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A GENERALIZED BARTHOLDI ZETA FUNCTION FORMULA FOR SIMPLE GRAPHS WITH BOUNDED DEGREE

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Abstract. We introduce a generalized Bartholdi zeta function for simple graphs with bounded degree. This zeta function is a generalization of both the Bartholdi zeta function which was introduced by L. Bartholdi and the Ihara zeta function which was introduced by G. Chinta, J. Jorgenson and A. Karlsson. Furthermore, we establish a Bartholdi type formula of this Bartholdi zeta function for simple graphs with bounded degree. Moreover, for regular graphs, we give a new expression of the heat kernel which is regarded as a one-parameter deformation of the expression obtained by G. Chinta, J. Jorgenson and A. Karlsson. By applying this formula, we give an alternative proof of the Bartholdi zeta function formula for regular graphs.

1. Introduction

All graphs in this paper are assumed to be connected, countable and simple. Let X be a graph with bounded degree and Δ_X be the combinatorial Laplacian of X. It is well-known that the spectrum of Δ_X is closely related to geometric properties and combinatorial properties of X at least from the view point of graph theory, number theory and probability theory. Classically, it is important to study the relationship between closed paths in X and the spectrum of Δ_X. In this paper, we study the relationship from the view point of number theory.

Especially for a finite graph X, it is well-known that closed geodesics of X are deeply related to the spectrum of Δ_X (cf. [25]). The relationship is described as the Ihara formula explicitly. The Ihara zeta function for a finite graph X is defined by

$$Z_X(u) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} u^m\right).$$

Here, N_m stands for the number of closed geodesics of length m in X. Then, the Ihara formula is described as follows (cf. [25]).

$$Z_X(u)^{-1} = (1-u^2)^{-\chi(X)} \det\big(I - u(D_X - \Delta_X) + u^2(D_X - I)\big).$$

Here, $\chi(X)$ stands for the Euler characteristic of X and D_X stands for the valency operator of X. For finite regular graphs, the above formula was originally established

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by Y. Ihara in the p-adic setting ([16]). Then, it has been generalized by T. Sunada, K. Hashimoto and H. Bass ([2], [15], [11], [12], [13], [14], [17], [21], [22], [23]). When X is regular, this formula gives an explicit relationship between the number of closed geodesic and the spectrum of Δ_X.

In 1999, L. Bartholdi introduced the Bartholdi zeta function for finite graphs and established a determinant expression of it ([1]). The Bartholdi zeta function is defined by

$$Z_X(u,t) = \exp\left(\sum_{C\in\mathcal{C}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)}\right).$$

Here, we denote by $\mathcal{C}$ the set of closed paths in X, by $\ell(C)$ the length of C and by $\mathrm{cbc}(C)$ the cyclic bump count of a closed path C. This is a generalization of the Ihara zeta function by adding a variable t which plays a role of counting back-trackings of a closed path. Indeed, if t is equal to 0, this zeta function coincides with the Ihara zeta function $Z_X(u)$. The determinant expression of $Z_X(u,t)$ is described as follows ([1]).

$$\begin{aligned} Z_X(u,t)^{-1} = &\big(1-(1-t)^2u^2\big)^{-\chi(X)} \\ &\times \det\big(I - u(D_X - \Delta_X) + (1-t)u^2(D_X-(1-t)I)\big). \end{aligned}$$

As in the case of the Ihara formula, when X is regular, this formula gives the explicit relationship between the number of closed paths and the spectrum of Δ_X.

Recently, several generalizations of the Ihara zeta function from finite graphs to infinite graphs have been considered (cf. [3], [4], [5], [6], [8], [9], [10], [20]). In this paper, we follow [3] essentially. In 2017, the author introduced the Ihara zeta function for a graph X with bounded degree as follows ([18]).

$$Z_X(u,x_0) = \left(\sum_{m=1}^{\infty} \frac{N_m(x_0)}{m} u^m\right).$$

Here, $N_m(x_0)$ stands for the number of closed geodesics of length m starting at x_0. If X is vertex-transitive, this zeta function coincides with the Ihara zeta function which was introduced in [3]. In [3], the definition of the Ihara zeta function for regular graphs is given (p. 185 in [3]). In general, however, when X is regular, the Ihara zeta function in [3] does not always coincide with our Ihara zeta function. In [3], G. Chinta, J. Jorgenson and A. Karlsson established the Ihara type formula for the Ihara zeta function for vertex-transitive graphs by giving a new expression of the heat kernel ([3]). This definition works well in the point of studying deeply the relationship between closed geodesics and the spectrum of Δ_X and also as an analogy with heat kernel analysis of rank one symmetric spaces. After that, the author established the Ihara type formula for the Ihara zeta function for graphs with bounded degree ([18]). His proof also gives an alternative proof of the formula for vertex-transitive graphs.

In this paper, we study the relationship between closed paths and the spectrum of Δ_X by introducing a Bartholdi zeta function for graphs with bounded degree. For a graph X with bounded degree and a vertex x_0, a Bartholdi zeta function is defined by as follows in this paper.

$$Z_X(u,t,x_0) = \exp\left(\sum_{C \in \mathcal{C}_{x_0}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)} \right).$$

Here, we denote by $\mathcal{C}_{x_0}$ the set of closed paths starting at x_0. We remark that we introduce a Bartholdi zeta function which is a generalization of the above. However, we do not introduce the definition in Introduction because the definition is a little technical in the sense of being based on a path counting formula. If t is equal to 0, this Bartholdi zeta function coincides with $Z_X(u,x_0)$. If X is a finite graph, by the definition of $Z_X(u,t,x_0)$, the following equality holds.

$$\prod_{x_0 \in VX} Z_X(u,t,x_0) = Z_X(u,t).$$

In this sense, this Bartholdi zeta function is a generalization of the original one. Furthermore, we present a Bartholdi type formula for this Bartholdi zeta function. Especially for regular graphs, this formula describes the relationship between the number of closed paths and the spectrum of Δ_X. We remark that for finite graphs, this formula can be regarded as a refined version of the original Bartholdi zeta function formula.

Moreover, for (possibly infinite) regular graphs, we give a new expression of the heat kernel which is regarded as a one-parameter deformation of the expression obtained in [3]. By applying this formula, we give an alternative proof of the Bartholdi formula for regular graphs. We note that our heat kernel approach to the Bartholdi formula is new even for finite regular graphs. This is an important application of our new heat kernel expression. Many applications of the heat kernel are well-known. Therefore, in addition to the above application, we believe that there should be more applications by using our new expression of the heat kernel.

This paper is organized as follows. In Section 2, we prepare some terminologies. In Section 3, we give a path counting formula which is a generalization of the formula obtained in [18]. In Section 4, we give a Bartholdi type formula for our Bartholdi zeta function. In Section 5, we give the Euler product expression of it. In Section 6, we give a new expression of the heat kernel for a regular graph (not necessarily finite). In Section 7, as an application of our heat kernel expression, we give an alternative proof of the Bartholdi type formula obtained in Section 4.

2. Preliminaries

2.1. **Graphs and Paths.** In this section, we give terminology of graphs and paths used throughout this paper (cf. [1], [21], [24]). A graph X is an ordered pair

(VX, EX) of disjoint sets VX and EX with two maps,

$$EX \to VX \times VX, e \mapsto (o(e), t(e)), \quad EX \to EX, e \mapsto \bar{e}$$

such that for each $e \in EX$, $\bar{e} \neq e$, $\bar{\bar{e}} = e$, $o(e) = t(\bar{e})$. For a graph $X = (VX, EX)$, two sets VX and EX are called vertex set and edge set respectively. A graph X is *simple* if X has no loops and multiple edges. For a vertex $x \in VX$, the *degree of* x is the cardinality of the set E_x, where $E_x = \{e \in EX | o(e) = x\}$. We denote the degree of x by $\deg(x)$. A graph X is *countable* if the vertex set is countable. A graph X *has bounded degree* if the supremum of the set of all degrees is not infinite. For a graph X, a *path of length* n is a sequence of edges

$$C = (e_1, \dots, e_n)$$

such that $t(e_i) = o(e_{i+1})$ for each i. We denote $o(e_1)$ by $o(C)$, $t(e_n)$ by $t(C)$ and the length of C by $\ell(C)$. A path C is *closed* if $o(C) = t(C)$. We regard a vertex as a path of length 0. A path $C = (e_1, \dots, e_n)$ *has a back-tracking or bump* if there exist i such that $e_{i+1} = \bar{e_i}$. A path $C = (e_1, \dots, e_n)$ *has a tail* if $e_n = \bar{e_1}$. For a path $C = (e_1, \dots, e_n)$, we define the *bump count* of C as follows.

$$\mathrm{bc}(C) = \sharp\{i \in \{1, \dots, n-1\} \mid e_i = \overline{e_{i+1}}\}.$$

For a closed path $C = (e_1, \dots, e_n)$, we define the *cyclic bump count* of C as follows.

$$\mathrm{cbc}(C) = \sharp\{i \in \mathbb{Z}/m\mathbb{Z} \mid e_i = \overline{e_{i+1}}\}.$$

For a closed path x_0, we define $\mathrm{bc}(x_0) = \mathrm{cbc}(x_0) = 0$. For a path $C = (e_1, \dots, e_m)$, we denote e_i by $e_i(C)$.

2.2. **The Laplacian of a graph.** For the vertex set VX of a graph X, we define *the* ℓ^2*-space on the vertex set* VX by

$$\ell^2(VX) = \left\{ f \colon VX \to \mathbb{C} \;\middle|\; \sum_{x \in VX} |f(x)|^2 < +\infty \right\}.$$

For a function $f \in \ell^2(VX)$ and a vertex $x \in VX$, we define the *adjacency operator* A_X on X and the *valency operator* D_X on X as follows respectively.

$$(A_X f)(x) = \sum_{e \in E_x} f(t(e)),$$
$$(D_X f)(x) = \deg(x) f(x).$$

Then, we define the *Laplacian* D_X on X by $\Delta_X = D_X - A_X$. The Laplacian is a semipositive and self-adjoint bounded operator if X has bounded degree.

2.3. **The heat kernel of a graph.** For a graph X with bounded degree and a fixed vertex x_0, the *heat kernel* $K_X(\tau, x_0, x)\colon \mathbb{R}_{\geq 0} \times VX \to \mathbb{R}$ on X is the solution of the heat equation

$$\begin{cases} \left(\Delta_X + \frac{\partial}{\partial \tau}\right) f(\tau, x) = 0, \\ f(0, x) = \delta_{x_0}(x). \end{cases}$$

Here, the function $f(\tau, x)$ is in the class C^1 on $\mathbb{R} \times VX$ for each $x \in VX$ and the function $\delta_{x_0}(x)$ is the Kronecker delta. The heat kernel on X uniquely exists among functions which are bounded on $[0, T] \times VX$ for each $T \in \mathbb{R}_{\geq 0}$ under our assumptions ([26]). By the uniqueness of the solution of the heat equation, it turns out that the heat kernel $K_X(\tau, x_0, x)$ is an invariant under the automorphism group $\mathrm{Aut}(X)$.

2.4. **The modified Bessel function.** In this section, we define the modified Bessel function and introduce some well-known properties of them. For $n \in \mathbb{Z}_{\geq 0}$ and $\tau \in \mathbb{R}$, we define the *modified Bessel function of the first kind* by the following power series.

$$\mathrm{I}_n(\tau) = \sum_{m=0}^{\infty} \frac{(\tau/2)^{n+2m}}{m!(m+n)!}.$$

For $-n \in \mathbb{Z}_{<0}$, we define $\mathrm{I}_{-n}(\tau)$ as follows.

$$\mathrm{I}_{-n}(\tau) = \mathrm{I}_n(\tau).$$

It is well-known that $\mathrm{I}_n(\tau)$ is the power series solution of the following differential equation.

$$\tau^2 \frac{d^2 w}{d\tau^2} + \tau \frac{dw}{d\tau} - (\tau^2 + n^2) w = 0.$$

Moreover, it is also well-known that $\mathrm{I}_n(\tau)$ satisfies the following formula.

$$2\frac{d}{d\tau} \mathrm{I}_n(\tau) = \mathrm{I}_{n-1}(\tau) + \mathrm{I}_{n+1}(\tau). \tag{1}$$

In addition, for $n \geq 0$ and $\tau \in \mathbb{R}_{\geq 0}$, $\mathrm{I}_n(\tau)$ has the following trivial bound.

$$\mathrm{I}_n(\tau) \leq \left(\frac{\tau}{2}\right)^n \frac{\mathrm{e}^{\tau}}{n!}. \tag{2}$$

2.5. $G(t)$**-transform.** For a real valued function $f(\tau)(0 < \tau < \infty)$ which is integrable in every finite interval, we define $G(t)f$ as follows.

$$G(t)f(u) = (u^{-2} - (q+t)(1-t)) \int_0^{\infty} \mathrm{e}^{-\left((q+t)(1-t)u + \frac{1}{u} - (q+1)\right)\tau} f(\tau) d\tau.$$

We call this transform *$G(t)$-transform.* The following formula holds (cf. [27]). If $0 < u < \frac{1}{\sqrt{(q+t)(1-t)}}$, then, for $k \geq 0$, we have

$$G(t)\left(\mathrm{e}^{-(q+1)\tau} \left((q+t)(1-t)\right)^{-\frac{k}{2}} \mathrm{I}_k\left(2\sqrt{(q+t)(1-t)}\tau\right)\right)(u) = u^{k-1}. \tag{3}$$

3. A generalized path counting formula

In this section, we give a generalization of the path counting formula obtained by T. Kousaka ([18]). First of all, we introduce several symbols. We take a vertex x_0 and $e \in E_{x_0}$. We denote by $\mathcal{C}_{x_0}$ the set of closed paths starting at x_0 and by $\mathcal{C}_{x_0}^{notail}$ the set of closed paths starting at x_0 which has no tail. For a complex variable t, we define $C_m(t, x_0)$, $N_m(t, x_0, e)$ as follows.

$$
\begin{aligned}
C_m(t, x_0) &= \sum_{C} t^{\mathrm{cbc}(C)}, \\
N_m(t, x_0, e) &= \sum_{C} t^{\mathrm{cbc}(C)}.
\end{aligned}
$$

Here, C runs through $\mathcal{C}_{x_0}$ such that $\ell(C) = m$ in the first equality and C runs through $\mathcal{C}_{x_0}^{notail}$ such that $e_1(C) = e, \ell(C) = m$ in the second equality. For $f \in \ell^2(VX)$ and $x \in VX$, we define $C_m(t)$ by

$$
C_m(t)f(x) = \sum_{C \in \mathcal{B}_x, \ell(C) = m} t^{\mathrm{bc}(C)} f(t(C)).
$$

Here, we denote by $\mathcal{B}_x$ the set of paths starting at x. We define $C_m(t)(x_0, e)$ as follows.

$$
C_m(t)(x_0, e) = \sum_{C} t^{\mathrm{bc}(C)}.
$$

Here, C runs through $\mathcal{C}_{x_0}$ such that $e_1(C) = e, \ell(C) = m$ in the above equality. Moreover, we define $C_m(t)(x_0, \cdot, \bar{e})$, $N_m(t, x_0, \cdot, \bar{e})$ and $C_m(t)(x_0, e, \bar{e})$ as follows.

$$
\begin{aligned}
C_m(t)(x_0, \cdot, \bar{e}) &= \sum_{C} t^{\mathrm{bc}(C)}, \\
N_m(t, x_0, \cdot, \bar{e}) &= \sum_{C} t^{\mathrm{bc}(C)}, \\
C_m(t)(x_0, e, \bar{e}) &= \sum_{C} t^{\mathrm{bc}(C)}.
\end{aligned}
$$

Here, C runs through $\mathcal{C}_{x_0}$ such that $e_m(C) = \bar{e}, \ell(C) = m$ in the first equality, C runs through $\mathcal{C}_{x_0}^{notail}$ such that $e_m(C) = \bar{e}, \ell(C) = m$ in the second equality and C runs through $\mathcal{C}_{x_0}$ such that $e_1(C) = e, e_m(C) = \bar{e}, \ell(C) = m$ in the third equality. We denote by $\mathcal{B}(\ell^2(VX))$ the set of bounded operators on $\ell^2(VX)$. We remark that $C_m(t)$ is in $\mathcal{B}(\ell^2(VX))$ for each t. For $B \in \mathcal{B}(\ell^2(VX))$ and $x_1, x_2 \in VX$, we define $B(x_1, x_2)$ as follows.

$$
B(x_1, x_2) = B\delta_{x_1}(x_2).
$$

Here, the symbol δ_{x_0} stands for the Kronecker delta. We define the following formal power series.

$$C^{\mathrm{cbc}}(t, x_0 : u) = \sum_{m=1}^{\infty} C_m(t, x_0) u^m,$$
$$C(t, x_0 : u) = \sum_{m=1}^{\infty} C_m(t)(x_0, x_0) u^m,$$
$$N(t, x_0 : u) = \sum_{m=1}^{\infty} N_m(t, x_0) u^m.$$

Here, we denote $\sum_{e \in E_{x_0}} N_m(t, x_0, e)$ by $N_m(t, x_0)$. In addition to this, for a vertex $x \in VX$, we denote $\deg(x) C(t, x : u) - \sum_{e \in E_x} C(t, t(e) : u)$ by $\Delta_X C(t, \cdot : u)(x)$ by regarding as an element of $\ell^2(VX)$ formally and for $m \geq 1$, $x \in VX$, we define $R_m(t)(x)$ as follows.

$$R_m(t)(x) = \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} \sum_{i=1}^{j} (1-t)^{2(j-i)} (1-t^2)^{i-1} [\Delta_X C_{m-2j}(t)](x, x).$$

First of all, we prove the following proposition.

Proposition 3.1. *For a vertex x_0, we have the following equality:*

$$\big(1 - (1-t)^2 u^2\big) N(t, x_0 : u) = \big(1 - (\deg(x_0) - (1 - t^2)) u^2\big) C(t, x_0 : u) - \deg(x_0) t u^2 + \frac{u^2}{1 - (1 - t^2) u^2} \Delta_X C(t, \cdot : u)(x_0).$$

Moreover, for $m \geq 3$, we have

$$N_m(t, x_0) = C_m(t)(x_0, x_0) - (\deg(x_0) - 2(1-t)) \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} (1-t)^{2(j-1)} C_{m-2j}(t)(x_0, x_0) + R_m(t)(x_0) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-2} t \deg(x_0).$$

Here, we denote the ceiling function by $\lceil \cdot \rceil$.

Proof. First of all, we prove the first identity. For $m \geq 3$, $x_0 \in VX$ and $e \in E_{x_0}$, we have

$$C_m(t)(x_0, e, \bar{e}) = \sum_{C=(e, e_2, \dots, e_{m-1}, \bar{e}), e_2 = \bar{e}, e_{m-1} \neq e} t^{\mathrm{bc}(C)} + \sum_{C=(e, e_2, \dots, e_{m-1}, \bar{e}), e_2 = \bar{e}, e_{m-1} = e} t^{\mathrm{bc}(c)} + \sum_{C=(e, e_2, \dots, e_{m-1}, \bar{e}), e_2 \neq \bar{e}, e_{m-1} \neq e} t^{\mathrm{bc}(C)} + \sum_{C=(e, e_2, \dots, e_{m-1}, \bar{e}), e_2 \neq \bar{e}, e_{m-1} = e} t^{\mathrm{bc}(C)}.$$

Then, we have

$$\begin{aligned}
&C_m(t)(x_0,e,\bar{e})\\
&=t\big(C_{m-2}(t)(t(e),\bar{e})-C_{m-2}(t)(t(e),\bar{e},e)\big)+t^2C_{m-2}(t)(t(e),\bar{e},e)\\
&\quad+\big(C_{m-2}(t)(t(e))-C_{m-2}(t)(t(e),\bar{e})-C_{m-2}(t)(t(e),\cdot,e)+C_{m-2}(t)(t(e),\bar{e},e)\big)\\
&\quad+t\big(C_{m-2}(t)(t(e),\cdot,e)-C_{m-2}(t)(t(e),\bar{e},e)\big)\\
&=C_{m-2}(t)(t(e),t(e))+(t-1)C_{m-2}(t)(t(e),\bar{e})\\
&\quad+(t-1)C_{m-2}(t)(t(e),\cdot,e)+(t-1)^2C_{m-2}(t)(t(e),\bar{e},e).
\end{aligned}$$

By this, we have

$$\begin{aligned}
&C_m(t)(x_0,e,\bar{e})\\
&=C_{m-2}(t)(t(e),t(e))+(t-1)\big(C_{m-2}(t)(t(e),\bar{e})-N_{m-2}(t,t(e),\bar{e})\big)\\
&\quad+(t-1)\big(C_{m-2}(t)(t(e),\cdot,e)-N_{m-2}(t,t(e),\cdot,e)\big)\\
&\quad+(t-1)\big(N_{m-2}(t,t(e),\bar{e})+N_{m-2}(t,t(e),\cdot,e)\big)+(t-1)^2C_{m-2}(t)(t(e),\bar{e},e)\\
&=C_{m-2}(t)(t(e),t(e))+2(t-1)N_{m-2}(t,t(e),\bar{e})+(t^2-1)C_{m-2}(t)(t(e),\bar{e},e).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
C_m(t)(x_0,e,\bar{e})&=C_{m-2}(t)(t(e),t(e))+2(t-1)N_{m-2}(t,t(e),\bar{e})\\
&\quad+(t^2-1)C_{m-2}(t)(t(e),\bar{e},e).
\end{aligned}\tag{4}$$

In the case that $m\geq 5$, by (4), we have

$$\begin{aligned}
&C_m(t)(x_0,e,\bar{e})\\
&=C_{m-2}(t)(t(e),t(e))+2(t-1)N_{m-2}(t,t(e),\bar{e})\\
&\quad+(t^2-1)\Big\{C_{m-4}(t)(x_0,x_0)+2(t-1)N_{m-4}(t,x_0,e)+(t^2-1)C_{m-4}(t)(x_0,e,\bar{e})\Big\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&C_m(t)(x_0,x_0)-N_m(t,x_0)\\
&=\big(\deg(x_0)C_{m-2}(t)(x_0,x_0)-\Delta_XC_{m-2}(t)(x_0,x_0)\big)\\
&\quad+2(t-1)N_{m-2}(t,x_0)+(t^2-1)\deg(x_0)C_{m-4}(t)(x_0,x_0)+(t^2-1)2(t-1)N_{m-4}(t,x_0)\\
&\quad+(t^2-1)^2\big(C_{m-4}(t)(x_0,x_0)-N_{m-4}(t,x_0)\big).
\end{aligned}\tag{5}$$

Hence, for $m\geq 1$, we have the following.

$$\begin{aligned}
&N_{m+4}(t,x_0)-2(1-t)N_{m+2}(t,x_0)+(1-t^2)(1-t)^2N_m(t,x_0)\\
&=C_{m+4}(t)(x_0,x_0)-\deg(x_0)C_{m+2}(t)(x_0,x_0)-(1-t^2)^2C_m(t)(x_0,x_0)\\
&\quad+\deg(x_0)(1-t^2)C_m(t)(x_0,x_0)+\Delta_XC_{m+2}(t)(x_0,x_0).
\end{aligned}$$

It is easy to check that this implies the following desired identity:

$$\big(1-(1-t)^2u^2\big)N(t,x_0:u)=\big(1-(\deg(x_0)-(1-t^2))u^2\big)C(t,x_0:u)\\ -\deg(x_0)tu^2+\frac{u^2}{1-(1-t^2)u^2}\Delta_X C(t,\cdot:u)(x_0).$$

Next, we prove the second identity. For $m=3$, by (4), we have

$$N_3(t,x_0)-(1-t)^2N_1(t,x_0)\\ =C_3(t)(x_0,x_0)-\big(\deg(x_0)-(1-t^2)\big)C_1(t)(x_0,x_0)+\Delta_X C_1(t)(x_0,x_0).$$

Then, we have

$$N_3(t,x_0)=C_3(t)(x_0,x_0)-\big(\deg(x_0)-2(1-t)\big)C_1(t)(x_0,x_0)+\Delta_X C_1(t)(x_0,x_0).$$

For $m=4$, by (4), we have

$$\begin{aligned}N_4(t,x_0)&=C_4(t)(x_0,x_0)-(\deg(x_0)-(1-t^2))C_2(t)(x_0,x_0)\\ &\quad+\Delta_X C_2(t)(x_0,x_0)+(1-t)^2N_2(t,x_0)\\ &=C_4(t)(x_0,x_0)-(\deg(x_0)-(1-t^2))C_2(t)(x_0,x_0)\\ &\quad+\Delta_X C_2(t)(x_0,x_0)+(1-t)^2(C_2(t)(x_0,x_0)-t\deg(x_0))\\ &=C_4(t)(x_0,x_0)-(\deg(x_0)-2(1-t))C_2(t)(x_0,x_0)\\ &\quad+\Delta_X C_2(t)(x_0,x_0)-(1-t)^2t\deg(x_0).\end{aligned}$$

Therefore, the second identity holds for $m=3,4$. In the case $m\geq 5$, the identity (5) is equivalent to the following identity.

$$N_m(t,x_0)-(1-t)^2N_{m-2}(t,x_0)-(1-t^2)\big(N_{m-2}(t,x_0)-(1-t)^2N_{m-4}(t,x_0)\big)\\ =C_m(t)(x_0,x_0)-(1-t^2)C_{m-2}(t)(x_0,x_0)\\ -\big(\deg(x_0-(1-t^2)\big)\big(C_{m-2}(t)(x_0,x_0)-(1-t^2)C_{m-4}(t)(x_0,x_0)\big)\\ +\Delta_X C_{m-2}(t)(x_0,x_0). \tag{6}$$

By summing the both sides of (6), we have the desired identity. □

Next, we prove the following theorem that is our goal in this section.

Theorem 3.2. *For a vertex x_0, we have the following identity:*

$$\big(1-(1-t^2)u^2\big)\big(1-(1-t)^2u^2\big)C^{\mathrm{cbc}}(t,x_0:u)\\ =\big\{1-(1-t)(\deg(x_0)-\Delta_X+2t)u^2+(1-t^2)(1-t)\big(\deg(x_0)-(1-t)\big)u^4\big\}C(t,\cdot:u)(x_0)\\ -\big(1-(1-t^2)\big)t\deg(x_0)(1-t)u^2.$$

Moreover, for $m \geq 3$, we have the following identity:

$$C_m(t,x_0) = C_m(t)(x_0,x_0) - \frac{\deg(x_0) - 2(1-t)}{1-t} \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} (1-t)^{2j} C_{m-2j}(t)(x_0,x_0)$$
$$+ (1-t)R_m(t)(x_0) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-1} t \deg(x_0).$$

Proof. For $m \geq 1$, $C_m(t,x_0) - N_m(t,x_0)$ (resp. $C_m(t)(x_0,x_0) - N_m(t,x_0)$) represents the number of closed paths with weight $t^{\mathrm{cbc}(\cdot)}$ (resp. $t^{\mathrm{bc}(\cdot)}$) of length m starting at x_0, which have no tail. Hence, we have

$$C_m(t,x_0) - N_m(t,x_0) = t\big(C_m(t)(x_0,x_0) - N_m(t,x_0)\big).$$

Then, we have

$$C^{\mathrm{cbc}}(t,x_0 : u) = tC(t,c_0 : u) + (1-t)N(t,x_0 : u).$$

By this and Proposition 3.1, we have

$$\begin{aligned}
&\big(1-(1-t)^2u^2\big)C^{\mathrm{cbc}}(t,x_0 : u) \\
&= \Bigg((1-t)(1-\big(\deg(x_0)-(1-t^2)\big)u^2) + t(1-(1-t)^2u^2)\Bigg)C(t,x_0 : u) \\
&\quad - (1-t)t\deg(x_0)u^2 + \frac{(1-t)u^2}{1-(1-t^2)u^2}\Delta_X C(t,\cdot : u)(x_0).
\end{aligned}$$

By simple calculation, this implies the first equality. Next, we verify the second equality. By Proposition 3.1, for $m \geq 3$, we have

$$\begin{aligned}
C_m(t,x_0) &= t\big(C_m(t)(x_0,x_0) - N_m(t,x_0)\big) \\
&\quad + \big(N_m(t,x_0) - C_m(t)(x_0,x_0)\big) + C_m(t)(x_0,x_0) \\
&= C_m(t)(x_0,x_0) - (1-t)\big(C_m(t)(x_0,x_0) - N_m(t,x_0)\big) \\
&= C_m(t)(x_0,x_0) \\
&\quad - (1-t)(\deg(x_0) - 2(1-t)) \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} (1-t)^{2(j-1)} C_{m-2j}(t)(x_0,x_0) \\
&\quad + (1-t)R_m(t)(x_0) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-1} t \deg(x_0).
\end{aligned}$$

□

In the end of this section, we note generating functions which we defined in this section. We define the generating function of $R_m(t)(x_0)$ as follows.

$$R(t,x_0 : u) = \sum_{m=1}^{\infty} R_m(t)(x_0) u^m.$$

It is straightforward to check that the following holds. If $|t| < 1$, $t \neq 0$ and $|u| < \frac{1}{2}$, then, we have

$$R(t, x_0 : u) = \frac{u^2}{\big(1-(1-t)^2u^2\big)\big(1-(1-t^2)u^2\big)} \Delta_X C(t, \cdot : u)(x_0).$$

Therefore, all generating functions which we defined in this section are expressed by $C(t, x_0 : u)$. We note that the above formula holds for $t = 0$.

4. A generalized Bartholdi zeta function formula for simple graphs with bounded degree

In this section, we introduce a Bartholdi zeta function for a graph with bounded degree. This zeta function is a generalization of the Bartholdi zeta function from a finite graph to a graph with bounded degree ([1]).

First of all, we define a Bartholdi zeta function for a graph with bounded degree. For a graph X with bounded degree, a vertex x_0 and complex variables t, u, we define a *Barholdi zeta function* as follows.

$$Z_X(u, t, x_0) = \exp\Bigg(\sum_{C \in \mathcal{C}_{x_0}} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)} \Bigg).$$

This is a natural generalization of the Ihara zeta function for a graph with bounded degree which was introduced in [3] in the spirit of L. Bartholdi although he introduced by using the Euler product expression. Before we give an Ihara type formula for this zeta function, we define several operators and give several properties of $C_m(t)$. We define $I(t)$, Q_X and $Q_X(t)$ as follows.

$$\begin{aligned} I(t) &= (1-t)I, \\ Q_X &= D_X - I, \\ Q_X(t) &= D_X - I(t). \end{aligned}$$

Then, we have the following proposition.

Proposition 4.1. *For $m \geq 2$, we have*

$$C_m(t) = \begin{cases} C_1(t)^2 - (1-t)(Q_X + I) & \textit{if } m = 2, \\ C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t) & \textit{if } m \geq 3. \end{cases}$$

We give the proof of this proposition although this proposition was proved in [19] because our proof is a little different from [19].

Proof. It is enough to show that for $x_0, x \in VX$,

$$C_m(t)(x_0, x) = \begin{cases} \big(C_1(t)^2 - (1-t)(Q_X + I)\big)(x_0, x) & \text{if } m = 2, \\ \big(C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t)\big)(x_0, x) & \text{if } m \geq 3. \end{cases}$$

For $m = 2$, it is obvious. In the case $m \geq 3$, for $x_0, x \in VX$, we have

$$C_{m-1}(t)C_1(t)(x_0, x) = C_{m-1}(t)C_1(t)\delta_{x_0}(x) = \sum_{C \in \mathcal{B}_x, \ell(C)=m-1} \sum_{e \in E^{x_0}_{t(C)}} t^{\mathrm{bc}(C)}.$$

By considering whether a path has backtracking at the last step and comparing $C_{m-1}(t)C_1(t)(x_0, x)$ to $C_m(t)(x_0, x)$, we have

$$\begin{aligned} &C_{m-1}(t)C_1(t)(x_0, x) - C_{m-2}(t)(x_0, x)t - C_{m-2}(t)(x_0, x)(\deg(x_0) - 1) \\ &\quad + C_{m-2}(t)(x_0, x)t^2 + C_{m-2}(t)(x_0, x)(\deg(x_0) - 1)t \\ &= C_m(t)(x_0, x). \end{aligned}$$

Therefore, we have

$$\begin{aligned} C_m(t) &= C_{m-1}(t)C_1(t) - C_{m-2}(t)t - C_{m-2}(t)Q_X + C_{m-2}(t)t^2 + C_{m-2}(t)Q_X t \\ &= C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t). \end{aligned}$$

□

For a complex valuable t, we define $\alpha(t)$ by

$$\alpha(t) = \frac{M + \sqrt{M^2 + 4(|t|+1)M}}{2}.$$

Here, we denote the maximum of all degrees of X by M. Then, we have the following Lemma.

Lemma 4.2. *For $|t| < 1$, then for $m \geq 0$, we have*

$$\|C_m(t)\| \leq \alpha(t)^m.$$

Proof. We prove this by induction on m. For $m = 0, 1$, there is nothing to do. We suppose that our assertion holds for $m - 1$. Then, we have

$$\begin{aligned} \|C_m(t)\| &= \|C_{m-1}(t)C_1(t) - (1-t)C_{m-2}(t)Q_X(t)\| \\ &\leq M\alpha(t)^{m-1} + (1+|t|)\alpha(t)^{m-2}(M - 1 + |t|) \\ &= \alpha(t)^{m-2}\big\{\alpha(t)M + (1+|t|)(M-1+|t|)\big\} \\ &= \alpha(t)^{m-2}(\alpha(t)^2 + |t|^2 - 1) < \alpha(t)^m. \end{aligned}$$

□

By Proposition 4.1 and Lemma 4.2, we have the following proposition.

Proposition 4.3. *For $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$, we have*

$$\Big(\sum_{m=0}^{\infty} C_m(t)u^m\Big)\Big(I - uA_X + (1-t)Q_X(t)u^2\Big) = (1 - (1-t)^2u^2)I,$$

$$\Big(\sum_{m=0}^{\infty} (\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j})u^m\Big)\Big(I - uA_X + (1-t)Q_X(t)u^2\Big) = I.$$

Next, for $m \leq 0$, t, $x \in VX$ and $f \in \ell^2(VX)$, we define an operator $R_m(t)$ by

$$R_m(t)f(x) = R_m(t)(x)f(x).$$

Then, we define an operator $C_m^{\mathrm{cbc}}(t)$ like the operator $N_{X,m}$ introduced in [18] by

$$C_m^{\mathrm{cbc}}(t) = \begin{cases} C_m(t) & \text{if } m = 0, 1, \\ tC_2(t) & \text{if } m = 2, \\ C_m(t) - \frac{Q_X(t)-I(t)}{1-t} \sum_{j=1}^{\lceil \frac{m}{2} \rceil - 1} (1-t)^{2j} C_{m-2j}(t) & \\ +(1-t)R_m(t) - \delta_{2\mathbb{Z}}(m)(1-t)^{m-1} tD_X & \text{if } m \geq 3. \end{cases}$$

We remark that this operator is also a bounded operator by our assumption. We also remark that $C_m^{\mathrm{cbc}}(t)(x_0, x_0) = C_m(t, x_0)$ and the following identity holds by Theorem 3.2.

$$Z_X(u, t, x_0) = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} C_m(t, x_0) u^m \right).$$

Therefore, for $x_0, x \in VX$, we define $Z_X(u, t, x_0, x)$ as follows.

$$Z_X(u, t, x_0, x) = \exp\left(\sum_{m=1}^{\infty} \frac{C_m^{\mathrm{cbc}}(t)(x_0, x)}{m} u^m \right).$$

Moreover, we define $f(z)$ as follows.

$$f(z) = zA_X - z^2(1-t)Q_X(t).$$

Then, we have the following Proposition ([18]).

Proposition 4.4. *For $|u| < \frac{1}{\alpha(t)}$, we have*

$$f'(u)(I - f(u))^{-1} = -\frac{d}{du}\log(I - f(u)) + (1-t)u^2 \sum_{n=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j f(u)^{n-1-j}(A_X Q_X(t) - Q_X(t)A_X) f(u)^{j-1}.$$

Under the above preparation, we give the following theorem.

Theorem 4.5. *For $|u| < \frac{1}{\alpha(t)}, |t| < 1$ and $x_0, x \in VX$, we have*

$$\begin{aligned} &Z_X(u, t, x_0, x) \\ &= (1 - (1-t)^2u^2)^{-\frac{\deg(x_0)-2}{2}\delta_{x_0}(x)} \exp\Big(- \left[\log(I - u(D_X - \Delta_X) + (1-t)Q_X(t)u^2)\right](x_0, x) \Big) \\ &\times \exp\left(\int_0^u (1-t)z^2 \sum_{n=2}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j \left[f(z)^{n-1-j}(A_X D_X - D_X A_X) f(z)^{j-1} \right](x_0, x) dz \right) \\ &\times \exp\left(\frac{\left[tD_X - C_2(t)\right](x_0, x)}{2}(1-t)u^2 \right) \exp\left(\sum_{m=3}^{\infty} \frac{(1-t)R_m(t)(x_0, x)}{m} u^m \right). \end{aligned}$$

Proof. We consider the following power series that converges in $|u| < \frac{1}{\alpha(t)}, |t| < 1$.

$$\sum_{m=0}^{\infty} C_m^{\mathrm{cbc}}(t) u^m.$$

By the definition of $C_m^{\mathrm{cbc}}(t)$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} C_m^{\mathrm{cbc}}(t) u^m &= \frac{Q_X(t)}{1-t} \sum_{m=0}^{\infty} C_m(t) u^m - \frac{Q_X(t)-I(t)}{1-t} \sum_{m=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j} u^m \\ &+ \big(Q_X(t)-I(t)\big) \frac{(1-t)u^2}{1-(1-t)^2u^2} + (1-t) \sum_{m=3}^{\infty} R_m(t) u^m \\ &+ C_2(t)(t-1)u^2 - \frac{t(1-t)^3u^4}{1-(1-t)^2u^2} D_X.\end{aligned}$$

Here, we denote the floor function by $\lfloor \cdot \rfloor$. In the right hand side of the above equation, the following equality holds.

$$\begin{aligned}&\big(Q_X(t)-I(t)\big) \frac{(1-t)u^2}{1-(1-t)^2u^2} - \frac{t(1-t)^3u^4}{1-(1-t)^2u^2} D_X \\ &= t(1-t)u^2 D_X + \frac{(1-t)^2u^2}{1-(1-t)^2u^2}(Q_X - I).\end{aligned}$$

Hence, we have

$$\begin{aligned}\sum_{m=0}^{\infty} C_m^{\mathrm{cbc}}(t) u^m &= \frac{Q_X(t)}{1-t} \sum_{m=0}^{\infty} C_m(t) u^m - \frac{Q_X(t)-I(t)}{1-t} \sum_{m=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j} u^m \\ &+ (Q_X - I) \frac{(1-t)^2u^2}{1-(1-t)^2u^2} + (1-t)u^2\big(tD_X - C_2(t)\big) \\ &+ (1-t) \sum_{m=3}^{\infty} R_m(t) u^m.\end{aligned}$$

By this, we have

$$\begin{aligned}\sum_{m=1}^{\infty} C_m^{\mathrm{cbc}}(t) u^m &= \frac{Q_X(t)}{1-t} \sum_{m=0}^{\infty} C_m(t) u^m - I - \frac{Q_X(t)-I(t)}{1-t} \sum_{m=0}^{\infty} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} C_{m-2j}(t)(1-t)^{2j} u^m \\ &+ (Q_X - I) \frac{(1-t)^2u^2}{1-(1-t)^2u^2} + (1-t)u^2\big(tD_X - C_2(t)\big) \\ &+ (1-t) \sum_{m=3}^{\infty} R_m(t) u^m.\end{aligned}$$

By Proposition 4.3, we have

$$\begin{aligned}
&\frac{Q_X(t)}{1-t}\sum_{m=0}^{\infty} C_m(t)u^m - I - \frac{Q_X(t)-I(t)}{1-t}\sum_{m=0}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} C_{m-2j}(t)(1-t)^{2j}u^m \\
&= \frac{Q_X(t)}{1-t}\big(1-(1-t)^2u^2\big)(I-f(u))^{-1} - I - \frac{Q_X(t)-I(t)}{1-t}(I-f(u))^{-1} \\
&= uf'(u)(I-f(u))^{-1}.
\end{aligned}$$

By Proposition 4.4, we have

$$\begin{aligned}
&\frac{Q_X(t)}{1-t}\sum_{m=0}^{\infty} C_m(t)u^m - I - \frac{Q_X(t)-I(t)}{1-t}\sum_{m=0}^{\infty}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} C_{m-2j}(t)(1-t)^{2j}u^m \\
&= -u\frac{d}{du}\log(I-f(u)) \\
&\quad + (1-t)u^3\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} jf(u)^{n-1-j}\big(A_XQ_X(t)-Q_X(t)A_X\big)f(u)^{j-1}.
\end{aligned}$$

Therefore, for $x_0, x \in VX$, we have

$$\begin{aligned}
&u\frac{d}{du}\sum_{m=1}^{\infty}\frac{C_m^{\mathrm{cbc}}(t)(x_0,x)}{m}u^m \\
&= -\left[u\frac{d}{du}\log(I-f(u))\right](x_0,x) \\
&\quad + (1-t)u^3\sum_{m=1}^{\infty}\frac{1}{n}\sum_{j=1}^{n-1} j\big[f(u)^{n-1-j}\big(A_XQ_X(t)-Q_X(t)A_X\big)f(u)^{j-1}\big](x_0,x) \\
&\quad - \frac{\deg(x_0)-2}{2}\delta_{x_0}(x)u\frac{d}{du}\big[\log(1-(1-t)^2u^2)\big] + \frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}u\frac{d}{du}\big[(1-t)u^2\big] \\
&\quad + u\frac{d}{du}\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m.
\end{aligned}$$

Dividing by u and integrating from 0 to u, we have

$$\sum_{m=1}^{\infty} \frac{C_m^{\mathrm{cbc}}(t)(x_0,x)}{m} u^m$$
$$= -\Big[\log(I-f(u))\Big](x_0,x_0)$$
$$+ \int_0^u (1-t)z^2 \sum_{m=1}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j\big[f(z)^{n-1-j}\big(A_X Q_X(t) - Q_X(t) A_X\big) f(z)^{j-1}\big](x_0,x)dz$$
$$- \frac{\deg(x_0)-2}{2}\delta_{x_0}(x)\log(1-(1-t)^2u^2) + \frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2$$
$$+ \sum_{m=3}^{\infty} \frac{(1-t)R_m(t)(x_0,x)}{m} u^m.$$

This implies the following identity

$$Z_X(u,t,x_0,x)$$
$$= (1-(1-t)^2u^2)^{-\frac{\deg(x_0)-2}{2}\delta_{x_0}(x)}$$
$$\times \exp\Bigg(- \big[\log(I - u(D_X - \Delta_X) + (1-t)Q_X(t)u^2)\big]\,(x_0,x)\Bigg)$$
$$\times \exp\Bigg(\int_0^u (1-t)z^2 \sum_{n=2}^{\infty} \frac{1}{n} \sum_{j=1}^{n-1} j\,\big[f(z)^{n-1-j}(A_X D_X - D_X A_X) f(z)^{j-1}\big]\,(x_0,x)dz\Bigg)$$
$$\times \exp\Bigg(\frac{\big[tD_X - C_2(t)\big](x_0,x)}{2}(1-t)u^2\Bigg) \exp\Bigg(\sum_{m=3}^{\infty} \frac{(1-t)R_m(t)(x_0,x)}{m} u^m\Bigg).$$

□

If X is a $(q+1)$-regular graph, we have

$$I - (D_X - \Delta_X)u + (1-t)Q_X(t)u^2 = I - \big((q+1)I - \Delta_X\big)u + (1-t)(q+t)u^2 I.$$

Since Δ_X is a self-adjoint bounded operator, there exists a unique spectral measure E such that

$$\Delta_X = \int_{\sigma(\Delta_X)} \lambda dE(\lambda).$$

Here, we denote the spectrum of the Laplacian Δ_X by $\sigma(\Delta_X)$. By Theorem 4.5 and the property of the spectral integral, we have

$$\begin{aligned} Z_X(u,t,x_0,x) &= (1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)} \\ &\times \exp\left(\int_{\sigma(\Delta_X)} -\log(1-(q+1-\lambda)u+(1-t)(q+t)u^2)d\mu_{x_0,x}(\lambda)\right) \\ &\times \exp\left(\frac{\left[tD_X-C_2(t)\right](x_0,x)}{2}(1-t)u^2\right) \\ &\times \exp\left(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m\right). \end{aligned}$$

Here, we denote $d\langle E(\lambda)\delta_{x_0},\delta_{x_0}\rangle$ by $d\mu_{x_0,x_0}(\lambda)$. Hence, we get the following corollary.

Corollary 4.6. *For* $|u|<\frac{1}{\alpha(t)}$, $|t|<1$, *we have*

$$\begin{aligned} Z_X(u,t,x_0,x) &= (1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)} \\ &\times \exp\left(\int_{\sigma(\Delta_X)} -\log(1-(q+1-\lambda)u+(1-t)(q+t)u^2)d\mu_{x_0,x}(\lambda)\right) \\ &\times \exp\left(\frac{\left[tD_X-C_2(t)\right](x_0,x)}{2}(1-t)u^2\right) \\ &\times \exp\left(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m\right). \end{aligned}$$

Moreover, we discuss the case that X is a finite $(q+1)$-regular graph. We introduce the notion of the local spectrum ([7]). For a vertex $x\in VX$, we denote x-local multiplicity of λ_i by $\mathrm{m}_x(\lambda_i)$. Here, the *x-local multiplicity* of λ_i is the xx-entry of the primitive idempotent E_{λ_i}. Let $\{\mu_0=\lambda_0,\mu_1,\dots,\mu_{d_x}\}$ be the set of eigenvalues whose local multiplicities are positive. For each vertex $x\in VX$, we denote the x-local spectrum by $\sigma_x(X)$. Here, the *x-local spectrum* is $\sigma_x(X)=\{\lambda_0^{\mathrm{m}_x(\lambda_0)},\mu_1^{\mathrm{m}_x(\mu_1)},\dots,\mu_{d_x}^{\mathrm{m}_x(\mu_{d_x})}\}$.

Then, we have the following corollary immediately by Corollary 4.6.

Corollary 4.7. *For* $|u|<\frac{1}{\alpha(t)}$, $|t|<1$, *we have*

$$\begin{aligned} Z_X(t,u,x_0,x) &= (1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)} \\ &\prod_{\lambda\in\sigma_{x_0}(\Delta_X)} \left(1-(q+1-\lambda)u+(1-t)(q+t)u^2\right)^{-m_{x_0}(\lambda)} \\ &\times \exp\left(\frac{\left[tD_X-C_2(t)\right](x_0,x)}{2}(1-t)u^2\right) \\ &\times \exp\left(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m\right). \end{aligned}$$

5. The Euler product expression

In this section, we give the Euler product expression of the Bartholdi zeta function which is introduced in Section 4. We have to introduce several terminologies to give the Euler product expression. We take a vertex x_0. A closed path C starting at x_0 is *primitive* if there is no closed paths starting at x_0 whose length is shorter than $\ell(C)$ and of which the multiple is C. We denote by $\mathcal{PK}_{x_0}$ the set of primitive closed paths starting at x_0. Then, the following theorem holds.

Theorem 5.1. *For $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$, we have*

$$Z_X(t, u, x_0) = \prod_{C \in \mathcal{PK}_{x_0}} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}}.$$

Proof. For $|u| < \frac{1}{\alpha(t)}$, $|t| < 1$ and $N \in \mathbb{Z}_{\geq 1}$, we have

$$\begin{aligned}
\log \prod_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}} &= - \sum_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} \frac{1}{\ell(C)} \log(1 - t^{\mathrm{cbc}(C)} u^{\ell(C)}) \\
&= \sum_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} \sum_{m=1}^{\infty} \frac{1}{\ell(C)m} t^{\mathrm{cbc}(C)m} u^{\ell(C)m} \\
&= \sum_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} \sum_{m=1}^{\infty} \frac{1}{\ell(C^m)} t^{\mathrm{cbc}(C^m)} u^{\ell(C^m)} \\
&= \sum_{C} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)}.
\end{aligned}$$

Here, the last sum runs through the set of closed paths starting at x_0 such that the length of primitive paths is less than or equal to N. Therefore, for any N, we have,

$$\exp\left(\sum_{C} \frac{1}{\ell(C)} t^{\mathrm{cbc}(C)} u^{\ell(C)} \right) = \prod_{C \in \mathcal{PK}_{x_0}, \ell(C) \leq N} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}}$$

Here, the sum runs through the same set as the above. Taking the limit of the both sides, we have

$$Z_X(t, u, x_0) = \prod_{C \in \mathcal{PK}_{x_0}} (1 - t^{\mathrm{cbc}(C)} u^{\ell(C)})^{-\frac{1}{\ell(C)}}.$$

□

6. The heat kernels on regular graphs

In this section, for a regular graph, we give a new expression of the heat kernels on regular graphs by using the modified Bessel function of the first kind. Let X be a

$(q+1)$-regular graph. We denote the heat kernel of X by $K_X(\tau, x_0, x)$. For $j \in \mathbb{Z}_{\geq 0}$ and real variable t which satisfies $|t| < 1$, we define the symbol $d_j(t)$ as follows.

$$d_j(t) = \begin{cases} 1 & \text{if } j = 0, \\ -\frac{q-1+2t}{1-t} & \text{if } j \geq 1. \end{cases}$$

Then, the following theorem holds.

Theorem 6.1. *For $\tau \in \mathbb{R}_{\geq 0}$, $x \in VX$ and $|t| < 1$, we have*

$$K_X(\tau, x_0, x) = \sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t)\, \mathrm{e}^{-(q+1)\tau} \\ \times (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j}{2}} \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right).$$

Proof. We define $g(\tau, x)$ as follows.

$$g(\tau, x) = \mathrm{e}^{(q+1)\tau} f(\tau, x).$$

Then, it turns out that the heat equation is equivalent to the following equation.

$$\begin{cases} \frac{\partial g}{\partial \tau}(\tau, x) - C_1(t) g(\tau, \cdot)(x) = 0, \\ g(0, x) = \delta_{x_0}(x). \end{cases}$$

Here, we remark that $C_1(t)$ is equal to A_X. It is sufficient to show that the following is the solution of the above equation.

$$g(\tau, x) = \sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t) \\ \times (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j}{2}} \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right).$$

It is obvious that $g(0, x) = \delta_{x_0}(x)$. Therefore, it remains to check that $g(\tau, x)$ is bounded on $[0, T] \times VX$ for each T and $g(\tau, x)$ satisfies the above equation indeed.

First, we check that $g(\tau, x)$ is bounded. By Proposition 4.1 and (2), we have

$$|g(\tau, x)| \leq \sum_{n=0}^{\infty} \alpha(t)^n \sum_{j=0}^{\infty} |d_j(t)|\, |1-t|^{2j}\, \tau^{n+2j} \frac{\mathrm{e}^{2\sqrt{(q+t)(1-t)}\tau}}{(n+2j)!}.$$

We denote the maximum of $d_j(t)$ by M_t. Then, we have

$$\begin{aligned} |g(\tau, x)| &\leq M_t\, \mathrm{e}^{2\sqrt{(q+t)(1-t)}\tau} \sum_{n=0}^{\infty} \alpha(t)^n \sum_{j=0}^{\infty} \frac{\tau^n (2\tau)^{2j}}{(n+2j)!} \\ &\leq M_t\, \mathrm{e}^{2\sqrt{(q+t)(1-t)}\tau} \sum_{n=0}^{\infty} \frac{\big(\alpha(t)\tau\big)^n}{n!} \sum_{j=0}^{\infty} \frac{(2\tau)^{2j}}{(2j)!} \\ &= M_t\, \mathrm{e}^{(2\sqrt{(q+t)(1-t)}+\alpha(t))\tau} \cosh(2\tau). \end{aligned}$$

Therefore, $g(\tau, x)$ is bounded on $[0, T] \times VX$ for each T.

Second, we check that $g(\tau, x)$ satisfies the equation. To prove this, we define $g(\tau)$ as follows.

$$g(\tau) = \sum_{n=0}^{\infty} C_n(t) \sum_{j=0}^{\infty} d_j(t) \times (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j}{2}} \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right).$$

Then,

$$\frac{\partial g}{\partial \tau}(\tau, x) - C_1(t) g(\tau, \cdot)(x) = \left\{ \frac{\partial g}{\partial \tau}(\tau) - C_1(t) g(\tau) \right\}(x_0, x).$$

Therefore, it is sufficient to check that

$$\frac{\partial g}{\partial \tau}(\tau) - C_1(t) g(\tau) = 0.$$

$$\begin{aligned}
&\frac{\partial g}{\partial \tau}(\tau) - C_1(t) g(\tau) \\
&= \sum_{n=0}^{\infty} C_n(t) \sum_{j=0}^{\infty} d_j(t) (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{n=0}^{\infty} C_n(t) \sum_{j=0}^{\infty} d_j(t) (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}} \mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad - \sum_{n=1}^{\infty} C_1(t) C_{n-1}(t) \\
&\qquad \times \sum_{j=0}^{\infty} d_j(t) (1-t)^{2j} \big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right).
\end{aligned}$$

Here, we used (1) and we remark that we are allowed to change the order of the differentiation and power series by (2). By Proposition 4.1, we have

$$C_n(t) = \begin{cases} C_1(t)^2 - (1-t)(q+1)I & \text{if } n = 2, \\ C_1(t) C_{n-1}(t) - (1-t)(q+t) C_{n-2}(t) & \text{if } n \geq 3. \end{cases}$$

Here, we remark that the operator $C_n(t)$ is a self-adjoint operator. By this relation, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} C_1(t)C_{n-1}(t) \\
&\quad \times \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&= C_1(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-j} \mathrm{I}_{2j}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + C_2(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + (1-t)(q+1)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{n=3}^{\infty} C_n(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{n=1}^{\infty} C_n(t) \\
&\qquad \times \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}} \mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right).
\end{aligned}$$

To explain our calculation clearly, we put

$$\begin{aligned}
(n=1) &= C_1(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-j} \mathrm{I}_{2j}\left(2\sqrt{(1-t)(q+t)}\tau\right), \\
(n=2-1) &= C_2(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right), \\
(n=2-2) &= (1-t)(q+1)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{1+2j}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right), \\
(n=3-1) &= \sum_{n=3}^{\infty} C_n(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right), \\
(n=3-2) &= \sum_{n=1}^{\infty} C_n(t)\sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}} \mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right).
\end{aligned}$$

By calculating $(n=1)+(n=2-1)+(n=3-1)$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} C_1(t)C_{n-1}(t) \\
&\quad\times \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&= \sum_{n=1}^{\infty} C_n(t) \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n-1+2j}{2}} \mathrm{I}_{n+2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + (n=2-2)+(n=3-2).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{\partial g}{\partial \tau}(\tau) - C_1(t)g(\tau) \\
&= \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{n=0}^{\infty} C_n(t) \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j-1}{2}} \mathrm{I}_{n+2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad - (n=2-2)-(n=3-2) \\
&= \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) - (n=2-2) \\
&= \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad + \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\left(1-\frac{q+1}{q+t}\right)\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&= \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j-1}\left(2\sqrt{(1-t)(q+t)}\tau\right) \\
&\quad - \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j}\left(\frac{1-t}{q+t}\right)\big((1-t)(q+t)\big)^{-\frac{2j-1}{2}} \mathrm{I}_{2j+1}\left(2\sqrt{(1-t)(q+t)}\tau\right).
\end{aligned}$$

Here, we note that $C_0(t)=I$. Therefore, we have

$$\frac{\partial g}{\partial \tau}(\tau) - C_1(t)g(\tau) = 0.$$

Here, we used $\mathrm{I}_{-1}(\tau) = \mathrm{I}_1(\tau)$. This completes the proof. □

7. An alternative proof of the Bartholdi zeta function formula

In this section, we give an alternative proof of the Bartholdi zeta function formula for a regular graph obtained in Section 4.

Since Δ_X is a self-adjoint bounded operator, there exists a unique spectral measure E such that

$$\Delta_X = \int_{\sigma(\Delta_X)} \lambda dE(\lambda).$$

Here, $\sigma(\Delta_X)$ stands for the spectrum of Δ_X. Therefore, for $x_0, x \in VX$, we have

$$K_X(\tau, x_0, x) = \int_{\sigma(\Delta)} \mathrm{e}^{-\tau\lambda}\, d\mu_{x_0,x}(\lambda).$$

Here, we denote $d\langle E(\lambda)\delta_{x_0}, \delta_x\rangle$ by $d\mu_{x_0,x}(\lambda)$. By applying the $G(t)$-transform and by easy culculation, for $0 < u < \frac{1}{\alpha(t)}$, we have

$$\begin{aligned} G(t)\big(K_X(\tau, x_0, x)\big)(u) &= G(t)\bigg(\int_{\sigma(\Delta)} \mathrm{e}^{-\tau\lambda}\, d\mu_{x_0,x}(\lambda)\bigg)(u) \\ &= \int_{\sigma(\Delta_X)} \frac{u^{-2} - (q+t)(1-t)}{(q+t)(1-t)u + \frac{1}{u} - (q+1-\lambda)} d\mu_{x_0,x}(\lambda). \end{aligned}$$

We remark that we are allowed to change the order of integrations in the above equation by Fubini's theorem. We also remark that if $0 < u < \frac{1}{\alpha(t)}$, then we have

$$(q+t)(1-t)u + \frac{1}{u} - (q+1-\lambda) > 0.$$

On the other hand, by Theorem 6.1, we have

$$\begin{aligned} G(t)\big(K_X(\tau, x_0, x)\big)(u) = G(t)\bigg(&\sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t)\, \mathrm{e}^{-(q+1)\tau} \\ &\times (1-t)^{2j}\big((1-t)(q+t)\big)^{-\frac{n+2j}{2}}\, \mathrm{I}_{n+2j}\left(2\sqrt{(1-t)(q+t)}\tau\right)\bigg)(u) \\ = &\sum_{n=0}^{\infty} C_n(t)(x_0, x) \sum_{j=0}^{\infty} d_j(t)(1-t)^{2j} u^{n+2j-1}. \end{aligned}$$

Here, we used (3) in the second equation. Therefore, we have

$$\begin{aligned} G(t)(K_X(\tau, x_0, x))(u) = &\sum_{n=0}^{\infty} C_n(x_0, x) u^{n-1} \\ &- \frac{q-1+2t}{1-t} \sum_{n=2}^{\infty} \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} C_{n-2j}(t)(x_0, x)(1-t)^{2j} u^{n-1}. \end{aligned}$$

We note that the following equality holds.

$$\sum_{n=2}^{\infty}\sum_{j=1}^{\lfloor\frac{n}{2}\rfloor} C_{n-2j}(t)(x_0,x)(1-t)^{2j}u^n = \sum_{n=3}^{\infty}\sum_{j=1}^{\lceil\frac{n}{2}\rceil-1}(1-t)^{2j}C_{n-2j}(t)(x_0,x)u^n + C_0(t)(x_0,x)\frac{(1-t)^4u^4}{1-(1-t)^2u^2} + C_0(t)(x_0,x)(1-t)^2u^2.$$

Hence, we have

$$\begin{aligned}
&G(t)(K_X(\tau,x_0,x))(u)\\
&= \sum_{n=3}^{\infty} C_n(t)(x_0,x)u^{n-1} - \frac{q-1+2t}{1-t}\sum_{n=3}^{\infty}\sum_{j=1}^{\lceil\frac{n}{2}\rceil-1}(1-t)^{2j}C_{n-2j}(t)(x_0,x)u^{n-1}\\
&\quad + \frac{1}{u}\big(C_0(t)(x_0,x) + C_1(t)(x_0,x)u + tC_2(t)(x_0,x)u^2\big)\\
&\quad + u(1-t)C_2(t)(x_0,x) - (q-1+2t)\frac{(1-t)u}{1-(1-t)^2u^2}C_0(t)(x_0,x).
\end{aligned}$$

By the definition of $C_m^{\mathrm{cbc}}(t)$, we have

$$\begin{aligned}
&G(t)\big(K_X(\tau,x_0,x)\big)(u)\\
&= \frac{1}{u}C_0(t)(x_0,x) + \frac{1}{u}\sum_{n=1}^{\infty}\big(C_n^{\mathrm{cbc}}(t)(x_0,x) - (1-t)R_m(t)(x_0,x)\big)u^m\\
&\quad + (1-t)u\big(C_2(t) - tD_X\big)(x_0,x) - (q-1)\frac{(1-t)^2u}{1-(1-t)^2u^2}C_0(t)(x_0,x).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{\sigma(\Delta_X)} \frac{u^{-2}-(q+t)(1-t)}{(q+t)(1-t)u + \frac{1}{u} - (q+1-\lambda)}d\mu_{x_0,x}(\lambda)\\
&= \frac{1}{u}C_0(t)(x_0,x) + \frac{1}{u}\sum_{n=1}^{\infty}\big(C_n^{\mathrm{cbc}}(t)(x_0,x) - (1-t)R_m(t)(x_0,x)\big)u^m\\
&\quad + (1-t)u\big(C_2(t) - tD_X\big)(x_0,x) - (q-1)\frac{(1-t)^2u}{1-(1-t)^2u^2}C_0(t)(x_0,x).
\end{aligned}$$

This is equivalent to the following equation.

$$\begin{aligned}&\frac{d}{du}\Bigg\{\frac{q-1}{2}C_0(t)\log(1-(1-t)^2u^2)+\sum_{n=1}^{\infty}\frac{C_n^{\mathrm{cbc}}(t)-(1-t)R_n(t)}{n}u^n\\&\quad+\frac{C_2(t)-tD_X}{2}(1-t)u^2\Bigg\}(x_0,x)\\&=\frac{d}{du}\int_{\sigma(\Delta_X)}-\log\big(1-(q+1-\lambda)u+(1-t)(q+t)u^2\big)d\mu_{x_0,x}(\lambda)\end{aligned}$$

By integrating both sides from 0 to u and determining the integrating constant, we have

$$\begin{aligned}&(1-(1-t)^2)^{\frac{q-1}{2}\delta_{x_0}(x)}Z_X(u,t,x_0,x)\exp\Bigg(-\sum_{n=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}\Bigg)\\&\quad\times\exp\Bigg(\frac{\big[C_2(t)-tD_X\big](x_0,x)}{2}(1-t)u^2\Bigg)\\&=\exp\Bigg(\int_{\sigma(\Delta_X)}-\log\big(1-(q+1-\lambda)u+(1-t)(q+t)u^2\big)d\mu_{x_0,x}(\lambda)\Bigg).\end{aligned}$$

Therefore, we have

$$\begin{aligned}Z_X(u,t,x_0,x)&=(1-(1-t)^2u^2)^{-\frac{q-1}{2}\delta_{x_0}(x)}\\&\quad\times\exp\Bigg(\int_{\sigma(\Delta_X)}-\log(1-(q+1-\lambda)u+(1-t)(q+t)u^2)d\mu_{x_0,x}(\lambda)\Bigg)\\&\quad\times\exp\Bigg(\frac{\big[tD_X-C_2(t)\big](x_0,x)}{2}(1-t)u^2\Bigg)\\&\quad\times\exp\Bigg(\sum_{m=3}^{\infty}\frac{(1-t)R_m(t)(x_0,x)}{m}u^m\Bigg).\end{aligned}$$

This gives an alternative proof of the Bartholdi zeta function formula obtained in Section 4.

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