Validity of Borodin and Kostochka Conjecture for $4K_1$–free Graphs

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Abstract: Problem of finding an optimal upper bound for the chromatic no. of even $3K_1$-free graphs is still open and pretty hard. Here we prove **Borodin & Kostochka Conjecture** for $4K_1$-free graphs G i.e. if $\Delta(G) \geq 9$ and G is $4K_1$-free, then $\chi(G) \leq \max\{\omega, \Delta-1\}$.

Introduction:

In [1], [2], [3], [4] chromatic bounds for graphs are considered especially in relation with ω and Δ. Gyárfás [5] and Kim [6] show that the optimal χ-binding function for the class of $4K_1$-free graphs has order $\omega^2/\log(\omega)$. If we forbid additional induced subgraphs, the order of the optimal χ-binding function drops below $\omega^2/\log(\omega)$. In 1941, Brooks' theorem stated that for any connected undirected graph *G* with maximum degree Δ, the chromatic number of *G* is at most Δ unless *G* is a complete graph or an odd cycle, in which case the chromatic number is $\Delta + 1$ [5]. In 1977, **Borodin & Kostochka** [6] conjectured that if $\Delta(G) \geq 9$, then $\chi(G) \leq \max\{\omega, \Delta-1\}$. In 1999, Reed proved the conjecture for $\Delta \geq 10^{14}$ [7]. Also D. W. Cranston and L. Rabern [8] proved it for claw-free graphs. Here we prove **Borodin & Kostochka** conjecture for $4K_1$-free graphs.

Notation: For a graph G, V(G), E(G), Δ, ω, χ denote the vertex set, edge set, maximum degree, size of a maximum clique, chromatic number of G resply. For $u \in V(G)$, $N(u) = \{v \in V(G) / uv \in E(G)\}$, and $\overline{N(u)} = N(u) \cup (u)$. If $S \subseteq V$, then $\langle S\rangle$ denotes subgraph of G induced by S. If C is some coloring of G and if $u \in V(G)$ is colored m in C, then u is called a m-vertex, if N(u) has a unique r-vertex, then r is called a unique color of u and if N(u) has more than one r-vertex, then r is called a repeat color of u. Also if P is a path in G s.t. vertices on P are alternately colored say i and j, then P is called an i-j path. All graphs considered henceforth are simple. We consider here simple and undirected graphs. For terms which are not defined herein we refer to Bondy and Murty [9].

Main Result: Let G be $4K_1$-free and $\Delta \geq 9$, then $\chi \leq \max\{\Delta-1, \omega\}$.
Proof: Let if possible G be a smallest, connected, $4K_1$-free graph with $\Delta \geq 9$ and $\chi > \max\{\Delta-1, \omega\}$. Then clearly as $G \neq C_{2n+1}$ or $K_{|V(G)|}$, $\chi = \Delta > \omega$. Let $u \in V(G)$. Then $G\text{-}u \neq K_{|V(G)|-1}$ (else $\chi = \omega$). **If $\Delta(G\text{-}u) \geq 9$**, then by minimality $\chi(G\text{-}u) \leq \max\{\omega(G\text{-}u), \Delta(G\text{-}u)-1\}$. Clearly if $\omega(G\text{-}u) \leq \Delta(G\text{-}u)-1$, then $\chi(G\text{-}u) = \Delta(G\text{-}u)-1 \leq \Delta-1$ and otherwise $\chi(G\text{-}u) = \omega(G\text{-}u) \leq \omega < \Delta$. In any case $\chi(G\text{-}u) \leq \Delta-1$. **Also if $\Delta(G\text{-}u) < 9$**, then as $G\text{-}u \neq C_{2n+1}$, by Brook's Theorem $\chi(G\text{-}u) \leq \Delta(G\text{-}u) < 9 \leq \Delta$. Thus always $\chi(G\text{-}u) \leq \Delta-1$ and in fact, $\chi(G\text{-}u) = \Delta-1$ and $\deg v \geq \Delta-1 \;\forall\, v \in V(G)$.

Let $u \in V(G)$ be s.t. $\deg u = \Delta$. Let $S = \{1,..., \Delta-1, \Delta\}$ be a Δ-coloring of G with only u colored Δ. Then N(u) has $\Delta-2$ vertices A_i with unique colors i ($1\leq i\leq\Delta-2$) and a pair of vertices say X, Y with the same color $\Delta-1$. Clearly A_i has a j-vertex for $1\leq i\neq j\leq\Delta-2$ (else color A_i by j, u by i).

Case 1: $\exists$ a $(\Delta-1)$-coloring of G-u s.t. $A_iA_j \notin E(G)$ for some i, j $\in \{1,.., \Delta-2\}$.

(A) For no m, A_m is the only m-vertex of both A_i and A_j for $1\leq i, j, m\leq\Delta-2$.
Let if possible A_i, A_k both have A_m as the only m-vertex. Then as A_m has at the most one repeat color, w.l.g. A_j be the only j-vertex of A_m. Then color A_i, A_j by m, A_m by j, u by i, a contradiction.
(B) A_i, A_j do not have more than two common adjacent A_ks in N(u).
Let A_i, A_j be both adjacent to say A_k, A_l, A_m $1\leq i, j, k, l, m\leq\Delta-2$. As each of A_i, A_j has at the most one repeat color, w.l.g. let A_m be the only m-vertex of both A_i and A_j, a contradiction to **(A)**.
(C) A_i is non-adjacent to at the most three A_k s $\Rightarrow$ As $\Delta-2 \geq 7$, A_i is adjacent to at least three A_m,$1\leq i$, k, $m\leq\Delta-2$.
Let if possible $A_1A_k \notin E(G)$ for $2\leq k\leq5$. As G is $4K_1$-free, $\exists$ at most two more 1-vertices a_{11}, a_{12} and as $\exists$ 1-k path from A_1 to A_k, either a_{11} or a_{12} is adjacent to A_k with two k-vertices for $2\leq k\leq5$. Again a_{1i} cannot have three repeat colors (else $N(a_{1i})$ has a color say r missing. Color a_{1i} by r. Then either **(i)**

some A_k ($2\leq k\leq 5$) has no 1-vertex, hence color A_k by 1, u by k or **(ii)** $a_{12}A_k \in E(G)$ ($2\leq k\leq 5$), a_{12} has four repeat colors and $N(a_{12})$ has color t missing. Color a_{12} by t, A_k by 1, u by k). Thus w.l.g. let A_ia_{11}, $A_ja_{12} \in E(G)$ for i = 2, 3 and j = 4, 5 s.t. a_{11} has two repeat colors 2, 3 and a_{12} has two repeat colors 4, 5. Clearly $A_1A_j \in E(G)$ $\forall$ $6\leq i\leq \Delta$-2 (else either a_{11} or a_{12} has three repeat colors).

Claim 1: Whenever A_1 has a unique i-vertex say B for $6\leq i\leq \Delta$-1, A_1 is the only 1-vertex of B.
Let if possible $Ba_{11} \in E(G)$. Then B has a unique m-vertex for $2\leq m\leq \Delta$-1 (else N(B) has some color r missing. Color B by r, A_1 by i, u by 1). As a_{11} has two repeat colors 2, 3, B is its only i-vertex. Then G has at the most one more i-vertex say b (else $<A_1, a_{11}, b_{11}, b_{12}> = 4K_1$). Again by **(A)**, B is not the only i–vertex of any A_k, for $2\leq k\leq 5$. Hence $A_kb \in E(G)$ for $2\leq k\leq 5$. Now $A_kB \notin E(G)$ for k = 2, 3 (else color A_k by 1, a_{11} by i, B by 2/3, A_1 by i, u by 1) and b has two k-vertices for k = 2, 3 (else color A_k by i, b by k, u by k) $\Rightarrow$ A_m is the only m-vertex of b for m = 4, 5 (else b has color r missing in N(b). Color b by r, A_2 by i, u by 2). Now A_m has two i-vertices (else b is the only i-vertex of A_m . Color b by m, A_m by i, u by m), m $\in$ {4, 5} $\Rightarrow$ a_{12} is the only 1-vertex of A_m, m $\in$ {4, 5}. Again $Ba_{12} \notin E(G)$ (else B has three 1-vertices and color say r missing in N(B). Color B by r, A_1 by i, u by 1) $\Rightarrow$ $a_{12}b \in E(G)$. Then color b by 4, A_4 by 1, a_{12} by i, u by 4, a contradiction. This proves **Claim 1**.

Now a_{1k} has an i-vertex for k = 1, 2 (else color a_{1k} by i. If a_{1k} is the only 1-vertex of A_m ($2\leq m\leq 5$), then color A_m by 1, u by m and if every A_m has two 1-vertices, then if k = 1 (2), color A_2 (A_4) by 1, a_{12} (a_{11}) by 2 (4), u by 2 (4)).

Let $a_{11}b_{i1} \in E(G)$. As a_{11} has two repeat colors 2, 3, b_{i1} is the only i-vertex of a_{11}.

Claim 2: a_{11} is the only 1-vertex of b_{i1}.
Let if possible $a_{12}b_{i1} \in E(G)$. As a_{12} has two repeat colors 4, 5, b_{i1} is the only i-vertex of a_{12}. Then G doesn't have an i-vertex say $b_{12} \notin \{B, b_{11}\}$ (else $<a_{11}, a_{12}, B, b_{12}> = 4K_1$). Again by **(A)**, B cannot be the only i-vertex of any A_m for $2\leq m\leq 5$. Hence $A_mb_{i1} \in E(G)$ for $2\leq m\leq 5$. **If A_k is the only k-vertex of b_{i1}** for some k, $2\leq k\leq 5$, then if b_{i1} is the only i-vertex of A_k, color A_k by i, b_{i1} by k, u by k and if A_k, has two i-vertices, then a_{1j} being the only 1-vertex of A_k, color A_k by 1, a_{1j} by i, b_{i1} by k, u by k, contradictions in both the cases. **Hence let b_{i1} have repeat colors k $\forall$ k, $2\leq k\leq 5$**. But then b_{i1} has color r missing in $N(b_{i1})$. Color b_{i1} by r and a_{11} by i. Then $A_2a_{12} \in E(G)$ (else color A_2 by 1, u by 2). Again as a_{12} has two repeat colors 4, 5, A_2 is its only 2-vertex and hence color A_2 by 1, a_{12} by 2, u by 2, a contradiction. This proves **Claim 2**.

Similarly if b_{i2} is an i-vertex of a_{12}, then a_{12} (b_{i2}) is the only 1-vertex (i-vertex) of b_{i2} (a_{12}). Now A_mb_{i1}, $A_nb_{i2}\in E(G)$ for m = 2, 3 and n = 4, 5 (else let $A_2b_{i1}\notin E(G)$. If a_{11} is the only 1-vertex of A_2, then color a_{11} by i, b_{i1} by 1, A_2 by 1, u by 2 and if $A_2a_{12}\in E(G)$, then color a_{11} by i, b_{i1} by 1, a_{12} by 2, A_2 by 1, u by 2).

As A_1 has at the most one repeat color, w.l.g. let A_1 have unique 2, 3, 4 vertices. Let P (R) be a 2-1 (4-1) path from A_2 (A_4) to A_1. As a_{12} (a_{11}) has a unique 2-vertex (4-vertex), clearly P = {A_2, a_{11}, a_{21}, A_1} and R = {A_4, a_{12}, a_{41}, A_1}.

Claim 3: $a_{21}a_{12}$, $A_2a_{12} \notin E(G)$. Similarly $a_{41}a_{11}$, $A_4a_{11} \notin E(G)$.
Let if possible $a_{21}a_{12} \in E(G)$. Then G has no other 2-vertex $a_{22} \notin \{A_2, a_{21}\}$ (else $<a_{22}, a_{12}, A_1, A_2> = 4K_1$). Also $a_{21}b_{i1} \in E(G)$ (else A_2 is the only 2-vertex of b_{i1}. If b_{i1} is the only i-vertex of A_2, then color b_{i1} by 2, A_2 by i, u by 2 and if A_2 has two i-vertices, then color b_{i1} by 2, A_2 by 1, a_{11} by i, u by 2). As a_{21} has three 1-vertices, $Ba_{21} \notin E(G)$ and hence $BA_2 \in E(G)$ (else color B by 2, A_1 by i, u by 1). Thus b_{i2} has no 2-vertex. Then if $A_4a_{11} \notin E(G)$, color b_{i2} by 2, a_{12} by i, A_4 by 1, u by 4 and if $A_4a_{11} \in E(G)$, color b_{i2} by 2, A_4 by i, u by 4, contradictions in both the cases. Hence $a_{21}a_{12} \notin E(G)$

$\Rightarrow$ $a_{21}b_{i1} \in E(G)$ (else color b_{i1} by 1, a_{11} by i, a_{21} by 1, A_1 by 2, u by 1).

Next let if possible $A_2a_{12} \in E(G)$. Then b_{i1} is the only i-vertex of A_2 and $A_2B \notin E(G)$. Also A_2 is the only 2-vertex of a_{12} and hence G has no other 2-vertex say a_{22} (else $<a_{22}, a_{12}, A_1, a_{11}> = 4K_1$) $\Rightarrow$ $Ba_{21} \in$

E(G) (else color B by 2, A_1 by i, u by 1). As a_{21} has two 1-vertices and i-vertices, $a_{21}b_{i2} \notin E(G)$. Also as A_2 has two 1-vertices $A_2b_{i2} \notin E(G)$. Color b_{i2} by 2, a_{12} by i. If a_{12} is the only 1-vertex of A_4, then color A_4 by 1, u by 4 and if $A_4a_{11} \in E(G)$, then color a_{11} by 4, A_4 by 1, u by 4, contradictions in both the cases. Hence $A_2a_{12} \notin E(G)$. This proves **Claim 3**.

Claim 4: Whenever A_1 has a unique i-vertex B for $6 \le i \le \Delta-1$, either A_2 or a_{21} has two i-vertices.
Let b_{i1} be the only i-vertex A_2. Now b_{i1} is not the only i-vertex of a_{21} (else $<a_{21}, A_2, b_{i2}, B> = 4K_1$). Thus a_{21} has two i-vertices. This proves **Claim 4.**

Now as $\Delta \ge 9$, and A_1 has at the most one repeat color, A_1 has at least two unique k-vertices for $k \in \{6, 7, ..., \Delta-1\}$. Let B, C be the unique i-vertex, k-vertex of A_1 resply for $i, k \in \{6, 7,..., \Delta-1\}$. Again as a_{21} has two 1-vertices, each of A_2 and a_{21} has at the most one other repeat color. By **Claim** 4, w.l.g. let A_2, a_{21} have two i-vetices, k-vertices resply. $\Rightarrow$ A_2, a_{21} has a unique 4-vertex each. Similarly A_4, a_{41} has a unique 2-vertex each. Now $A_2a_{41} \notin E(G)$ (else color a_{41} by 2, A_2 by 4, u by 2). Also $A_2A_4 \notin E(G)$ (else color A_4 by 2, A_2 by 4, a_{11} by 2, a_{21} by 1, A_1 by 2, u by 1) $\Rightarrow a_{21}a_{41} \in E(G)$ (else $<a_{21}, a_{41}, A_2, A_4> = 4K_1$). As a_{11} is the unique 1-vertex of A_2, color a_{41} by 2, a_{21} by 4, a_{11} by 2, A_2 by 1, u by 2, a contradiction.

This proves **(C).**

If $A_iA_j \notin E(G)$ $(1 \le i, j \le \Delta-2)$, then as $\Delta-2 \ge 7$, by **(C)**, $\exists$ m $(1 \le m \le \Delta-2)$ s.t. A_iA_m $A_jA_m \in E(G)$. Also by **(B)**, $\exists$ maximum two such m's $(1 \le m \le \Delta-2)$.

Case 1.1: $\exists$ i, j s.t. $A_iA_j \notin E(G)$ and $A_iA_k, A_iA_m, A_jA_k, A_jA_m \in E(G)$, $1 \le i, j, k, m \le \Delta-2$.
W.l.g. let i = 1, j = 2, k = 5, j = 6. Also by **(C)**, let $A_1A_4, A_2A_7 \in E(G)$. Then by **(B)**, $A_1A_7, A_2A_j \notin E(G)$. By **(A)**, w.l.g. let A_1, A_2 have two 5-vertices, 6-vertices resply. Clearly A_4 (A_7) is the unique 4-vertex (7-vertex) of A_1 (A_2). Also by **(C)**, A_7 is adjacent to at least one of A_i, $i \in \{3, 4, 6\}$ and if $A_7A_i \in E(G)$, $i \in \{3, 4, 6\}$, then A_7 has two i-vertices (else A_7, A_1 have a unique i-vertex A_i, a contradiction to **(A)**) and hence A_2 is the unique 2-vertex of A_7. Now A_3A_1 or $A_3A_2 \in E(G)$ (else by **(C)**, A_3 is adjacent to at least three of A_4, A_5, A_6, A_7 and either A_3, A_1 or A_3, A_2 have a common adjacent A_i s.t. A_i is their only i-vertex, a contradiction to **(A)**). W.l.g. let $A_3A_1 \in E(G)$. Again A_3 is the unique 3-vertex of A_1. Now $\exists$ 2-i paths from A_2 to A_i (i = 1, 3, 4). Also as G is $4K_1$-free, $\exists$ at most two more 2-vertices a_{21}, a_{22} and at least one of them say a_{21} has two repeat colors from {1, 3, 4}.

Case 1.1.1: $a_{21}A_3, a_{21}A_4 \in E(G)$ and a_{21} has two repeat colors 3, 4.
Then $a_{22}A_1 \in E(G)$ and a_{22} has two 1-vertices and a_{22} is the only 2-vertex of A_1. W.l.g. let a_{22} have a unique 3–vertex (else a_{22} has a color r missing in $N(a_{22})$. Color a_{22} by r, A_1 by 2, u by 1). Then $a_{22}A_3 \notin E(G)$ (else color a_{22} by 3, A_3 by 1, A_1 by 2, u by 3). Consider a 3-2 path T from A_3 to A_2 with a_{31} being the 3-vertex of A_2 on T. As a_{22} has a unique 3–vertex, clearly $a_{21}a_{31} \in E(G)$. Now $a_{22}a_{31} \in E(G)$ (else alter colors along $\{A_2, a_{31}, a_{21}, A_3\}$, color A_1 by 3, u by 1). Then G does not have a 3-vertex $a_{32} \notin \{A_3, a_{31}\}$ (else $<A_2, a_{22}, a_{32}, A_3> = 4K_1$). Now $A_7a_{31} \in E(G)$ (else A_3 is the only 3-vertex of both A_1 and A_7, contrary to **(A)**). But as a_{31} has three 2-vertices, A_7 is its only 7-vertex. Also by **(C)**, A_7 is adjacent to at least one A_j ($j \in \{3, 4, 6\}$ and has two j-vertices (else A_j is the only j-vertex of $A_1$and A_7, contrary to **(A)**). Hence A_2 is the only 2-vertex of A_7. Then color a_{31} by 7, A_7 by 2, A_2 by 3, u by 7, a contradiction. This proves **Case 1.1.1**.

Case 1.1.2: A_3, A_4 do not have a common adjacent 2-vertex.
W.l.g. let $a_{21}A_1, a_{21}A_3 \in E(G)$ and $a_{22}A_4 \in E(G)$. Then $a_{21}A_4, a_{22}A_3 \notin E(G)$. Clearly a_{21} has two 1-vertices and 3–vertices and hence a unique 4-vertex. Let a_{41} be the unique 4-vertex of A_2. Then as $\exists$ a 2-4 path S from A_2 to A_4, clearly $a_{22}a_{41} \in E(G)$. Now $a_{21}a_{41} \notin E(G)$ (else G does not have a 4-vertex $a_{42} \notin \{a_{41}, A_4\}$, as otherwise $<A_2, a_{21}, a_{42}, A_4> = 4K_1 \Rightarrow A_7a_{41} \in E(G)$ as otherwise A_7 and A_1 have a common unique 4-vertex A_4, a contradiction to **(A)**. But then color A_7 by 2, A_2 by 4, a_{41} by 7, u by 7). Let a_{42} be the unique 4-vertex of a_{21}. Then $a_{42}a_{22} \in E(G)$ (else alter colors along $\{A_4, a_{22}, a_{41}, A_2\}$, color A_1 by 4, u by 1). Thus a_{22} has three 4-vertices and hence a unique i-vertex for $1 \le i \le \Delta-1$, $i \notin \{2, 4\}$. Now A_4 has a unique j-vertex for j = 1 or 3. Consider a 2-j path T from A_2 to A_j and let a_{j1} be the

unique j-vertex of A_2. Clearly $a_{21}a_{j1} \in E(G)$. Again $a_{22}a_{j1} \notin E(G)$ (else G does not have a j-vertex $a_{j2} \notin \{a_{j1}, A_j\}$, as otherwise $<A_2, a_{22}, a_{j2}, A_j> = 4K_1 \Rightarrow A_7a_{j1} \in E(G)$. Color A_7 by 2, A_2 by j, a_{j1} by 7, u by 7). Hence $\exists$ a_{j2} s.t. $a_{22}a_{j2} \in E(G)$ (else color a_{22} by j, A_4 by 2, u by 4). Now clearly a_{22} is the unique 2-vertex of a_{j2} and vice versa $\Rightarrow A_4a_{j2} \in E(G)$ (else color a_{22} by j, a_{j2} by 2, A_4 by 2, u by 4). Clearly A_4 has two 1-vertices (else A_1 is its unique 1-vertex. Alter colors along $\{A_4, a_{22}, a_{41}, A_2\}$, color A_1 by 4, u by 1) $\Rightarrow$ j = 3 and a_{32} is the unique 3-vertex of $A_4 \Rightarrow A_3A_4 \notin E(G)$. Then by **(C)**, A_4 is adjacent to at least two A_ks for $k \in \{5, 6, 7\}$. Let $A_4A_m \in E(G)$, for m = 5 or 7. Then A_m is the unique m-vertex of A_4 and A_2, a contradiction to **(A).**

Case 1.2: $\forall$ i, j s.t. $A_iA_j \notin E(G)$, A_i, A_j have only one common adjacent A_k in N(u), $1 \le i, j, k \le \Delta-2$.
W.l.g. let i = 1, j = 2 and k = 3. By **(C)**, let $A_1A_m \in E(G)$ for m = 4, 5, $A_2A_l \in E(G)$ for l = 6, 7. **Let if possible $A_3A_4 \notin E(G)$**. Now A_4 is adjacent to at the most one of A_6, A_7 (else we get **Case 1.1** with A_2 and A_4) and hence by **(C)**, $A_4A_5 \in E(G)$. Also by **(C)**, w.l.g. let $A_4A_6 \in E(G)$. Again A_3A_5 , $A_3A_6 \notin E(G)$ (else we get **Case 1.1** with A_3 and A_4) and hence by **(C)**, $A_3A_7 \in E(G) \Rightarrow A_5A_7 \notin E(G)$ (else we get **Case 1.1** with A_3 and A_5) and $A_5A_6 \in E(G)$. But then we get **Case 1.1** with A_1 and A_6, a contradiction. **Hence $A_3A_i \in E(G)$ for $4 \le i \le 7$**. Again A_4A_5, $A_6A_7 \in E(G)$ (else we get **Case 1.1** with A_4, A_5 or A_6, A_7). Also either all A_i have two 3-vertices for $1 \le i \neq 3 \le 7$ or say A_1 has a unique 3-vertex. Again if A_1 has a unique 3-vertex, then A_2, A_6, A_7 all have two 3-vertices (else a contradiction to **(A)**). Hence w.l.g. let A_1, A_4, A_5 have two 3-vertices. As G is $4K_1$-free, G has at the most two 2-vertices say a_{2i} (i = 1, 2). W.l.g. let $A_1a_{21} \in E(G)$. Now a_{21} has at the most two repeat colors (else a color say r is missing in $N(a_{21})$. Color a_{21} by r, A_1 by 2, u by 1). Also as $\exists$ i-2 paths from A_i to A_2 for i = 1, 4, 5, either a_{21} or a_{22} has two j-vertices for j = 1, 4, 5. W.l.g. let a_{21} have two repeat colors 1, 4 with A_1a_{21}, $A_4a_{21} \in E(G) \Rightarrow A_5a_{22} \in E(G)$ and a_{22} has two 5-vertices. Again at least two of {1, 4, 5} are unique colors of A_2.

Case 1.2.1. A_2 has a unique 1-vertex and 5-vertex.
Let A_2a_{11}, $A_2a_{51} \in E(G)$. As a_{21} has two repeat colors 1, 4, it has a unique 5-vertex and clearly as $\exists$ 2-5 path from A_2 to A_5, $a_{22}a_{51} \in E(G)$. Now $a_{21}a_{51} \notin E(G)$ (else G doesn't have a 5-vertex $a_{52} \notin \{A_5, a_{51}\}$ as otherwise $<A_5, a_{52}, A_2, a_{21}> = 4K_1$. As a_{51} has three 2-vertices, A_6 is its only 6-vertex. Also a_{51} is the only 5-vertex of A_6. Color a_{51} by 6, A_6 by 5, u by 6) $\Rightarrow a_{21}a_{52} \in E(G)$. Also $a_{52}a_{22} \in E(G)$ (else color A_2 by 5, a_{51} by 2, a_{22} by 5, A_5 by 2, A_1 by 5, u by 1).. But then $a_{22}a_{11} \notin E(G)$ (else G doesn't have a 1-vertex $a_{12} \notin \{A_1, a_{11}\}$ as otherwise $<A_1, a_{12}, A_2, a_{22}> = 4K_1$ and a_{11} has three repeat colors 2, 6, 7 with color say r missing in $N(a_{11})$. Color a_{11} by r, A_2 by 1, u by 2) $\Rightarrow a_{11}a_{21} \in E(G)$. Let $a_{22}a_{12} \in E(G)$. Then a_{22} (a_{12}) is the only 2-vertex (1-vertex) of a_{12} (a_{22}). Color a_{22} by 1, a_{12} by 2, A_5 by 2, u by 5, a contradiction.

Case 1.2.2. A_2 has a unique 1-vertex and 4-vertex.
Let A_2a_{11}, $A_2a_{41} \in E(G)$. As a_{22} has two 5-vertices, w.l.g. let a_{22} have a unique 1-vertex. Then $a_{22}a_{11} \notin E(G)$ (else if $\exists$ a_{12}, then $<A_1, a_{12}, A_2, a_{22}> = 4K_1$ and if a_{12} doesn't exist, then a_{11} has three repeat colors 2, 6, 7 and color say r is missing in $N(a_{11})$. Color a_{11} by r, A_2 by 1, u by 2) $\Rightarrow a_{22}a_{12} \in E(G)$ and $a_{21}a_{11} \in E(G)$. Then a_{22} (a_{12}) is the only 2-vertex (1-vertex) of a_{12} (a_{22}). Color a_{22} by 1, a_{12} by 2, A_5 by 2, u by 5, a contradiction.

This proves **Case 1**.

Case 2: In every (Δ-1)-coloring of G-u, all vertices with unique colors in N(u) are adjacent.

Clearly $\Delta-1 \le \omega$ and hence $\Delta-1 = \omega \ge 8 \Rightarrow <\bigcup_{i=1}^{\Delta-2} A_i>$ is a maximum clique in G-u and $\{X, Y\} = N(u) - \bigcup_{i=1}^{\Delta-2} A_i$.

I. At most two vertices in $\bigcup_{i=1}^{\Delta-2} A_i$ are non-adjacent to both X and Y.

Let if possible A_1, A_2, A_3 be non-adjacent to both X and Y. Then clearly $\exists$ a (Δ-1)-vertex say Z in V(G) s.t. $ZA_i \in E(G)$ for i = 1, 2, 3. Moreover, as G is $4K_1$-free, Z is their only (Δ-1)-vertex. If A_i is the only i-vertex of Z for some i ($1 \le i \le 3$), then color A_i by Δ-1, Z by i, u by i, a contradiction. Hence Z has at least two i-vertices for i = 1, 2, 3. But then Z has some color r missing in N(Z). Color Z by r, A_i by Δ-1, u by i, a contradiction.

II. Every vertex A_i of N(u) has at least one j-vertex $j \neq i$ (else color A_i by j and u by i), $1 \le i, j \le \Delta-2$.

III. X (Y) has a k-vertex for every k = 1,.., Δ-2.

Let if possible X not have a k-vertex. Also as $<u \cup \bigcup_{i=1}^{\Delta-2} A_i >$ is a maximum clique in G, $\exists$ i ($1 \le i \le \Delta-2$) s.t. $YA_i \notin E(G)$. Then color X by k. Now i = k (else we get **Case 1** as Y and A_i are unique vertices in N(u)). As $\Delta \ge 9$ and each of Y and A_i has at the most one repeat color, clearly $\exists$ j ($1 \le j \le \Delta-2$) s.t. A_j is the only j-vertex of both Y and A_i. Also A_j has either a unique i-vertex A_i or (Δ-1)-vertex Y. Color Y and A_i by j, A_j by i (Δ-1), u by Δ-1 (j), a contradiction.

IV. X (Y) is adjacent to at least ω-5 vertices in $\bigcup_{i=1}^{\Delta-2} A_i$.

Let if possible X be non-adjacent to A_i, i = 1,.., 5. By **I**, w.l.g. let $YA_i \in E(G)$ for i = 1, 2, 3. Also let $YA_k \notin E(G)$ for some $k \ge 4$. By **II** and **III**, Y and A_k each has at the most one repeat color and hence w.l.g. let A_1 be the unique 1-vertex of Y and A_k. Now A_1 has two (Δ-1)-vertices (else color Y and A_k by 1, A_1 by Δ-1, u by k) $\Rightarrow$ A_k is the unique k-vertex of A_1. Then color Y and A_k by 1, A_1 by k and we get **Case 1** with two non-adjacent, unique vertices X, A_1, a contradiction.

V. X (Y) is not the only (Δ-1)-vertex of any A_i.

Let if possible X be the only (Δ-1)-vertex of some A_i. By **IV**, $\exists$ k, j s.t. XA_k, $XA_j \in E(G)$. Also let $XA_m \notin E(G)$ for some m. If A_i is the only i-vertex of X and A_m, then color X, A_m by i, A_i by Δ-1, u by m, a contradiction. Hence let A_i be not the only i-vertex of either X or A_m. As X and A_m have at the most one repeat color, w.l.g. let A_k be the only k-vertex of X and A_m. Again if X is the only (Δ-1)-vertex of A_k, then as before we get a contradiction. Hence let A_k have two (Δ-1)-vertices. But then color A_k by i, A_i by Δ-1, X by k, A_m by k, u by m, a contradiction.

By **IV**, w.l.g. let $XA_k \in E(G)$ for k = 1, 2, 3 and $XA_4 \notin E(G)$. Also w.l.g. let A_1 be the only 1-vertex of X and A_4. By **V**, A_1 has two (Δ-1)-vertices. If any A_i ($1 \le i \le \Delta-2$, $i \neq 4$) is non-adjacent to Y, then as before by coloring X, A_4 by 1 and A_1 by 4, we get **Case 1** and hence $YA_k \in E(G)$ for every $k \neq 4$. Similarly $XA_k \in E(G)$ for every $k \neq 4$. As $\Delta \ge 9$, $\exists$ i s.t. A_i is the only i-vertex of X, Y and A_4. Color X, Y, A_4 by i, A_i by Δ-1, u by 4, a contradiction.

This proves **Case 2** and completes the proof of the Main Result.

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