

# THE NUMBER OF REPRESENTATIONS BY A TERNARY SUM OF TRIANGULAR NUMBERS

MINGYU KIM AND BYEONG-KWEON OH

ABSTRACT. For positive integers  $a, b, c$ , and an integer  $n$ , the number of integer solutions  $(x, y, z) \in \mathbb{Z}^3$  of  $a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2} + c\frac{z(z-1)}{2} = n$  is denoted by  $t(a, b, c; n)$ . In this article, we prove some relations between  $t(a, b, c; n)$  and the numbers of representations of integers by some ternary quadratic forms. In particular, we prove various conjectures given by Z. H. Sun in [6].

## 1. INTRODUCTION

For a positive integer  $x$ , a non negative integer of the form  $T_x = \frac{x(x-1)}{2}$  is called a *triangular number*. For example,  $0, 1, 3, 6, 10, 15, \dots$  are triangular numbers. Since  $T_x = T_{1-x}$ ,  $T_x$  is a triangular number for any integer  $x$ . For positive integers  $a_1, a_2, \dots, a_k$ , a polynomial of the form

$$\mathcal{T}_{(a_1, \dots, a_k)}(x_1, \dots, x_k) = a_1 T_{x_1} + a_2 T_{x_2} + \dots + a_k T_{x_k}$$

is called a  $k$ -ary sum of triangular numbers. For a non negative integer  $n$ , we define

$$T(a_1, \dots, a_k; n) = \{(x_1, \dots, x_k) \in \mathbb{Z}^k : \mathcal{T}_{(a_1, \dots, a_k)}(x_1, \dots, x_k) = n\}$$

and  $t(a_1, \dots, a_k; n) = |T(a_1, \dots, a_k; n)|$ . One may easily show that

$$t(a_1, \dots, a_k; n) = |\{(x_1, \dots, x_k) \in (\mathbb{Z}_o)^k : a_1 x_1^2 + \dots + a_k x_k^2 = 8n + a_1 + \dots + a_k\}|,$$

where  $\mathbb{Z}_o$  is the set of all odd integers. Hence  $t(a_1, \dots, a_k; n)$  is closely related with the number of representations by some diagonal quadratic form of rank  $k$ . For example, if  $k = 3$  and  $a_1 = a_2 = a_3 = 1$ , then every integer solution  $(x, y, z)$  of  $x^2 + y^2 + z^2 = 8n + 3$  is in  $(\mathbb{Z}_o)^3$ . Therefore, for any positive integer  $n$ , we have

$$t(1, 1, 1; n) = |\{(x, y, z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = 8n + 3\}| = 24H(-(8n + 3)),$$

where  $H(-D)$  is the Hurwitz class number with discriminant  $-D$ . For various results in this direction, see [1], [2], [5] and [8].

Recently, Sun proved in [6] various relations between  $t(a_1, \dots, a_k; n)$  and the numbers of representations of integers by some diagonal quadratic forms. He also conjectured various relations between  $t(a, b, c; n)$  and the numbers of representations by some ternary diagonal quadratic forms.

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In this article, we consider the number  $t(a, b, c; n)$  of representations by a ternary sum of triangular numbers. We show that for any positive integers  $a, b, c$  such that  $(a, b, c) = 1$ ,  $t(a, b, c; n)$  is equal to the number of representations of a subform of the ternary diagonal quadratic form  $ax^2 + by^2 + cz^2$ , if  $a + b + c$  is not divisible by 8, or a difference of the numbers of representations of two ternary quadratic forms otherwise.

In Section 3, we prove all conjectures in [6] on ternary sums of triangular numbers, which are Conjectures 6.1~6.4 and 6.7. In fact, we generalize Conjectures 6.1 and 6.2 in [6], and prove these generalized conjectures. Note that Conjectures 6.5 and 6.6 in [6] are about quaternary sums of triangular numbers, which will be treated later.

An integral quadratic form  $f(x_1, x_2, \dots, x_k)$  of rank  $k$  is a degree 2 homogeneous polynomial

$$f(x_1, x_2, \dots, x_k) = \sum_{1 \leq i, j \leq k} a_{ij} x_i x_j \quad (a_{ij} = a_{ji} \in \mathbb{Z}).$$

We always assume that  $f$  is positive definite, that is, the corresponding symmetric matrix  $(a_{ij}) \in M_{k \times k}(\mathbb{Z})$  is positive definite. If  $a_{ij} = 0$  for any  $i \neq j$ , then we simply write  $f = \langle a_{11}, \dots, a_{kk} \rangle$ . For an integer  $n$ , if the Diophantine equation  $f(x_1, x_2, \dots, x_k) = n$  has an integer solution, then we say  $n$  is *represented by  $f$* . We define

$$R(f, n) = \{(x_1, \dots, x_k) \in \mathbb{Z}^k : f(x_1, \dots, x_k) = n\},$$

and  $r(f, n) = |R(f, n)|$ . Since we are assuming that  $f$  is positive definite, the above set is always finite. The genus of  $f$ , denoted by  $\text{gen}(f)$ , is the set of all quadratic forms that are locally isometric to  $f$ . The number of isometry classes in  $\text{gen}(f)$  is called the class number of  $f$ .

Any unexplained notations and terminologies on integral quadratic forms can be found in [3] or [4].

## 2. REPRESENTATIONS OF TERNARY SUMS OF TRIANGULAR NUMBERS

Let  $a, b$  and  $c$  be positive integers such that  $(a, b, c) = 1$ . Throughout this section, we assume, without loss of generality, that  $a$  is odd. We show that the number  $t(a, b, c; n)$  is equal to the number of representations of a subform of the ternary diagonal quadratic form  $ax^2 + by^2 + cz^2$ , if  $a + b + c$  is not divisible by 8, or a difference of the numbers of representations of two ternary quadratic forms otherwise.

Let  $f(x, y, z) = ax^2 + by^2 + cz^2$  be a ternary diagonal quadratic form. Recall that

$$t(a, b, c; n) = |\{(x, y, z) \in \mathbb{Z}^3 : f(x, y, z) = n, xyz \equiv 1 \pmod{2}\}|.$$

**Lemma 2.1.** *Assume that  $a + b + c$  is odd. For any positive integer  $n$ , we have*

$$t(a, b, c; n) = r(f(x, x - 2y, x - 2z), 8n + a + b + c).$$

*In particular, if  $a \equiv b \equiv c \pmod{4}$ , then we have*

$$t(a, b, c; n) = r(f(x, y, z), 8n + a + b + c).$$

*Proof.* Let  $g(x, y, z) = f(x, x - 2y, x - 2z)$ . Define a map  $\phi : T(a, b, c; n) \rightarrow R(g, n)$  by  $\phi(x, y, z) = (x, \frac{x-y}{2}, \frac{x-z}{2})$ . Then one may easily show that it is a bijective map.

Now, assume that  $a \equiv b \equiv c \pmod{4}$ . If  $ax^2 + by^2 + cz^2 = 8n + a + b + c$  for some integers  $x, y$  and  $z$ , then one may easily show that  $x, y$  and  $z$  are all odd. The lemma follows directly from this.  $\square$

**Lemma 2.2.** *Assume that  $S = a + b + c$  is even, and without loss of generality, we also assume that both  $a$  and  $b$  are odd and  $c$  is even. Then, for any positive integer  $n$ , we have*

$$t(a, b, c; n) = \begin{cases} r(f(x, y, z), 8n + S) & \text{if } S \equiv 2 \pmod{4} \text{ and } c \equiv 4 \pmod{8}, \\ r(f(x, y, y - 2z), 8n + S) & \text{if } S \equiv 2 \pmod{4} \text{ and } c \not\equiv 4 \pmod{8}, \\ 2r(f(x, x - 4y, z), 8n + S) & \text{if } S \equiv 4 \pmod{8} \text{ and } c \equiv 2 \pmod{4}, \\ 2r(f(x, x - 4y, x - 2z), 8n + S) & \text{if } S \equiv 4 \pmod{8} \text{ and } c \equiv 0 \pmod{4}, \end{cases}$$

and if  $S \equiv 0 \pmod{8}$ , then

$$t(a, b, c; n) = r(f(x, x - 2y, x - 2z), 8n + S) - r\left(f(x, y, z), 2n + \frac{S}{4}\right).$$

*Proof.* Since the proof is quite similar to each other, we only provide the proof of the fourth case, that is, the case when  $S \equiv 4 \pmod{8}$  and  $c \equiv 0 \pmod{4}$ . Let  $g(x, y, z) = f(x, x - 4y, x - 2z)$ . We define a map

$$\begin{aligned} \psi : \{(x, y, z) \in (\mathbb{Z}_o)^3 : f(x, y, z) = 8n + S, x \equiv y \pmod{4}\} \\ \rightarrow \{(x, y, z) \in \mathbb{Z}^3 : g(x, y, z) = 8n + S\} \text{ by } \psi(x, y, z) = \left(x, \frac{x-y}{4}, \frac{x-z}{2}\right). \end{aligned}$$

From the assumption, it is well defined. Conversely, assume that  $g(x, y, z) = 8n + S$  for some  $(x, y, z) \in \mathbb{Z}^3$ . Since

$$f(x, x - 4y, x - 2z) = ax^2 + b(x - 4y)^2 + c(x - 2z)^2 \equiv ax^2 + bx^2 + cx^2 \equiv Sx^2 \equiv S \pmod{8}$$

and  $S \equiv 4 \pmod{8}$ , the integer  $x$  is odd. Therefore, the map  $(x, y, z) \rightarrow (x, x - 4y, x - 2z)$  is an inverse map of  $\psi$ . The lemma follows from this and the fact that

$$t(a, b, c; n) = 2|\{(x, y, z) \in (\mathbb{Z}_o)^3 : f(x, y, z) = 8n + S, x \equiv y \pmod{4}\}|.$$

This completes the proof.  $\square$

### 3. SUMS OF TRIANGULAR NUMBERS AND DIAGONAL QUADRATIC FORMS

In this section, we generalize some conjectures given by Sun in [6] on the relations between  $t(a, b, c; n)$  and the numbers of representations of integers by some ternary quadratic forms, and prove these conjectures.

Let  $f(x_1, x_2, \dots, x_k)$  be an integral quadratic form of rank  $k$  and let  $n$  be an integer. For a vector  $\mathbf{d} = (d_1, \dots, d_k) \in (\mathbb{Z}/2\mathbb{Z})^k$ , we define

$$R_{\mathbf{d}}(f, n) = \{(x_1, \dots, x_k) \in R(f, n) : (x_1, \dots, x_k) \equiv (d_1, \dots, d_k) \pmod{2}\}.$$

The cardinality of the above set will be denoted by  $r_{\mathbf{d}}(f, n)$ . Note that

$$t(a, b, c; n) = r_{(1,1,1)}(ax^2 + by^2 + cz^2, 8n + a + b + c).$$

We also define

$$\tilde{R}_{(1,1)}(ax^2 + by^2, N) = \{(x, y) \in R_{(1,1)}(ax^2 + by^2, N) : x \not\equiv y \pmod{4}\}.$$

Note that if we define the cardinality of  $\tilde{R}_{(1,1)}(ax^2 + by^2, N)$  by  $\tilde{r}_{(1,1)}(ax^2 + by^2, N)$ , then we have

$$r_{(1,1)}(ax^2 + by^2, N) = 2 \cdot \tilde{r}_{(1,1)}(ax^2 + by^2, N).$$

**Lemma 3.1.** *Let  $m$  be a positive integer.*

(i) *If  $m \equiv 1 \pmod{4}$ , then we have*

$$2r_{(1,0)}(x^2 + 3y^2, m) = r_{(1,1)}(x^2 + 3y^2, 4m).$$

(ii) *If  $m \equiv 3 \pmod{4}$ , then we have*

$$2r_{(0,1)}(x^2 + 3y^2, m) = r_{(1,1)}(x^2 + 3y^2, 4m).$$

(iii) *If  $m \equiv 4 \pmod{8}$ , then we have*

$$2r_{(0,0)}(x^2 + 3y^2, m) = r_{(1,1)}(x^2 + 3y^2, m).$$

*Proof.* (i) Note that the map

$$\psi_1 : R_{(1,0)}(x^2 + 3y^2, m) \rightarrow \tilde{R}_{(1,1)}(x^2 + 3y^2, 4m) \text{ defined by } \psi_1(x, y) = (x + 3y, -x + y)$$

is a bijective map.

(ii) If we define a map

$$\psi_2 : R_{(0,1)}(x^2 + 3y^2, m) \rightarrow \tilde{R}_{(1,1)}(x^2 + 3y^2, 4m) \text{ by } \psi_2(x, y) = (x + 3y, -x + y),$$

then one may easily check that it is a bijective map.

(iii) One may easily show that if we define a map

$$\psi_3 : R_{(0,0)}(x^2 + 3y^2, m) \rightarrow \tilde{R}_{(1,1)}(x^2 + 3y^2, m) \text{ by } \psi_3(x, y) = \left( \frac{x + 3y}{2}, \frac{-x + y}{2} \right),$$

then it is a bijective map.  $\square$

**Lemma 3.2.** *Let  $a, b$  ( $a < b$ ) be positive odd integers such that  $\gcd(a, b) = 1$  and  $a + b \equiv 0 \pmod{8}$ . Then*

$$(3.1) \quad r_{(1,1)}(ax^2 + by^2, m) = r_{(1,1)}(ax^2 + by^2, 4m)$$

*for any integer  $m$  divisible by 8 if and only if  $(a, b) \in \{(3, 5), (1, 7), (1, 15)\}$ .*

*Proof.* Assume that Equation (3.1) holds for any integer  $m$  divisible by 8. Let  $a + b = 2^u k$  for some integer  $u \geq 3$  and an odd integer  $k$ . Note that  $1 \leq a < 2^{u-1}k$ .

First, we assume  $u \geq 5$ . Since

$$a \cdot 1^2 + (2^u k - a) \cdot 1^2 = 4 \cdot 2^{u-2}k \quad \text{and} \quad 2^{u-2}k \equiv 0 \pmod{8},$$

there exist odd integers  $x$  and  $y$  satisfying  $ax^2 + (2^u k - a)y^2 = 2^{u-2}k$ , which is a contradiction.

Next, assume that  $u = 4$ . Since

$$a \cdot 7^2 + (16k - a) \cdot 1^2 = 4(4k + 12a) \quad \text{and} \quad 4k + 12a \equiv 0 \pmod{8},$$

there exist two odd integers  $x_1, y_1$  such that  $ax_1^2 + (16k - a)y_1^2 = 4k + 12a$ . Thus,  $4k + 12a \geq 16k$  and hence  $k \leq a$ . Now, since  $a \cdot 1^2 + (16k - a) \cdot 1^2 = 16k$ , there are

two positive odd integers  $x_2, y_2$  with  $ax_2^2 + (16k - a)y_2^2 = 64k$ . Since  $16k - a > 8k$  by assumption, we have  $y_2^2 = 1$ . Furthermore, since  $ax_2^2 = a + 48k \leq 49a$ ,  $(x_2, a) = (3, 6k), (5, 2k)$  or  $(7, k)$ . Since  $a$  is odd, we have  $(a, b) = (1, 15)$  in this case.

Finally, we assume that  $u = 3$ . Since  $a \cdot 1^2 + (8k - a) \cdot 1^2 = 8k$ , there are positive odd integers  $x_3, y_3$  such that  $ax_3^2 + (8k - a)y_3^2 = 32k$ . Hence we have

$$(3.2) \quad y_3^2 = 1 \quad \text{and} \quad ax_3^2 = a + 24k.$$

Note that if  $x_3 = 3$ , then  $(a, b) = (3, 5)$  and if  $x_3 = 5$ , then  $(a, b) = (1, 7)$ . Assume that  $x_3 \geq 7$ , that is,  $2a \leq k$ . Since  $a \cdot 3^2 + (8k - a) \cdot 1^2 = 8k + 8a$ , there are two odd integers  $x_4, y_4$  such that  $ax_4^2 + (8k - a)y_4^2 = 32k + 32a$ . If  $y_4^2 \geq 9$ , then  $a + 72k - 9a \leq 32k + 32a$ , which is a contradiction to the assumption that  $2a \leq k$ . Hence we have

$$(3.3) \quad y_4^2 = 1 \quad \text{and} \quad ax_4^2 = 33a + 24k.$$

Now, by Equations (3.2) and (3.3), we have  $x_4^2 - x_3^2 = 32$ . Therefore,  $x_3^2 = 49$ ,  $x_4^2 = 81$ , and  $k = 2a$ . which is a contradiction to the assumption that  $k$  is odd.

To prove the converse, we define three maps

$$\chi_1 : \tilde{R}_{(1,1)}(3x^2 + 5y^2, m) \rightarrow \tilde{R}_{(1,1)}(3x^2 + 5y^2, 4m) \quad \text{by} \quad \chi_1(x, y) = \left( \frac{x - 5y}{2}, \frac{3x + y}{2} \right),$$

$$\chi_2 : \tilde{R}_{(1,1)}(x^2 + 7y^2, m) \rightarrow \tilde{R}_{(1,1)}(x^2 + 7y^2, 4m) \quad \text{by} \quad \chi_2(x, y) = \left( \frac{3x - 7y}{2}, \frac{x + 3y}{2} \right),$$

and

$$\chi_3 : \tilde{R}_{(1,1)}(x^2 + 15y^2, m) \rightarrow \tilde{R}_{(1,1)}(x^2 + 15y^2, 4m) \quad \text{by} \quad \chi_3(x, y) = \left( \frac{x + 15y}{2}, \frac{-x + y}{2} \right).$$

One may easily show that the above three maps are all bijective.  $\square$

**Theorem 3.3.** *Let  $a, b, c$  be positive integers such that  $(a, b, c) \neq (1, 1, 1)$  and  $\gcd(a, b, c) = 1$ . Assume that two of three fractions  $\frac{b}{a}, \frac{c}{b}, \frac{c}{a}$  are contained in  $\{1, \frac{5}{3}, 7, 15\}$ . Then, for any positive integer  $n$ , we have*

$$2t(a, b, c; n) = r(ax^2 + by^2 + cz^2, 4(8n + a + b + c)) - r(ax^2 + by^2 + cz^2, 8n + a + b + c).$$

*Proof.* Note that all of  $a, b$  and  $c$  are odd. Furthermore, from the assumption, one may easily show that

$$-a \equiv b \equiv c \pmod{8}, \quad a \equiv -b \equiv c \pmod{8} \quad \text{or} \quad a \equiv b \equiv -c \pmod{8}.$$

By switching the roles of  $a, b$  and  $c$  if necessary, we may assume  $a \equiv b \equiv -c \pmod{8}$ .

Then we have

$$\left( \frac{a}{(a, c)}, \frac{c}{(a, c)} \right), \left( \frac{b}{(b, c)}, \frac{c}{(b, c)} \right) \in \{(3, 5), (5, 3), (1, 7), (7, 1), (1, 15), (15, 1)\}.$$

Let

$$f = f(x, y, z) = ax^2 + by^2 + cz^2 \quad \text{and} \quad N = 8n + a + b + c.$$

One may easily show that if  $f(x, y, z) = 4N$ , then

$$(ax^2, by^2, cz^2) \equiv (0, 0, 4), (0, 4, 0), (a, 4, c), (4, 0, 0), (4, b, c), \quad \text{or} \quad (4, 4, 4) \pmod{8}.$$

Let

$$\begin{aligned} A &= \{(x, y, z) \in R(f, 4N) : y \equiv 2 \pmod{4}, xz \equiv 1 \pmod{2}\}, \\ B &= \{(x, y, z) \in R(f, 4N) : x \equiv 2 \pmod{4}, yz \equiv 1 \pmod{2}\}. \end{aligned}$$

Note that

$$r(f, 4N) - r(f, N) = |A| + |B|.$$

Thus it is sufficient to show  $t(a, b, c; n) = |A|$  and  $t(a, b, c; n) = |B|$ . To show the first equality, we apply Lemma 3.2 to show that

$$r_{(1,1,1)}(f, N) = \sum_{y \in \mathbb{Z}} r_{(1,1)}(ax^2 + cz^2, N - by^2) = \sum_{y \in \mathbb{Z}} r_{(1,1)}(ax^2 + cz^2, 4(N - by^2)) = |A|.$$

The proof of  $t(a, b, c; n) = |B|$  is quite similar to this. This completes the proof.  $\square$

*Remark 3.4.* All triples  $(a, b, c)$  satisfying the assumption of Theorem 3.3 are listed in Table 1 below. The triples marked with asterisk are exactly those that are listed in Conjecture 6.1 of [6].

$(1, 1, 7)^*$ , $(1, 1, 15)^*$ , $(3, 3, 5)$ , $(1, 7, 7)^*$ , $(3, 5, 5)$ , $(1, 7, 15)^*$ , $(1, 9, 15)^*$
$(1, 15, 15)^*$ , $(3, 5, 21)$ , $(1, 7, 49)$ , $(1, 15, 25)^*$ , $(3, 5, 35)$ , $(3, 5, 45)$ , $(1, 7, 105)$
$(3, 5, 75)$ , $(1, 15, 105)$ , $(3, 21, 35)$ , $(1, 15, 225)$ , $(9, 15, 25)$ , $(5, 21, 35)$ , $(7, 15, 105)$

TABLE 1.

**Theorem 3.5.** *Let  $a, b$  be relatively prime positive odd integers such that one of four fractions  $\frac{b}{a}, \frac{a}{b}, \frac{3a}{b}, \frac{b}{3a}$  is contained in  $\{\frac{5}{3}, 7, 15\}$ . Then, for any positive integer  $n$ , we have*

$$2t(a, 3a, b; n) = 3r(\langle a, 3a, b \rangle, 8n + 4a + b) - r(\langle a, 3a, b \rangle, 4(8n + 4a + b)).$$

*Proof.* Since all the other cases can be treated in a similar manner, we only consider the case when  $\frac{b}{3a} = \frac{5}{3}$ , that is,  $(a, 3a, b) = (1, 3, 5)$ . One may easily show that if  $x^2 + 3y^2 + 5z^2 = 4(8n + 9)$ , then

$$(x^2, 3y^2, 5z^2) \equiv (0, 0, 4), (1, 3, 0), (4, 0, 0), (4, 3, 5), \text{ or } (4, 4, 4) \pmod{8}.$$

Let

$$f = f(x, y, z) = x^2 + 3y^2 + 5z^2 \quad \text{and} \quad N = 8n + 9.$$

From the above observation, we have

$$\begin{aligned} 3r(f, N) - r(f, 4N) &= 3r_{(0,0,0)}(f, 4N) - r(f, 4N) \\ &= 2r_{(0,0,0)}(f, 4N) - r_{(1,1,0)}(f, 4N) - r_{(0,1,1)}(f, 4N). \end{aligned}$$

Therefore, it suffices to show that

$$2r_{(1,1,1)}(f, N) = 2r_{(0,0,0)}(f, 4N) - r_{(1,1,0)}(f, 4N) - r_{(0,1,1)}(f, 4N).$$

Since  $r_{(0,0,0)}(f, 4N) = r(f, N)$  and

$$r(f, N) = r_{(1,1,1)}(f, N) + r_{(1,0,0)}(f, N) + r_{(0,0,1)}(f, N),$$

it is enough to show that

$$r_{(1,0,0)}(f, N) = \frac{1}{2}r_{(1,1,0)}(f, 4N) \quad \text{and} \quad r_{(0,0,1)}(f, N) = \frac{1}{2}r_{(0,1,1)}(f, 4N).$$

To prove the first assertion, we apply (i) of Lemma 3.1 to show that

$$\begin{aligned} r_{(1,0,0)}(f, N) &= \sum_{z \in \mathbb{Z}} r_{(1,0)}(x^2 + 3y^2, N - 5z^2) \\ &= \frac{1}{2} \sum_{z \in \mathbb{Z}} r_{(1,1)}(x^2 + 3y^2, 4(N - 5z^2)) = \frac{1}{2}r_{(1,1,0)}(f, 4N). \end{aligned}$$

For the second assertion, we apply (iii) of Lemma 3.1 and Lemma 3.2 to show that

$$\begin{aligned} r_{(0,0,1)}(f, N) &= \sum_{z \in \mathbb{Z}} r_{(0,0)}(x^2 + 3y^2, N - 5z^2) \\ &= \frac{1}{2} \sum_{z \in \mathbb{Z}} r_{(1,1)}(x^2 + 3y^2, N - 5z^2) = \frac{1}{2}r_{(1,1,1)}(x^2 + 3y^2 + 5z^2, N) \\ &= \frac{1}{2} \sum_{x \in \mathbb{Z}} r_{(1,1)}(3y^2 + 5z^2, N - x^2) = \frac{1}{2} \sum_{x \in \mathbb{Z}} r_{(1,1)}(3y^2 + 5z^2, 4(N - x^2)) \\ &= \frac{1}{2}r_{(0,1,1)}(f, 4N). \end{aligned}$$

This completes the proof.  $\square$

*Remark 3.6.* All triples  $(a, 3a, b)$  satisfying the assumption of the Theorem 3.5 are listed in Table 2 below. Those triples marked with asterisk are exactly those that are listed in Conjecture 6.2 of [6].

$(1, 3, 5)^*$ , $(1, 3, 7)^*$ , $(1, 3, 15)^*$ , $(1, 3, 21)^*$ , $(1, 5, 15)^*$ , $(1, 3, 45)$
$(3, 5, 9)^*$ , $(1, 7, 21)^*$ , $(3, 5, 15)^*$ , $(3, 7, 21)^*$ , $(1, 15, 45)$ , $(5, 9, 15)$

TABLE 2.

**Theorem 3.7.** *Let  $(a, b, c) \in \{(1, 2, 15), (1, 15, 18), (1, 15, 30)\}$ . For any positive even integer  $n$ , we have*

$$(3.4) \quad 2t(a, b, c; n) = r(\langle a, b, c \rangle, 4(8n + a + b + c)) - r(\langle a, b, c \rangle, 8n + a + b + c).$$

*Proof.* First, assume that  $(a, b, c) = (1, 2, 15)$ . Let

$$f = f(x, y, z) = x^2 + 2y^2 + 15z^2 \quad \text{and} \quad N = 8n + 18.$$

One may easily show that if  $f(x, y, z) = 4N$ , then

$$(x^2, 2y^2, 15z^2) \equiv (0, 0, 0), (1, 0, 7), \quad \text{or} \quad (4, 0, 4) \pmod{8}.$$

Hence the right-hand side of Equation (3.4) is

$$r(f, 4N) - r(f, N) = r_{(1,0,1)}(f, 4N).$$

Note that

$$\begin{aligned} r_{(1,1,1)}(f, N) &= \sum_{y \in \mathbb{Z}} r_{(1,1)}(x^2 + 15z^2, (N - 2y^2)) \\ &= \sum_{y \in \mathbb{Z}} r_{(1,1)}(x^2 + 15z^2, 4(N - 2y^2)) = r_{(1,1,1)}(x^2 + 8y^2 + 15z^2, 4N) \\ &= |\{(x, y, z) \in R(f, 4N) : xz \equiv 1 \pmod{2}, y \equiv 2 \pmod{4}\}| \end{aligned}$$

by Lemma 3.2. Since

$$|\{(x, y, z) \in R(f, 4N) : xz \equiv 1 \pmod{2}, y \equiv 0 \pmod{4}\}| = r(x^2 + 32y^2 + 15z^2, 4N),$$

it suffices to show that

$$(3.5) \quad r_{(1,1,1)}(f, N) = r(x^2 + 32y^2 + 15z^2, 4N).$$

It is well known that

$$\text{gen}(f_1 = 4x^2 + 4y^2 + 8z^2 + 2xy) = \{f_1, f_2, f_3\},$$

where  $f_2 = 4x^2 + 6y^2 + 6z^2 + 4yz + 2xz + 2xy$ ,  $f_3 = 2x^2 + 6y^2 + 12z^2 + 6yz + 2xz$ , and

$$\text{gen}(g_1 = 4x^2 + 8y^2 + 18z^2 + 8yz + 4xz) = \{g_1, g_2 = 2x^2 + 10y^2 + 24z^2\}.$$

Note that

$$r_{(1,1,1)}(f, N) = r(x^2 + 2(x - 2y)^2 + 15(x - 2z)^2, N) = r(g_1, N).$$

On the other hand, the right-hand side of Equation (3.5) is

$$\begin{aligned} r(x^2 + 15y^2 + 32z^2, 4N) &= r((3x + y)^2 + 15(x + y)^2 + 32z^2, 4N) \\ &= r(12x^2 + 8y^2 + 16z^2 + 18xy, 2N) \\ &= r(48x^2 + 8y^2 + 16z^2 + 36xy, 2N) + r(12x^2 + 32y^2 + 16z^2 + 36xy, 2N) \\ &= 2r(f_1, N). \end{aligned}$$

Therefore, it suffices to show that for any positive even integer  $n = 2m$ ,

$$(3.6) \quad 2r(f_1, 16m + 18) = r(g_1, 16m + 18).$$

By the Minkowski-Siegel formula, we have

$$r(f_1, 16m+18) + 2r(f_2, 16m+18) + r(f_3, 16m+18) = r(g_1, 16m+18) + r(g_2, 16m+18).$$

If  $f_1(x, y, z) = 16m + 18$ , then one may easily check that  $x + 3y - 4z \equiv 0 \pmod{8}$ , and if  $f_2(x, y, z) = 16m + 18$ , then  $x - 6y + 2z \equiv 0 \pmod{8}$ . If we define a map

$$\begin{aligned} \phi_1 : \{(x, y, z) \in R(f_1, 16m + 18) : x + 3y - 4z \equiv 0 \pmod{16}\} \\ \rightarrow \{(x, y, z) \in R(f_2, 16m + 18) : x - 6y + 2z \equiv 0 \pmod{16}\} \end{aligned}$$



by  $\phi_1(x, y, z) = \left( \frac{12x + 4y + 16z}{16}, \frac{-11x - y + 12z}{16}, \frac{x - 13y - 4z}{16} \right)$ , then it is a bijective map. Furthermore, the map

$$\begin{aligned} \phi_2 : \{ (x, y, z) \in R(f_1, 16m + 18) : x + 3y - 4z \equiv 8 \pmod{16} \} \\ \rightarrow \{ (x, y, z) \in R(f_2, 16m + 18) : x - 6y + 2z \equiv 8 \pmod{16} \} \end{aligned}$$

defined by  $\phi_2(x, y, z) = \left( \frac{4x + 12y - 16z}{16}, \frac{-13x + y + 4z}{16}, \frac{-x - 11y - 12z}{16} \right)$  is also bijective. Therefore, we have

$$(3.7) \quad r(f_1, 16m + 18) = r(f_2, 16m + 18).$$

Note that the above equation does not hold, in general, if  $n$  is odd. If we define two maps

$$\phi_3 : R(\langle 8, 10, 24 \rangle, 16m + 18) \rightarrow R(f_1, 16m + 18) \text{ by } \phi_3(x, y, z) = (y + 2z, y - 2z, x)$$

and

$$\phi_4 : R(\langle 2, 24, 40 \rangle, 16m + 18) \rightarrow R(f_3, 16m + 18) \text{ by } \phi_4(x, y, z) = (x + z, 2y + z, -2z),$$

then one may easily check that both of them are bijective. Hence we have

$$\begin{aligned} r(g_2, 16m + 18) &= r(\langle 8, 10, 24 \rangle, 16m + 18) + r(\langle 2, 24, 40 \rangle, 16m + 18) \\ &= r(f_1, 16m + 18) + r(f_3, 16m + 18), \end{aligned}$$

for any non negative integer  $m$ . Therefore, from the Minkowski-Siegel formula given above, we have  $2r(f_2, 16m + 18) = r(g_1, 16m + 18)$  for any non negative integer  $m$ . Equation (3.6) follows directly from this and Equation (3.7).

For the other two cases, one may easily show Equation (3.4) by replacing  $N, f_i, g_i$  and  $\phi_i$  with the following data:

(1)  $(a, b, c) = (1, 15, 18)$ . In this case, we let  $N = 8n + 34$  and

$$\begin{aligned} f_1 &= 4x^2 + 4y^2 + 72z^2 + 2xy, \\ f_2 &= 4x^2 + 16y^2 + 22z^2 + 14yz - 2xz + 4xy, \\ f_3 &= 6x^2 + 16y^2 + 16z^2 - 8yz + 6xz + 6xy, \end{aligned}$$

and

$$g_1 = 4x^2 + 34y^2 + 34z^2 + 8yz + 4xz + 4xy, \quad g_2 = 10x^2 + 18y^2 + 24z^2.$$

Define

$$\begin{aligned} \phi_1 : \{ (x, y, z) \in R(f_1, 16m + 34) : 3x + y + 4z \equiv 0 \pmod{16} \} \\ \rightarrow \{ (x, y, z) \in R(f_2, 16m + 34) : 3x - y + 2z \equiv 0 \pmod{16} \} \end{aligned}$$

by

$$\phi_1(x, y, z) = \left( \frac{x - 5y - 68z}{16}, \frac{-5x - 7y + 20z}{16}, \frac{-4x + 4y - 16z}{16} \right),$$

$$\begin{aligned} \phi_2 : \{ (x, y, z) \in R(f_1, 16m + 34) : 3x + y + 4z \equiv 8 \pmod{16} \} \\ \rightarrow \{ (x, y, z) \in R(f_2, 16m + 34) : 3x - y + 2z \equiv 8 \pmod{16} \} \end{aligned}$$

by

$$\phi_2(x, y, z) = \left( \frac{9x - 5y - 52z}{16}, \frac{3x + 9y + 4z}{16}, \frac{4x - 4y + 16z}{16} \right),$$

and

$$\begin{aligned} \phi_3 : R(10x^2 + 24y^2 + 72z^2, 16m + 34) \\ \rightarrow R(f_1, 16m + 34) \text{ by } \phi_3(x, y, z) = (x - 2y, x + 2y, z), \\ \phi_4 : R(18x^2 + 24y^2 + 40z^2, 16m + 34) \\ \rightarrow R(f_3, 16m + 34) \text{ by } \phi_4(x, y, z) = (x + 2y, -x + z, -x - z). \end{aligned}$$

(2)  $(a, b, c) = (1, 15, 30)$ . In this case, we let  $N = 8n + 46$  and

$$\begin{aligned} f_1 &= 4x^2 + 4y^2 + 120z^2 + 2xy, \\ f_2 &= 4x^2 + 16y^2 + 34z^2 + 14yz - 2xz + 4xy, \\ f_3 &= 10x^2 + 16y^2 + 16z^2 + 8yz + 10xz + 10xy, \end{aligned}$$

and

$$g_1 = 4x^2 + 46y^2 + 46z^2 + 32yz + 4xz + 4xy, \quad g_2 = 6x^2 + 30y^2 + 40z^2.$$

Define

$$\begin{aligned} \phi_1 : \{(x, y, z) \in R(f_1, 16m + 46) : 3x - y - 4z \equiv 0 \pmod{16}\} \\ \rightarrow \{(x, y, z) \in R(f_2, 16m + 46) : 3x - y + 2z \equiv 8 \pmod{16}\} \end{aligned}$$

by

$$\phi_1(x, y, z) = \left( \frac{7x - 13y - 4z}{16}, \frac{-3x + y - 44z}{16}, \frac{-4x - 4y + 16z}{16} \right),$$

$$\begin{aligned} \phi_2 : \{(x, y, z) \in R(f_1, 16m + 46) : 3x - y - 4z \equiv 8 \pmod{16}\} \\ \rightarrow \{(x, y, z) \in R(f_2, 16m + 46) : 3x - y + 2z \equiv 0 \pmod{16}\} \end{aligned}$$

by

$$\phi_2(x, y, z) = \left( \frac{9x - 11y + 20z}{16}, \frac{3x + 7y + 28z}{16}, \frac{-4x - 4y + 16z}{16} \right),$$

and

$$\begin{aligned} \phi_3 : R(6x^2 + 40y^2 + 120z^2, 16m + 46) \\ \rightarrow R(f_1, 16m + 46) \text{ by } \phi_3(x, y, z) = (x + 2y, -x + 2y, z), \\ \phi_4 : R(24x^2 + 30y^2 + 40z^2, 16m + 46) \\ \rightarrow R(f_3, 16m + 46) \text{ by } \phi_4(x, y, z) = (-y - 2z, x + y, -x + y). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 3.8.** *For any positive integer  $n$  such that  $n \not\equiv 1 \pmod{3}$ , we have*

$$(3.8) \quad 2t(1, 1, 27; n) = r(x^2 + y^2 + 27z^2, 4(8n + 29)) - r(x^2 + y^2 + 27z^2, 8n + 29).$$

*Proof.* Let  $N = 8n + 29$  and

$$\begin{aligned} f &= f(x, y, z) = x^2 + y^2 + 27z^2, \\ g &= g(x, y, z) = 8x^2 + 20y^2 + 29z^2 + 4yz + 8xz + 8xy, \\ h &= h(x, y, z) = 2x^2 + 5y^2 + 27z^2 + 2xy. \end{aligned}$$

For any positive integer  $m \not\equiv 1 \pmod{3}$ , we let

$$\delta_m = \begin{cases} 1 & \text{if } m \equiv 0 \pmod{3}, \\ 2 & \text{if } m \equiv 2 \pmod{3}. \end{cases}$$

Note that

$$(3.9) \quad r(f, m) = \delta_m |\{(x, y, z) \in R(f, m) : x \equiv y \pmod{3}\}|.$$

Since

$$r(f, 4N) = \delta_N \cdot r(x^2 + (x - 3y)^2 + 27z^2, 4N) = \delta_N \cdot r(h, 4N)$$

and

$$\begin{aligned} & |\{(x, y, z) \in R(f, 4N) : y \equiv 0 \pmod{2}\}| \\ &= \delta_N \cdot r(x^2 + 4(x - 3y)^2 + 27z^2, 4N) = \delta_N \cdot r(8x^2 + 5y^2 + 27z^2 + 4xy, 4N) \\ &= \delta_N |\{(x, y, z) \in R(h, 4N) : x \equiv 0 \pmod{2}\}|, \end{aligned}$$

we have

$$(3.10) \quad |\{(x, y, z) \in R(f, 4N) : y \text{ is odd}\}| = \delta_N |\{(x, y, z) \in R(h, 4N) : x \text{ is odd}\}|.$$

One may easily show that if  $(x, y, z) \in R(f, 4N)$ , then

$$(x^2, y^2, 27z^2) \equiv (0, 0, 4), (0, 1, 3), (0, 4, 0), (1, 0, 3), (4, 0, 0), (4, 4, 4) \pmod{8}.$$

From this and Equation (3.10), the right hand side of Equation (3.8) becomes

$$r(f, 4N) - r(f, N) = 2\delta_N |\{(x, y, z) \in R(h, 4N) : x \equiv 1 \pmod{2}\}|.$$

On the other hand, by Equation (3.9),

$$\begin{aligned} t(1, 1, 27; n) &= r_{(1,1,1)}(f, N) \\ &= \delta_N |\{(x, y, z) \in R(f, N) : x \equiv y \pmod{3}, x \equiv y \equiv z \pmod{2}\}| \\ &= \delta_N \cdot r(x^2 + (x - 6y)^2 + 27(x - 2z)^2, N) = \delta_N \cdot r(g, N). \end{aligned}$$

Therefore, it is enough to show that

$$r(g, N) = |\{(x, y, z) \in R(h, 4N) : x \equiv 1 \pmod{2}\}|.$$

Now, we let

$$\begin{aligned} A &= \{(x, y, z) \in R(g, N) : x \equiv 0 \pmod{2}\}, \\ B &= \{(x, y, z) \in R(h, 4N) : x \equiv 1 \pmod{2}, x + z \equiv 0 \pmod{8}\}. \end{aligned}$$

Note that  $x + z \equiv 8 \pmod{16}$  if  $(x, y, z) \in B$ . Define a map  $\phi : A \rightarrow B$  by

$$\phi(x, y, z) = (x - 7z, -x - 4y + z, -x - z).$$

Then, one may easily show that  $\phi$  is a bijection. Since  $g(x + z, y, -z) = g(x, y, z)$  and  $z_0$  is odd for any  $(x_0, y_0, z_0) \in R(g, N)$ , we have

$$|\{(x, y, z) \in R(g, N) : x \equiv 0 \pmod{2}\}| = |\{(x, y, z) \in R(g, N) : x \equiv 1 \pmod{2}\}|$$

and thus

$$r(g, N) = 2 |\{(x, y, z) \in R(g, N) : x \equiv 0 \pmod{2}\}|.$$

Now, we are ready to prove the assertion. Note that if  $(x, y, z) \in R(h, 4N)$  and  $x \equiv 1 \pmod{2}$ , then  $z \equiv \pm x \pmod{8}$ . Therefore, we have

$$\begin{aligned} & |\{(x, y, z) \in R(h, 4N) : x \equiv 1 \pmod{2}\}| \\ &= 2 |\{(x, y, z) \in R(h, 4N) : x \equiv 1 \pmod{2}, x + z \equiv 0 \pmod{8}\}| \\ &= 2|B| = 2|A| = r(g, N). \end{aligned}$$

This completes the proof.  $\square$

Finally, we prove the Conjecture 6.7 in [6].

**Theorem 3.9.** *For a positive integer  $n$ , the Diophantine equation*

$$\mathcal{T}_{(1,1,6)}(x, y, z) = \frac{x(x-1)}{2} + \frac{y(y-1)}{2} + 6\frac{z(z-1)}{2} = n$$

*has an integer solution if and only if  $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$  for any positive integer  $r$ .*

*Proof.* Note that  $\mathcal{T}_{(1,1,6)}(x, y, z) = n$  has an integer solution if and only if  $f(x, y, z) = x^2 + y^2 + 6z^2 = 8n + 8$  has an integer solution  $x, y, z$  such that  $xyz \equiv 1 \pmod{2}$ . Since the ternary quadratic form  $f(x, y, z)$  has class number one, it represents every integer that is locally represented (see 102.5 of [4]). Therefore, one may easily check that  $f(x, y, z) = 8n + 8$  has an integer solution if and only if  $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$  for any positive integer  $r$ .

Now, assume that  $n$  is a positive integer such that  $n \not\equiv 2 \cdot 3^{2r-1} - 1 \pmod{3^{2r}}$  for any positive integer  $r$ . Note that  $f(x, y, z) = 8n + 8$  has an integer solution  $x, y, z$  such that  $xyz \equiv 1 \pmod{2}$  if and only if  $r(f, 8n + 8) - r(f, 2n + 2) > 0$ . By the Minkowski-Siegel formula, we have

$$\frac{r(f, 8n + 8)}{r(f, 2n + 2)} = 2 \frac{\alpha_2(f, 8n + 8)}{\alpha_2(f, 2n + 2)},$$

where  $\alpha_2$  is the local density over  $\mathbb{Z}_2$  (for details, see, for example, [3]). For a positive integer  $s$  and a positive odd integer  $t$ , one may easily compute by using the result of [7] that

$$\alpha_2(f, 2^{st}) = \begin{cases} 2 - 3 \cdot 2^{-s/2} & \text{if } s \equiv 0 \pmod{2}, \\ 2 - 2^{(1-s)/2} & \text{if } s \equiv 1 \pmod{2}, t \equiv 1 \pmod{8}, \\ 2 & \text{if } s \equiv 1 \pmod{2}, t \equiv 5 \pmod{8}, \\ 2 - 3 \cdot 2^{(-s-1)/2} & \text{if } s \equiv 1 \pmod{2}, t \equiv 3 \text{ or } 7 \pmod{8}. \end{cases}$$

Therefore, we have  $2\alpha_2(f, 8n + 8) > \alpha_2(f, 2n + 2)$  for any positive integer  $n$ . This completes the proof.  $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SEOUL 151-747,  
KOREA

*E-mail address:* `kmg2562@snu.ac.kr`

DEPARTMENT OF MATHEMATICAL SCIENCES AND RESEARCH INSTITUTE OF MATHEMATICS, SEOUL  
NATIONAL UNIVERSITY, SEOUL 151-747, KOREA

*E-mail address:* `bkoh@snu.ac.kr`