, 2)$ -obstruction of G , a contrary. \square

We note that the bound 4 in the above Lemma is the best possible. As D_8 and Q_8 are $\mathcal{T}(4, 1)$ -groups.

Proposition 2.3. *Assume that a finite group G is not $\mathcal{T}(m, n)$ -group, for two positive integers m, n . Then $mn \leq |G| - |Z(G)|$. Moreover, if $mn = |G| - |Z(G)|$, then for every $a \in G \setminus Z(G)$, $|a| \leq n + |Z(G)|$.*

Proof. As G is not an $\mathcal{T}(m, n)$ -group, then there exists (m, n) -obstruction for G like $\{A_1, A_2, \dots, A_m\}$. It follows that $A_i \cap A_j = \emptyset$ for every $i \neq j$ and $Z(G) \cap A_i = \emptyset$. Therefore $\bigcup_{i=1}^m A_i \subseteq G \setminus Z(G)$ and so $mn \leq |G| - |Z(G)|$.

Now if $mn = |G| - |Z(G)|$, then we can see that, for every noncentral element $a \in G$, there exists $1 \leq i \leq m$ such that $C_G(a) \setminus Z(G) \subseteq A_i$. Thus $|C_G(a)| \leq n + |Z(G)|$ and so $|a| \leq n + |Z(G)|$. \square

Corollary 2.4. *S_3 is an $\mathcal{T}(2, 3)$ and $\mathcal{T}(3, 2)$ -group. Also the dihedral group of order $2n$, D_{2n} is an $\mathcal{T}(2, n)$. If n is even integer, then D_{2n} is an $\mathcal{T}(2, n-1)$ but not an $\mathcal{T}(2, n-2)$ -group. For this, if we put $D_{2n} = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1} \rangle$, $A_1 = \langle a \rangle \setminus Z(D_{2n})$ and $A_2 = \bigcup_{j=1}^{\frac{n}{2}-1} C_G(ba^j) \setminus Z(D_{2n})$. Then we can see that $\{A_1, A_2\}$ is a $(2, n-2)$ -obstruction. Moreover it is easy to see that, if n is even integer, then D_{2n} is an $\mathcal{T}(\frac{n}{2} + 2, 1)$ -group. Also, if n is odd integer, then D_{2n} is an $\mathcal{T}(n+2, 1)$ -group. Finally, every finite p -group of order p^n is an $\mathcal{T}(p^{n-1}, p)$ and also an $\mathcal{T}(p, p^{n-1})$ -group.*

Example 2.5. *Let $G = S_4$. It is not difficult to see that, according to the centralizers of G , G is an $\mathcal{T}(14, 1)$, $\mathcal{T}(11, 2)$, $\mathcal{T}(6, 3)$, $\mathcal{T}(4, 5)$ and $\mathcal{T}(3, 7)$ -group.*

Lemma 2.6. *Every finite group G is an $\mathcal{T}(m, \lceil \frac{|G|}{m} \rceil)$ and so is an $\mathcal{T}(m, \lceil \frac{|G|}{2} \rceil)$, for every $m \geq 2$.*

Proof. The result follows from Proposition 2.3. \square

Remark 2.7. *Assume that G_1 is an $\mathcal{T}(m_1, n_1)$ -group and G_2 is an $\mathcal{T}(m_2, n_2)$ -group. Then the group $G_1 \times G_2$ need not to be an $\mathcal{T}(m, n)$ -group, where $m = \max\{m_1, m_2\}$ and $n = \max\{n_1, n_2\}$. For example, S_3 is an $\mathcal{T}(3, 2)$ -group but $S_3 \times S_3$ is not an $\mathcal{T}(3, 2)$ -group (note that $S_3 \times S_3$ is an $\mathcal{T}(7, 3)$ -group). In particular, the group $G_1 \times G_2$ need not to be even an $\mathcal{T}(m_1 m_2, n_1 n_2)$ -group. For example, it is easy to see that the quaternion group Q_8 is an $\mathcal{T}(4, 1)$ -group, but $Q_8 \times S_3$ is not an $\mathcal{T}(12, 2)$ -group (in fact, $Q_8 \times S_3$ is an $\mathcal{T}(13, 2)$ -group). For, if we consider the subsets of $Q_8 \times S_3$ as follows:*

$$A_1 = \{(i, (1, 2)), (-i, (1, 2))\}, A_2 = \{(i, (1, 3)), (-i, (1, 3))\},$$

$$A_3 = \{(i, (2, 3)), (-i, (2, 3))\}, A_4 = \{(i, (1, 2, 3)), (-i, (1, 2, 3))\},$$

$$A_5 = \{(j, (1, 2)), (-j, (1, 2))\}, A_6 = \{(j, (1, 3)), (-j, (1, 3))\},$$

$$A_7 = \{(j, (2, 3)), (-j, (2, 3))\}, A_8 = \{(j, (1, 2, 3)), (-j, (1, 2, 3))\},$$

$$A_9 = \{(k, (1, 2)), (-k, (1, 2))\}, A_{10} = \{(k, (1, 3)), (-k, (1, 3))\},$$

$$A_{11} = \{(k, (2, 3)), (-k, (2, 3))\}, A_{12} = \{(k, (1, 2, 3)), (-k, (1, 2, 3))\}.$$

Then it is easy to see that the subsets $\{A_1, A_2, \dots, A_{12}\}$ is a $(12, 2)$ -obstruction for the group $Q_8 \times S_3$.

Lemma 2.8. *Let G_1 be an $\mathcal{T}(m, 1)$ -group, G_2 be an abelian group, then $G_1 \times G_2$ is an $\mathcal{T}(m, 1)$ -group.*

Remark 2.9. Clearly every finite group is an $\mathcal{T}(m, n)$ -group for some m, n . But it is not true that every infinite group is an $\mathcal{T}(m, n)$ -group. For instance, every group which contain a free subgroup, is not an $\mathcal{T}(m, n)$ -group, for every $m \geq 2, n \geq 1$. Moreover, it is well-known that every free group is a residually finite group (even though the converse is not necessarily true). But there exist some residually finite groups that are not again an $\mathcal{T}(m, n)$ -group, for every $m \geq 2, n \geq 1$. For example, the group $SL_2(\mathbb{Z})$ is a residually finite group. The subgroup of $SL_2(\mathbb{Z})$ generated by matrixes $a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ is a free group of rank 2. So $SL_2(\mathbb{Z})$ is not an $\mathcal{T}(m, n)$ -group, for every $m \geq 2$ and $n \geq 1$.

For any nonempty set X , $|X|$ denotes the cardinality of X . Let A be a subset of a group G . Then a subset X of A is a set of pairwise non-commuting elements if $xy \neq yx$ for any two distinct elements x and y in X . If $|X| \geq |Y|$ for any other set of pairwise non-commuting elements Y in A , then the cardinality of X (if it exists) is denoted by $w(A)$ and is called the clique number of A (for more information concerning the clique number of groups, see for example [12] and [2]).

Lemma 2.10. *Let G is not an $\mathcal{T}(m, n)$ -group and $\{A_1, A_2, \dots, A_m\}$ be a (m, n) -obstruction for G . Then*

$$m + \max\{w(A_i) \mid 1 \leq i \leq m\} \leq w(G).$$

Proof. Clearly. □

Lemma 2.11. *Let G be an $\mathcal{T}(m, n)$ -group. Then $w(G) < mn$ and G is center-by-finite.*

Proof. We show that for any set X of pairwise non-commuting elements of G , we have $|X| < mn$. Suppose that $|X| \geq mn$, then we can take m n -subsets of X that is a (m, n) -obstruction for G . It is a contradiction. By the famous theorem of B. H. Neumann [8], since every set of non-commuting elements of $\mathcal{T}(m, n)$ -group G is finite, therefore it is center-by-finite. □

Now we show that for $\mathcal{T}(m, n)$ -groups with $|Z(G)| \geq n$, we get even $w(G) < m$. In fact, we have

Proposition 2.12. *If G is an $\mathcal{T}(m, n)$ -group, then $|Z(G)| < n$ or $w(G) < m$.*

Proof. Let G be an $\mathcal{T}(m, n)$ -group and $|Z(G)| \geq n$. We may assume $Z_1 \subseteq Z(G)$ and $|Z_1| = n$. Now if $w(G) \geq m$ and $\{x_1, x_2, \dots, x_m\}$ be a pairwise non-commuting set of G , then $\{x_1Z_1, x_2Z_1, \dots, x_mZ_1\}$ is a (m, n) -obstruction for G , which is a contradiction. □

It is easy to see that a group G is an $\mathcal{T}(m, 1)$ -group, if and only if $w(G) < m$.

Corollary 2.13. *Assume that G is a nilpotent finite $\mathcal{T}(m, n)$ -group and p is a prime divisor of $|G|$ such that $n \leq p$. Then G is an $\mathcal{T}(m, 1)$ -group. In particular, every nilpotent $\mathcal{T}(m, 2)$ -group is an $\mathcal{T}(m, 1)$ -group.*

If G is a non-Abelian group, then G is not an $\mathcal{T}(3, z)$, which $z = |Z(G)|$, since $w(G) \geq 3$.

If G is an $\mathcal{T}(w(G), 2)$ -group, then $Z(G) = 1$.

Corollary 2.14. *Let G be a non-Abelian $\mathcal{T}(m, n)$ -group with at least m pairwise non-commuting elements, then G is a finite group.*

Lemma 2.15. *Let G be a non-Abelian $\mathcal{T}(2, n)$ or $\mathcal{T}(3, n)$ -group and N be a normal subgroup of G such that G/N is non-Abelian. Then $|N| < n$.*

Proof. Suppose that $N = \{a_1, a_2, \dots, a_t\}$ and $t \geq n$. It is enough to prove the theorem for non-Abelian $\mathcal{T}(3, n)$ -groups. We chose elements x, y in $G \setminus N$, and we consider three subsets of G , as follows:

$$A_1 = \{xa_1, xa_2, \dots, xa_t\}, \quad A_2 = \{ya_1, ya_2, \dots, ya_t\}$$

and

$$A_3 = \{xya_1, xya_2, \dots, xya_t\}.$$

Now as G is an $\mathcal{T}(3, n)$ -group, we can follow that $[x, y] \in N$, that is G/N is abelian, which is a contradiction. \square

Theorem 2.16. *Let G be a non-Abelian group and its clique number is finite. Then there exist a natural number m such that G is an $\mathcal{T}(m, n)$ -group for all $n \in N$.*

Proof. As the clique number of G is finite, so according to the famous Theorem of B. H. Neumann [8], G is center-by-finite. So we put $[G : Z(G)] = m$. We claim that for every $n \in N$, G is an $\mathcal{T}(m, n)$ -group. There exists $m - 1$ elements g_1, g_2, \dots, g_{m-1} in G , such that $Z(G), g_1Z(G), g_2Z(G), \dots, g_{m-1}Z(G)$ are distinct cosets of $Z(G)$ in G and

$$G = Z(G) \bigcup \left(\bigcup_{j=1}^{m-1} (g_j Z(G)) \right).$$

Let $\{A_1, A_2, \dots, A_m\}$ be an (m, n) -obstruction of G . Now as every $g_i Z(G)$ is abelian, therefore if $g_i Z(G) \cap A_r \neq \emptyset$ for some $1 \leq i \leq m - 1$ and $1 \leq r \leq m$, then $g_i Z(G) \cap A_j = \emptyset$ for every $j \neq r$. On the other hand $Z(G) \cap A_i = \emptyset$ for every $1 \leq i \leq m$, thus $A_i \subseteq \bigcup_{j=1}^{m-1} (g_j Z(G))$ for every $1 \leq i \leq m$. From this one can follow that there exist two subsets like A_r and A_s such that $A_r \cup A_s \subseteq g_j Z(G)$ for some $1 \leq j \leq m - 1$, a contradiction. Therefore G is an $\mathcal{T}(m, n)$ -group, for all $n \in N$. \square

Remark 2.17. In the above Theorem, the finiteness of clique number is necessary. For example, it should be borne in mind that infinite p -groups can easily have trivial center. The group $G = C_p \wr C_{p^\infty}$, the regular wreath product C_p by C_{p^∞} , is an infinite centerless p -group, where C_p is a cyclic group of order p and C_{p^∞} is a quasi-cyclic (or Prüfer) group. So $[G : Z(G)]$ is infinite and so $w(G)$ is infinite and G is not an $\mathcal{T}(m, n)$ -group.

B. H. Neumann [8] showed that if every set of non-commuting elements of group G is finite, then G is center-by-finite. Moreover, it is not difficult to see that every center-by-finite group has finite clique number. Here, by using the above Theorem, we will obtain the following result.

Corollary 2.18. *If G be a group and $[G : Z(G)] = m$. Then $w(G) < m$.*

Proof. As $[G : Z(G)] = m$ and G is an $\mathcal{T}(m, 1)$ -group, if and only if $w(G) < m$, the result follows by Theorem 2.16. \square

Corollary 2.19. *Every infinite $\mathcal{T}(m, n)$ -group with $m \leq 3$, is an abelian group.*

Proof. If G is non-Abelian group, then there exists x, y such that $xy \neq yx$. So $\{x, y, xy\}$ is a subset of pairwise non-commuting elements of G . Therefore by Proposition 2.12, $|Z(G)| < n$ and by Lemma 2.11 G is center-by-finite and so G is a finite group, a contradiction. \square

Theorem 2.20. *Let G be a finite $\mathcal{T}(m, n)$ -group, $m \leq 4$, $n > 1$ and $(p, |G|) = 1$, for every prime number $p \leq n$. Then G is abelian.*

Proof. It is enough to prove the theorem for the case $m = 4$. Suppose, a contrary, that G is a non-Abelian $\mathcal{T}(4, n)$ -group. Then there exists elements x and y in G , such that $xy \neq yx$. Now we consider four subsets of G as follows:

$$A_1 = \{x, x^2, \dots, x^n\}, \quad A_2 = \{y, y^2, \dots, y^n\},$$

$$A_3 = \{xy, (xy)^2, \dots, (xy)^n\} \text{ and } A_4 = \{xy, xy^2, \dots, xy^{n-1}, x^2y\}.$$

Then it is not difficult to see that $\{A_1, A_2, A_3, A_4\}$ is a $(4, n)$ -obstruction for G , a contradiction (note that if $a \in G$ and $(i, |a|) = 1$, then $C_G(a^i) = C_G(a)$). \square

As a corollary, for $p > 2$ every finite p -group, $G \in \mathcal{T}(4, p-1)$ is abelian.

Note that the group D_8 is a non-Abelian $\mathcal{T}(4, 1)$ -group. This example suggests that it may be necessary to restrict ourselves to $\mathcal{T}(m, n)$ -groups with $n > 1$ in the above Theorem.

Proposition 2.21. *Assume that G is a non-Abelian group. Then*

- (1) *If there exist a positive integer n such that $(p, |G|) = 1$, for every prime number $p \leq n$. Then $w(G) \geq n + 2$.*
- (2) *If p is the smallest prime divisor of $|G|$. Then $w(G) \geq p + 1$. Moreover, if G is a finite p -group and $G \in \mathcal{T}(m, p-1)$, then $p + 1 \leq w(G) < m$.*

Proof. (1) Since G is non-Abelian group, there exists elements x, y in G , such that $xy \neq yx$. Now $X = \{x, y, xy, xy^2, \dots, xy^n\}$ is a set of pairwise non-commuting elements of G of cardinality $n + 2$.

(2) For every prime number $q \leq p - 1$, $(q, |G|) = 1$, then by part (1), $w(G) \geq (p - 1) + 2$. Thus $w(G) \geq p + 1$. \square

Lemma 2.22. *Let G be a finite $\mathcal{T}(m, n)$ -group where $Z(G) \neq 1$ and p be smallest prime divisor of $|G|$. Then*

- (1) $p \leq \max\{m - 2, n - 1\}$.
- (2) *If G is a nilpotent, then $p \leq \max\{m - 2, \sqrt[t]{n - 1}\}$, where t is the number of prime divisors of order G , $|\pi(G)|$.*

Proof. (1) As G is an $\mathcal{T}(m, n)$ -group, by Proposition 2.12, $p \leq |Z(G)| < n$ or $p + 1 \leq w(G) < m$. Therefore $p \leq n - 1$ or $p \leq m - 2$ and so $p \leq \max\{m - 2, n - 1\}$.

(2) In this case it is enough to note that the set of prime divisors of the center of G is equal to $\pi(G)$, so $p^t \leq \prod_{i=1}^t p_i \leq |Z(G)| < n$. Thus $p \leq \sqrt[t]{n - 1}$ or $p \leq m - 2$. \square

Corollary 2.23. (1) *If G is a finite p -group and $G \in \mathcal{T}(p, p)$, then G is abelian.*

(2) *Every finite non-Abelian nilpotent $\mathcal{T}(m, n)$ -group with $3 \leq n \leq 6$ and $m \leq 3$ is a p -group.*

(3) *If G is a finite nilpotent $\mathcal{T}(4, n)$ -group with $n \leq 3$ and odd order, then G is an abelian group.*

(4) *Every finite nilpotent $\mathcal{T}(4, n)$ -group with $n \leq 3$, is an abelian-by-2-group.*

Theorem 2.24. *Let G be a non-Abelian nilpotent $\mathcal{T}(3, n)$ -group. Then*

$$|\pi(G)| \leq \log_3(n+2).$$

Proof. Use induction on $|\pi(G)|$, the case $3 \leq n \leq 6$ being clear by Case (2) of Corollary 2.23. Assume that $n \geq 7$ and the result holds for $|\pi(G)| - 1$. Since G is finite non-Abelian nilpotent, then there exist a Sylow subgroup P of G , such that $\frac{G}{P}$ is non-Abelian and $\frac{G}{P} \in \mathcal{T}(3, n-t)$ -group, for every $2t \leq n$. So $|\pi(\frac{G}{P})| \leq \log_3(n-t+2)$, therefore $|\pi(G)| - 1 \leq \log_3(n-t+2) < \log_3(n+2)$ and hence $|\pi(G)| \leq \log_3(n+2)$, as wanted. \square

Remark 2.25. By argument similar to the one in the proof of Theorem 2.24, we can follow that if G is a non-Abelian nilpotent $\mathcal{T}(4, n)$ -group with odd order, then $|\pi(G)| \leq \log_4(n+6)$ (in this case note that if $4 \leq n \leq 9$, then G is a p -group, for some prime number p).

3. On solvable $\mathcal{T}(m, n)$ -groups

In this section we investigate solvable $\mathcal{T}(m, n)$ -groups. At first we obtain the derived length of a solvable $\mathcal{T}(m, n)$ -group in terms n , for $m = 3$ or 4 and then give a solvability criterion for $\mathcal{T}(m, n)$ -groups in terms m and n . To prove our results it is necessary to establish a technical lemma.

Lemma 3.1. *Let G be an $\mathcal{T}(m, n)$ -group for some integers $m \geq 2, n > 1$ and N be a proper non-trivial normal subgroup of G , then $\frac{G}{N}$ is an $\mathcal{T}(m, n-t)$ -group, where $n \geq 2t$.*

Proof. Suppose that G is an $\mathcal{T}(m, n)$ -group and $N \triangleleft G$, but $\frac{G}{N}$ is not an $\mathcal{T}(m, n-t)$ -group. We can take m subsets $X_i = \{x_{i1}N, x_{i2}N, \dots, x_{in-t}N\}$, $1 \leq i \leq m$ of $\frac{G}{N}$ of cardinality $n-t$, such that for every $1 \leq i, j \leq m$ and $1 \leq k, l \leq n-t$, $[x_{ik}, x_{jl}]$ is not belongs to N . Let a be a nontrivial element of N , then we can obtain m n -subsets $Y_i = \{ax_{i1}, ax_{i2}, \dots, ax_{in-t}, x_{i1}, x_{i2}, \dots, x_{it}\}$ of G , for some $2t \leq n$. Thus $\{Y_1, Y_2, \dots, Y_m\}$ is a (m, n) -obstruction for G , a contrary. \square

Corollary 3.2. *Let G be a non-simple $\mathcal{T}(w(G), 2)$ -group. Then for proper non-trivial normal subgroup N of G , $\frac{G}{N}$ is an $\mathcal{T}(w(G), 1)$ -group and so $w(\frac{G}{N}) < w(G)$.*

Theorem 3.3. *Let G be a solvable $\mathcal{T}(3, n)$ -group (or $\mathcal{T}(4, n)$ -group and the order of G is odd). Then the derived length of G , d is at most $\log_2(2n)$.*

Proof. We argue by induction on d . The case $d = 1$ being obvious. Assume that $d \geq 2$ and so, by Lemma 3.1, the group $\frac{G}{G^{(d-1)}}$, has solvability length $d-1$, is an $\mathcal{T}(3, n-t)$ -group, where $2t \leq n$. Therefore $d-1 \leq \log_2(2(n-t)) < \log_2(2n)$. Thus $d-1 < \log_2(2n)$, so $d \leq \log_2(2n)$, as wanted. (We note that the bound $\log_2(2n)$ is the best possible, as S_3 is an $\mathcal{T}(3, 2)$ -group and $d(S_3) = 2 = \log_2(4)$.) Now if G is $\mathcal{T}(4, n)$ -group and the order of G is odd, then by argument similar, the result follows (for proof it is enough to note that $\mathcal{T}(4, 1)$ -group of odd order is abelian). \square

Note that the group D_8 is a solvable $\mathcal{T}(4, 1)$ -group with solvability length 2, but $2 \not\leq \log_2(2)$. This example suggests that it may be necessary to restrict ourselves to groups with odd order in the above Theorem.

If G is a finite group, then for each prime divisor p of $|G|$, we denote by $v_p(G)$ the number of Sylow p -subgroups of G .

Lemma 3.4. *Let G be a finite $\mathcal{T}(m, n)$ -group and p be a prime number dividing $|G|$ such that every two distinct Sylow p -subgroups of G have trivial intersection, then $v_p(G) \leq mn - 1$.*

Proof. Since G is an $\mathcal{T}(m, n)$ -group, we have $w(G) < mn$. Now Lemma 3 of [6] completes the proof. \square

Now we obtain a solvability criteria for $\mathcal{T}(m, n)$ -groups in terms of m and n .

Theorem 3.5. *Let G be an $\mathcal{T}(m, n)$ -group. Then we have*

- (i) *If $mn \leq 21$, then G (not necessarily finite) is solvable and this estimate is sharp.*
- (ii) *If $mn \leq 60$ and G is non-solvable finite group, then $G = A_5$ (in fact, A_5 is the only non-solvable $\mathcal{T}(m, n)$ -group, which $mn \leq 60$).*

Proof. (i) Since G is an $\mathcal{T}(m, n)$ -group, we have $w(G) < mn$. Then by Theorem 1.2 of [12], G is solvable. This estimate is sharp, because A_5 is an $\mathcal{T}(22, 1)$ -group. (ii) We know that the alternating group A_5 has five Sylow 2-subgroups of order 4, ten Sylow 3-subgroups of order 3 and six Sylow 5-subgroups of order 5, that their intersections are trivial. From this we can follow that A_5 is an $\mathcal{T}(22, 1)$, $\mathcal{T}(22, 2)$, $\mathcal{T}(17, 3)$ and $\mathcal{T}(14, 4)$ -group. For uniqueness, suppose, on the contrary, that there exists a non-Abelian finite simple group not isomorphic to A_5 and of least possible order which is an $\mathcal{T}(m, n)$ -group, which $mn \leq 60$. Then by Proposition 3 of [5], Lemma 3.4 and by argument similar to the one in the proof of Theorem 1.3 of [1] gives a contradiction in each case. \square

Corollary 3.6. *Every arbitrary $\mathcal{T}(21, i)$ -group G with $i \leq z$ is solvable, where $z = |Z(G)|$.*

In the follow we show that the influence the value of n for the solvability of $\mathcal{T}(m, n)$ -groups is more important than the value of m .

Corollary 3.7. *Let G be an $\mathcal{T}(m, n)$ -group. Each of the following conditions implies that G is solvable.*

- (a) $n = 2$ and $m \leq 21$. (b) $n = 3$ and $m \leq 16$. (c) $n = 4$ and $m \leq 13$. (d) $n = 5$ and $m \leq 8$. (e) $n = 6$ and $m \leq 8$. (f) $n = 7$ and $m \leq 7$. (g) $n = 8$ and $m \leq 7$.

Proof. Suppose, on the contrary, that G is a non-solvable group. By Theorem 3.5, $G \cong A_5$. This is a contradiction, since A_5 is an $\mathcal{T}(22, 2)$ -group but is not an $\mathcal{T}(21, 2)$ -group.

By similar argument of Case (a) the rest Cases would be proved, since

$$A_5 \in \mathcal{T}(17, 3) \cap \mathcal{T}(14, 4) \cap \mathcal{T}(9, 5) \cap \mathcal{T}(9, 6) \cap \mathcal{T}(8, 7) \cap \mathcal{T}(8, 8) \text{ but}$$

$$A_5 \notin \mathcal{T}(16, 3) \cup \mathcal{T}(13, 4) \cup \mathcal{T}(8, 6) \cup \mathcal{T}(7, 8).$$

\square

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