Reductions of modular Galois representations of Slope (2,3)

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We compute the semisimplifications of the mod p reductions of 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of slope $(2,3)$ and arbitrary weight, building on work of Bhattacharya-Ghate.

Contents

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0 Introduction

Let p be a prime number. Via the mod p local Langlands correspondence, the reductions of 2-dimensional crystalline representations were computed

- for slope $0 < v(a_p) < 1$ and weight $k > 2p+2$ in [BG09], and

- for $p \geq 3$, slope $1 < v(a_p) < 2$ (with a condition on a_p when $v(a_p) = 3/2$) and weight $2p + 2 \leq k \leq p^2 - p$ in [GG15] and for all weights in [BG15].

In this article, we extend these computations to slope $2 < v(a_p) < 3$. We will follow the notation of [GG15] and [BG15].

0.1 Situation

Let $\mathbf{E}$ be a finite extension of $\mathbb{Q}_p$ and let v be the additive valuation on $\mathbf{E}$ satisfying $v(p) = 1$.

Let $\mathscr{G}_{\mathbb{Q}_p}$ be the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of $\mathbb{Q}_p$. A *p-adic Galois representation* is a continuous action of $\mathscr{G}_{\mathbb{Q}_p}$ on a finite-dimensional vector space defined over $\mathbf{E}$.

Among all p-adic Galois representations the *crystalline* Galois representations admit an explicit parameterization: Every crystalline representation V of dimension 2 is, up to twist by a crystalline character, uniquely determined by,

- a *weight*, an integer $k \geq 2$, and
- an *eigenvalue* a_p in $\mathbf{E}$ with $v(a_p) > 0$.

The rational number $v(a_p)$ is called the *slope* of V.

Inside V the compact group $\mathscr{G}_{\mathbb{Q}_p}$ stabilizes a lattice upon which, by the Brauer-Nesbitt principle, the *semisimplified* induced representation $\bar{\mathrm{V}}$ of $\mathscr{G}_{\mathbb{Q}_p}$ over $\overline{\mathbb{F}}_p$ does not depend ($\bar{\mathrm{V}}$ is the *mod p reduction* of V). Let V_{k,a_p} be the crystalline representation of weight k and eigenvalue a_p, that is, the crystalline representation attached to the (admissible) ϕ-module of basis $\{e_1, e_2\}$ whose Frobenius ϕ and filtration $\mathrm{V}_\bullet$ is given (as in [Ber11, Paragraph 2.3]) by

$$\phi = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix} \quad \text{and} \quad \ldots = \mathrm{V}_0 = \mathrm{V} \supset \mathrm{V}_1 = \ldots = \mathrm{V}_{k-1} = \mathbf{E} \cdot e_1 \supset 0 = \mathrm{V}_k = \ldots$$

Let $\bar{\mathrm{V}}_{k,a_p}$ be its semisimplified mod p reduction. In conjecture 4.1.1 of [BG16], they conjecture that if p is odd, k is even and $v(a_p) \notin \mathbb{Z}$, then $\bar{\mathrm{V}}_{k,a_p}$ is irreducible.

The finite-dimensional irreducible Galois representation over $\overline{\mathbb{F}}_p$ are classified and, up to twists by unramified characters, parametrized by integers, as follows: For n in $\mathbb{N}$, let $\mathbb{Q}_{p^n}$ (respectively $\mathbb{Q}_{p^{-n}}$) be the smallest field extension of $\mathbb{Q}_p$ that contains a primitive $(p^n - 1)$-th root ζ_n (respectively p_n) of 1 (respectively of $-p$). The *fundamental character* $\omega_n\colon\ \mathrm{Gal}(\mathbb{Q}_{p^{-n}}/\mathbb{Q}_{p^n}) \longrightarrow \mathbb{F}_{p^n}^*$ is defined by

$$\sigma \mapsto \zeta_n \qquad \text{where } \zeta_n \text{ is determined by } \sigma(p_n) = \zeta_n \cdot p_n.$$

Let $\omega := \omega_1$. For λ in $\overline{\mathbb{F}}_p$, let $\mathrm{u}(\lambda)\colon \mathscr{G}_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^*$ be the unramified character that sends the (arithmetic) Frobenius to λ. For a in $\mathbb{Z}$, let

$$\mathrm{ind}_{\mathscr{G}_{\mathbb{Q}_{p^n}}}^{\mathscr{G}_{\mathbb{Q}_p}} \omega_n^a := \overline{\mathbb{F}}_p[\mathscr{G}_{\mathbb{Q}_p}] \otimes_{\overline{\mathbb{F}}_p[\mathscr{G}_{\mathbb{Q}_{p^n}}]} \omega_n^a \otimes_{\overline{\mathbb{F}}_p}$$

be the induction of ω_n^a from $\mathscr{G}_{\mathbb{Q}_{p^n}}$ to $\mathscr{G}_{\mathbb{Q}_p}$. The conjugated characters $\omega_n(g \cdot g^{-1})$ for g in $\mathscr{G}_{\mathbb{Q}_p}$ are $\omega_n, \omega_n^p, \ldots, \omega_n^{p^{n-1}}$ and all distinct; therefore, by Mackey's criterion, $\mathrm{ind}_{\mathscr{G}_{\mathbb{Q}_{p^n}}}^{\mathscr{G}_{\mathbb{Q}_p}} \omega_n^a$ is irreducible and its determinant is ω^a on $\mathscr{G}_{\mathbb{Q}_{p^n}}$. Let $\mathrm{ind}(\omega_n^a)$ denote the twist of $\mathrm{ind}_{\mathscr{G}_{\mathbb{Q}_{p^n}}}^{\mathscr{G}_{\mathbb{Q}_p}} \omega_n^a$ by the unramified character that turns its determinant into ω^a on all of $\mathscr{G}_{\mathbb{Q}_p}$.

Every irreducible n-dimensional representation of $\mathscr{G}_{\mathbb{Q}_p}$ over $\overline{\mathbb{F}}_p$ is of the form

$$\mathrm{ind}(\omega_n^a) \otimes \mathrm{u}(\lambda)$$

for some a in $\mathbb{Z}$ and λ in $\overline{\mathbb{F}}_p^*$ (cf. [op. cit., Paragraph 1.1]). In particular, every mod p reduction of dimension 2 is either of the form

$$\mathrm{ind}(\omega_2^a) \otimes \mathrm{u}(\lambda) \quad \text{or} \quad (\omega_1^a \otimes \mathrm{u}(\lambda)) \oplus (\omega_1^b \otimes \mathrm{u}(\mu))$$

for some a, b in $\mathbb{Z}$ and λ, μ in $\overline{\mathbb{F}}_p^*$.

The powers a and b of the fundamental character ω_2 are not unique in $\mathbb{Z}$ but satisfy the following congruences: ω_2 has order $p^2 - 1$, so $w^{p^2} = w$, and ω_2^i and ω_2^{ip} are conjugate under $\mathscr{G}_{\mathbb{Q}_p}$, thus have isomorphic inductions.

There are also restrictions on the exponents occurring in the mod p reduction: We recall that the Galois representation V_{k,a_p} is obtained from a filtered ϕ-module by a functor; which is a tensor functor, in particular, it is compatible with taking the determinant. This way, the determinant of the Galois representation V_{k,a_p} is known and can be made explicit, and so its mod p reduction. It is ω^{k-1}. At the same time, we recall that the determinant of $\mathrm{ind}(\omega_2^i)$ is (by definition) ω^i.

For a weight k and an eigenvalue a_p that parametrize a crystalline representation V_{k,a_p}, we compute a in $\mathbb{Z}$ and λ in $\overline{\mathbb{F}}_p^*$ that parametrize the mod p reduction $\bar{V}_{k,a_p}$ for

- a weight k in certain mod $(p-1)$ and mod p congruence classes, and
- a slope $2 < v(a_p) < 3$.

By applying [BG09, Lemma 3.3] to the results of Section 5 and Section 6, we obtain:

Theorem 0.1. *Let $r := k-2$ and a in $\{3,\ldots,p+1\}$ such that $r \equiv a \bmod p-1$. If $p \geq 5$, $r \geq 3p+2$ and $v(a_p)$ in $]2,3[$ (and, if $v(a_p) = 5/2$, then $v(a_p^2 - p^5) = 5$), then*

$$\overline{\mathrm{V}}_{k,a_p} \cong \begin{cases} \mathrm{ind}(\omega_2^{a+1}), & \textit{for } a = 3 \textit{ and } r \not\equiv 0,1,2 \bmod p \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 3 \textit{ and } r \equiv 0 \bmod p \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = 3 \textit{ and } p \parallel r-(p+1) \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 3 \textit{ and } r \equiv p+1 \bmod p^2 \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = 3 \textit{ and } r \equiv 2 \bmod p \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 4 \textit{ and } r \not\equiv 2,3,4 \bmod p \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 4 \textit{ and } r \equiv 2 \bmod p \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = 4 \textit{ and } r \equiv 3 \bmod p \\ \mathrm{ind}(\omega_2^{a+1}), & \textit{for } a = 4 \textit{ and } r \equiv 4 \bmod p \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 5 \textit{ and } r \equiv 2,3 \bmod p \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = 5,\ldots,p-1 \textit{ and } p \parallel r-a \\ \mathrm{ind}(\omega_2^{a+1}), & \textit{for } a = 5,\ldots,p-1 \textit{ and } r \equiv a \bmod p^2 \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 6,\ldots,p \textit{ and } r \not\equiv a, a-1 \bmod p \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = 6,\ldots,p \textit{ and } p \parallel r-a+1 \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = 6,\ldots,p \textit{ and } r \equiv a-1 \bmod p^2 \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = p \textit{ and } p \parallel r-p \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = p \textit{ and } p^2 \parallel r-p \\ \mathrm{u}(\sqrt{-1})\omega \oplus \mathrm{u}(-\sqrt{-1})\omega, & \textit{for } a = p \textit{ and } r \equiv p \bmod p^3 \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = p+1 \textit{ and } r \not\equiv 0,1 \bmod p \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = p+1 \textit{ and } p \parallel r-p \\ \mathrm{ind}(\omega_2^{a+2p-1}), & \textit{for } a = p+1 \textit{ and } r \equiv p \bmod p^2 \\ \mathrm{ind}(\omega_2^{a+p}), & \textit{for } a = p+1 \textit{ and } r \equiv 1 \bmod p \end{cases}$$

where $\parallel$ denotes exact divisibility.

This result is as predicted by the main theorem of [BG15]: Since the slope increases by a unit, here the reducible case occurs when $p^3 \mid p-r$ (whereas, in [BG15], when $p^2 \mid p-r$). In [Ars15], Arsovski examines whether the representation is irreducible or not, for a large class of slopes (integral and non-integral) and even weights, but does not specify it. Here we deal with all weights and compute the exact shape of the representation, but we could not address:

- the case $a = 5$ when $v(a_p^2) > v(\binom{r-2}{3}p^5)$ (to determine reducibility in Section 6), and

- the case $v(a_p) = 5/2$ when $v(a_p^2 - p^5) \neq 5$.

These cases are part of Ghate's zig-zag conjecture (see [Gha19]), which has been addressed in recent work (see [GR19a]) for $a = 3$ and $v(a_p) = 3/2$. Based on op.cit., we hope to address the condition imposed for $a = 5$ in future work.

0.2 Outline

We refer to [BG09], [GG15] and [BG15] for a more detailed exposition. Let **L** be the 2-*dimensional mod p local Langlands correspondence*, an injection

$$\left\{\begin{matrix}\text{continuous actions of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \\ \text{on 2-dimensional } \overline{\mathbb{F}}_p\text{-vector spaces}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{semisimple } \textit{smooth} \\ \text{actions of } \mathrm{GL}_2(\mathbb{F}_p) \text{ on} \\ \overline{\mathbb{F}}_p\text{-vector spaces}\end{matrix}\right\}$$

Since **L** is injective, to determine $\bar{\mathrm{V}}_{k,a_p}$, it suffices to determine $\mathbf{L}(\bar{\mathrm{V}}_{k,a_p})$. As **L** and the p-adic local Langlands correspondence (the analog of the mod p local Langlands correspondence that attaches actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over 2-dimensional $\mathbb{Q}_p$-vector spaces to actions of $\mathrm{GL}_2(\mathbb{Q}_p)$) on Banach spaces) are compatible with taking the mod p reduction,

$$\mathbf{L}(\bar{\mathrm{V}}_{k,a_p}) = \bar{\Theta}^{\mathrm{ss}}_{k,a_p}$$

where the right-hand side is the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over the (infinite dimensional) $\overline{\mathbb{F}}_p$-vector space given by

- the semisimplification $\bar{\Theta}^{\mathrm{ss}}_{k,a_p}$ of

- the reduction modulo p $\bar{\Theta}_{k,a_p}$ of

- the canonical lattice $\overline{\mathbb{Z}}_p$-lattice Θ_{k,a_p} of

- the base extension Π_{k,a_p} from **E** to $\overline{\mathbb{Q}}_p$ of

- the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ that corresponds to V_{k,a_p} under the p-adic local Langlands correspondence. Explicitly,

  $$\Pi_{k,a_p} = \mathrm{ind}^{\mathrm{G}}_{\mathrm{KZ}} \mathrm{Sym}^r\, \overline{\mathbb{Q}}_p^2/(\mathrm{T} - a_p)$$

  where

- $r = k - 2$,
- $G = GL_2(\mathbb{Q}_p)$, $K = GL_2(\mathbb{Z}_p)$ and $Z = \mathbb{Q}_p^*$ is the center of G,
- $\mathrm{Sym}^r\, \overline{\mathbb{Q}}_p^2$ is the representation of $GL_2(\mathbb{Q}_p)$ given by all homogeneous polynomials of total degree r, and
- T is the *Hecke operator* that generates the endomorphism algebra of all $\overline{\mathbb{Q}}_p[G]$-linear maps on $\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r\, \overline{\mathbb{Q}}_p^2$.

The canonical $\overline{\mathbb{Z}}_p$-lattice Θ_{k,a_p} of Π_{k,a_p} is given by the image

$$\Theta_{k,a_p} := \mathrm{im}(\mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r\, \overline{\mathbb{Z}}_p^2 \to \Pi_{k,a_p})$$

and the mod p-reduction $\bar{\Theta}_{k,a_p}$ by $\Theta_{k,a_p}/p\Theta_{k,a_p}$.

Let $V_r := \mathrm{Sym}^r\, \overline{\mathbb{F}}_p^2$. It is a representation of $GL_2(\mathbb{F}_p)$ that extends to one of KZ by letting p and Z act trivially. We note that there is a natural $\overline{\mathbb{F}}_p[G]$-linear surjection

$$\mathrm{ind}_{KZ}^{G} V_r \twoheadrightarrow \bar{\Theta}_{k,a_p}. \qquad (*)$$

Our main result will be that, generally, there is a single Jordan-Hölder factor J of V_r whose induction surjects onto the right-hand side. Then [BG09, Proposition 3.3] uniquely determines $\bar{V}_{k,a_p}$.

To find the Jordan-Hölder factor J of V_r, we first define a quotient Q of V_r whose induction surjects onto the right-hand side. For this, let $X(k, a_p)$ denote the kernel of the above epimorphism. Put $\Gamma := GL_2(\mathbb{F}_p)$.

Let $\theta := X^pY - XY^p \in V_{p+1}$ and let V_r^{***} be the image of the multiplication map $\theta^3\cdot$ from V_{r-3p-3} to V_r. For $i = 0, \ldots, r$, let

$$X_{r-i} := \text{ the } \overline{\mathbb{F}}_p[\Gamma]\text{-submodule of } V_r \text{ generated by } X^iY^{r-i}\ .$$

Observation. Mistakably, the notation X_{r-i} involves *two* parameters,

- r in $\mathbb{N}$ for the surrounding, and
- i in $\{0, \ldots, r\}$ for the inner submodule.

For example, put $r' = r - 1$.

- Then $X_{r'}$ is the submodule of V_{r-1}, homogeneous polynomials of two variables of total degree $r - 1$, generated by Y^{r-1},
- whereas X_{r-1} is the submodule of V_r, homogeneous polynomials of two variables of total degree r, generated by XY^{r-1}.

By [BG09, Remark 4.4],

- if $2 < v(a_p)$, then $\operatorname{ind}_{KZ}^{G} X_{r-2} \subseteq X(k, a_p)$, and
- if $v(a_p) < 3$, then $\operatorname{ind} V_r^{***} \subseteq X(k, a_p)$.

Finally put

$$Q := V_r/(X_{r-2} + V_r^{***})$$

Thence, if $2 < v(a_p) < 3$, then the epimorphism $(*)$ induces an epimorphism

$$\operatorname{ind}_{KZ}^{G} Q \twoheadrightarrow \bar{\Theta}_{k,a_p}.$$

Thus we need to understand the modules X_{r-2}, V_r^{***} and their intersection $X_{r-2}^{***} := X_{r-2} \cap V_r^{***}$:

- In Lemma 1.3, the Jordan-Hölder series of the module V_r/V_r^{***} is computed,
- In Section 2, the Jordan-Hölder series of the modules X_{r-2} and X_{r-2}/X_{r-2}^{*} is computed (where $X_{r-2}^{*} := X_{r-2} \cap V_r^{*}$), and
- in Section 3 that of $X_{r-2}^{*}/X_{r-2}^{***}$, and
- in Section 4 that of Q.

See the introduction of Section 4 for further digression. These computations depend on the congruence class of r modulo $p - 1 = \#\mathbb{F}_p^*$ and of that modulo $p = \#\mathbb{F}_p$; combinatorial conditions on $\Sigma(r)$, the sum of the digits of the p-adic expansion of r enter.

We then compute in Section 4 the Jordan-Hölder factors of Q: A priori, Q has at most 6 Jordan-Hölder factors.

- If Q happens to have a *single* Jordan-Hölder factor, that is, if there is a homomorphism of an irreducible module onto $\bar{\Theta}_{k,a_p}$, then [BG09, Proposition 3.3] describes $\bar{\Theta}_{k,a_p}$ completely.
- Otherwise, that is, if Q happens to have *more than one* Jordan-Hölder factor J, then in Section 5 we show, for all but a single Jordan-Hölder factor J_0 of Q, there are functions f_J in $\operatorname{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$ such that
  - its image $(T - a_p)(f_J)$ under the Hecke operator lies in $\operatorname{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Z}}_p^2$, and
  - its mod p reduction $\bar{f}_J$

- ▹ lies in $\mathrm{ind}_{KZ}^{G} J$, and
- ▹ generates the entire $\overline{\mathbb{F}}_p[G]$-module $\mathrm{ind}_{KZ}^{G} J$ (this holds, for example, when it is supported on a single coset of G/KZ).

Then again [BG09, Proposition 3.3] applied to

$$\mathrm{ind}_{KZ}^{G} J_0 \twoheadrightarrow \bar{\Theta}_{k,a_p}$$

describes $\bar{\Theta}_{k,a_p}$ completely.

In Section 6, if the only remaining Jordan-Hölder factor is $V_{p-2} \otimes D^n$ for some n, we need to distinguish between the irreducible and reducible case: To this end we construct additional functions and observe whether the map $\mathrm{ind}_{KZ}^{G} V_{p-2} \otimes D^n \to \overline{\Theta}_{k,a_p}$ factors through the cokernel of either T (in which case irreducibility holds) or of $T^2 - cT + 1$ for some $c \in \overline{\mathbb{F}}_p$ (in which case reducibility holds).

1 Groundwork

We restate key results of [Glo78] in our notation (which follows that of [GG15], [BG15] and [BGR18]). Let M be the multiplicative monoid of all 2×2-matrices with coefficients in $\mathbb{F}_p$. Inside the M-representation of all homogeneous polynomials of two variables,

- here, as in *op. cit.*, V_r denotes the subrepresentation given by all those of (total) degree r, a vector space of dimension $r + 1$,
- whereas in [Glo78], it denotes the subrepresentation given by all those of (total) degree $r - 1$, a vector space of dimension r.

That is, there is a one-dimensional offset.

1.1 The Jordan-Hölder series of $V_m \otimes V_n$ for $m = 2, 3$

For an M-representation U, let σU and φU denote the socle and cosocle of U.

Lemma 1.1 (The Jordan-Hölder series of a Tensor product of two irreducible representations). *Let* $0 \le m \le n \le p - 1$.

(i) If $0 \le m + n \le p - 1$, *then*

$$V_m \otimes V_n \cong \bigoplus_{i=0,\dots,m} V_{m+n-2i} \otimes D^i.$$

(ii) If $p \leq m + n \leq 2p - 2$, then

$$V_m \otimes V_n \cong V_{p(m+n+2-p)-1} \oplus (V_{p-n-2} \otimes V_{p-m-2} \otimes D^{m+n+2-p})$$

where the second summand equals

$$(V_{p-n-2} \otimes V_{p-m-2} \otimes D^{m+n+2-p}) \cong \bigoplus_{i=0,\ldots,p-n-2} V_{(p-m-2)+(p-n-2)-2i} \otimes D^{m+n+2-p-i}$$

and the first summand $V = V_{(k+1)p-1}$ *for* k *in* $\{1, \ldots, p-1\}$ *is a direct sum*

$$V = \bigoplus_{m=0,\ldots,\lfloor k/2 \rfloor} U_{k-2m} \otimes D^m$$

where $U = U_l$ *for* l *in* $\{1, \ldots, p\}$ *has Jordan-Hölder series*

$$0 \subset \sigma U \subset \varphi U \subset U$$

whose successive semisimple Jordan-Hölder factors $\bar{U} = \sigma U$, $\bar{U}' = \varphi U / \sigma U$ *and* $\bar{U}'' = U/\varphi U$ *are*

- $\bar{U} = \bar{U}'' = V_{p-l-1} \otimes D^l$, *and*
- $\bar{U}' = (V_l \otimes D) \oplus V_{l+2}$.

with the convention that $V_k = 0$ *for* $k < 0$.

Proof:

(i): By [Glo78, (5.5)].

(ii): The equality for the second summand follows by (i), which is the case $0 \leq m + n \leq p$. The Jordan-Hölder series of V_{kp} for $k = 1, \ldots, p$ is given in [Glo78, (5.9)]. □

Corollary 1.2 (of Lemma 1.1). *As* $\mathbb{F}_p[M]$*-modules*

(i) we have $V_2 \otimes V_{p-2} = V_{p-4} \otimes D^2 \oplus V_{2p-1}$ *where* V_{2p-1} *has successive semisimple Jordan-Hölder factors* $V_{p-2} \otimes D$, V_1 *and* $V_{p-2} \otimes D$, *and*

(ii) we have $V_2 \otimes V_{p-1} = [(V_{p-1} \otimes D) \oplus V_{p+1}] \oplus (V_{p-3} \otimes D^2)$ *where*

- V_{p-1} *is irreducible,*
- V_{p+1} *has successive semisimple Jordan-Hölder factors* $(V_0 \otimes D) \oplus V_2$ *and* $V_{p-3} \otimes D^4$, *and*
- V_{p-3} *is irreducible.*

1.2 The singular submodules of V_r

We recall that $\Gamma := \mathrm{GL}_2(\mathbb{F}_p)$.

Lemma 1.3 (Extension of [GG15, Proposition 2.2]). *Let $p > 2$. The Jordan-Hölder series of $\mathbf{F}[\Gamma]$-modules*

(i) of V_r/V_r^*, for $r \geq p$, and $r \equiv a \bmod (p-1)$ with $a \in \{1, \dots, p-1\}$ is

$$0 \to V_a \to V_r/V_r^* \to V_{p-a-1} \otimes D^a \to 0,$$

and this sequence splits if and only if $a = p-1$;

(ii) of V_r^*/V_r^{**} for $r \geq 2p+1$, and $r \equiv a \bmod (p-1)$ with $a \in \{3, \dots, p+1\}$ is

$$0 \to V_{a-2} \otimes D \to V_r^*/V_r^{**} \to V_{p-a+1} \otimes D^{a-1} \to 0$$

and this sequence splits if and only if $a = p+1$;

(iii) of V_r^{**}/V_r^{***}, for $r \geq 3p+2$, and $r \equiv a \bmod (p-1)$ with $a \in \{5, \dots, p+3\}$ is

$$0 \to V_{a-4} \otimes D^2 \to V_r^{**}/V_r^{***} \to V_{p-a+3} \otimes D^{a-2} \to 0$$

and this sequence splits if and only if $a = p+3$.

Proof:

(i): This is [GG15, Proposition 2.1]. (Note that the exact structure of V_{a+p-1}/V_{a+p-1}^* can also be read off by [Glo78, (5.7)].)

(ii): This is [GG15, Proposition 2.2].

(iii): Follow [GG15, Proposition 2.2] and use $V_r^{**}/V_r^{***} \cong (V_{r-p-1}^*/V_{r-p-1}^{**}) \otimes D$.

The sequences in (i), (ii), (iii) split for $a = p-1, p+1, p+3$ respectively because V_{p-1} is an injective module over $\mathbb{F}_p[\Gamma]$. □

In V_r we have $\theta \equiv 0$ if and only if $X^pY \equiv XY^p$. Therefore, θ divides F, equivalently $F \equiv 0 \bmod \theta$, if and only if

- the indices of all nonzero coefficients of F are congruent mod $p-1$,
- their sum vanishes, and
- the initial coefficients as well.

Similarly, one checks that:

Lemma 1.4 (Extension of [BG15, Lemma 2.3]). *Let* $F(X, Y) = \sum_{0 \le j \le r} c_j X^{r-j} Y^j$ *in* V_r. *If the indices of all nonzero coefficients are congruent mod* $p-1$, *that is,* $c_j, c_k \neq 0$ *implies* $j \equiv k \bmod p-1$, *then*

(i) $F \in V_r^*$ *if and only if* $c_0 = 0 = c_r$ *and* $\sum c_j = 0$,

(ii) $F \in V_r^{**}$ *if and only if*

$$c_0 = c_1 = 0 = c_{r-1} = c_r \quad \textit{and} \quad \sum c_j = \sum j c_j = 0, \ \textit{and}$$

(iii) $F \in V_r^{***}$ *if and only if*

$$c_0 = c_1 = c_2 = 0 = c_{r-2} = c_{r-1} = c_r \ \textit{and} \sum c_j = \sum j c_j = \sum j(j-1) c_j = 0.$$

1.3 Some combinatorial Lemmas

The following Lemma, known as Lucas's Theorem, is a key combinatorial lemma used throughout the paper.

Lemma 1.5 (Lucas's Theorem). *Let* r *and* n *be natural numbers and* $r = r_0 + r_1 p + r_2 p^2 + \cdots$ *and* $n = n_0 + n_1 p + n_2 p^2 + \cdots$ *their* p*-adic expansions. Then*

$$\binom{r}{n} \equiv \binom{r_0}{n_0}\binom{r_1}{n_1}\binom{r_2}{n_2} \cdots \bmod p.$$

Lemma 1.6 (Extension of [BG15, Lemmas 2.5 and 2.6]). *For* $i = 0, 1, 2$, *let* a *in* $\{1+i, \ldots, p-1+i\}$ *be such that* $r \equiv a \bmod p-1$. *Then*

$$\sum_{\substack{j \equiv a-i \\ 0<j<r-i}} \binom{r}{j} \equiv \begin{cases} 0, & \textit{for } i = 0 \\ a - r, & \textit{for } i = 1 \\ \dfrac{(a-r)(a+r-1)}{2}, & \textit{for } i = 2. \end{cases}$$

Proof: For $i = 0, 1$, see [BG15, Lemmas 2.5 and 2.6]. For $i = 2$, we apply induction on r: We have

$$\binom{x+2}{n} = \binom{x}{n-2} + 2\binom{x}{n-1} + \binom{x}{n}.$$

Applying this identity for $i = 2$, and using the known cases ($i = 0, 1$) and the induction hypothesis,

$$\begin{aligned}\sum_{\substack{j\equiv a-2\\0<j<r-2}}\binom{r}{j} &= \sum_{\substack{j\equiv a-2\\0<j<r-2}}\binom{r-2}{j-2} + 2\sum_{\substack{j\equiv a-2\\0<j<r-2}}\binom{r-2}{j-1} + \sum_{\substack{j\equiv a-2\\0<j<r-2}}\binom{r-2}{j}\\ &\equiv \frac{(a-r)(a+r-5)}{2} + 2(a-r) + 0 \bmod p\\ &\equiv \frac{(a-r)(a+r-5+4)}{2} = \frac{(a-r)(a+r-1)}{2} \bmod p. \square\end{aligned}$$

Remark. More generally:

$$\sum_{\substack{j\equiv a-i\\0<j<r-i}}\binom{r}{j} \equiv \binom{a}{i} - \binom{r}{i} \bmod p$$

Since we do not go beyond $i = 2$, we will not prove the above identity.

Lemma 1.7 (Analog of [BG15, Lemma 2.7]). *If* $r \equiv 1 \bmod p-1$ *and* $p^2 \mid p - r$*, then*

$$\sum_{\substack{j\equiv 1 \bmod p-1\\1<j<r}}\binom{r}{j} \equiv p - r \bmod p^3.$$

Proof: Let $r = p + np^t(p-1) = 1 + (np^t + 1)(p-1)$ for $n > 0$ and $t \geq 0$. If $p^2 \mid r - p = n(p-1)p^t$, then $t = 2$.

In the proof of [BGR18][Proposition 2.8.(1)] (where $r = 2 + np^t(p-1)$) we replace 2 with 1 and np^t with $np^t + 1$, obtaining

$$\sum_{j\equiv 1 \bmod p-1}\binom{r}{j} \equiv 1 + p + np^{t+1} \bmod p^{t+2},$$

and therefore

$$\sum_{\substack{j\equiv 1 \bmod p-1\\1<j<r}}\binom{r}{j} \equiv p + np^{t+1} - r \bmod p^{t+2}.$$

In particular for $t = 2$,

$$\sum_{\substack{j\equiv 1 \bmod p-1\\1<j<r}}\binom{r}{j} \equiv p - r \bmod p^3. \qquad \square$$

Lemma 1.8. *Let $r \equiv a \bmod p-1$ with a in $\{3,\dots,p+1\}$. There are integers $\{\alpha_j : a \le j < r$ and $j \equiv a \bmod p-1\}$ such that*

(i) *we have $\alpha_j \equiv \binom{r}{j} \bmod p$, and*

(ii) *for $n = 0,1,2$, we have $\sum_{j\ge n} \binom{j}{n}\alpha_j \equiv 0 \bmod p^{4-n}$ and, for $n = 3$, we have*
   - *if $a = 4,\dots,p+1$, then $\sum_{j\ge 3}\binom{j}{3}\alpha_j \equiv 0 \bmod p^{4-n}$, and*
   - *if $a = 3$, then $\sum_{j\ge3}\binom{j}{3}\alpha_j \equiv \binom{r}{3} \bmod p$.*

Proof: If $r \le ap$, then $\binom{r}{j} \equiv 0 \bmod p$ for all $0 < j < r$ such that $j \equiv a \bmod p-1$. Therefore, we can put $\alpha_j = 0$, and the proposition trivially holds true.

Let $r > ap$. By Lemma 1.7 and noting that $j(j-1)(j-2)\binom{r}{j} = r(r-1)(r-2)\binom{r-3}{j-3}$ we see that

$$\sum_{j\ge3}\binom{j}{3}\binom{r}{j} \equiv \begin{cases} \binom{r}{3}, & \text{for } a = 3 \\ 0, & \text{otherwise.} \end{cases}$$

This solves the case $n = 3$.

By Lemma 1.7 again, $\sum_{j\ge2}\binom{j}{2}\binom{r}{j}$, $\sum_{j\ge1} j\binom{r}{j}$ and $\sum_j \binom{r}{j} \equiv 0 \bmod p$ for $j \equiv a \bmod p-1$. Put

$$s_0 = -p^{-1}\sum_j \binom{r}{j}, \quad s_1 = -p^{-1}\sum_{j\ge1} j\binom{r}{j} \quad \text{and} \quad s_2 = -p^{-1}\sum_{j\ge2}\binom{j}{2}\binom{r}{j}.$$

and $\alpha_j = \binom{r}{j} + p\delta_j$.

Thus we have to solve for 3 equations ($n = 0,1,2$) in δ_j's. So we can take all but three δ_j's to be 0. Thus we need to choose 3 j's wisely so that such a solution exists.

There are δ_j such that

$$\sum \alpha_j \equiv 0 \bmod p^4, \quad \sum j\alpha_j \equiv 0 \bmod p^3, \quad \text{and} \quad \sum \binom{j}{2}\alpha_j \equiv 0 \bmod p^2$$

if and only if the following system of linear equations $(*)$ in the three unknowns δ_k, δ_l and δ_m is solvable:

$$\begin{matrix} 1 & 1 & 1 \equiv s_0 \bmod p^4, \\ k & l & m \equiv s_1 \bmod p^3, \\ \binom{k}{2} & \binom{l}{2} & \binom{m}{2} \equiv s_2 \bmod p^2. \end{matrix} \qquad (*)$$

It suffices to solve all equations modulo p^4. For this, we show that there are k, l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that the determinant of $(*)$ is invertible in $\mathbb{Z}/p^4\mathbb{Z}$, or equivalently, that it is nonzero $\mod p$.

Since $r > ap$, we can put $k = ap$. Then $(*)$ is modulo p given by an upper triangular matrix whose upper left coefficient is 1, and therefore its determinant equals that of its lower right 2×2-matrix

$$\begin{pmatrix} l & m \\ \binom{l}{2} & \binom{m}{2} \end{pmatrix}$$

which is 0 if and only if l, m or $l - m$ vanishes modulo p. Therefore, choosing l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that $l, m, l-m \not\equiv 0 \mod p$, the system of linear equations $(*)$ is solvable. □

Lemma 1.9. *Let a in $\{4, \ldots, p+1\}$ such that $r \equiv a \mod p-1$. If $r \equiv a \mod p$, then there are integers $\{\beta_j : a-1 \leq j < r-1$ and $j \equiv a-1 \mod p-1\}$ such that*

(i) *we have $\beta_j \equiv \binom{r}{j} \mod p$, and*

(ii) *for $n = 0, 1, 2, 3$, we have $\sum_{j \geq n} \binom{j}{n} \beta_j \equiv 0 \mod p^{4-n}$.*

Proof: If $r \leq (a-1)p$ and $r \equiv a \mod p-1$, then $\Sigma(r) = a$. Therefore, because $r \equiv a \mod p$, we have $r = a$. Hence, $\{j : a-1 \leq j < r-1 \text{ and } j \equiv a-1 \mod p-1\} = \emptyset$ and the proposition trivially holds true.

Let $r > (a-1)p$. By Lemma 1.6 for $i = 1$ and noting that $r - a \equiv 0 \mod p$, we have $\sum_{j \geq 3} \binom{j}{3}\binom{r}{j}$, $\sum_{j \geq 2} \binom{j}{2}\binom{r}{j}$, $\sum_{j \geq 1} j\binom{r}{j}$ and $\sum_j \binom{r}{j} \equiv 0 \mod p$.

Therefore, we are in a situation analogous to that of the proof of Lemma 1.8, and we can proceed analogously (where we put $k = (a-1)p$ instead of $k = ap$): Put

$$s_0 = -p^{-1} \sum_{j \geq 0} \binom{r}{j}, \quad s_1 = -p^{-1} \sum_{j \geq 1} j \binom{r}{j} \quad \text{and} \quad s_2 = -p^{-1} \sum_{j \geq 2} \binom{j}{2}\binom{r}{j}.$$

and $\beta_j = \binom{r}{j} + p\delta_j$. There are δ_j such that

$$\sum \beta_j \equiv 0 \mod p^4, \quad \sum j\beta_j \equiv 0 \mod p^3 \quad \text{and} \quad \sum \binom{j}{2}\beta_j \equiv 0 \mod p^2$$

if and only if the following system of linear equations $(*)$ in the three unknowns

δ_k, δ_l and δ_m is solvable:

$$\begin{matrix} 1 & 1 & 1 \equiv s_0 \bmod p^4, \\ k & l & m \equiv s_1 \bmod p^3, \\ \binom{k}{2} & \binom{l}{2} & \binom{m}{2} \equiv s_2 \bmod p^2. \end{matrix} \qquad (*)$$

It suffices to solve all equations modulo p^4. For this, we show that there are k, l and m in $\{a-1, a+(p-2), \dots, r-p\}$ such that the determinant of $(*)$ is invertible in $\mathbb{Z}/p^4\mathbb{Z}$, or equivalently, that it is nonzero $\bmod p$.

Because $r > (a-1)p$, we have $0 < (a-1)p < r$; we may, and will, therefore put $k = (a-1)p$. Then $(*)$ is modulo p given by an upper triangular matrix whose upper left coefficient is 1, and therefore its determinant equals that of its lower right 2×2-matrix

$$\begin{pmatrix} l & m \\ \binom{l}{2} & \binom{m}{2} \end{pmatrix}$$

which is 0 if and only if l, m or $l - m$ vanishes modulo p. Therefore, choosing l and m in $\{a-1, a-1+(p-1), \dots, r-1-(p-1)\}$ such that $l, m, l-m \not\equiv 0 \bmod p$, the system of linear equations $(*)$ is solvable. □

Lemma 1.10. *Let a in $\{4, \dots, p+1\}$ such that $r \equiv a \bmod p-1$. If $r \equiv a \bmod p$, then there are integers $\{\gamma_j : a-2 \leq j < r-2$ and $j \equiv a-2 \bmod p-1\}$ such that*

(i) *we have $\gamma_j \equiv \binom{r}{j} \bmod p$, and*

(ii) *for $n = 0, 1, 2, 3$, we have $\sum_{j \geq n} \binom{j}{n} \gamma_j \equiv 0 \bmod p^{4-n}$.*

Proof: If $r \leq (a-2)p$ and $r \equiv a \bmod p-1$, then $\Sigma(r) = a$. Therefore, because $r \equiv a \bmod p$, we have $r = a$. Therefore, $\{j : a-2 \leq j < r-2$ and $j \equiv a-2 \bmod p-1\} = \emptyset$ and the proposition trivially holds true.

Let $r > (a-2)p$. By Lemma 1.6 for $i = 2$ and noting that $a - r \equiv 0 \bmod p$, we have $\sum_{j \geq 3} \binom{j}{3}\binom{r}{j}$, $\sum_{j \geq 2} \binom{j}{2}\binom{r}{j}$, $\sum_{j \geq 1} j\binom{r}{j}$ and $\sum_{j \geq 0} \binom{r}{j} \equiv 0 \bmod p$.

Therefore, we are in a situation analogous to that of the proof of Lemma 1.8, and we can proceed analogously (where we put $k = (a-2)p$ instead of $k = ap$): Put

$$s_0 = -p^{-1} \sum_{j \geq 0} \binom{r}{j}, \quad s_1 = -p^{-1} \sum_{j \geq 1} j \binom{r}{j} \quad \text{and} \quad s_2 = -p^{-1} \sum_{j \geq 2} \binom{j}{2}\binom{r}{j}.$$

and $\gamma_j = \binom{r}{j} + p\delta_j$. There are δ_j such that

$$\sum \gamma_j \equiv 0 \bmod p^4, \sum j\gamma_j \equiv 0 \bmod p^3, \text{ and } \sum \binom{j}{2}\gamma_j \equiv 0 \bmod p^2$$

if and only if the following system of linear equations (∗) in the three unknowns δ_k, δ_l and δ_m is solvable:

$$\begin{matrix} 1 & 1 & 1 \equiv s_0 \bmod p^4, \\ k & l & m \equiv s_1 \bmod p^3, \\ \binom{k}{2} & \binom{l}{2} & \binom{m}{2} \equiv s_2 \bmod p^2. \end{matrix} \qquad (*)$$

It suffices to solve all equations modulo p^4. For this, we show that there are k, l and m in $\{a-2, a-2+(p-1), \dots, r-p-1\}$ such that the determinant of (∗) is invertible in $\mathbb{Z}/p^4\mathbb{Z}$, or equivalently, that it is nonzero $\bmod p$.

Because $r > (a-2)p$, we have $0 < (a-2)p < r$; we may, and will, therefore put $k = (a-2)p$. Then (∗) is modulo p given by an upper triangular matrix whose upper left coefficient is 1, and therefore its determinant equals that of its lower right 2×2-matrix

$$\begin{pmatrix} l & m \\ \binom{l}{2} & \binom{m}{2} \end{pmatrix}$$

which is 0 if and only if l, m or $l-m$ vanishes modulo p. Therefore, choosing l and m in $\{a-2, a-2+(p-1), \dots, r-2-(p-1)\}$ such that $l, m, l-m \not\equiv 0 \bmod p$, the system of linear equations (∗) is solvable. □

Lemma 1.11. *Let $a = p$ and $r \equiv a \bmod p-1$.*

(i) *If $p^2 \mid p-r$, then there are integers $\{\gamma_j : p-1 \le j < r-1$ and $j \equiv 0 \bmod p-1\}$ such that*

- *we have $\gamma_j \equiv \binom{r}{j} \bmod p^2$, and*
- *for $0 \le n \le 4$, we have $\sum_{j \ge n} \binom{j}{n}\gamma_j \equiv 0 \bmod p^{5-n}$.*

(ii) *If $p^2 \mid p-r$, then there are integers $\{\gamma_j : p \le j < r$ and $j \equiv 1 \bmod p-1\}$ such that*

- *we have $\gamma_j \equiv \binom{r}{j} \bmod p^2$, and*
- *for $0 \le n \le 4$, we have $\sum_{j \ge n} \binom{j}{n}\gamma_j \equiv 0 \bmod p^{5-n}$.*

(iii) *If* $p^3 \mid p - r$, *then there are integers* $\{\gamma_j : p \leq j < r$ *and* $j \equiv 1 \bmod p-1\}$ *such that*

- *we have* $\gamma_j \equiv \binom{r}{j} \bmod p^3$, *and*
- *for* $0 \leq n \leq 5$, *we have* $\sum_{j \geq n} \binom{j}{n} \gamma_j \equiv 0 \bmod p^{6-n}$.

Proof: We adapt Lemma 1.8 by referring, in (i) and (ii), to [BG15, Lemma 2.7], respectively, in (iii), to Lemma 1.7:

Ad (i): Let $a = p$ and $r \equiv a \bmod p-1$. Because $p^2 \mid p - r$, we have $r > (a-1)p$. By [BG15, Lemma 2.7], because by assumption $a - r \equiv 0 \bmod p^2$, we have $\sum_{j \geq 3} \binom{j}{3}\binom{r}{j}$, $\sum_{j \geq 2} \binom{j}{2}\binom{r}{j}$, $\sum_{j \geq 1} j\binom{r}{j}$ and $\sum_{j \geq 0} \binom{r}{j} \equiv 0 \bmod p^2$.

Therefore, we are in a situation analogous to that of the proof of Lemma 1.8, and we can proceed analogously (where we put $k = (a-1)p$ instead of $k = ap$): Put

$$s_0 = -p^{-2} \sum_{j \geq 0} \binom{r}{j}, \quad s_1 = -p^{-2} \sum_{j \geq 1} j \binom{r}{j} \quad \text{and} \quad s_2 = -p^{-2} \sum_{j \geq 2} \binom{j}{2}\binom{r}{j}.$$

and $\gamma_j = \binom{r}{j} + p^2 \delta_j$. There are δ_j such that

$$\sum \gamma_j \equiv 0 \bmod p^5, \sum j\gamma_j \equiv 0 \bmod p^4, \text{ and } \sum \binom{j}{2}\gamma_j \equiv 0 \bmod p^3$$

if and only if the following system of linear equations $(*)$ in the three unknowns δ_k, δ_l and δ_m is solvable:

$$\begin{matrix} 1 & 1 & 1 \equiv s_0 \bmod p^5, \\ k & l & m \equiv s_1 \bmod p^4, \\ \binom{k}{2} & \binom{l}{2} & \binom{m}{2} \equiv s_2 \bmod p^3. \end{matrix} \tag{$*$}$$

It suffices to solve all equations modulo p^5. For this, we show that there are k, l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that the determinant of $(*)$ is invertible in $\mathbb{Z}/p^5\mathbb{Z}$, or equivalently, that it is nonzero $\bmod p$.

Because $r > (a-1)p$, we have $a - 1 \leq (a-1)p < r - 1$; we may, and will, therefore put $k = (a-1)p$. Then $(*)$ is modulo p given by an upper triangular matrix whose upper left coefficient is 1, and therefore its determinant equals that of its lower right 2×2-matrix

$$\begin{pmatrix} l & m \\ \binom{l}{2} & \binom{m}{2} \end{pmatrix}$$

which is 0 if and only if l, m or $l-m$ vanishes modulo p. Therefore, choosing l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that $l, m, l-m \not\equiv 0 \bmod p$, the system of linear equations $(*)$ is solvable.

Ad (ii): Let $a = p$ and $r \equiv a \bmod p-1$. Because $p^2 \mid a-r$, we have $r > ap$. By [BG15, Lemma 2.7], because by assumption $p - r \equiv 0 \bmod p^2$, we have $\sum_{j\geq 3}\binom{j}{3}\binom{r}{j}$, $\sum_{j\geq 2}\binom{j}{2}\binom{r}{j}$, $\sum_{j\geq 1} j\binom{r}{j}$ and $\sum_{j\geq 0}\binom{r}{j} \equiv 0 \bmod p^2$.

Therefore, we are in a situation analogous to that of the proof of Lemma 1.8, and we can proceed analogously: Put

$$s_0 = -p^{-2}\sum_{j\geq 0}\binom{r}{j}, \quad s_1 = -p^{-2}\sum_{j\geq 1} j\binom{r}{j} \quad \text{and} \quad s_2 = -p^{-2}\sum_{j\geq 2}\binom{j}{2}\binom{r}{j}.$$

and $\gamma_j = \binom{r}{j} + p^2\delta_j$. There are δ_j such that

$$\sum \gamma_j \equiv 0 \bmod p^5, \sum j\gamma_j \equiv 0 \bmod p^4, \text{ and } \sum \binom{j}{2}\gamma_j \equiv 0 \bmod p^3$$

if and only if the following system of linear equations $(*)$ in the three unknowns δ_k, δ_l and δ_m is solvable:

$$\begin{array}{ccl} 1 & 1 & 1 \equiv s_0 \bmod p^5, \\ k & l & m \equiv s_1 \bmod p^4, \\ \binom{k}{2} & \binom{l}{2} & \binom{m}{2} \equiv s_2 \bmod p^3. \end{array} \tag{$*$}$$

It suffices to solve all equations modulo p^5. For this, we show that there are k, l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that the determinant of $(*)$ is invertible in $\mathbb{Z}/p^5\mathbb{Z}$, or equivalently, that it is nonzero $\bmod p$.

Because $r > ap$, we have $a \leq ap < r$; we may, and will, therefore put $k = ap$. Then $(*)$ is modulo p given by an upper triangular matrix whose upper left coefficient is 1, and therefore its determinant equals that of its lower right 2×2-matrix

$$\begin{pmatrix} l & m \\ \binom{l}{2} & \binom{m}{2} \end{pmatrix}$$

which is 0 if and only if l, m or $l-m$ vanishes modulo p. Therefore, choosing l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that $l, m, l-m \not\equiv 0 \bmod p$, the system of linear equations $(*)$ is solvable.

Ad (iii): Let $a = p$ and $r \equiv a \bmod p-1$. Because $p^3 \mid p - r$, we have $r > ap$. By Lemma 1.7, because by assumption $a - r \equiv 0 \bmod p^3$, we have $\sum_{j\geq 3}\binom{j}{3}\binom{r}{j}$, $\sum_{j\geq 2}\binom{j}{2}\binom{r}{j}$, $\sum_{j\geq 1} j\binom{r}{j}$ and $\sum_{j\geq 0}\binom{r}{j} \equiv 0 \bmod p^3$.

Therefore, we are in a situation analogous to that of the proof of Lemma 1.8, and we can proceed analogously: Put

$$s_0 = -p^{-3}\sum_{j\geq 0}\binom{r}{j}, \quad s_1 = -p^{-3}\sum_{j\geq 1} j\binom{r}{j} \quad \text{and} \quad s_2 = -p^{-3}\sum_{j\geq 2}\binom{j}{2}\binom{r}{j}.$$

and $\gamma_j = \binom{r}{j} + p^3\delta_j$. There are δ_j such that

$$\sum \gamma_j \equiv 0 \bmod p^6, \sum j\gamma_j \equiv 0 \bmod p^5, \text{ and } \sum \binom{j}{2}\gamma_j \equiv 0 \bmod p^4$$

if and only if the following system of linear equations $(*)$ in the three unknowns δ_k, δ_l and δ_m is solvable:

$$\begin{matrix} 1 & 1 & 1 \equiv s_0 \bmod p^6, \\ k & l & m \equiv s_1 \bmod p^5, \\ \binom{k}{2} & \binom{l}{2} & \binom{m}{2} \equiv s_2 \bmod p^4. \end{matrix} \qquad (*)$$

It suffices to solve all equations modulo p^6. For this, we show that there are k, l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that the determinant of $(*)$ is invertible in $\mathbb{Z}/p^6\mathbb{Z}$, or equivalently, that it is nonzero $\bmod p$.

Because $r > ap$, we have $a \leq ap < r$; we may, and will, therefore put $k = ap$. Then $(*)$ is modulo p given by an upper triangular matrix whose upper left coefficient is 1, and therefore its determinant equals that of its lower right 2×2-matrix

$$\begin{pmatrix} l & m \\ \binom{l}{2} & \binom{m}{2} \end{pmatrix}$$

which is 0 if and only if l, m or $l - m$ vanishes modulo p. Therefore, choosing l and m in $\{a, a+(p-1), \ldots, r-(p-1)\}$ such that $l, m, l-m \not\equiv 0 \bmod p$, the system of linear equations $(*)$ is solvable. □

2 The Jordan-Hölder series of X_{r-2}

The next statement about under which conditions $X_{r-2} \supset X_{r-1}$ is a proper inclusion is useful to obtain an additional Jordan-Hölder factor in X_{r-2}. In contrast to the inclusion $X_{r-1} \supseteq X_r$, however, not always $X_{r-2} \neq X_{r-1}$ for r sufficiently big. To exemplify this, there is by Lemma 2.2 the natural epimorphism

$$X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$$

given by multiplication. Let $r'' = r - 2$. If $\Sigma(r'')$ is minimal, then by Proposition 2.11 the left-hand side of

$$0 \to X^*_{r''} \to X_{r''} \to X_{r''}/X^*_{r''} \to 0$$

vanishes. Let a in $\{3, \ldots, p+1\}$ such that $r \equiv a \bmod p-1$. If $a = 3$, then the right-hand side is $X_{r''}/X^*_{r''} = V_1$. Therefore,

$$X_{r''} \otimes V_2 = V_1 \otimes V_2 = V_1 \otimes D \oplus V_3 \twoheadrightarrow X_{r-2}.$$

That is, there is an epimorphism with only two Jordan-Hölder factors onto X_{r-2}. Therefore, necessarily $X_{r-2} = X_{r-1}$.

This equality happens in other cases as well: For $r = p+2, \ldots, p+(p-2)$, that is, $r = (p-1) + a$ for $a = 3, \ldots, p-1$, by Proposition 2.6.(iii),

$$X_{r-2}/X^*_{r-2} = V_a/V^*_a = V_a$$

where the equality on the right-hand side stands because V_a is irreducible when $a = 3, \ldots, p-1$; thus, $X^*_{r-2} = V_{p-1}$ is irreducible. Therefore, $X_{r-2} = X_{r-1}$ because both have two Jordan-Hölder factors. For $r = 2p-1$, by Proposition 2.6.(iii),

$$X_{r-2}/X^*_{r-2} = V_p/V^*_p$$

has by Lemma 1.3.(i) two Jordan-Hölder factors. Therefore, since $\dim X^*_{r-2} \leq p$, there are exactly three Jordan-Hölder factors in X_{r-2}; still, $X_{r-2} = X_{r-1}$, because by [BG15, Proposition 3.3.(i)] $X_{r-1} = V_{2p-1}$, in particular, X_{r-2} has exactly three Jordan-Hölder factors.

By the next statement, $X_{r-2} = X_{r-1}$ if and only if $r = p^n + r_0$ where $r_0 = 2, \ldots, p-1$ and n in $\mathbb{N}$. (The preceding discussion showed this only for $r_0 = 2$ or $n = 1$.)

Lemma 2.1. *Let r in $\mathbb{N}$. We have $0 \subset X_r \subseteq X_{r-1} \subseteq X_{r-2}$ and*

- *the inclusion $X_r \subseteq X_{r-1}$ is an equality if and only if $r \leq p$, and*
- *for $p > 2$, the inclusion $X_{r-1} \subseteq X_{r-2}$ is an equality if and only if $r \leq p$ or $r = p^n + r_0$ where r_0 in $\{2, \ldots, p-1\}$ and $n > 0$.*

Proof: For $X_r \subseteq X_{r-1}$ and when this inclusion is an inequality, see [BG15, Lemma 4.1]. Note that $X_r = X_{r-1} = V_r$ for $r < p$.

We have $X_{r-1} \subseteq X_{r-2}$, because $4X^{r-1}Y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X^{r-2}Y^2 - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} X^{r-2}Y^2$.

If $r < p$, then V_r is irreducible. In particular, $X_{r-2} = X_{r-1}$.

If $r = p$, then $X_{r-2} = V_p = X_{r-1}$ by Proposition 2.6.(ii) and (iii).

We may hence assume $r > p$. We have $X_{r-1} = X_{r-2}$ if and only if there are coefficients C, $c_0, \ldots, c_{p-1}$, $d_0, \ldots, d_{p-1}$ and D in $\mathbb{F}_p$ such that

$$X^2Y^{r-2} = CX^r + \sum c_k(kX + Y)^{r-1}X + \sum d_l(X + lY)^{r-1}Y + DY^r.$$

For $T \in \{0, \ldots, r-2\}$, put

$$C_T = \sum c_k k^T \quad \text{and} \quad D_T = \sum d_l l^{r-2-T}.$$

Comparing the coefficients on both sides of the equation, this equation is satisfied if and only if

- $C + \sum C_{r-1} = 0$ (by the coefficient of X^r),
- $D_{r-1} + D = 0$ (by the coefficient of Y^r), and,
- by the coefficients of $X^{T+1}Y^{r-(T+1)}$ for $T = 0, \ldots, r-2$,

$$\binom{r-1}{T}C_T + \binom{r-1}{T+1}D_{r-2-T} = \begin{cases} 0, & \text{for } T = 0, \\ 1, & \text{for } T = 1, \\ 0, & \text{for } T = 2, \ldots, r-2. \end{cases} \tag{+}$$

In the following, we will show that there are coefficients $c_0, \ldots, c_{p-1}$, and $d_0, \ldots, d_{p-1}$ in $\mathbb{F}_p$ such that (+) is satisfied if and only if the stated conditions on r are satisfied. That is, we show that if the stated conditions on r are not satisfied, then (+) cannot be satisfied, but if they are satisfied, then there are such coefficients.

Because $\#\mathbb{F}_p^* = p - 1$, if $T' \equiv T'' \bmod p-1$, then $C_{T'} = C_{T''}$ and $D_{T'} = D_{T''}$. In particular, for $T \equiv 1 \bmod p-1$,

$$\binom{r-1}{T}C_T + \binom{r-1}{T+1}D_T = \binom{r-1}{T}C_1 + \binom{r-1}{T+1}D_1.$$

Expand $r - 1 = r_0 + r_1 p + r_2 p^2 + \cdots$ with $r_0, r_1, \ldots \in \{0, \ldots, p-1\}$.

Case 1. $r_0 = 0$.

Then by Lucas's Theorem modulo p,

$$\binom{r-1}{1} = r_0 = 0 \quad \text{and} \quad \binom{r-1}{2} = \binom{r_0}{2} = 0$$

This equation contradicts that of (+) for $T = 1$! Therefore $X_{r-2} \supset X_{r-1}$.

Case 2. $r_0 > 0$.

Case 2.1. There is a digit $r_j > 1$. Let j be the minimal index of all digits with that property.

For $T = p^j, p^j + p^j - 1$ with $j \geq 1$, by Lucas's Theorem modulo p,

$$\binom{r-1}{p^j} = \binom{r_j}{1} \quad \text{and} \quad \binom{r-1}{p^j+1} = \binom{r_j}{1}\binom{r_0}{1}$$

$$\binom{r-1}{p^j+p^j-1} = \binom{r_j}{1}\binom{r_{j-1}}{p-1}\cdots\binom{r_0}{p-1} \quad \text{and} \quad \binom{r-1}{2p^j} = \binom{r_j}{2}$$

Because $p^j, p^j + p^j - 1 \equiv 1 \bmod p - 1$,

$$r_j C_1 + r_j r_0 D_1 = 0$$

$$r_j\binom{r_{j-1}}{p-1}\cdots\binom{r_0}{p-1}C_1 + \binom{r_j}{2}D_1 = 0.$$

The determinant of the matrix M of this system of equations is

$$|M| = r_j \cdot \begin{vmatrix} 1 & r_0 \\ r_j\binom{r_{j-1}}{p-1}\cdots\binom{r_0}{p-1} & \binom{r_j}{2} \end{vmatrix} = r_j \cdot \left[\binom{r_j}{2} - r_j r_0 \binom{r_{j-1}}{p-1}\cdots\binom{r_0}{p-1}\right].$$

Case 2.1.1. $j > 1$.

By minimality of j, we have $\binom{r_{j-1}}{p-1} = 0$. Thence $|M| = r_j\binom{r_j}{2} \neq 0$, that is, $C_1 = D_1 = 0$. This equation contradicts that of (+) for $T = 1$! Therefore $X_{r-2} \supset X_{r-1}$.

Case 2.1.2. $j = 1$.

Then

$$|M| = r_1\left[\binom{r_1}{2} - r_1 r_0\binom{r_0}{p-1}\right]$$

We obtain $|M| \neq 0$,

- if $r_0 < p-1$, because $|M| = r_1\binom{r_1}{2}$, and
- if $r_0 = p-1$, because $\binom{r_1}{2} \neq r_1(p-1)$

That is, $C_1 = D_1 = 0$. This equation contradicts that of (+) for $T = 1$! Therefore $X_{r-2} \supset X_{r-1}$.

Case 2.2. All $r_1, r_2, \ldots \leq 1$. That is, $r-1$ is of the form $r-1 = r_0 + p^{n_1} + \cdots + p^{n_m}$ for $0 < n_1 < \ldots < n_m$ in $\mathbb{N}$.

For $T = 1$, we have

$$\binom{r-1}{T}C_1 + \binom{r-1}{T+1}D_1 = r_0C_1 + \binom{r_0}{2}D_1 = 1.$$

Case 2.2.1. We have $r_0 = p-1$. By Lucas's Theorem,

- for $T = p^{n_1}$, we have, because $T \equiv 1 \bmod p-1$,

$$\binom{r-1}{T}C_1 + \binom{r-1}{T+1}D_1 = r_{n_1}C_1 + r_{n_1}r_0D_1 = C_1 + r_0D_1 = 0;$$

- for $T = p^{n_1} + r_0$, then $T+1 = 2p$ if $n_1 = 1$, and $T+1 = p^{n_1} + p$ if $n_1 > 1$. Thus, if $n_1 = 1$ we have $\binom{r_1}{2} = 0$ because $r_1 \leq 1$, and if $n_1 > 1$, we have $\binom{r_1}{1} = 0$ because $r_1 = 0$. Therefore, because $T \equiv 1 \bmod p-1$,

$$\binom{r-1}{T}C_1 + \binom{r-1}{T+1}D_1 = r_{n_1}\binom{r_0}{r_0}C_1 = C_1 = 0,$$

Therefore $C_1 = 0$, thus $D_1 = 0$. Thus

$$r_0C_1 + \binom{r_0}{2}D_1 = 1$$

is impossible to satisfy.

Case 2.2.2. We have $r_0 < p-1$.

Case 2.2.2.1. We have $m > 1$. By Lucas's Theorem,

- for $T = p^{n_1}$, we have, because $T \equiv 1 \bmod p-1$,

$$\binom{r-1}{T}C_1 + \binom{r-1}{T+1}D_1 = r_{n_1}C_1 + r_{n_1}r_0D_1 = C_1 + r_0D_1 = 0;$$

- for $T = p^{n_2} + p^{n_1} - 1$, we have $\binom{r-1}{T} = 0$ because $\binom{r_0}{p-1} = 0$. Therefore, because $T \equiv 1 \bmod p - 1$,

$$\binom{r-1}{T} C_1 + \binom{r-1}{T+1} D_1 = r_{n_2} r_{n_1} D_1 = D_1 = 0$$

Therefore $D_1 = 0$, thus $C_1 = 0$. Thus

$$r_0 C_1 + \binom{r_0}{2} D_1 = 1$$

is impossible to satisfy.

Case 2.2.2.2. We have $m = 1$. In this case, r satisfies the stated conditions for $X_{r-1} = X_{r-2}$, and we show, equivalently, that (+) can be solved. We have:

- the only T in $\{0, \dots, r-2\}$ such that $T \equiv 1 \bmod p - 1$ and $\binom{r-1}{T} \not\equiv 0 \bmod p$ are $T = p^0, p^{n_1}$,
- the only T in $\{0, \dots, r-2\}$ such that $T \equiv 1 \bmod p - 1$ and $\binom{r-1}{T+1} \not\equiv 0 \bmod p$ are $T = p^0, p^{n_1}$ for $r_0 > 1$, and, $T = p^{n_1}$ for $r_0 = 1$.

Therefore, to solve (+), it suffices to choose $C_1, \dots, C_{p-1}$ and $D_1, \dots, D_{p-1}$ such that they resolve (+) for $T = p^0$ and p^{n_1}; that is, by Lucas's Theorem, such that for $T = 1$,

$$r_0 C_1 + \binom{r_0}{2} D_1 = 1$$

and

$$\binom{r-1}{p^{n-1}} C_{p^{n_1}} + \binom{r-1}{p^{n_1}+1} D_{p^{n_1}} = C_1 + r_0 D_1 = 0.$$

That is, such that

$$C_1 = -r_0 D_1 \quad \text{and} \quad D_1 = \frac{1}{\binom{r_0}{2} - r_0^2} \tag{*}$$

(where the denominator is nonzero because $r_0 \neq p - 1$) and where

$$C_2, \dots, C_{p-1} \quad \text{and} \quad D_2, \dots, D_{p-1}$$

are arbitrary. We can therefore choose $c_1, \dots, c_{p-1}$ respectively $d_1, \dots, d_{p-1}$ such that C_1 respectively D_1 satisfy Equation (∗).

□

2.1 Tensor Product Epimorphism

Lemma 2.2 (Extension of [BG15, Lemma 3.6]). *Let $r \geq 2$. Put $r'' = r - 2$. The map*

$$\phi\colon X_{r''} \otimes V_2 \to X_{r-2}$$
$$f \otimes g \mapsto f \cdot g$$

is an epimorphism of $\mathbb{F}_p[M]$-modules.

Proof: By [Glo78, (5.1)], the map $\phi_{r'',2}\colon V_{r''} \otimes V_2 \twoheadrightarrow V_r$ defined by $u \otimes v \mapsto uv$ is M-linear. Let ϕ be its restriction to the M-submodule $X_{r''} \otimes V_2$. The submodule $X_{r''} \otimes V_2$ is generated by $X^{r''} \otimes X^2$, $X^{r''} \otimes Y^2$ and $X^{r''} \otimes XY$, which map to X^r, $X^{r-2}Y^2$ and $X^{r-1}Y$. Therefore the image of ϕ is included in $X_{r-2} \subseteq V_r$. Because $X^{r-2}Y^2$ generates X_{r-2}, surjectivity follows. □

Corollary 2.3. *We have* $\dim X_{r-2} \leq 3p+3$. *If* $\dim X_{r-2} = 3p+3$, *then the epimorphism* $\phi\colon X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$ *is an isomorphism.*

Proof: Because $\dim X_{r''} \leq p+1$ and $\dim V_2 = 3$, the left-hand side of the epimorphism $\phi\colon X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$ in Lemma 2.2 has dimension $\leq 3(p+1) = 3p+3$. Therefore its kernel is 0. □

Corollary 2.4 (Extension of [BG15, Lemma 3.5]). *If* $\dim X_{r-2} = 3p+3$, *then* $\dim X_{r-1} = 2p+2$ *is maximal and* $\dim X_r = \dim X_{r'} = \dim X_{r''} = p+1$ *are maximal.*

Proof: If $\dim X_{r-2} = 3p+3$, then by Corollary 2.3 the epimorphism $\phi\colon X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$ in Lemma 2.2 is an isomorphism; in particular, the left-hand side in $\phi\colon X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$ has dimension $3(p+1)$. Therefore, as $\dim V_2 = 3$, we have $\dim X_{r''} = p+1$.

That $\dim X_{r-1} = 2p+2$ (that is, is maximal) is seen as in the proof of [BG15, Lemma 3.5]. Therefore $\dim X_r = p+1$ (that is, is maximal) by [BG15, Lemma 3.5].

If $\dim X_{r-1} = 2p+2$, then by the epimorphism $X_{r'} \otimes V_2 \twoheadrightarrow X_{r-1}$, given by $f \otimes g \mapsto f \cdot g$, also $\dim X_{r'} = p+1$ is maximal. □

Lemma 2.5 (Extension of [GG15, Lemma 3]). *Let $p > 2$ and $r \geq 2$. The $\mathbb{F}_p$-module X_{r-2} is generated by*

$$\{X^r, Y^r, X^{r-1}Y, X^2(jX+Y)^{r-2}, Y^2(X+kY)^{r-2}, XY(lX+Y)^{r-2} : j,k,l \in \mathbb{F}_p\}.$$

Proof: We have $X_{r-2} = \langle X^{r-2}Y^2\rangle$. We compute

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^{r-2}Y^2 = (aX + cY)^{r-2}(bX + dY)^2$$
$$= b^2X^2(aX + cY)^{r-2} + d^2Y^2(aX + cY)^{r-2} + 2bdXY(aX + cY)^{r-2}.$$

If $a = 0$, then the right-hand side is in the span of $X^2Y^{r-2}, Y^r, XY^{r-1}$. If $c = 0$, then the right-hand side is in the span of $X^r, X^{r-2}Y^2, X^{r-1}Y$. If $ac \neq 0$, then the right-hand side is in the span of

$$\{X^r, Y^r, X^{r-1}Y, X^2(jX + Y)^{r-2}, Y^2(X + kY)^{r-2}, XY(lX + Y)^{r-1}\}$$

where j, k, l in $\mathbb{F}_p$. We conclude as in [GG15, Lemma 3]. □

2.2 Singular Quotient of X_r, X_{r-1} and X_{r-2}

We generalize [Glo78, (4.5)] by computing the quotients of X_r, X_{r-1} and X_{r-2} by its largest singular module: We denote by

$$N = \{ \text{ all } m \text{ in } M \text{ such that } \det m = 0\},$$

all singular matrices and, for every module V with an action of M, its largest singular submodule by

$$V^* = \{ \text{ all } v \text{ in } V \text{ such that } n \cdot v = 0 \text{ for all } n \text{ in } N\}.$$

Proposition 2.6 (Extension of [Glo78, (4.5)]). *Let* $r > 0$.

(i) For the unique a *in* $\{1, \ldots, p-1\}$ *such that* $r \equiv a \bmod p-1$,

$$X_r/X_r^* = X_a/X_a^* = V_a.$$

(ii) For the unique a *in* $\{2, \ldots, p\}$ *such that* $r \equiv a \bmod p-1$,

$$X_{r-1}/X_{r-1}^* = X_{a-1}/X_{a-1}^* = V_a/V_a^* = \begin{cases} V_a, & \text{for } a = 2, \ldots, p-1 \\ V_a/V_a^*, & \text{for } a = p \text{ and } r \geq p \end{cases}$$

(iii) For the unique a *in* $\{3, \ldots, p+1\}$ *such that* $r \equiv a \bmod p-1$,

$$X_{r-2}/X_{r-2}^* = X_{a-2}/X_{a-2}^* = V_a/V_a^* = \begin{cases} V_a, & \text{for } a = 3, \ldots, p-1 \\ V_a/V_a^*, & \text{for } a = p, p+1 \text{ and } r \geq p \end{cases}$$

Proof:

(i) To prove $X_r/X_r^* = X_a/X_a^*$, we adapt the proof of [Glo78, (4.5)] so that it readily generalizes to X_{r-1}: Let U_r (denoted X in *op. cit.*) be the vector space of dimension $p+1$ with basis vectors x_0, x_1, ..., x_p. Let $\rho_r \colon U_r \to X_r$ be given by

$$x_0 \mapsto x^r \quad \text{and} \quad x_i \mapsto (ix+y)^r.$$

In particular,

$$\rho_r x_i = (\rho_1 x_i)^r.$$

For every v in $X_1 = V_1$, there is γ in $\mathbb{F}_p$ and a unique i in $\{0,1,\ldots,p\}$ such that $v = \gamma\rho_1(x_i)$. In particular, for every $v = m \cdot \rho_1(x_i)$ for $i = 0, 1, \ldots, p$. Let M act on U_r by

$$m \cdot x_i = \begin{cases} 0, & \text{if } m \cdot \rho_1(x_i) = 0 \\ \gamma^r x_j, & \text{if } m \cdot \rho_1(x_i) = \gamma\rho_1(x_j). \end{cases}$$

With this action of M, the proof of [Glo78, (4.5)] shows that ρ_r is M-linear. Also, $\#\mathbb{F}_p^* = p-1$, the $\mathbb{F}_p[M]$-modules U_r and U_a are isomorphic.

We claim

$$\rho_a^{-1}(X_a^*) = \rho_r^{-1}(X_r^*),$$

that is: For every n in N and x in $U_a = U_r$, we have $n \cdot \rho_a(x) = 0$ if and only if $n \cdot \rho_r(x) = 0$.

To see this, note that the image of n on V_1 is at most one-dimensional, $\dim n(V_1) \leq 1$, that is, there is v_n in V_1 such that for every v in V_1 there is γ_v in $\mathbb{F}_p$ such that $n \cdot v = \gamma_v v_n$. Therefore, by definition of the M-linear homomorphism ρ_r, for every $i = 0,1,\ldots,p$ there is γ_i in $\mathbb{F}_p$ such that

$$n \cdot \rho_r(x_i) = \gamma_i^r v_n^r.$$

Writing $x = \sum_i b_i x_i$, therefore

$$n \cdot \rho_r(x) = \left[\sum b_i \gamma_i^r\right] v_n^r$$

Similarly,

$$n \cdot \rho_a(x) = \left[\sum b_i \gamma_i^a\right] v_n^a$$

Because $r \equiv a \bmod p-1$ and $\#\mathbb{F}_p^* = p-1$,

$$\sum b_i \gamma_i^a = \sum b_i \gamma_i^r.$$

Therefore,

$$n \cdot \rho_r(x) = 0 \quad \text{if and only if} \quad n \cdot \rho_a(x) = 0,$$

that is,

$$\rho_r^{-1}(X_r^*) = \rho_a^{-1}(X_a^*).$$

Therefore

$$X_r/X_r^* \xleftarrow{\sim} U_r/\rho_r^{-1}(X_r^*) = U_a/\rho_a^{-1}(X_a^*) \xrightarrow{\sim} X_a/X_a^*.$$

(As observed in the proof of [Gl078, (4.5)], indeed $X_a^* = 0$ because $a < p$ and V_a is irreducible.)

(ii) To prove $X_{r-1}/X_{r-1}^* = X_{a-1}/X_{a-1}^*$, we adapt the above proof: Put $r' = r-1$.

- Let $U_{r-1} = U_{r'} \otimes V_1$ be the $\mathbb{F}_p[M]$-module given by the tensor product of the $\mathbb{F}_p[M]$-modules $U_{r'}$ and V_1: If $x_0, x_1, \ldots, x_p$ is a basis of $U_{r'}$ and v' and v'' one of V_1, then the basis vectors of U_{r-1} are $x_0 \otimes v', \ldots, x_p \otimes v'$ and $x_0 \otimes v'', \ldots, x_p \otimes v''$.
- let $\rho_{r-1}\colon U_{r-1} \to X_{r-1}$ be the composition

$$U_{r-1} = U_{r'} \otimes V_1 \xrightarrow{\rho_{r'} \otimes \mathrm{id}} X_{r'} \otimes V_1 \to X_{r-1}$$

where the right-hand side homomorphism sends $f \otimes g$ to $f \cdot g$.

Because the $\mathbb{F}_p[M]$-modules $U_{r'}$ and $U_{a'}$ are isomorphic, so are U_{r-1} and U_{a-1}. We claim

$$\rho_{a-1}^{-1}(X_{a-1}^*) = \rho_{r-1}^{-1}(X_{r-1}^*),$$

that is: For every n in N and x in $U_{a-1} = U_{r-1}$, we have $n \cdot \rho_{a-1}(x) = 0$ if and only if $n \cdot \rho_{r-1}(x) = 0$. Because the image of n on V_1 is at most one-dimensional, $\dim n(V_1) \leq 1$, there is v_n in V_1 such that

- for every $i = 0, 1, \ldots, p$ there is γ_i in $\mathbb{F}_p$ such that

$$n \cdot x_i = \gamma_i^{r'} v_n^{r'}, \quad \text{and}$$

- there are γ' and γ'' in $\mathbb{F}_p$ such that $n \cdot v' = \gamma' v_n$ and $n \cdot v'' = \gamma'' v_n$.

Writing $x = \sum_i b'_i x_i \otimes v' + \sum_i b''_i x_i \otimes v''$, therefore

$$\begin{aligned} n \cdot \rho_{r-1}(x) &= \left[\gamma' \sum b'_i \gamma_i^{r'}\right] v_n^{r'} \cdot v_n + \left[\gamma'' \sum b''_i \gamma_i^{r'}\right] v_n^{r'} \cdot v_n \\ &= \left[\sum_i (\gamma' b'_i + \gamma'' b''_i)\gamma_i^{r'}\right] v_n^r \end{aligned}$$

Similarly,

$$n \cdot \rho_{a-1}(x) = \left[\sum_i (\gamma' b'_i + \gamma'' b''_i)\gamma_i^{a'}\right] v_n^a.$$

Because $r' \equiv a' \bmod p-1$ and $\#\mathbb{F}_p^* = p-1$,

$$\sum_i (\gamma' b'_i + \gamma'' b''_i)\gamma_i^{r'} = \sum_i (\gamma' b'_i + \gamma'' b''_i)\gamma_i^{a'}$$

Therefore,

$$n \cdot \rho_{r-1}(x) = 0 \quad \text{if and only if} \quad n \cdot \rho_{a-1}(x) = 0,$$

that is,

$$\rho_{r-1}^{-1}(X_{r-1}^*) = \rho_{a-1}^{-1}(X_{a-1}^*).$$

Therefore

$$X_{r-1}/X_{r-1}^* \xleftarrow{\sim} U_{r-1}/\rho_{r-1}^{-1}(X_{r-1}^*) = U_{a-1}/\rho_{a-1}^{-1}(X_{a-1}^*) \xrightarrow{\sim} X_{a-1}/X_{a-1}^*.$$

(iii) To prove $X_{r-2}/X_{r-2}^* = X_{a-2}/X_{a-2}^*$, we adapt the above proof: Put $r'' = r-2$.

- Let $U_{r-2} = U_{r''} \otimes V_2$ be the $\mathbb{F}_p[M]$-module given by the tensor product of the $\mathbb{F}_p[M]$-modules $U_{r''}$ and V_2: If $x_0, x_1, \ldots, x_p$ is a basis of $U_{r'}$ and v_0, v_1 and v_2 one of V_2, then the basis vectors of U_{r-2} are $x_0 \otimes v_1, \ldots, x_p \otimes v_1, x_0 \otimes v_1, \ldots, x_p \otimes v_1$ and $x_0 \otimes v_2, \ldots, x_p \otimes v_2$.
- let $\rho_{r-2} \colon U_{r-2} \to X_{r-2}$ be the composition

$$U_{r-2} = U_{r''} \otimes V_2 \xrightarrow{\rho_{r''} \otimes \mathrm{id}} X_{r''} \otimes V_2 \to X_{r-2}$$

where the right-hand side homohomomorphism sends $f \otimes g$ to $f \cdot g$.

Because the $\mathbb{F}_p[\mathrm{M}]$-modules $\mathrm{U}_{r''}$ and $\mathrm{U}_{a''}$ are isomorphic, so are U_{r-2} and U_{a-2}.

Let n in N and x in $\mathrm{U}_{a-2} = \mathrm{U}_{r-2}$. It suffices to prove that $n \cdot \rho_{a-2}(x) = 0$ if and only if $n \cdot \rho_{r-2}(x) = 0$, and we will prove this as above: Because the image of n on V_1 is at most one-dimensional, $\dim n(\mathrm{V}_1) \leq 1$, there is v_n in V_1 such that

- by definition of the M-action and ρ_r on U_r, for every $i = 0, 1, \ldots, p$ there is γ_i in $\mathbb{F}_p$ such that

$$n \cdot \rho_{r''}(x_i) = \gamma_i^{r''} v_n^{r''}, \quad \text{and}$$

- by definition of the M-action on V_2 with basis $v_0 = x^2$, $v_1 = xy$ and $v_2 = y^2$, there are Γ_0, Γ_2 and Γ_1', Γ_1'' in $\mathbb{F}_p$ such that

$$n \cdot v_0 = \Gamma_0^2 v_n^2, \quad n \cdot v_1 = \Gamma_1' \Gamma_1'' v_n^2, \quad \text{and} \quad n \cdot v_2 = \Gamma_2^2 v_n^2.$$

Writing $x = \sum_{i=0,1,\ldots,p, j=0,1,2} b_{i,j} x_i \otimes v_j$, therefore

$$\begin{aligned} & n \cdot \rho_{r-2}(x) \\ = & \left[\sum b_{i,0} \gamma_i^{r''} \Gamma_0^2\right] v_n^{r''} \cdot v_n^2 + \left[\sum b_{i,1} \gamma_i^{r''} \Gamma_1' \Gamma_1''\right] v_n^{r''} \cdot v_n^2 \\ & + \left[\sum b_{i,2} \gamma_i^{r''} \Gamma_2^2\right] v_n^{r''} \cdot v_n^2 \\ = & \left[\sum_i \gamma_i^{r''} (\Gamma_0^2 b_{i,0} + \Gamma_1' \Gamma_1'' b_{i,1} + \Gamma_2^2 b_{i,2})\right] v_n^r \end{aligned}$$

Similarly,

$$n \cdot \rho_{a-2}(x) = \left[\sum_i \gamma_i^{a''} (\Gamma_0^2 b_{i,0} + \Gamma_1' \Gamma_1'' b_{i,1} + \Gamma_2^2 b_{i,2})\right] v_n^a.$$

Because $r'' \equiv a'' \bmod p-1$ and $\#\mathbb{F}_p^* = p-1$, the result follows as above. □

Lemma 2.7 (Jordan-Hölder series of X_r). *There is a short exact sequence*

$$0 \to \mathrm{X}_r^* \to \mathrm{X}_r \to \mathrm{X}_r/\mathrm{X}_r^* \to 0.$$

Let $r \geq p$. For a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$,

- *we have $\mathrm{X}_r/\mathrm{X}_r^* = \mathrm{V}_a$, and*

- $\dim X_r = p+1$ *if and only if* $X_r^* \neq 0$; *if so, then* $X_r^* = V_{p-a-1} \otimes D^a$.

Proof: We have $\dim X_r \leq p+1$ and $X_r/X_r^* = X_a/X_a^* = V_a$ by Proposition 2.6.(i). By [BG15, Lemma 4.6], either $X_r^* = V_{p-a-1} \otimes D^a$ (if and only if $\dim X_r = p+1$) or $X_r^* = 0$ (if and only if $\dim X_r < p+1$). □

Lemma 2.8 (Extension of [BG15, Lemma 4.7]). *Let* $p \geq 3$ *and* $r \geq p$. *Let* a *in* $\{1, \ldots, p-1\}$ *such that* $r \equiv a \bmod p-1$.

(i) *If* $a = 1 \bmod p-1$, *then* $X_r^* = X_r^{**}$ *if and only if* $p \mid r$, *and* $X_r^{**} = X_r^{***}$.

(ii) *If* $a = 2$, *then* $X_r^* = X_r^{**}$, *and* $X_r^{**} = X_r^{***}$ *if and only if* $r \equiv 0, 1 \bmod p$.

(iii) *If* $a \geq 3$, *then* $X_r^* = X_r^{**} = X_r^{***}$.

Proof: Regarding the equality between X_r^* and X_r^{**}: If $a = 1$, then by [BG15, Lemma 3.1], we have $X_r^* = X_r^{**}$ if and only if $p|r$. If $a \geq 2$, then $X_r^* = X_r^{**}$ by [BG15, Lemma 4.7].

Regarding the equality between X_r^{**} and X_r^{***}: If $X_r^{**}/X_r^{***} \neq 0$, then $X_r^{**}/X_r^{***} = V_{p-a-1} \otimes D^a$ by Lemma 2.7. By Lemma 1.3.(iii), we find that $V_{p-a-1} \otimes D^a$ is a Γ-submodule of V_r^{**}/V_r^{***} if and only if $a = 2$. (Beware of the shift from a to $a+p-1$ for $a = 1, \ldots, 4$!) Therefore, if $a \neq 2$, then $X_r^{**}/X_r^{***} = 0$.

For $a = 2$, recall the polynomial in the proof of [BG15, Lemma 3.1.(i)]:

$$F(X, Y) = \sum_{j=0,\ldots,r} \binom{r}{j} \sum_{k \in \mathbb{F}_p} k^{r-j} X^{r-j} Y^j \equiv \sum_{\substack{j=0,\ldots,r \\ j \equiv 2 \bmod p-1}} -\binom{r}{j} X^{r-j} Y^j \bmod p.$$

It is in X_r^{**} by Lemma 1.4. If $r \not\equiv 0, 1 \bmod p$, then $\binom{r}{2} = r(r-1)/2 \not\equiv 0$; therefore, by the same token, $F(X, Y)$ is not in X_r^{***}. Thus $X_r^{**}/X_r^{***} \neq 0$.

If $r \equiv 0 \bmod p$, then we follow the proof of [BG15, Lemma 3.1.(ii)]: Write $r = p^n u$ for $n \geq 1$ and $p \nmid u$. Let $\iota \colon X_u \to X_r$ be given by

$$f(X, Y) \mapsto f(X^{p^n}, Y^{p^n}) = f(X, Y)^{p^n}.$$

It induces an isomorphism

$$X_u^{**}/X_u^{***} \xrightarrow{\sim} X_r^{**}/X_r^{***}.$$

For F in X_r^{**}, let f in X_u^* such that $F = \iota(f)$. Because $p \nmid u$ and $p-1|u-2$,

- either $u = p+1$, in which case $V_u^{**} = 0$, thus $X_r^{**} = X_u^{**} = 0$ in particular, $X_r^{**} = X_r^{***}$, or

- $u \geq 2p+1$, in which case $V_u^{**} = V_{u-2(p+1)} \otimes D^2$ by [Glo78, (4.1)]:

Therefore, in this case, there is g in $V_{u-2(p+1)}$ such that $f = \theta g$. Because $n \geq 1$, we have

$$F = \iota(\theta g)) = \theta^{p^n} \iota(g).$$

Because $n \geq 1$ and $p \geq 3$, we have $\theta^3 | F$. That is, $X_r^{**} \subseteq V_r^{***}$, that is $X_r^{**} = X_r^{***}$.

If $r \equiv 1 \bmod p$, then let $\phi \colon V_{r-p-1} \to V_r$ be given by $f \mapsto \theta f$ inducing, by [Glo78, (4.1)], as $r \geq p$,

$$V_{r-p-1}/V_{r-p-1}^{*} \otimes D \xrightarrow{\sim} V_r^{*}/V_r^{**},$$

yielding the isomorphism

$$X_u^{**}/X_u^{***} \xrightarrow{\sim} X_r^{**}/X_r^{***}$$

where $r - p - 1 = p^n u$ for $n \geq 1$ and $p \nmid u$, and we can conclude as above.

(Alternatively, the proof of [BG15, Proposition 5.4] shows $X_{r'}^{**} = X_{r'}^{***}$, hence $X_{r-1}^{**} \subseteq V_r^{***}$; that is, $X_{r-1}^{**} = X_{r-1}^{***}$; In particular $X_r^{**} = X_r^{***}$. In fact, Lemma 3.8 will show that even $X_{r-2}^{**} = X_{r-2}^{***}$.) □

2.3 Jordan-Hölder series of X_{r-2}

To compute the Jordan-Hölder series of $Q := V_r/(V_r^{***} + X_{r-2})$, it would help to know that of X_{r-2}. However, to this end, the exact Jordan-Hölder series of X_{r-2} will turn out dispensable, but that of $X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$ sufficient. Therefore, the following Proposition 2.9 will serve as fulcrum of all subsequent computations of the Jordan-Hölder factors of Q:

Proposition 2.9. *Let $r \geq p+1$. Let $r \equiv a \bmod p-1$ for a in $\{3, \ldots, p+1\}$. Put $r'' = r - 2$. We have the following short exact sequences (where, by convention, $V_i = 0$ for $i < 0$):*

- *If $X_{r''}^{*} \neq 0$,*
  - *For $a = 3$,*

$$\begin{aligned} 0 &\to (V_{2p-1} \otimes D) \oplus (V_{p-4} \otimes D^3) \\ &\to X_{r''} \otimes V_2 \\ &\to (V_1 \otimes D) \oplus V_3 \to 0 \end{aligned}$$

where V_{2p-1} has Jordan-Hölder series $V_{p-2} \otimes D$, V_1 and $V_{p-2} \otimes D$.

– *For a in $\{4,\ldots,p-1\}$,*

$$\begin{aligned}0 &\to (V_{p-a+3}\otimes D^{a-2})\oplus(V_{p-a+1}\otimes D^{a-1})\oplus(V_{p-a-1}\otimes D^{a})\\ &\to X_{r''}\otimes V_2\\ &\to (V_{a-4}\otimes D^2)\oplus(V_{a-2}\otimes D)\oplus V_a\to 0\end{aligned}$$

– *For $a=p$,*

$$\begin{aligned}0 &\to (V_3\otimes D^{p-2})\oplus(V_1\otimes D^{p-1})\\ &\to X_{r''}\otimes V_2\\ &\to (V_{p-4}\otimes D^2)\oplus V_{2p-1}\to 0\end{aligned}$$

where V_{2p-1} has Jordan-Hölder series $V_{p-2}\otimes D$, V_1 and $V_{p-2}\otimes D$.

– *For $a=p+1$,*

$$0\to V_2\otimes D^{p-1}\to X_{r''}\otimes V_2\to V_{3p-1}\to 0$$

where $V_{3p-1}=(V_{p-1}\otimes D^2)\oplus U$ and U has successive semisimple Jordan-Hölder factors $V_{p-3}\otimes D^2$, $(V_0\otimes D)\oplus V_2$ and $V_{p-3}\otimes D^2$.

- *If $X^*_{r''}=0$, then all summands on the left-hand sides vanish.*

Proof: By Lemma 2.7, for the unique $a''\in\{1,\ldots,p-1\}$ such that $r''=r-2\equiv a''$ mod $p-1$, (that is, $a''=a-2$ for the unique $a\in\{3,\ldots,p+1\}$ such that $r\equiv a$ mod $p-1$),

$$0\to V_{p-a''-1}\otimes D^{a''}\to X_{r''}\to V_{a''}\to 0.$$

By flatness of the $\mathbf{F}[M]$-module V_2,

$$0\to(V_{p-a''-1}\otimes D^{a''})\otimes V_2\to X_{r''}\otimes V_2\to V_{a''}\otimes V_2\to 0$$

We regard the left-hand side of the short exact sequence, that is, $(V_{p-a''-1}\otimes D^{a''})\otimes V_2$:

- if $a''=1$, then by Lemma 1.1.(ii),

$$V_2\otimes V_{p-a''-1}=V_{2p-1}\oplus V_{p-4}\otimes D^2;$$

- if $a''=2,\ldots,p-3$, then by Lemma 1.1.(i),

$$\begin{aligned}V_2\otimes V_{p-a''-1}&=[V_1\otimes V_{p-a''}]\oplus V_{p-a''-3}\otimes D^2\\ &=[(V_{p-a''-1}\otimes D)\oplus V_{p-a''+1}]\oplus V_{p-a''-3}\otimes D^2;\end{aligned}$$

- if $a'' = p-2$, that is, $p - a'' - 1 = 1$, then $V_2 \otimes V_1 = (V_1 \otimes D) \oplus V_3$ by Lemma 1.1.(i);

- if $a'' = p-1$, that is, $p - a'' - 1 = 0$, then $V_2 \otimes V_0 = V_2$.

We regard the right-hand side of the short exact sequence, that is, $V_{a''} \otimes V_2$:

- if $a'' = 1$, then $V_1 \otimes V_2 = (V_1 \otimes D) \oplus V_3$ by Lemma 1.1.(i).

- if $a'' = 2, \ldots, p-3$, then by Lemma 1.1.(i) (where we recall $V_{-1} = 0$),

$$\begin{aligned} V_2 \otimes V_{a''} &= [V_1 \otimes V_{a''+1}] \oplus V_{a''-2} \otimes D^2 \\ &= [(V_{a''} \otimes D) \oplus V_{a''+2}] \oplus V_{a''-2} \otimes D^2. \end{aligned}$$

- if $a'' = p-2$, then, like for $a'' = 1$ on the left-hand side of the short exact sequence,

$$V_2 \otimes V_{p-2} = (V_1 \otimes V_{p-1}) \oplus V_{p-4} \otimes D^2 = (V_{2p-1}) \oplus V_{p-4} \otimes D^2,$$

  where V_{2p-1} has by Lemma 1.1.(ii) (for $k = 1$) Jordan-Hölder series $V_{p-2} \otimes D$, V_1 and $V_{p-2} \otimes D$;

- if $a'' = p-1$, then by Lemma 1.1.(i),

$$V_2 \otimes V_{a''} = V_{3p-1} = (V_{p-1} \otimes D^2) \oplus U,$$

  where U has successive semisimple Jordan-Hölder factors $V_{p-3} \otimes D^2$, $(V_0 \otimes D) \oplus V_2$ and $V_{p-3} \otimes D^2$. □

Let us collect what we can infer about the Jordan-Hölder factors of X_{r-2} by Lemma 1.1 from looking at the short exact sequence

$$0 \to X^*_{r''} \otimes V_2 \to X_{r''} \otimes V_2 \to X_{r''}/X^*_{r''} \otimes V_2 \to 0.$$

- The left-hand side has minimal dimension 3 for $a'' = p+1$, the right-hand side has minimal dimension $2 \cdot 3 = 6$ for $a'' = 1$.

- Regarding the number of Jordan-Hölder factors,
  - the left-hand side has the minimal number of Jordan-Hölder factors 1 for $a'' = p+1$,
  - whereas the right-hand side has minimal number of Jordan-Hölder factors 2 for $a'' = 1$, and

- in the generic case $a'' \in \{2, \ldots, p-3\}$, both sides have 3 Jordan-Hölder factors.

- Under the conditions of Lemma 2.1, there are at least 3 Jordan-Hölder factors in X_{r-2}. Because $X_{r''} \otimes V_2$ has by Proposition 2.9 only 6 Jordan-Hölder factors, X_{r-2} has by the epimorphism $X_{r''} \otimes V_2 \twoheadrightarrow X_{r-2}$ between 3 and 6 Jordan-Hölder factors.

Corollary 2.10. *Let a in $\{5, \ldots, p+3\}$ such that $r \equiv a \bmod p-1$. Put $r' = r-1$ and $r'' = r-2$. If $X^*_{r'}$ and $X^*_{r''}$ are nonzero, then there is a short exact sequence*

$$0 \to V_{p-a+3} \otimes D^{a-2} \to X_{r''} \otimes V_2 / X_{r'} \otimes V_1 \to V_{a-4} \otimes D^2 \to 0.$$

Proof: Compare the short exact sequences of Proposition 2.9 with those of [BG15, Proposition 3.13.(ii) and 4.9(iii)]. □

2.4 Sum of the Digits

For a natural number r, let

$$\Sigma(r) := \text{ the sum of the digits of the } p\text{-adic expansion of } r\ .$$

Since $p \equiv 1 \bmod p-1$, we have $\Sigma(r) \equiv r \bmod p-1$. Thus, if a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$, then $\Sigma(r) = a$ is smallest possible. In other words, $\Sigma(r) = a$ holds if and only if $\Sigma(r) < p$. If $\Sigma(r) < p$, we say $\Sigma(r)$ is *minimal*, otherwise $\Sigma(r)$ is *non-minimal*.

By Observation 4.1, to compute the Jordan-Hölder series of $Q := V_r/(V_r^{***} + X_{r-2})$, we can always assume that the multiplication map $\phi\colon X_{r''} \otimes V_2 \to X_{r-2}$ of Lemma 2.2 is an isomorphism; the Jordan-Hölder series of the right-hand side was described in Proposition 2.9. However, for completeness, we will now describe the Jordan-Hölder series of X_{r-2} depending on minimality of $\Sigma(r)$, $\Sigma(r')$ and $\Sigma(r'')$. It turns out that the kernel of ϕ is given by the Jordan-Hölder factors of $X^*_{r''} \otimes V_1$, $X^*_{r'} \otimes V_1$ respectively X^*_r when $\Sigma(r'')$, $\Sigma(r')$ respectively $\Sigma(r)$ is minimal.

The following Proposition 2.11 states (and proves more directly) results contained in [BG15, Sections 3 and 4], in particular [BG15, Lemma 3.10, Proposition 3.11, Lemma 4.5 and Lemma 4.6].

Proposition 2.11. *Let $p \geq 3$ and $r \geq p$. We have $X^*_r = 0$ if and only if $\Sigma(r)$ is minimal.*

Proof: If $\Sigma(r)$ is minimal, that is, $\Sigma(r) = a$, and

- if $a = 1$, that is, $r = p^n$, then by the $\mathbb{F}[\mathrm{M}]$-homomorphism $\mathrm{X} \mapsto \mathrm{X}^{p^n}$, we have $\mathrm{X}_1 \xrightarrow{\sim} \mathrm{X}_r$, in particular, $\mathrm{X}_r^* = 0$;

- if a in $\{2, \ldots, p-1\}$, then $\dim \mathrm{X}_r < p+1$ by [BG15, Lemma 4.5], therefore $\mathrm{X}_r^* = 0$ by Proposition 2.6.

Let $\Sigma(r)$ be non-minimal, that is, $\Sigma(r) \geq p$. We have $\mathrm{X}_r^* = 0$ if and only if $\dim \mathrm{X}_r < p+1$ if and only if the standard generating set of X_r is linearly dependent: That is, there is $b_0, \ldots, b_{p-1}$ and b_p in $\mathbb{F}_p$, not all zero, such that

$$b_0 \mathrm{Y}^r + \sum_{k=1,\ldots,p-1} b_k (k\mathrm{X} + \mathrm{Y})^r + b_p \mathrm{X}^r = 0. \tag{$*$}$$

We show that if $\Sigma(r) \geq p$, then $(*)$ implies $b_0, \ldots, b_{p-1}, b_p$ to vanish. It suffices to show that $b_1, \ldots, b_{p-1}$ vanish. Because $\#\mathbb{F}_p^* = p-1$,

$$\sum_{k=1,\ldots,p-1} b_k (k\mathrm{X} + \mathrm{Y})^r = \sum_{k=1,\ldots,p-1} b_k \sum_{i=1,\ldots,p-1} k^i \sum_{j \equiv i \bmod p-1} \binom{r}{j} \mathrm{X}^j \mathrm{Y}^{r-j}. \tag{$**$}$$

For $i = 1, \ldots, p-1$, let

$$\mathrm{B}_i = \sum_{k=1,\ldots,p-1} b_k k^i.$$

By the nonzero Vandermode determinant of $(k^i)_{i,j=1,\ldots,p-1}$, if $\mathrm{B}_1 = \cdots = \mathrm{B}_{p-1} = 0$, then $b_1 = \cdots = b_{p-1} = 0$. Thus, it suffices to show $\mathrm{B}_1 = \cdots = \mathrm{B}_{p-1} = 0$. Comparing the coefficients of $\mathrm{X}^t \mathrm{Y}^{r-t}$, by $(*)$ and $(**)$, for every t such that $t \equiv i$,

$$\mathrm{B}_i \binom{r}{t} = 0. \tag{$***$}$$

Let t in $\{1, \ldots, p-1\}$. Write $r = r_0 + r_1 p + \cdots$. Since $\Sigma(r) = r_0 + r_1 + \cdots \geq p$, we can write $t = t_0 + t_1 + \cdots$ with $0 \leq t_j \leq r_j$ for $j = 0, 1, \ldots$. Put $t' = t_0 + t_1 p + \cdots$. Then $t' \equiv t \bmod p-1$, and, by Lucas's Theorem, $\binom{r}{t'} \neq 0$. By $(***)$

$$0 = \binom{r}{t'} \mathrm{B}_{t'} = \binom{r}{t'} \mathrm{B}_t;$$

that is, $\mathrm{B}_t = 0$. We conclude that $\mathrm{B}_1, \ldots, \mathrm{B}_{p-1}$, (and therefore $b_1, \ldots, b_{p-1}$) vanish. □

Let a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$. The following Lemma 2.12 shows that, with few exceptions for $a = 1, 2$, the minimality of $\Sigma(r'')$ implies that of $\Sigma(r')$; likewise, the minimality of $\Sigma(r')$ implies that of $\Sigma(r)$.

Lemma 2.12. *Let a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$. Put $r' = r-1$ and $r'' = r-2$.*

- *For a in $\{3, \ldots, p-1\}$,*
  - *if $\Sigma(r'')$ is minimal, then $\Sigma(r')$ and $\Sigma(r)$ are minimal;*
  - *if $\Sigma(r')$ is minimal, then $\Sigma(r)$ is minimal.*
- *For $a = 2$, we have $\Sigma(r')$ is minimal if and only if $r' = p^n$; moreover*
  - *If $\Sigma(r'')$ is minimal (and $r > p$), then neither $\Sigma(r')$ nor $\Sigma(r)$ is minimal;*
  - *If $\Sigma(r')$ is minimal, then $\Sigma(r)$ is minimal.*
- *for $a = 1$, we have $\Sigma(r)$ is minimal if and only if $r = p^n$; moreover*
  - *If $\Sigma(r'')$ is minimal (and $r > p$), then $\Sigma(r')$ is minimal but $\Sigma(r)$ is not minimal;*
  - *If $\Sigma(r')$ is minimal, then $\Sigma(r)$ is not minimal.*

  For every a, if $\Sigma(r'')$ and $\Sigma(r')$ are non-minimal, then $\Sigma(r)$ can be either minimal or non-minimal.

Proof: We use the definition of minimality of $\Sigma(r'')$ and that $\Sigma(r') = \Sigma(r'') + 1$ (respectively $\Sigma(r) = \Sigma(r'') + 2$) if $p \nmid r'$ (respectively $p \nmid r$):

- For a in $\{3, \ldots, p-1\}$:
  - Because $r'' \equiv a-2$ and $a-2 \leq p-3$, we have $\Sigma(r'') < p$ if and only if $\Sigma(r'') \leq p-3$. Therefore, if $\Sigma(r'') < p$, then both $\Sigma(r') = \Sigma(r'') + 1$ and $\Sigma(r) = \Sigma(r'') + 2 < p$.
  - Because $r' \equiv a-1$ and $a-1 \leq p-2$, if $\Sigma(r') \leq a-1 \leq p-2 < p$, then $\Sigma(r) \leq p-1 < p$.
- For $a = 2$:
  - We have $\Sigma(r'')$ is minimal if and only if $\Sigma(r'') = p-1$. Therefore, if $\Sigma(r'') = p-1$, then both $\Sigma(r') = \Sigma(r'') + 1$ and $\Sigma(r) = \Sigma(r'') + 2 \geq p$.
  - We have $\Sigma(r')$ is minimal if and only if $\Sigma(r') = 1$. Therefore $r = p^n + 1$ and $\Sigma(r)$ is minimal.
- For $a = 1$:
  - We have $\Sigma(r'')$ is minimal if and only if $\Sigma(r'') = p-2$. Therefore $\Sigma(r') = \Sigma(r'') + 1 < p$ is minimal but $\Sigma(r) = \Sigma(r'') + 2 = p$ is non-minimal.

– If $\Sigma(r') = p - 1$ is minimal, then $\Sigma(r) = p$ is not-minimal. □

For a in $\{2, \ldots, p-1\}$ such that $r \equiv a \mod p - 1$, let us keep for the record how the minimality conditions on $\Sigma(r)$ and $\Sigma(r)'$ are equivalent to those in [BG15, Section 4]: Write $r = up^n$ such that $p \nmid u$ and put $u' = u - 1$. Then $\Sigma(r)$ is minimal, if and only if $\Sigma(u)$ is minimal, if and only if $\Sigma(u')$ is minimal because $p \nmid u$. Putting $r' = r - 1$,

- If $\Sigma(u')$ is minimal, then $\Sigma(r')$ is minimal if and only $n = 0$, because $\Sigma(r') = \Sigma(u') - 1 + d$ where $d = 1$ if $n = 0$, that is, $p \nmid r$, and $d > p - 1$ if $n > 0$, that is, $p \mid r$.

- If $\Sigma(u')$ is not minimal, then $\Sigma(r')$ is not minimal, because $\Sigma(r') = \Sigma(u') - 1 + d$ where $d = 1$ if and only if $p \nmid r$, that is, $n = 0$, and $d > p - 1$ if and only if $p \mid r$, that is, $n > 0$.

2.5 Sum of the Digits of $r - 2$ is *non*-minimal

Let a in $\{3, \ldots, p+1\}$ such that $r \equiv a \mod p-1$. Let $r'' = r-2$. Then $\Sigma(r'') > a-2$, that is, is non-minimal, if and only if $\Sigma(r'') \geq p$. We assume in this Section 2.5 that $\Sigma(r'')$ is non-minimal, that is, $\Sigma(r'') \geq p$ and will show that X_{r-2}/X_{r-1} has two Jordan-Hölder factors.

By Lemma 2.1, we have $X_{r-2} = X_{r-1}$ if and only if $r = p^n + r_0$ with r_0 in $\{2, \ldots, p-1\}$. That is, $r'' = p^n + r_0''$ with $0 \leq r_0 \leq p - 3$; in particular, $\Sigma(r'')$ is minimal. By the same token, $X_{r-1} = X_r$ if and only if $r < p$.

We conclude that if $r \geq p$ and $\Sigma(r'')$ non-minimal, then

$$0 \subseteq X_r^* \subset X_r \subset X_{r-1} \subset X_{r-2}$$

where

- the two inclusions to the right of X_r are proper by Lemma 2.1,

- we have $X_r/X_r^* = V_a$, in particular a proper inclusion $X_r^* \subset X_r$ by Proposition 2.6 (which in this case is [Glo78, (4.5)]), and

- we have $X_r^* = 0$ if and only if $\Sigma(r)$ is minimal by Proposition 2.11.

By Lemma 2.7 and Proposition 2.11 the Jordan-Hölder series of X_r is known. Therefore, by [BG15, Proposition 3.13 and 4.9]:

- Let $r \equiv a \mod p - 1$ for $1 \leq a \leq p - 1$.

- Either $\Sigma(r)$ is non-minimal, then the Jordan-Hölder series

$$0 \to V_{p-a-1} \otimes D^a \to X_r \to V_a \to 0, \tag{$*$}$$

(which is dual to, that is, inverts the directions of the arrows of

$$0 \to V_a \to V_r/V_r^* \to V_{p-a-1} \otimes D^a \to 0,)$$

- or it is minimal, in which case the right-hand side of the short exact sequence $(*)$ around X_r vanishes.

- Let $r \equiv a \bmod p-1$ for $2 \leq a \leq p$.
  - Either $\Sigma(r')$ is non-minimal, then the Jordan-Hölder series is

$$0 \to V_{p-a+1} \otimes D^{a-1} \to X_{r-1}/X_r \to V_{a-2} \otimes D \to 0, \tag{$**$}$$

  (which is dual to that of V_r^*/V_r^{**} for $a = 3, \ldots, p$)

  - or it is minimal, in which case
    - ▹ either $r < p$ and $X_{r-1}/X_r = 0$,
    - ▹ or, otherwise, the right-hand side of the short exact sequence $(**)$ around X_{r-1} vanishes.

Regarding $\Sigma(r'')$, let $r \equiv a$ for $3 \leq a \leq p+1$.

- Either $\Sigma(r'')$ is non-minimal, then we show in Section 2.5.3 for $\Sigma(r')$ and $\Sigma(r)$ *non*-minimal that the Jordan-Hölder series is

$$0 \to V_{p-a+3} \otimes D^{a-2} \to X_{r-2}/X_{r-1} \to V_{a-4} \otimes D^2 \to 0, \tag{$***$}$$

(which is dual to that of V_r^*/V_r^{**} for $a = 5, \ldots, p+1$)

- or it is minimal, and
  - either $r = p^n + r_0$ with $r_0 \in \{2, \ldots, p-1\}$, then we proved in Lemma 2.1 that $X_{r-2}/X_{r-1} = 0$,
  - or, otherwise, we will prove in Section 2.6 that the right-hand side of the short exact sequence $(***)$ around X_{r-2}/X_{r-1} vanishes.

Independently of whether one of $\Sigma(r')$ or $\Sigma(r)$ is minimal or not, if $\Sigma(r'')$ is non-minimal, then, except when $r \equiv 3 \bmod p-1$, a specific fourth Jordan-Hölder factor appears in X_{r-2}:

Lemma 2.13. *Let a in $\{4, \ldots, p+1\}$ such that $r \equiv a \bmod p-1$. If $\Sigma(r'') \geq p$ and $r \geq 3p+2$, then $\mathrm{V}_{p-a+3} \otimes \mathrm{D}^{a-2}$ is a Jordan-Hölder factor of X_{r-2}.*

Proof: Let a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$. Because $\Sigma(r'')$ is non-minimal, by Lemma 2.7 and Proposition 2.11,

$$\psi\colon \mathrm{V}_{p-a+1} \otimes \mathrm{D}^{a-2} \xrightarrow{\sim} \mathrm{X}^*_{r''}.$$

Composing this isomorphism with the epimorphism

$$\begin{aligned} \mathrm{X}_{r''} \otimes \mathrm{V}_2 &\twoheadrightarrow \mathrm{X}_{r-2} \\ f \otimes g &\mapsto f \cdot g, \end{aligned}$$

we obtain

$$(\mathrm{V}_{p-a+1} \otimes \mathrm{D}^{a-2}) \otimes \mathrm{V}_2 \xrightarrow{\sim} \mathrm{X}^*_{r''} \otimes \mathrm{V}_2 \twoheadrightarrow \mathrm{X}_{r-2}. \qquad (*)$$

For $n = 0, \ldots, p-3$ in $\mathbb{N}$, let us construct an $\mathbb{F}_p[\mathrm{M}]$-linear section

$$\mathrm{V}_{n+2} \to \mathrm{V}_n \otimes \mathrm{V}_2.$$

Given f in V_{n+2}, let f_{xx}, f_{xy} and f_{yy} in V_n denote its partial derivatives of second order. By the proof of [Glo78, (5.2)], the $\mathbb{F}_p$-linear map

$$\begin{aligned} \phi_n\colon \mathrm{V}_{n+1} &\to \mathrm{V}_n \otimes \mathrm{V}_1 \\ f &\mapsto f_x \otimes x + f_y \otimes y \end{aligned}$$

is M-linear, and so is its iteration $(\phi_n \otimes \mathrm{id}) \circ \phi_{n+1}$, given by

$$\begin{aligned} \mathrm{V}_{n+2} &\to \mathrm{V}_n \otimes (\mathrm{V}_1 \otimes \mathrm{V}_1) \\ f &\mapsto f_{xx} \otimes x \otimes x + f_{xy} \otimes x \otimes y + f_{yx} \otimes y \otimes x + f_{yy} \otimes y \otimes y. \end{aligned}$$

By composing with $\mathrm{id} \otimes \pi$ where π is the $\mathbb{F}_p[\mathrm{M}]$-linear homomorphism

$$\mathrm{V}_1 \otimes \mathrm{V}_1 \to \mathrm{V}_2$$

given by $f \otimes g \mapsto f \cdot g$, we obtain that the $\mathbb{F}_p$-linear map

$$\begin{aligned} \mathrm{V}_{n+2} &\to \mathrm{V}_n \otimes \mathrm{V}_2 \\ f &\mapsto f_{xx} \otimes x^2 + f_{xy} \otimes 2xy + f_{yy} \otimes y^2, \end{aligned}$$

is M-linear. In particular, we obtain for $a > 3$ an $\mathbb{F}_p[\mathrm{M}]$-linear section

$$\mathrm{V}_{p-a+3} \otimes \mathrm{D}^{a-2} \to (\mathrm{V}_{p-a+1} \otimes \mathrm{D}^{a-2}) \otimes \mathrm{V}_2 \xrightarrow{\sim} \mathrm{X}^*_{r''} \otimes \mathrm{V}_2$$

which sends

$$X^{p-a+3} \mapsto \binom{p-a+3}{2} X^{p-a+1} \otimes X^2.$$

If $a > 3$, then $\binom{p-a+3}{2} \not\equiv 0 \bmod p$, that is, the right-hand side is nonzero. Under the map $(*)$,

$$X^{p-a+1} \otimes X^2 \mapsto \psi(X^{p-a+1}) \cdot X^2 \neq 0.$$

Therefore, $V_{p-a+3} \otimes D^{a-2}$ is a nonzero Jordan-Hölder factor of X_{r-2}. □

2.5.1 Sum of the Digits of $r-1$ is minimal

Because $\Sigma(r')$ is minimal, by [BG15, Proposition 3.13 and 4.9] we have $\dim X_{r-1} < 2p+2$, therefore, by Corollary 2.4, we have $\dim X_{r-2} < 3p+3$; that is, X_{r-2} has at most five Jordan-Hölder factors.

Let $r \geq 2p+1$ and $\Sigma(r') < p$, that is, the sum of the digits of $r-1$ is minimal. Let a in $\{3,\dots,p+1\}$ such that $r \equiv a \bmod p-1$. Recall the Jordan-Hölder series of X_{r-1}:

- If $a = 3,\dots,p$, then by [BG15, Proposition 4.9.(i)],

$$X_{r-1} = V_{a-2} \otimes D \oplus V_a. \tag{2.1}$$

- Otherwise, if $a = p+1$, then by [BG15, Proposition 3.13.(i)],

$$X_{r-1} = V_{2p-1}$$

  where we recall that V_{2p-1} has successive semisimple Jordan-Hölder factors $V_{p-2} \otimes D$, V_1 and $V_{p-2} \otimes D$ as stated in Corollary 1.2.(i).

Proposition 2.14. *Let a in $\{5,\dots,p\}$ such that $r \equiv a \bmod p-1$. Let $\Sigma(r'') \geq p$ and $\Sigma(r') < p$. If $r \geq 3p+2$, then*

$$0 \to V_{p-a+3} \otimes D^{a-2} \to X_{r-2}/X_{r-1} \to V_{a-4} \otimes D^2 \to 0.$$

Proof: By Lemma 2.13,

$$X_{r-2} \hookleftarrow V_{p-a+3} \otimes D^{a-2}.$$

Expand $r = r_0 + r_1 p + \cdots$ p-adically. Because $\Sigma(r') = a-1$ in $\{4,\dots,p-1\}$ (and $r \geq p$), we have $r_0 \leq a-1$. Therefore $r \equiv r_0 \neq a \bmod p$. If $r_0 = a-1$ in $\{4,\dots,p-1\}$, then $r = r_0 + p^n$; in particular, $\Sigma(r'')$ would be minimal. Therefore $r_0 \neq a-1 \bmod p$.

Thus we can apply Lemma 3.6, yielding by Lemma 1.3.(iii),

$$X_{r-2}^{**}/X_{r-2}^{***} \hookleftarrow V_{a-4} \otimes D^2.$$

By Lemma 2.2, the Jordan-Hölder series of X_{r-2} is included in that of Proposition 2.9.

We conclude by Corollary 2.4 and (2.1) that the Jordan-Hölder series of X_{r-2}/X_{r-1} is

$$0 \to V_{p-a+3} \otimes D^{a-2} \to X_{r-2}/X_{r-1} \to V_{a-4} \otimes D^2 \to 0. \qquad \square$$

As the Jordan-Hölder series of $Q = V_r/(X_{r-2} + V_r^{***})$ (and thus our main theorem) does not depend on whether $\Sigma(r'')$, $\Sigma(r')$ or $\Sigma(r)$ are minimal or not, we dispense with the cases $a = 2, 3, 4$ at this point.

2.5.2 Sum of the Digits of $r - 1$ is *non*-minimal but that of r is minimal

Because $\Sigma(r)$ is minimal, by Proposition 2.11 we have $\dim X_r < p$, therefore, by Corollary 2.4, we have $\dim X_{r-2} < 3p + 3$; that is, X_{r-2} has at most five Jordan-Hölder factors. We will show that all occur.

Let a in $\{3, \ldots, p + 1\}$ such that $r \equiv a \bmod p - 1$. Let $r \geq 2p + 1$ and $\Sigma(r) < p$, that is, the sum of the digits of r is minimal. Recall the Jordan-Hölder series of X_{r-1}:

(i) For $a = 3, \ldots, p - 1$ and $a = p + 1$, by [BG15, Proposition 4.9.(ii)],

$$0 \to V_{p-a+1} \otimes D^{a-1} \to X_{r-1} \to V_{a-2} \otimes D \oplus V_a \to 0 \qquad (2.2)$$

(ii) For $a = p$, we have $r = p^n$ for $n > 1$ and by [BG15, Proposition 3.13.(iii)],

$$0 \to V_1 \otimes D^{p-1} \to X_{r-1} \to W \to 0 \qquad (2.3)$$

where $W = V_{2p-1}/V_{2p-1}^{*}$, that is, $0 \to V_{p-2} \otimes D \to W \to V_1 \to 0$.

Proposition 2.15. *Let $\Sigma(r'') \geq p$, $\Sigma(r') \geq p$ and $\Sigma(r) < p$. Let $r \equiv a \bmod p - 1$. If a in $\{4, \ldots, p - 1\}$ and $r \geq 3p + 2$, then*

$$0 \to V_{p-a+3} \otimes D^{a-2} \to X_{r-2}/X_{r-1} \to V_{a-4} \otimes D^2 \to 0$$

Proof: By Lemma 2.13,

$$X_{r-2} \hookleftarrow V_{p-a+3} \otimes D^{a-2}.$$

Expand $r = r_0 + r_1 p + \cdots$ p-adically.

If $\Sigma(r-1)$ is *non*-minimal but $\Sigma(r)$ is minimal, then $r \equiv 0 \bmod p$. In particular, for $a = \{4, \ldots, p-1\}$, we have $r \not\equiv a, a-1 \bmod p$.

Thus we can apply Lemma 3.6 yielding by Lemma 1.3.(iii),

$$X_{r-2}^{**}/X_{r-2}^{***} \hookleftarrow V_{a-4} \otimes D^2.$$

By Lemma 2.2, the Jordan-Hölder series of X_{r-2} is included in that of Proposition 2.9. Because $\Sigma(r)$ is minimal, by Proposition 2.11 we have $\dim X_r < p$, therefore, by Corollary 2.4, we have $\dim X_{r-2} < 3p+3$; that is, X_{r-2} has at most five Jordan-Hölder factors; whereas X_{r-1} has three Jordan-Hölder factors by (2.2).

We conclude by Corollary 2.4 that the Jordan-Hölder series of X_{r-2}/X_{r-1} is

$$0 \to V_{p-a+3} \otimes D^{a-2} \to X_{r-2}/X_{r-1} \to V_{a-4} \otimes D^2 \to 0.$$ □

Lemma 2.16 (Extension of [BG15, Lemma 3.10]). *If $r = p^n$ for some $n > 1$, then* $\dim X_{r-2} = 2p+4$.

Proof: By Lemma 2.5,

$$\{X^2(kX+Y)^{r-2}, XY(lX+Y)^{r-2}, Y^2(X+mY)^{r-2}, X^r, Y^r, X^{r-1}Y, XY^{r-1} : k, l, m \in \mathbb{F}_p\}$$

is a set of generators of X_{r-2}. Because

$$(X+kY)^2 = X^2 + 2kXY + Y^2,$$

and therefore

$$(X+kY)^r = X^2(X+kY)^{r-2} + 2kXY(X+kY)^{r-2} + Y^2(X+kY)^{r-2},$$

the span over $\mathbb{F}_p$ of the sets

$$\{X^2(kX+Y)^{r-2}, XY(lX+Y)^{r-2}, Y^2(X+mY)^{r-2}, X^r, Y^r, X^{r-1}Y, XY^{r-1} : k, l, m \in \mathbb{F}_p\}$$

and

$$\{X^2(kX+Y)^{r-2}, XY(lX+Y)^{r-2}, (X+mY)^r, X^r, Y^r, X^{r-1}Y, XY^{r-1} : k, l, m \in \mathbb{F}_p\}$$

are equal. Because $r = p^n$, we have $(X+mY)^r = X^r + m^rY^r$, and therefore the span of

$$\{(X+mY)^r : m \in \mathbb{F}_p\}$$

equals that of X^r and Y^r. Therefore the span over $\mathbb{F}_p$ of

$$\{X^2(kX+Y)^{r-2}, XY(lX+Y)^{r-2}, Y^2(X+mY)^{r-2}, X^r, Y^r, X^{r-1}Y, XY^{r-1} : k, l, m \in \mathbb{F}_p\}$$

equals that of

$$\{X^2(kX+Y)^{r-2}, XY(lX+Y)^{r-2}, X^r, Y^r, X^{r-1}Y, XY^{r-1} : k, l \in \mathbb{F}_p\}.$$

We show that the elements of the latter set are linearly independent, that is, if

$$AX^r + BY^r + CX^{r-1}Y + DXY^{r-1} + \sum_{k \in \mathbb{F}_p} e_k X^2(kX+Y)^{r-2} + \sum_{l \in \mathbb{F}_p} f_l XY(lX+Y)^{r-2} = 0, \quad (*)$$

then the coefficients A, B, C, D and e_k, f_l for f, l in $\mathbb{F}_p$ all vanish: Let t in $\{2, \ldots, r\}$. Comparing the coefficients of $X^{t+2}Y^{r-2-t}$ on both sides of $(*)$ gives

$$\binom{r-2}{t} \sum_{k=1,\ldots,p-1} e_k k^t + \binom{r-2}{t+1} \sum_{l=1,\ldots,p-1} f_l l^{t+1} = 0 \qquad (**)$$

Let

$$E_t := \sum_{k=1,\ldots,p-1} e_k k^t \quad \text{and} \quad F_{t+1} := \sum_{l=1,\ldots,p-1} f_l l^{t+1}$$

Because $\# = \mathbb{F}_p^* = p-1$, the sums E_t and F_{t+1} only depend on t mod $p-1$. Because the Vandermonde determinant is nonzero, if $E_1, \ldots, E_{p-1} = 0$ then $e_1, \ldots, e_{p-1} = 0$; likewise if $F_1, \ldots, F_{p-1} = 0$ then $f_1, \ldots, f_{p-1} = 0$. It therefore suffices to show that $E_1, \ldots, E_{p-1} = 0$.

Write

$$r - 2 = p^n - 2 = r_{n-1}p^{n-1} + \cdots + r_1 p + r_0 = (p-1)p^{n-1} + \cdots + (p-1)p + p - 2.$$

For $t = 1, \ldots, p-2$, put $t' = t + p - 1$. Then $t' \le r$ and $t' \equiv t$ mod $p-1$. By $(**)$,

$$\binom{r-2}{t} E_t + \binom{r-2}{t+1} F_t = 0$$
$$\binom{r-2}{t'} E_t + \binom{r-2}{t'+1} F_t = 0$$

The determinant of this linear equation system is

$$\binom{r_0}{t+1}\binom{r_1}{1}\binom{r_0}{t-1} - \binom{r_0}{t}\binom{r_1}{1}\binom{r_0}{t} \equiv -\binom{r_1}{1}\binom{r_0}{t-1}\binom{r_0}{t}\frac{r_0+1}{t(t+1)} \not\equiv 0 \bmod p$$

because $0 < r_0 + 1, r_1 \le p - 1$. Therefore $E_t, F_t = 0$.

For $t = p - 1$, put $t' = r_0 + p$. Then $t' \le r - 2$ and $t' \equiv t$ mod p. We compute

$$\binom{r-2}{t'+1} \equiv \binom{r_1}{1}\binom{r_0}{p-1} \equiv 0 \bmod p \quad \text{and} \quad \binom{r-2}{t'} \equiv \binom{r_1}{1}\binom{r_0}{p-2} \not\equiv 0 \bmod p.$$

Therefore $(**)$ gives $E_t = 0$. □

Proposition 2.17. *Let* $\Sigma(r'') \geq p$, $\Sigma(r') \geq p$ *and* $\Sigma(r) < p$. *If* $r \equiv p \bmod p-1$ *and* $r \geq 3p+2$, *then the Jordan-Hölder series of* X_{r-2}/X_{r-1} *is*

$$0 \to V_3 \otimes D^{p-2} \to X_{r-2}/X_{r-1} \to V_{p-4} \otimes D^2 \to 0$$

Proof: By Proposition 2.9 for $a = p$, we have

$$\begin{aligned} 0 &\to (V_3 \otimes D^{p-2}) \oplus (V_1 \otimes D^{p-1}) \\ &\to X_{r''} \otimes V_2 \\ &\to (V_{p-4} \otimes D^2) \oplus V_{2p-1} \to 0 \end{aligned} \qquad (*)$$

where V_{2p-1} has Jordan-Hölder series $V_{p-2} \otimes D$, V_1 and $V_{p-2} \otimes D$. By Lemma 2.16, we have $\dim X_{r-2} = 2p+4$. By comparing Equation (2.3) with $(*)$, the Jordan-Hölder factors $V_3 \otimes D^{p-2}$ and $V_{p-4} \otimes D^2$ must appear in the Jordan-Hölder series of X_{r-2}. □

As the Jordan-Hölder series of $Q = V_r/(X_{r-2} + V_r^{***})$ (and thus our main theorem) does not depend on whether $\Sigma(r'')$, $\Sigma(r')$ or $\Sigma(r)$ are minimal or not, we dispense with the cases $a = 2, 3, 4$ at this point.

2.5.3 Sum of the Digits of $r-1$ and r are *non*-minimal

We show that if $\Sigma(r'')$, $\Sigma(r')$ and $\Sigma(r)$ are all non-minimal, then X_{r-2} is maximal, that is, $\dim X_{r-2} = 3p+3$.

We recall that $\Sigma(r)$ is non-minimal if and only if, for a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$, we have $\Sigma(r) > a$, that is, if and only if $\Sigma(r) \geq p$. Therefore, in analogy to [BG15, Lemma 4.3], we conclude that $\Sigma(r'')$ and $\Sigma(r')$ and $\Sigma(r)$ non-minimal if and only if

(i) either $p \nmid r', r$ and $\Sigma(r'')$ non-minimal,

(ii) or $r = p^n u$ for $n \geq 1$, and $\Sigma(u)$ non-minimal,

(iii) or $r' = p^n u'$ for $n \geq 1$ and $\Sigma(u')$ non-minimal.

We will prove successively that $\dim X_{r-2} = 3p+3$ is maximal in each one of these possibilities:

Lemma 2.18 (Analogue of [BG15, Lemma 4.2]). *Let* $p > 3$ *and let* $r \geq 3p+2$. *If* $\Sigma(r'') \geq p$ *and* $p \nmid r', r$, *then* $\dim X_{r-2} = 3p+3$.

Proof: We need to show that the spanning set

$$\{X^r, Y^r, X^{r-1}Y, X^2(jX+Y)^{r-2}, Y^2(X+kY)^{r-2}, XY(lX+Y)^{r-2} : j,k,l \in \mathbb{F}_p\}$$

is linearly independent; that is, if there are constants A, B, C and $d_j, e_k, f_l \in \mathbb{F}_p$ for $j,k,l = 0,1,\ldots,p-1$ satisfying

$$\begin{aligned} 0 = \quad & AX^r + BY^r + CX^{r-1}Y \\ & + \sum_j d_j Y^2(X+jY)^{r-2} + \sum_k e_k XY(kX+Y)^{r-2} + \sum_l f_l X^2(lX+Y)^{r-2} \end{aligned} \qquad (*)$$

then $A, B, C = 0$ and $d_j, e_k, f_l = 0$ for $j,k,l = 0,1,\ldots,p-1$.

Let us assume $(*)$. Put

$$D_i := \sum d_j j^i, \; E_i := \sum e_k k^{r-3-i}, \; F_i := \sum f_l l^{r-4-i} \quad \text{for } i = 0,\ldots,r-4$$

Since $\#\mathbb{F}_p^* = p-1$ and $i' \equiv i''$ mod $p-1$, then $D_{i'} \equiv D_{i''}$. If $D_1, \ldots, D_{p-1} = 0$, then $d_1, \ldots, d_{p-1} = 0$ (and therefore $d_0 = 0$), because the system of linear equations of $D_1, \ldots, D_{p-1} = 0$ has full rank (by its nonzero Vandermonde determinant). Likewise if $E_1, \ldots, E_{p-1} = 0$, then $e_1, \ldots, e_{p-1} = 0$ and if $F_1, \ldots, F_{p-1} = 0$, then $f_1, \ldots, f_{p-1} = 0$. To show that all coefficients A, B, C e d_j, e_k and f_l for $j,k,l = 0, \ldots, p-1$ vanish, it therefore suffices to show $D_1, \ldots, D_{p-1} = 0$ and $E_1, \ldots, E_{p-1} = 0$.

By comparing the coefficient of $X^{r-2-t}Y^{t+2}$ on both sides of $(*)$ for t in $\{0, \ldots, r-2\}$,

$$0 = \binom{r-2}{t} D_t + \binom{r-2}{t+1} E_t + \binom{r-2}{t+2} F_t. \qquad (2.4)$$

We will show that Equation (2.4) forces $D_{t'}$ and $E_{t''}$ to vanish for t' and t'' in full sets of representatives of $\{1, \ldots, p-1\}$. That is, for every t in $\{1, \ldots, p-1\}$ there is t' and t'' with $t' \equiv t$ and $t'' \equiv t$ mod $p-1$ such that $D_{t'}$ and $E_{t''}$ vanish.

Expand $r - 2 = r_0 + r_1 p + r_2 p^2 + \cdots$ with $r_0, r_1, \ldots \in \{0, \ldots, p-1\}$. Let i be the smallest index such that $r_i \neq 0$. Fixate t in $\{1, \ldots, p-1\}$.

Case 1. Suppose $t \in \{1, \ldots, r_i - 1\}$.

If $r_0 = 0$, then $i > 0$. By Lucas's Theorem,

- for $t' := tp^i$, we have $\binom{r-2}{t'} \not\equiv 0$ and $\binom{r-2}{t'+1}, \binom{r-2}{t'+2} \equiv 0 \bmod p$, thus Equation (2.4) yields $D_{t'} = 0$;
- for $t'' := (t+1)p^i - 1$, we have $\binom{r-2}{t''+1} \not\equiv 0$ and $\binom{r-2}{t''+2}, \binom{r-2}{t''} \equiv 0 \bmod p$, thus Equation (2.4) yields $E_{t''} = 0$.

Because $t', t'' \equiv t \bmod p$, we have $D_t = D_{t'} = 0$ and $E_t = E_{t''} = 0$. We can therefore assume that $r_0 > 1$; in particular, $i = 0$.

In the following, we choose $t', t'' \equiv t \bmod p-1$ such that (+) yields modulo p the system of equations:

$$\begin{aligned}
\binom{r-2}{t} D_t + \binom{r-2}{t+1} E_t + \binom{r-2}{t+2} F_t &\equiv 0 \\
\binom{r-2}{t'} D_t + \binom{r-2}{t'+1} E_t + \binom{r-2}{t'+2} F_t &\equiv 0 \\
\binom{r-2}{t''} D_t + \binom{r-2}{t''+1} E_t + \binom{r-2}{t''+2} F_t &\equiv 0
\end{aligned}$$

We show $D_t = E_t = F_t = 0$ by proving that the determinant of the matrix M attached to this system of equations is nonzero, that is,

$$|M| = \begin{vmatrix} \binom{r-2}{t} & \binom{r-2}{t+1} & \binom{r-2}{t+2} \\ \binom{r-2}{t'} & \binom{r-2}{t'+1} & \binom{r-2}{t'+2} \\ \binom{r-2}{t''} & \binom{r-2}{t''+1} & \binom{r-2}{t''+2} \end{vmatrix} \not\equiv 0 \bmod p.$$

Case 1.1. There is an index $i > 0$ such that $r_i > 1$. Put $t' := t + p^i - 1$ and $t'' := t + 2p^i - 2$

Case 1.1.1. Suppose $t \in \{2, \ldots, r_0 - 2\}$. By Lucas's Theorem $\binom{r-2}{t'} = \binom{r_0}{t-1}\binom{r_i}{1}$, $\binom{r-2}{t'+1} = \binom{r_0}{t}\binom{r_i}{1}$ and $\binom{r-2}{t'+1} = \binom{r_0}{t+1}\binom{r_i}{1}$, as well as $\binom{r-2}{t''} = \binom{r_0}{t-2}\binom{r_i}{2}$, $\binom{r-2}{t''+1} = \binom{r_0}{t-1}\binom{r_i}{2}$ and $\binom{r-2}{t''+2} = \binom{r_0}{t}\binom{r_i}{2}$. Thus,

$$\begin{aligned}
|M| &\equiv \begin{vmatrix} \binom{r_0}{t} & \binom{r_0}{t+1} & \binom{r_0}{t+2} \\ \binom{r_0}{t-1}\binom{r_i}{1} & \binom{r_0}{t}\binom{r_i}{1} & \binom{r_0}{t+1}\binom{r_i}{1} \\ \binom{r_0}{t-2}\binom{r_i}{2} & \binom{r_0}{t-1}\binom{r_i}{2} & \binom{r_0}{t}\binom{r_i}{2} \end{vmatrix} \\
&= r_i \binom{r_i}{2} \cdot \begin{vmatrix} \binom{r_0}{t} & \binom{r_0}{t+1} & \binom{r_0}{t+2} \\ \binom{r_0}{t-1} & \binom{r_0}{t} & \binom{r_0}{t+1} \\ \binom{r_0}{t-2} & \binom{r_0}{t-1} & \binom{r_0}{t} \end{vmatrix}
\end{aligned}$$

mod p

By [Kra99, (2.17)] (for $a = t$ and $a + b = r_0$ in the notation of *loc. cit.*),

$$\begin{vmatrix} \binom{r_0}{t} & \binom{r_0}{t+1} & \binom{r_0}{t+2} \\ \binom{r_0}{t-1} & \binom{r_0}{t} & \binom{r_0}{t+1} \\ \binom{r_0}{t-2} & \binom{r_0}{t-1} & \binom{r_0}{t} \end{vmatrix} \equiv \prod_{i=1,2,3} \prod_{j=1,\ldots,t} \prod_{k=1,\ldots,r_0-t} \frac{i+j+k-1}{i+j+k-2} \bmod p$$

For this product to be nonzero, every factor has to be nonzero. Because $j + k \leq r_0$, we have $i + j + k - 1$ in $\{2, \ldots, r_0 + 2\}$. This set does not contain 0 in $\mathbb{F}_p$ if and only if $r_0 < p - 2$. Because $p \nmid r', r$, we have $r_0 < p - 2$, and conclude $|M| \neq 0$ in $\mathbb{F}_p$. That is, $D_t = E_t = F_t = 0$.

Case 1.1.2. Suppose $t = 1$. Then $t'' = 2p^i - 1 = p^i + p^i - 1 = p^i + (p-1)(1 + p + \ldots + p^{i-1})$. Because $r_0 < p - 2$, by Lucas's Theorem, $\binom{r-2}{t''} \equiv 0 \mod p$. Therefore

$$|M| \equiv r_i \binom{r_i}{2} \cdot \begin{vmatrix} \binom{r_0}{1} & \binom{r_0}{2} & \binom{r_0}{3} \\ \binom{r_0}{0} & \binom{r_0}{1} & \binom{r_0}{2} \\ 0 & \binom{r_0}{0} & \binom{r_0}{1} \end{vmatrix} = r_i \binom{r_i}{2} \frac{r_0(r_0+1)(r_0+2)}{6} \neq 0 \mod p,$$

because $r_0 < p - 2$. This determinant is well-defined because by assumption $p > 3$.

Case 1.1.3. Suppose $t = r_0 - 1$. Then $\binom{r-2}{t+2} \equiv 0 \mod p$ by Lucas's Theorem. Therefore, similarly to the case $t = 1$,

$$|M| \equiv r_i \binom{r_i}{2} \cdot \begin{vmatrix} \binom{r_0}{t} & \binom{r_0}{t+1} & 0 \\ \binom{r_0}{t'} & \binom{r_0}{t'+1} & \binom{r_0}{t'+2} \\ \binom{r_0}{t''} & \binom{r_0}{t''+1} & \binom{r_0}{t''+2} \end{vmatrix} \neq 0 \mod p.$$

Case 1.2. All $r_1, r_2, \ldots \leq 1$. Because $\Sigma(r'') \geq p$ and $r_0 < p - 1$, there are $0 < i' < i''$ such that $r_{i'}$ and $r_{i''} = 1$. Put $t' := t + p^{i'} - 1$ and $t'' := t + p^{i''} + p^{i'} - 2$.

Case 1.2.1. Suppose $t \in \{2, \ldots, r_0 - 2\}$. Then, similar to Case 1.1.1.,

$$|M| \equiv r_{i'} r_{i''} \begin{vmatrix} \binom{r_0}{t} & \binom{r_0}{t+1} & \binom{r_0}{t+2} \\ \binom{r_0}{t-1} & \binom{r_0}{t} & \binom{r_0}{t+1} \\ \binom{r_0}{t-2} & \binom{r_0}{t-1} & \binom{r_0}{t} \end{vmatrix} \neq 0 \mod p.$$

Case 1.2.2. Suppose $t = 1$. Because $\Sigma(r'') \geq p$ and $r_0 < p - 1$, there are $0 < i' < i''$ such that $r_{i'}$ and $r_{i''} = 1$. Put $t' := t + p^{i'} - 1$ and $t'' := t + p^{i''} + p^{i'} - 2$. Then $t'' = p^{i''} + p^{i'} - 1 = p^{i''} + (p-1)(1 + p + \ldots + p^{i'-1})$. Then, similar to Case 1.1.2.,

$$|M| \equiv r_{i''} r_{i'} \cdot \begin{vmatrix} \binom{r_0}{1} & \binom{r_0}{2} & \binom{r_0}{3} \\ \binom{r_0}{0} & \binom{r_0}{1} & \binom{r_0}{2} \\ 0 & \binom{r_0}{0} & \binom{r_0}{1} \end{vmatrix} \neq 0 \mod p.$$

Case 1.2.3. Suppose $t = r_0 - 1$. Then $\binom{r-2}{t+2} \equiv 0 \bmod p$. Then, similar to Case 1.1.3.,

$$|\mathrm{M}| \equiv r_{i''} r_{i'} \cdot \begin{vmatrix} \binom{r_0}{t} & \binom{r_0}{t+1} & 0 \\ \binom{r_0}{t'} & \binom{r_0}{t'+1} & \binom{r_0}{t'+2} \\ \binom{r_0}{t''} & \binom{r_0}{t''+1} & \binom{r_0}{t''+2} \end{vmatrix} \not\equiv 0 \bmod p.$$

Case 2. Suppose $t \in \{r_i, \ldots, p-1\}$.

- By assumption $\Sigma(r'') = r_i + \cdots + r_m \geq p$, so we can write $t = r_i + s_{i+1} + \cdots + s_m$ with s_j in $\{0, \ldots, r_j\}$ for $j = i+1, \ldots, m$. Put $t' = r_i + s_{i+1}p + \cdots + s_m p^m$. Then $t' \equiv t \bmod p-1$ and $\binom{r-2}{t'} \not\equiv 0 \bmod p$ by Lucas's Theorem. If
  - either $i = 0$, then, because $p \nmid r-1, r$, we have $r_0 < p-2$. Therefore $\binom{r-2}{t'+1}, \binom{r-2}{t'+2} \equiv 0 \bmod p$ by Lucas's Theorem.
  - or $i > 0$, then $r_0 = 0$. Therefore $\binom{r-2}{t'+1}, \binom{r-2}{t'+2} \equiv 0 \bmod p$ by Lucas's Theorem.

  By Equation (2.4), in either case $\mathrm{D}_t = \mathrm{D}_{t'} = 0$.
- To show $\mathrm{E}_t = 0$, we choose t' with $t' \equiv t \bmod p-1$ as follows:
  - If $i = 0$, then let $r'_0 = r_0 - 1$. Because by assumption $\Sigma(r'') = r_0 + \cdots + r_m \geq p$ and $t \leq p-1$, we can write $t = r'_0 + s'_1 + \cdots$ with s'_j in $\{0, \ldots, r_j\}$ for $j = 1, 2, \ldots$. Put $t' = r'_0 + s'_1 p + \cdots$. Then $t' \equiv t \bmod p-1$.

    Because $i = 0$ and $p \nmid r-1, r$, we have $r_0 < p-2$. Therefore $\binom{r-2}{t'+1} \not\equiv 0$ and $\binom{r-2}{t'+2} \equiv 0 \bmod p$ by Lucas's Theorem.
  - If $i > 0$, then let $r'_i = r_i - 1$. Because by assumption $\Sigma(r'') = r_i + \cdots + r_m \geq p$ and $t \leq p-1$, we can write $t = r'_i + s'_{i+1} + \ldots$ with s'_j in $\{0, \ldots, r_j\}$ for $j = 1, 2, \ldots$. Put $t' = (p-1) + \cdots + (p-1)p^{i-1} + r'_i p^i + s'_{i+1} p^{i+1} + \cdots$. Then $t' \equiv t \bmod p-1$. Because $t'+1 = r_i + s'_{i+1} p^{i+1} + \cdots$, by Lucas's Theorem $\binom{r-2}{t'+1} \not\equiv 0 \bmod p$.

    Since $i > 0$, in particular $r_0 = 0$, that is, $t'+2 = 1 + r_i p^i + s'_{i+1} p^{i+1} + \cdots$. By Lucas's Theorem, $\binom{r-2}{t'+2} \equiv 0 \bmod p$.

  Since $\mathrm{D}_t = 0$, we conclude by Equation (2.4), that in either case $\mathrm{E}_t = 0$. □

Lemma 2.19. *Let $p > 3$ and write $r = p^n u$ for $n \geq 1$ such that $p \nmid u$. If $\Sigma(u)$ is non-minimal, then* $\dim \mathrm{X}_{r-2} = 3p + 3$.

Proof: For every x in $\mathbb{N}$ such that $p \nmid x$, define

$$r(x) := xp^n - 2 = xp^n - p^n + p^n - 2 = p^n(x-1) + (p-1)[p^{n-1} + \cdots + p] + (p-2).$$

We notice that $r(x) \equiv x - 2 \mod p-1$. Expand p-adically $u = u_0 + u_1 p + u_2 p^2 + \cdots$ with u_0, u_1, u_2, … in $\{0, \ldots, p-1\}$ and $u_0 > 0$. Then

$$r - 2 = r(u) = [(u_0 - 1) + u_1 p + u_2 p^2 + \cdots]p^n + (p-1)(p^{n-1} + \cdots + p) + (p-2).$$

Using the notation of Lemma 2.18, we will show that Equation (2.4) forces $D_{t'}$ and $E_{t''}$ or $F_{t''}$ to vanish for t' and t'' in full sets of representatives of $\{1, \ldots, p-1\}$. That is, for every t in $\{0, \ldots, p-2\}$ there is t' and t'' with $t' \equiv t$ and $t'' \equiv t \mod p-1$ such that $D_{t'}$ and $E_{t''}$ vanish.

Case 1. Suppose $t \in \{0, \ldots, u_0 - 3\}$. Let i be the smallest index > 0 such that $u_i > 0$ (which exists because $u_0 \leq p-1$ and $\Sigma(u) \geq p$). Put $t' = r(t+2)$ and $t'' = r(t+1+p^i)$ Then t' and $t'' \equiv t \mod p-1$. By Lucas's Theorem,

- we have $\binom{r-2}{t'} \equiv \binom{u_0-1}{t+1} \neq 0$ and $\binom{r-2}{t''} \equiv u_i \binom{u_0-1}{t} \neq 0$,
- we have $\binom{r-2}{t'+2} \equiv \binom{u_0-1}{t+2} \neq 0$ and $\binom{r-2}{t''+2} \equiv u_i \binom{u_0-1}{t+1} \neq 0$, and
- we have $\binom{r-2}{t'+1} \equiv 0$ and $\binom{r-2}{t''+1} \equiv 0$.

Therefore (+) yields modulo p the system of equations:

$$\binom{r-2}{t'} D_t + \binom{r-2}{t'+2} F_t \equiv 0$$
$$\binom{r-2}{t''} D_t + \binom{r-2}{t''+2} F_t \equiv 0$$

To see that $D_t = F_t = 0$, we will prove that the determinant of the matrix M attached to this system of equations is nonzero, that is,

$$|M| \equiv \begin{vmatrix} \binom{r-2}{t'} & \binom{r-2}{t'+2} \\ \binom{r-2}{t''} & \binom{r-2}{t''+2} \end{vmatrix} \neq 0 \mod p.$$

Putting $u_0' = u_0 - 1$, by [Kra99, (2.17)],

$$\begin{vmatrix} \binom{r-2}{t'} & \binom{r-2}{t'+2} \\ \binom{r-2}{t''} & \binom{r-2}{t''+2} \end{vmatrix} \equiv u_i^2 \begin{vmatrix} \binom{u_0'}{t+1} & \binom{u_0'}{t+2} \\ \binom{u_0'}{t} & \binom{u_0'}{t+1} \end{vmatrix}$$
$$= u_i^2 \prod_{i=1,2} \prod_{j=1,\ldots,t+1} \prod_{k=1,\ldots,u_0'-(t+1)} \frac{i+j+k-1}{i+j+k-2} \mod p$$

For this product to be nonzero, every factor has to be nonzero. Because $j+k \leq u_0'$, we have $i+j+k-1$ in $\{2,\dots,u_0'+1\}$. This set does not contain 0 in $\mathbb{F}_p$ if and only if $u_0' < p-1$. Because $u_0 \leq p-1$, we have $u_0' = u_0-1 < p-1$, and conclude $|\mathrm{M}| \neq 0 \bmod p$. That is, $\mathrm{D}_t = \mathrm{F}_t = 0$.

Case 2. Suppose either $u_0 = 1$ or, otherwise, $t \in \{u_0-2,\dots,p-2\}$.

- To show $\mathrm{D}_t = 0$, we choose t' with $t' \equiv t \bmod p-1$ as follows: Because by assumption $\Sigma(u) = u_0 + u_1 + \cdots + u_m \geq p$ and $t \leq p-2$, we can write $t+2 = u_0 + s_1 + \cdots + s_m$ with s_j in $\{0,\dots,u_j\}$ for $j = 1,\dots,m$. Put $t' = r(u_0 + s_1 p + \cdots + s_m p^m)$. Then $t' \equiv t \bmod p-1$. We have $\binom{r-2}{t'} \not\equiv 0 \bmod p$ and $\binom{r-2}{t'+1}, \binom{r-2}{t'+2} \equiv 0 \bmod p$ by Lucas's Theorem. By Equation (2.4), we conclude $\mathrm{D}_t = \mathrm{D}_{t'} = 0$.
- To show E_t or $\mathrm{F}_t = 0$, we choose t' with $t' \equiv t \bmod p-1$ as follows:

Case 2.1. We have $t \leq p-3$: Because by assumption $\Sigma(u) = u_0 + \cdots + u_m \geq p$ and $t \leq p-3$, we can write $t+3 = u_0 + s_1' + \cdots$ with s_j' in $\{0,\dots,u_j\}$ for $j = 1,\dots,m$. Put $t' = r(u_0' + s_1' p + \cdots + s_m' p^m) - 1$. Then $t' \equiv t \bmod p-1$. By Lucas's Theorem, $\binom{r-2}{t'+1} \not\equiv 0$ and $\binom{r-2}{t'+2} \equiv 0 \bmod p$.

Case 2.2. We have $t = p-2$:

Case 2.2.1. If $n = 1$ and $u_0 > 1$ or $n > 1$, then $\binom{r-2}{t+2} \not\equiv 0 \bmod p$ by Lucas's Theorem. In addition, $\binom{r-2}{t+1} \equiv 0$.

Case 2.2.2. If $n = 1$ and $u_0 = 1$, then let i be the smallest index > 0 such that $u_i > 0$ (which exists because $\Sigma(u) \geq p$). Let $t' = r(p^i)$. Then $t' \equiv p-2 = t \bmod p-1$. We have $\binom{r-2}{t+1} \equiv 0$ and $\binom{r-2}{t+2} \not\equiv 0 \bmod p$ by Lucas's Theorem.

Because $\mathrm{D}_t = 0$, we conclude by Equation (2.4) that $\mathrm{F}_t = 0$. □

Lemma 2.20. *Let $p > 3$ and write $r-1 = p^n u$ for $n \geq 1$ such that $p \nmid u$. If $\Sigma(u)$ is non-minimal, then* $\dim \mathrm{X}_{r-2} = 3p+3$.

Proof: For every x in $\mathbb{N}$ such that $p \nmid x$, define

$$r(x) := xp^n - 1 = (xp^n - p^n) + p^n - 1 = (p^n(x-1)) + (p-1)[p^{n-1} + \cdots + p + 1]$$

We notice that $r(x) \equiv x-1 \bmod p-1$. Expand p-adically $u = u_0 + u_1 p + u_2 p^2 + \cdots$ with $u_0, u_1, u_2, \dots$ in $\{0,\dots,p-1\}$ and $u_0 > 0$. Then

$$r-2 = r(u) = [(u_0-1) + u_1 p + u_2 p^2 + \cdots]p^n + (p-1)(p^{n-1} + \cdots + p + 1).$$

Using the notation of Lemma 2.18, we will show that Equation (2.4) forces $D_{t'}$ and $E_{t''}$ to vanish for t' and t'' in full sets of representatives of $\{1, \ldots, p-1\}$. That is, for every t in $\{0, \ldots, p-2\}$ there is t' and t'' with $t' \equiv t$ and $t'' \equiv t \bmod p-1$ such that $D_{t'}$ and $E_{t''}$ vanish.

Case 1. Suppose $t \in \{0, \ldots, u_0 - 2\}$.

As in Lemma 2.18, we choose t', t'' and $t''' \equiv t \bmod p-1$ such that Equation (2.4) yields modulo p the system of equations

$$\binom{r-2}{t'} D_t + \binom{r-2}{t'+1} E_t + \binom{r-2}{t'+2} F_t \equiv 0$$
$$\binom{r-2}{t''} D_t + \binom{r-2}{t''+1} E_t + \binom{r-2}{t''+2} F_t \equiv 0$$
$$\binom{r-2}{t'''} D_t + \binom{r-2}{t'''+1} E_t + \binom{r-2}{t'''+2} F_t \equiv 0$$

and prove that the determinant of the matrix M attached to this system of equations is nonzero, that is,

$$|M| = \begin{vmatrix} \binom{r-2}{t'} & \binom{r-2}{t'+1} & \binom{r-2}{t'+2} \\ \binom{r-2}{t''} & \binom{r-2}{t''+1} & \binom{r-2}{t''+2} \\ \binom{r-2}{t'''} & \binom{r-2}{t'''+1} & \binom{r-2}{t'''+2} \end{vmatrix} \not\equiv 0 \bmod p.$$

Put $t' = p^n t$, $t'' = r(p^i + t + 1) - 1$, $t''' = r(t+1)$ for the smallest $i > 0$ such that $u_i > 0$ (which exists because $u_0 \leq p-1$ and $\Sigma(u) \geq p$). Then t', t'' and $t''' \equiv t \bmod p-1$. By Lucas's Theorem, with $u' = u_0 - 1$,

- we have $\binom{r-2}{t'} \equiv \binom{u'}{t}$, $\binom{r-2}{t'+1} \equiv (p-1)\binom{u'}{t}$ and $\binom{r-2}{t'+2} \equiv \binom{p-1}{2}\binom{u'}{t}$,
- we have $\binom{r-2}{t''} \equiv u_i(p-1)\binom{u'}{t}$, $\binom{r-2}{t''+1} \equiv u_i\binom{u'}{t}$ and $\binom{r-2}{t''+2} \equiv u_i\binom{u'}{t+1}$,
- we have $\binom{r-2}{t'''} \equiv \binom{u'}{t}$, $\binom{r-2}{t'''+1} \equiv \binom{u'}{t+1}$ and $\binom{r-2}{t'''+2} \equiv (p-1)\binom{u'}{t+1}$,

Therefore,

$$
\begin{aligned}
|\mathrm{M}| &\equiv u_i^3 \begin{vmatrix} \binom{u'}{t} & (p-1)\binom{u'}{t} & \binom{p-1}{2}\binom{u'}{t} \\ (p-1)\binom{u'}{t} & \binom{u'}{t} & \binom{u'}{t+1} \\ \binom{u'}{t} & \binom{u'}{t+1} & (p-1)\binom{u'}{t+1} \end{vmatrix} \\
&= u_i^3 \begin{vmatrix} \binom{u'}{t} & -\binom{u'}{t} & \binom{u'}{t} \\ -\binom{u'}{t} & \binom{u'}{t} & \binom{u'}{t+1} \\ \binom{u'}{t} & \binom{u'}{t+1} & -\binom{u'}{t+1} \end{vmatrix} \\
&= u_i^3 \begin{vmatrix} 0 & 0 & \binom{u'+1}{t+1} \\ -\binom{u'}{t} & \binom{u'}{t} & \binom{u'}{t+1} \\ \binom{u'}{t} & \binom{u'}{t+1} & -\binom{u'}{t+1} \end{vmatrix} \\
&= u_i^3 \binom{u'+1}{t+1}\binom{u'}{t+1}\left[-\binom{u'}{t} - \binom{u'}{t+1}\right] = -u_i^3 \binom{u'}{t+1}\binom{u'+1}{t+1}^2 \bmod p.
\end{aligned}
$$

Because $t < u' < p-1$, we have $|\mathrm{M}| \neq 0$.

Case 2. Suppose $t \in \{u_0 - 1, \dots, p-2\}$.

- To show $\mathrm{D}_t = 0$, we choose t' with $t' \equiv t \bmod p-1$ as follows: Because by assumption $\Sigma(u) = u_0 + u_1 + \cdots \geq p$ and $u_0 \leq t+1 \leq p-1 \leq p$, we can write $t+1 = u_0 + s_1 + \cdots$ with s_j in $\{0, \dots, u_j\}$ for $j = 1, 2, \dots$. Put $t' = r(u_0 + s_1 p + \cdots)$. Then $t' \equiv t \bmod p-1$. By Lucas's Theorem, $\binom{r-2}{t'} \not\equiv 0$ but $\binom{r-2}{t'+1}$ and $\binom{r-2}{t'+2} \equiv 0 \bmod p$. By Equation (2.4), we conclude $\mathrm{D}_t \equiv \mathrm{D}_{t'} = 0 \bmod p$.
- To show E_t or $\mathrm{F}_t = 0$, we choose t' with $t' \equiv t \bmod p-1$ as follows: Because by assumption $\Sigma(u) = u_0 +_1 + \cdots \geq p$ and $u_0 \leq t+2 \leq p$, we can write $t+2 = u_0 + s_1 + \cdots + s_m$ with s_j in $\{0, \dots, u_j\}$ for $j = 1, 2, \dots$. Put $t' = r(u_0 + s_1' p + \cdots) - 1$. Then $t' \equiv t \bmod p-1$. by Lucas's Theorem, $\binom{r-2}{t'}$ and $\binom{r-2}{t'+1} \not\equiv 0$, but $\binom{r-2}{t'+2} \equiv 0 \bmod p$. Because $\mathrm{D}_t \equiv 0 \bmod p$, we conclude by Equation (2.4) that $\mathrm{E}_t \equiv \mathrm{E}_{t'} = 0 \bmod p$. □

Proposition 2.21. *Let $p > 3$. Let $\Sigma(r''), \Sigma(r')$ and $\Sigma(r) < p$. If $r \geq 3p+2$ and $\Sigma(r'')$, $\Sigma(r')$ and $\Sigma(r)$ are non-minimal, then $\mathrm{X}_{r-2} \xrightarrow{\sim} \mathrm{X}_{r''} \otimes \mathrm{V}_2$ and its Jordan-Hölder series is that of Proposition 2.9.*

Proof: If $\Sigma(r'')$, $\Sigma(r')$ and $\Sigma(r)$, then by the preceding Lemma 2.18, Lemma 2.19 and Lemma 2.20, the dimension of X_{r-2} is equal to that of $\mathrm{X}_{r''} \otimes \mathrm{V}_2$, hence the natural epimorphism

$$\mathrm{X}_{r''} \otimes \mathrm{V}_2 \twoheadrightarrow \mathrm{X}_{r-2}$$

is an isomorphism. □

2.6 Sum of the Digits of $r - 2$ is minimal

Let a in $\{3, \ldots, p+1\}$ such that $r \equiv a \bmod p-1$. Let $r'' = r-2$. We assume in this Section 2.6 that $\Sigma(r'')$ is minimal, that is, $\Sigma(r'') < p$, or, equivalently, $\Sigma(r'') = a-2$.

If r satisfies the conditions of Lemma 2.1, that is, $r \leq p$ or $r = p^n + r_0$ where r_0 in $\{2, \ldots, p-1\}$ and $n > 0$, then the inclusion $X_{r-1} \subseteq X_{r-2}$ is an equality. Therefore, the Jordan-Hölder series of $X_{r-2} = X_{r-1}$ is known

- for $a = p$ by [BG15, Proposition 3.13], and
- for $a = 3, \ldots, p-1, p+1$ by [BG15, Proposition 4.9].

Otherwise, X_{r-2} has at least three distinct Jordan-Hölder factors by Lemma 2.1: By Proposition 2.11 and Lemma 2.7,

$$X_{r''} = V_{a-2} \quad \text{and } X^*_{r''} = 0.$$

By Lemma 2.2, there is thus an $\mathbb{F}_p[M]$-linear surjection

$$\phi : V_{a-2} \otimes V_2 \twoheadrightarrow X_{r-2} \tag{2.5}$$

2.6.1 $r \equiv 3 \bmod p-1$

Proposition 2.22. *Let $r \geq p$. If $r \equiv 3 \bmod p-1$ and $\Sigma(r'') < p$, then*

$$V_1 \otimes V_2 \cong V_1 \otimes D \oplus V_3 \xrightarrow{\sim} X_{r-2}.$$

Proof: For $a = 3$ the right-hand side of Equation (2.5) is $V_1 \otimes V_2$. By Proposition 2.9,

$$V_1 \otimes V_2 = V_1 \otimes D \oplus V_3 \twoheadrightarrow X_{r-2}.$$

That is, there is an epimorphism with only two Jordan-Hölder factors onto X_{r-2}. Because $r \geq p$, there are by Lemma 2.1.(i) (which is [BG15, Lemma 4.1]) at least two Jordan-Hölder factors in X_{r-2}; therefore this epimorphism must be an isomorphism. □

Alternatively, if $r \equiv 3 \bmod p-1$ and $\Sigma(r'')$ is minimal, that is, $\Sigma(r'') = 1$, then $r = p^n + 2$. In particular, r satisfies the conditions of Lemma 2.1, and the inclusion $X_{r-1} \subseteq X_{r-2}$ is an equality. By [BG15, Proposition 4.9],

$$V_1 \otimes V_2 = V_1 \otimes D \oplus V_3 \xrightarrow{\sim} X_{r-1}.$$

2.6.2 $r \equiv 4, \ldots, p-1 \bmod p-1$

Let a in $\{4, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$. By Lemma 2.12, if $\Sigma(r'')$ is minimal, then $\Sigma(r')$ and $\Sigma(r)$ are minimal, too.

Proposition 2.23. *Let $p > 2$. Let a in $\left\{4, \ldots, p-1\right\}$ such that $r-2 \equiv a-2 \bmod p-1$ and $r \geq p$. Let $\Sigma(r'') < p$.*

(i) *If $r = p^n + r_0$ where $r_0 = a-1$ and $n > 0$, then*

$$X_{r-2} = V_{a-2} \otimes D \oplus V_a,$$

(ii) *otherwise,*

$$X_{r-2} \cong V_a \oplus (V_{a-2} \otimes D) \oplus (V_{a-4} \otimes D^2).$$

Proof: Let $p > 2$. Let a in $\left\{4, \ldots, p-1\right\}$ such that $r-2 \equiv a-2 \bmod p-1$. If $r = p^n + r_0$ where r_0 in $\{2, \ldots, p-1\}$ and $n > 0$, then the inclusion $X_{r-1} \subseteq X_{r-2}$ is an equality and, by [BG15, Proposition 4.9.(i)]

$$X_{r-2} = X_{r-1} = V_{a-2} \otimes D \oplus V_a$$

Otherwise, Equation (2.5) becomes by Proposition 2.9,

$$V_{a-2} \otimes V_2 = V_a \oplus (V_{a-2} \otimes D) \oplus (V_{a-4} \otimes D^2) \twoheadrightarrow X_{r-2}.$$

By Lemma 2.1 the right-hand side has at least three Jordan-Hölder factors. Because the map is surjective, these are exhausted by those of the left-hand side. Thus the surjection is a bijection. □

2.6.3 $r \equiv p \bmod p-1$

If $a = p$, then $\Sigma(r'')$ is minimal if and only if $\Sigma(r'') = p-2$. Therefore, as observed in Lemma 2.12, indeed $\Sigma(r') = p-1$ is minimal, but $\Sigma(r) = p$ is non-minimal!

Proposition 2.24. *Let $r \geq p$ and $r \equiv p \bmod p-1$. Let $\Sigma(r'') < p$.*

(i) *If $r = p^n + (p-1)$, then*

$$0 \to V_1 \otimes D^{p-1} \to X_{r-2} \to V_{2p-1} \to 0,$$

(ii) *otherwise,*

$$X_{r-2} \cong V_{p-4} \otimes D^2 \oplus V_{2p-1}.$$

Proof: If $r = p^n + (p-1)$, then the inclusion $X_{r-1} \subseteq X_{r-2}$ is an equality. Because $\Sigma(r') = p-1 < p$ is minimal, by [BG15, Proposition 4.9.(i)]

$$0 \to V_1 \otimes D^{p-1} \to X_{r-1} \to V_{2p-1} \to 0.$$

Otherwise, by Proposition 2.11, we have $X^*_{r''} = 0$. Therefore Equation (2.5) becomes by Proposition 2.9,

$$V_{p-4} \oplus V_{2p-1} \twoheadrightarrow X_{r-2}$$

where V_{2p-1} has successive semisimple Jordan-Hölder factors $V_{p-2} \otimes D$, V_1 and $V_{p-2} \otimes D$. By Proposition 2.6.(iii), we have

$$X_{r-2}/X^*_{r-2} = V_p/V^*_p.$$

By Lemma 1.3,

$$0 \to V_1 \to V_p/V^*_p \to V_{p-2} \to 0.$$

In particular, X_{r-2}/X^*_{r-2} has 2 Jordan-Hölder factors.

Because $\Sigma(r) = p$ is non-minimal, $X^*_r \neq 0$ by Proposition 2.11. Therefore, by Lemma 2.1, we have

$$0 \subset X^*_r \subset X_r \subset X_{r-1} \subset X_{r-2}.$$

That is, X_{r-2} has at least 4 Jordan-Hölder factors. Therefore, all 4 Jordan-Hölder factors of the left-hand side must appear on the right-hand side of the epimorphism $V_{p-4} \otimes D^2 \oplus V_{2p-1} \twoheadrightarrow X_{r-2}$; therefore, it must be an isomorphism. □

2.6.4 $r \equiv p+1 \mod p-1$

If $a = p+1$, then $\Sigma(r'')$ is minimal if and only if $\Sigma(r'') = p-1$. Therefore, as observed in Lemma 2.12, neither $\Sigma(r') = p$ nor $\Sigma(r) = p+1$ are minimal!

Proposition 2.25 (Extension of [BG15, Proposition 3.3]). *Let $r \geq p$ and $r \equiv p+1 \mod p-1$. If $\Sigma(r'') < p$, then*

$$X_{r-2} \cong V_{3p-1}.$$

Proof: By Proposition 2.11, we have $X^*_{r''} = 0$. Therefore Equation (2.5) becomes by Proposition 2.9,

$$V_{3p-1} \twoheadrightarrow X_{r-2}$$

We recall that by Lemma 1.1.(ii), the successive semisimple Jordan-Hölder factors of the $\mathbb{F}_p[M]$-module V_{3p-1} are $V_{3p-1} = U_2 \oplus (U_0 \otimes D)$ where

- we have $U_0 = V_{p-1} \otimes D$, and
- the $\mathbb{F}_p[M]$-module U_2 has successive semisimple Jordan-Hölder factors $V_{p-3} \otimes D^2$, $(V_0 \otimes D) \oplus V_2$ and $V_{p-3} \otimes D^2$.

In particular, V_{3p-1} has 5 Jordan-Hölder factors.

By [BG15, Proposition 4.9], because $\Sigma(r') = p$ is non-minimal,

$$0 \to V_{p-3} \otimes D^2 \oplus V_{p-1} \otimes D \to X_{r-1} \to V_0 \otimes D \oplus V_2.$$

In particular, X_{r-1} has 4 Jordan-Hölder factors.

Because $r \equiv p+1 \bmod p-1$, impossibly $r = p^n + r_0$ for $1 < r_0 < p$. Therefore, by Lemma 2.1,

$$X_{r-1} \subset X_{r-2}.$$

Therefore X_{r-2} has at least 5 Jordan-Hölder factors. Hence, all 5 Jordan-Hölder factors of the left-hand side must appear on the right-hand side of the epimorphism $V_{3p-1} \twoheadrightarrow X_{r-2}$ and thus it is an isomorphism. □

3 Vanishing conditions on the singular quotients of X_{r-2}

In this section we study the singular quotients of X_{r-2}, that is, whether X^{*}_{r-2}/X^{**}_{r-2}, $X^{**}_{r-2}/X^{***}_{r-2}$, or $X^{*}_{r-2}/X^{***}_{r-2}$ are zero or not by applying Lemma 1.4 and Lemma 1.6. In correspondence with Lemma 1.3, we will choose a such that $r \equiv a \bmod p-1$ for X^{*}_{r-2}/X^{**}_{r-2} in the range $\{3, \ldots, p+1\}$, whereas for $X^{**}_{r-2}/X^{***}_{r-2}$ in $\{5, \ldots, p+3\}$.

Lemma 3.1. *Let $a \in \{4, \ldots, p\}$. If $r > p$ and $r \equiv a \bmod p-1$ and $r \equiv a \bmod p$, then*

$$0 = \begin{cases} X^{*}_{r-2}/X^{**}_{r-2}, & \text{if } a = 4 \\ X^{*}_{r-2}/X^{***}_{r-2}, & \text{if } 5 \le a \le p \end{cases}$$

Proof: We follow Lemma 6.2 of [BG15].

Consider $\sum_{k \in \mathbb{F}_p} k^{p-2}(kX+Y)^{r-1}X \in X_{r-1}$. Working $\bmod p$:

$$\sum_{k \in \mathbb{F}_p} k^{p-2}(kX+Y)^{r-1}X \equiv -(r-1)X^2Y^{r-2} - \sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j.$$

Claim:

$$G(X,Y) = \sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j \in \begin{cases} V^{**}_r, & \text{for } a = 4, \\ V^{***}_r, & \text{for } 5 \le a \le p. \end{cases}$$

Proof of our claim: Let c_j denote the coefficients of G. If $a \geq 5$, then $c_j = 0$ for $j = 0, 1, 2$ and $j = r-2, r-1, r$. If $a = 4$, then $c_j = 0$ for $j = 0, 1$ and $j = r-2, r-1, r$, but $c_2 \neq 0$. We now show $\sum c_j, \sum j c_j, \sum j(j-1)c_j \equiv 0 \bmod p$ to obtain $G(X, Y) \in V_r^{***}$ for $a \geq 5$ and $G(X, Y) \in V_r^{**}$ for $a = 4$:

By Lemma 1.6 for $i = 1$, we have $\sum_j c_j \equiv (a-1)-(r-1) = a-r \equiv 0 \bmod p$ by our assumption $r \equiv a \bmod p$. Likewise, computing $\sum_j j c_j \bmod p$ and $\sum_j j(j-1)c_j \bmod p$:

$$\sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} j\binom{r-1}{j} = (r-1) \sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} \binom{r-2}{j-1}$$
$$\equiv (r-1)((a-2)-(r-2)) \equiv 0 \bmod p$$

and

$$\sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} j(j-1)\binom{r-1}{j} = (r-1)(r-2) \sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} \binom{r-3}{j-2}$$
$$\equiv (r-1)(r-2)((a-3)-(r-3)) \equiv 0 \bmod p.$$

Therefore $(r-1)X^2Y^{r-2}$ is in $X_{r-1} + V_r^{***}$. Since the case $a = p+1$ is excluded, $r \equiv a \not\equiv 1 \bmod p$, and we conclude $X_{r-2} \subseteq X_{r-1} + V_r^{***}$.

Claim:

$$X_{r-1}^{*} = X_{r-1}^{***}.$$

Proof of our claim: In the proof of Lemma 6.2 of [BG15], we can show as above that $\sum_j j(j-1)c_j \equiv 0 \bmod p$ by proving

$$\sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} j(j-1)\binom{r}{j} \equiv 0 \bmod p;$$

which means their $F(X, Y) \in V_r^{***}$ (and not just in V_r^{**}). By Lemma 2.8, we have $X_r^{*} = X_r^{***}$. We therefore conclude $X_{r-1}^{*} = X_{r-1}^{***}$.

Now by following the argument at the end of the proof of [BG15, Lemma 6.2], we conclude $X_{r-2}^{*} \subseteq X_{r-2}^{***}$. □

3.1 X_{r-2}^{*}/X_{r-2}^{**}

Lemma 3.2. *Let $a = 4, \ldots, p$ and $r \equiv a \bmod p-1$. If $r \geq 2p+1$ and $r \not\equiv a \bmod p$, then*

$$X_{r-2}^{*}/X_{r-2}^{**} \neq 0.$$

Proof: Consider the polynomial

$$\begin{aligned}F(X,Y) &= (a-2)X^{r-1}Y + \sum_{k\in\mathbb{F}_p} k^{p+2-a}(kX+Y)^{r-2}X^2 \in X_{r-2}\\ &\equiv (a-r)X^{r-1}Y - \sum_{\substack{0<j<r-3\\ j\equiv a-3 \bmod p-1}} \binom{r-2}{j} X^{j+2}Y^{r-2-j} \bmod p.\end{aligned}$$

By Lemma 1.4 we see $F(X,Y) \in V_r^*$ but the coefficient c_1 of $X^{r-1}Y$ in $F(X,Y)$ is $a-r \not\equiv 0 \bmod p$ by the hypothesis, so $F(X,Y) \notin V_r^{**}$. Thus $X_{r-2}^*/X_{r-2}^{**} \neq 0$. □

Since V_r^*/V_r^{**} splits if and only if $a = p+1$, this is the only value of a for which X_{r-2}^*/X_{r-2}^{**} can be different from V_r^*/V_r^{**}, its socle or 0 (and indeed it is if $r \equiv a \bmod p$):

Lemma 3.3. *If $r \geq 2p+1$ and $r \equiv p+1 \bmod p-1$, then*

$$X_{r-2}^*/X_{r-2}^{**} = X_{r-1}^*/X_{r-1}^{**} = \begin{cases} V_r^*/V_r^{**}, & \text{if } r \not\equiv 0,1 \bmod p\\ V_{p-1}\otimes D, & \text{if } r \equiv 0 \bmod p\\ V_0 \otimes D, & \text{if } r\equiv 1 \bmod p.\end{cases}$$

Proof: Consider

$$F(X,Y) := XY^{r-1} - X^{r-1}Y \in X_{r-1} \subseteq X_{r-2}.$$

By Lemma 1.4, we have $F(X,Y) \in V_r^*$ but $F(X,Y) \notin V_r^{**}$ as the coefficient c_1 of $X^{r-1}Y$ is not zero. Thus, $X_{r-2}^*/X_{r-2}^{**} \neq 0$. Since the polynomial $F(X,Y) \in X_{r-1}$ and V_r^*/V_r^{**} splits for $a = p+1$, we can determine the structure of X_{r-1}^*/X_{r-1}^{**} by checking if the image of the polynomial $F(X,Y)$ maps to zero or not. This has been studied already in Section 5 of [BG15]. □

By Lemma 2.8, for $a = 3$ and $p \nmid r-2$, we have $X_{r''}^* \neq X_{r''}^{**}$, so not necessarily $\phi(X_{r''}^* \otimes V_2) \subseteq X_{r-2}^{**}$. (We observe in particular that $r \equiv 3 \bmod p-1$ and $r \not\equiv 2 \bmod p$ imply $\Sigma(r'') \geq p$ (otherwise $\Sigma(r'') = 1$, that is, $r'' = p^n$ for some n), thus $X_{r''}^* \neq 0$.) Indeed, there is no inclusion:

Lemma 3.4. *If $r \geq 3p+2$ and $r \equiv 3 \bmod p-1$ and $r \not\equiv 2 \bmod p$, then $\phi(X_{r''}^* \otimes V_2) \not\subseteq X_{r-2}^{**}$.*

Proof: We adapt Lemma 5.2 of [BG15]: Consider $F(X,Y) = \sum_{k=0}^{p-1}(kX+Y)^{r''} \in X_{r''}$. As in the proof of Lemma 3.10,

$$F(X,Y) \equiv - \sum_{\substack{0<j<r'' \\ j\equiv 1 \bmod p-1}} \binom{r''}{j} X^{r''-j}Y^j.$$

By Lemma 1.4 we have $\phi(F\otimes X^2) \in V_r^*$, but $\phi(F\otimes X^2) \notin V_r^{**}$ as the coefficient c_2 of $X^{r-2}Y^2$ in $\phi(F\otimes X^2)$ is $\binom{r''}{1} = r-2 \not\equiv 0 \bmod p$. $\square$

Lemma 3.5. *If* $r \geq 2p+1$ *and* $r \equiv 3 \bmod p-1$, *then*

$$X_{r-2}^*/X_{r-2}^{**} = \begin{cases} V_r^*/V_r^{**}, & \text{if } r \not\equiv 1,2 \bmod p \\ V_1 \otimes D, & \text{if } r \equiv 1,2 \bmod p. \end{cases}$$

Proof:

- Let $r \not\equiv 1,2 \bmod p$. Consider

  $$F(X,Y) := X^2Y^{r-2} + \frac{1}{2}\sum_{k\in\mathbb{F}_p} k^{p-2}(kX+Y)^{r-1}X \in X_{r-2}.$$

  Working modulo p:

  $$F(X,Y) = X^2Y^{r-2} - \frac{1}{2}\sum_{\substack{0<j\leq r-2 \\ j\equiv 1 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j,$$

  So,

  $$F(X,Y) = (1-\frac{1}{2}(r-1))X^2Y^{r-2} - \frac{1}{2}\sum_{\substack{0<j<r-2 \\ j\equiv 1 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j,$$

  By Lemma 1.6 we see that $\sum c_j = (1-\frac{1}{2}(r-1)) - \frac{1}{2}(3-r) \equiv 0 \bmod p$. By Lemma 1.4, we have $F(X,Y) \in V_r^*$, but $F(X,Y) \notin V_r^{**}$ because the coefficient c_1 of $X^{r-1}Y$ is $-\frac{1}{2}(r-1) \not\equiv 0 \bmod p$. Since $F \in X_{r-2}$, we conclude $X_{r-2}^*/X_{r-2}^{**} \neq 0$. Dividing the polynomial $F(X,Y)$ by θ yields

  $$F(X,Y) \equiv (r-2)\theta(X^{r-2p}Y^{p-1}) \bmod V_r^{**}$$

  which by Lemma 5.1 projects onto a non-zero element in $V_{p-2}\otimes D^2$. This Jordan-Hölder factor is on the right-hand side of the short exact sequences of Lemma 1.3.(ii) which does not split. Therefore the inclusion $X_{r-2}^*/X_{r-2}^{**} \subseteq V_r^*/V_r^{**}$ must be an equality, that is, $X_{r-2}^*/X_{r-2}^{**} = V_r^*/V_r^{**}$.

- For the case $r = 2 \bmod p$, we have by Proposition 2.9 the short exact sequence:

$$0 \to (V_{2p-1} \otimes D) \oplus (V_{p-4} \otimes D^3) \to X_{r-2} \to (V_1 \otimes D) \oplus V_3 \to 0.$$

  When we restrict this exact sequence to the largest singular submodules, we obtain

$$0 \to (V_{2p-1} \otimes D) \oplus (V_{p-4} \otimes D^3) \to X^*_{r-2} \to V_1 \otimes D \to 0.$$

  where V_{2p-1} has $V_{p-2} \otimes D, V_{p-2} \otimes D$ and V_1 as factors. Because $p|r''$, by Lemma 2.8, we have $X^*_{r''} = X^{**}_{r''}$, so $\phi(X^*_{r''} \otimes V_2) \subseteq X^{**}_{r-2}$. We see that $0 \neq F(X, Y) = X^2Y^{r-2} - X^{r-1}Y \in X^*_{r-2}/X^{**}_{r-2}$. Thus X^*_{r-2}/X^{**}_{r-2} contains exactly one Jordan-Hölder factor, that is, $X^*_{r-2}/X^{**}_{r-2} = V_1 \otimes D$.

- For the case $r = 1 \bmod p$, we consider (as above) the short exact sequence:

$$0 \to (V_{2p-1} \otimes D) \oplus (V_{p-4} \otimes D^3) \to X^*_{r-2} \to V_1 \otimes D \to 0.$$

  First, by Lemma 2.7 and Lemma 2.8 the factor $V_{p-4} \otimes D^3 = X^{***}_r$ is included in X^{**}_{r-2}. By Lemma 3.4, the factors from $V_{2p-1} \otimes D$ cannot all be contained in X^{**}_{r-2}. By [BG15, Lemma 6.1] we know that $X^*_{r-1}/X^{**}_{r-1} = V_1 \otimes D$, and we also know that $X^*_{r-1}/X^{**}_{r-1} \subseteq X^*_{r-2}/X^{**}_{r-2}$. By looking at the structure of V^*_r/V^{**}_r, the factor $V_1 \otimes D$ on the right of the short exact sequence above cannot be contained in X^{**}_{r-2}. By [BG15, Proposition 6.4] we know that $\phi(X_{r'} \otimes V_1)$ is contained inside X^{**}_{r-1}, hence the factors $V_{p-2} \otimes D^2$ are also inside X^{**}_{r-2}. Thus $X^*_{r-2}/X^{**}_{r-2} = V_1 \otimes D$. □

3.2 $X^{**}_{r-2}/X^{***}_{r-2}$

Lemma 3.6. *Let $a = 5, \ldots, p-1, p$. If $r \geq 3p+2$ and $r \equiv a \bmod p-1$ and $r \not\equiv a, a-1 \bmod p$, then*

$$X^{**}_{r-2}/X^{***}_{r-2} \neq 0.$$

Proof: For A, B and C constants in $\mathbb{F}_p$, let $F(X, Y)$ in X_{r-2} be given by:

$$\begin{aligned}
\mathrm{F}(\mathrm{X},\mathrm{Y}) =&\mathrm{A}\cdot\left[(r-2)\mathrm{X}^2\mathrm{Y}^{r-2}+\sum_{k\in\mathbb{F}_p}k^{p-2}(k\mathrm{X}+\mathrm{Y})^{r-2}\mathrm{XY}\right]\\
&+\mathrm{B}\cdot\left[\frac{(r-1)(r-2)}{2}\mathrm{X}^2\mathrm{Y}^{r-2}+\sum_{k\in\mathbb{F}_p}k^{p+3-a}(\mathrm{X}+k\mathrm{Y})^{r-1}\mathrm{Y}\right]\\
&+\mathrm{C}\cdot\mathrm{X}^2\mathrm{Y}^{r-2}\\
\equiv&\mathrm{A}\cdot\left[-\sum_{\substack{0<j<r-3\\ j\equiv a-3 \bmod p-1}}\binom{r-2}{j}\mathrm{X}^{r-j-1}\mathrm{Y}^{j+1}\right]\\
&+\mathrm{B}\cdot\left[-\sum_{\substack{0<j<r-3\\ j\equiv a-3 \bmod p-1}}\binom{r-1}{j}\mathrm{X}^{r-j-1}\mathrm{Y}^{j+1}\right]\\
&+\mathrm{C}\cdot\mathrm{X}^2\mathrm{Y}^{r-2} \bmod p.
\end{aligned}$$

By Lemma 1.6 for $i=1$, we obtain the following system of linear equations for $\sum_j c_j$ and $\sum_j jc_j$ to simultaneously vanish:

$$\sum_j c_j = \mathrm{C}+\alpha\mathrm{A}+\frac{\alpha\beta}{2}\mathrm{B}=0$$

and

$$\sum_j jc_j = (r-2)\mathrm{C}+\alpha(r-1)\mathrm{A}+\frac{\alpha((\beta-2)r+2)\mathrm{B}}{2}=0.$$

where $\alpha = a-r$ and $\beta = a+r-3$. For F not to be in V_r^{***}, we need $\mathrm{C}\neq 0$.

The determinant given by the rightmost two columns is

$$\frac{\alpha^2((\beta-2)r+4)}{2}-\frac{\alpha^2\beta(r-1)}{2}=\frac{\alpha^2(\beta-2r+2)}{2}$$

and thus is nonzero if and only if $\alpha = a-r\not\equiv 0 \bmod p$ and $2r-2-\beta = r-a+1\not\equiv 0 \bmod p$, that is, $r\not\equiv a-1 \bmod p$. Thus, if $r\not\equiv a, a-1 \bmod p$, then we can choose C to be an arbitrary nonzero number so that F is in $\mathrm{X}_{r-2}^{**}\setminus\mathrm{X}_{r-2}^{***}$. □

We recall that the case $r\equiv a \bmod p$ was examined in Lemma 3.1. It remains to examine the case $r\equiv a-1 \bmod p$. We do not show here that $\mathrm{X}_{r-2}^{**}/\mathrm{X}_{r-2}^{***}\cong 0$, equivalently, that both factors from V_r^{**}/V_r^{***} are in the Jordan-Hölder series of Q. However, in Section 5 we show that either both factors are in the kernel of $\mathrm{ind}_{KZ}^{G} Q \twoheadrightarrow \bar{\Theta}_{k,a_p}$ or only one of them appears as the final factor. (In fact, the recent preprint [GR19b, Lemma 4.15] shows that $X_{r-2}^{**}/X_{r-2}^{***} \cong 0$.)

We will now compute $X_{r-2}^{**}/X_{r-2}^{***}$ for the remaining cases $p+1, p+2$ and $p+3$:

Lemma 3.7. *If $r \geq 3p+2$ and $r \equiv p+1 \bmod p-1$ and $r \not\equiv 0, 1 \bmod p$, then*

$$X_{r-2}^{**}/X_{r-2}^{***} \neq 0.$$

Proof: Consider

$$F(X, Y) = X^r + \sum_{k \in \mathbb{F}_p} (kX + Y)^r \in X_r \subseteq X_{r-2}.$$

Working mod p:

$$-F(X, Y) \equiv \sum_{\substack{0<j<r \\ j \equiv 2 \bmod p-1}} \binom{r}{j} X^j Y^{r-j}.$$

Let c_j denote the coefficients of $-F$. By Lemma 1.6 we see that $\sum_j c_j = \sum_j \binom{r}{j} \equiv 0 \bmod p$. We compute

$$\begin{aligned}\sum_j j c_j &= \sum_{0<j\equiv 2<r} j\binom{r}{j} \\ &= r \sum_{0<j'\equiv 1<r'} \binom{r'}{j'} \equiv 0 \bmod p\end{aligned}$$

by Lemma 1.6. Therefore, by Lemma 1.4, we have $F(X, Y) \in V_r^{**}$, but $F(X, Y) \notin V_r^{***}$ because the coefficient c_{r-2} of X^2Y^{r-2} is $\binom{r}{2} \not\equiv 0 \bmod p$ by hypothesis. Thus, $X_{r-2}^{**}/X_{r-2}^{***} \neq 0$. (In fact, we have shown that even $X_r^{**}/X_r^{***} \neq 0$.) □

Lemma 3.8. *Let $r \geq 3p+2$ and $r \equiv p+1 \bmod p-1$ and $r \equiv 0, 1 \bmod p$. Then*

$$X_{r-2}^{**}/X_{r-2}^{***} = 0.$$

Proof: We may assume that $\Sigma(r''), \Sigma(r')$ and $\Sigma(r) \geq p$. By Proposition 2.9 we have the short exact sequence:

$$0 \to V_2 \to X_{r-2} \to V_{3p-1} \to 0.$$

where $V_{3p-1} = (V_{p-1} \otimes D^2) \oplus U$ and U has successive semisimple Jordan-Hölder factors $V_{p-3} \otimes D^2$, $(V_0 \otimes D) \oplus V_2$ and $V_{p-3} \otimes D^2$. If we restrict this exact sequence to the largest singular submodules, then by Proposition 2.6

$$0 \to X^*_{r-2} \to V^*_{3p-1}$$

where $V^*_{3p-1} \cong V_{2p-2} \otimes D$ by [Glo78]. Note that the dimension of V^*_{3p-1} is $2p-1$, which means the possible factors in X^*_{r-2} are $V_{p-3} \otimes D^2$,$V_{p-1} \otimes D$ and $V_0 \otimes D$.

If $r \equiv 1 \mod p$, then by [BG15, Proof of Proposition 5.4]

$$0 \to \phi(X^*_{r'} \otimes V_1) \to X^*_{r-1} \to V_0 \otimes D \to 0$$

where $\phi(X^*_{r'} \otimes V_1) = V_{p-3} \otimes D^2 \oplus V_{p-1} \otimes D \subseteq X^{**}_{r-1} \subset X^{**}_{r-2}$. By Lemma 3.3, we know that $X^*_{r-2}/X^{**}_{r-2} = V_0 \otimes D$. Hence, X^{**}_{r-2} contains exactly the two factors $V_{p-3} \otimes D^2 \oplus V_{p-1} \otimes D$. Now, as $p \mid r' = r-1$, by Lemma 3.1(ii) of [BG15], we can see that $X^*_{r'} = X^{***}_{r'}$ (not just $X^{**}_{r'}$), so the two factors from $\phi(X^*_{r'} \otimes V_1)$ are also contained inside V^{***}_r. Therefore $X^{**}_{r-2} \subseteq V^{***}_r$, that is, $X^{**}_{r-2} = X^{***}_{r-2}$.

If $p|r$, then by [BG15, Proof of Proposition 5.5]

$$0 \to \phi(X^*_{r'} \otimes V_1) \to X^*_{r-1} \to V_0 \otimes D \to 0$$

where $\phi(X^*_{r'} \otimes V_1) = V_{p-3} \otimes D^2 \oplus V_{p-1} \otimes D$. By [BG15, Proof of Proposition 5.5]

$$0 \to \phi(X^*_{r'} \otimes V_1) \cap X^{**}_{r-1} \to X^{**}_{r-1} \to V_0 \otimes D \to 0$$

where the left-hand side is $V_{p-3} \otimes D^2$. By Lemma 2.8.(i), we have $X^*_r = X^{**}_r = X^{***}_r$, hence $V_{p-3} \otimes D^2 = X^*_r \subseteq V^{***}_r$.

By looking at the structure of V^{**}_r/V^{***}_r, we see that $V_0 \otimes D$ cannot be a factor of $X^{**}_{r-2}/X^{***}_{r-2}$, hence $V_0 \otimes D \in X^{***}_{r-2}$. We conclude $X^{**}_{r-2} = X^{***}_{r-2}$. □

Lemma 3.9. *If $r \geq 3p+2$ and $r \equiv p+2 \mod p-1$ and $r \not\equiv 0, 1, 2 \mod p$, then*

$$X^{**}_{r-2}/X^{***}_{r-2} = V^{**}_r/V^{***}_r.$$

Proof: Consider

$$F(X, Y) := A_1 X^{r-2} Y^2 - A_2 \sum_{k \in \mathbb{F}_p} k^{p-2} (kX+Y)^r - \sum_{k \in \mathbb{F}_p} (kX+Y)^{r-1} X \in X_{r-2},$$

where A_1 and A_2 are constants chosen such that:

$$A_1 + 3A_2 \equiv -1 \mod p, A_2 r \equiv -1 \mod p \text{ and } 2A_1 + r - 3 \equiv 0 \mod p.$$

Working $\mod p$, we obtain

$$F(X,Y) \equiv A_1 X^{r-2}Y^2 + A_2 \sum_{\substack{0<j\leq r-1 \\ j\equiv 2 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j + \sum_{\substack{0<j\leq r-1 \\ j\equiv 2 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j.$$

Denote the coefficients of F by c_j. First, we note that c_0, c_1, c_r cannot occur. The coefficient c_{r-1} is given by $A_2 r + 1 \equiv 0$ by the conditions above on A_1 and A_2.

By Lemma 1.6 we see that $\sum_j c_j = A_1 + A_2((3-r)+r)+1 = A_1 + 3A_2 + 1 \equiv 0 \mod p$ and, again by Lemma 1.6 we see that $\sum jc_j = 2A_1 + A_2(r(3-r)+r)+1 \equiv 2A_1 + r - 3 \equiv 0 \mod p$.

Therefore, by Lemma 1.4, we have $F(X,Y) \in V_r^{**}$, but $F(X,Y) \notin V_r^{***}$ because the coefficient c_2 of $X^{r-2}Y^2$ is $\not\equiv 0 \mod p$ because $r \not\equiv 0,1,2 \mod p$ by assumption.

Following an argument similar to Lemma 3.14, we also see that $F(X,Y) \equiv \binom{r-1}{2}\theta^2 X^{r-3p-1}Y^{p-1} \mod V_r^{***}$, which by Lemma 5.1 maps to a non-zero element in $V_1 \otimes D$ as $r \not\equiv 0,1,2 \mod p$. Hence $X_{r-2}^{**}/X_{r-2}^{***} = V_r^{**}/V_r^{***}$ as the short exact sequence of Lemma 1.3.(iii) does not split. □

Lemma 3.10. *If $r \geq 3p+2$ and $r \equiv p+2 \mod p-1$ and $r \equiv 0 \mod p$, then*

$$X_{r-2}^{**}/X_{r-2}^{***} \neq 0.$$

Proof: Consider

$$F(X,Y) := XY^{r-1} + \sum_{k\in\mathbb{F}_p} (kX+Y)^{r-1}.$$

Working $\mod p$, we obtain

$$F(X,Y) \equiv -\sum_{\substack{0<j<r-1, \\ j\equiv 2 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j.$$

Denote the coefficients of $-F$ by c_j. First, we note that c_0, c_{r-1}, c_r do not occur, and $c_1 = r \equiv 0 \mod p$. By Lemma 1.6 for $i=0$ we see that $\sum_j c_j = \sum_j \binom{r-1}{j} \equiv 0 \mod p$ and, again by Lemma 1.6 for $i=0$,

$$\begin{aligned}\sum_j jc_j &= \sum_{0<j\equiv a-1<r-1} j\binom{r-1}{j} \\ &= (r-1)\sum_{0<j'\equiv r-2<r-2} \binom{r-2}{j'} \equiv 0 \mod p.\end{aligned}$$

Therefore, by Lemma 1.4, we have $F(X,Y) \in V_r^{**}$, but $F(X,Y) \notin V_r^{***}$ because the coefficient c_2 of $X^{r-2}Y^2$ is $\binom{r-1}{2} \not\equiv 0 \mod p$ as $r \equiv 0 \mod p$ by assumption. □

Lemma 3.11. *If* $r \geq 3p+2$ *and* $r \equiv p+2 \bmod p-1$ *and* $r \equiv 1, 2 \bmod p$, *then*

$$X^{**}_{r-2}/X^{***}_{r-2} = 0.$$

Proof: By Proposition 2.9 we have the short exact sequence:

$$0 \to \phi(X^{*}_{r''} \otimes V_2) = (V_{2p-1} \otimes D) \oplus (V_{p-4} \otimes D^3) \to X_{r-2} \to (V_1 \otimes D) \oplus V_3 \to 0.$$

When we restrict this exact sequence to the largest singular submodules, we obtain

$$0 \to (V_{2p-1} \otimes D) \oplus (V_{p-4} \otimes D^3) \to X^{*}_{r-2} \to V_1 \otimes D \to 0.$$

For $r \equiv 2 \bmod p$, because $p | r''$, by Lemma 2.8 we have $X^{*}_{r''} = X^{**}_{r''} = X^{***}_{r''}$, so $\phi(X^{*}_{r''} \otimes V_2) \subseteq X^{***}_{r-2} \subseteq X^{**}_{r-2}$. By Lemma 3.5 we know $X^{*}_{r-2}/X^{**}_{r-2} = V_1 \otimes D$, hence the Jordan-Hölder factor $V_1 \otimes D$ on the right-hand side is not in X^{**}_{r-2}. Thus X^{**}_{r-2} and X^{***}_{r-2} both contain $\phi(X^{*}_{r''} \otimes V_2)$ and no other factors, which means they are equal.

For $r \equiv 1 \bmod p$, by Lemma 3.5, we have:

$$0 \to (V_{p-2} \otimes D \oplus V_{p-2} \otimes D) \oplus (V_{p-4} \otimes D^3) \to X^{**}_{r-2} \to V_1 \otimes D \to 0.$$

By Lemma 2.8 we have that $\phi(X_{r'} \otimes V_1) \subset X^{***}_{r-1} \subset X^{***}_{r-2}$. Hence, the factors $V_{p-2} \otimes D$ are contained in X^{***}_{r-2}. By Lemma 2.7, we have $V_{p-4} \otimes D^3 = X^{*}_{r}$. We have shown that for $a \geq 2$, we have $X^{*}_{r} = X^{***}_{r}$, hence $V_{p-4} \otimes D^3 \subset X^{***}_{r-2}$. Finally, we observe from Proposition 2.9 that the factor $V_1 \otimes D$ on the right of the short exact sequence corresponds to the factor $V_{a-2} \otimes D$ (for $a = 3$). By looking at the structure of V^{**}_{r}/V^{***}_{r}, we know that it cannot be a factor of $X^{**}_{r-2}/X^{***}_{r-2}$. Hence $X^{**}_{r-2}/X^{***}_{r-2} = 0$. □

Lemma 3.12. *If* $r \geq 3p+2$ *and* $r \equiv p+3 \bmod p-1$ *and* $r \not\equiv 2, 3 \bmod p$, *then*

$$X^{**}_{r-2}/X^{***}_{r-2} \neq 0.$$

Proof: Consider

$$F(X, Y) = X^2 Y^{r-2} + \sum_{k \in \mathbb{F}_p} (kX + Y)^{r-2} X^2 \in X_{r-2}.$$

Working $\bmod p$:

$$-F(X, Y) \equiv \sum_{\substack{0<j<r-2 \\ j \equiv 2 \bmod p-1}} \binom{r-2}{j} X^j Y^{r-j}.$$

Let c_j denote the coefficients of F. By Lemma 1.6 we see that $\sum c_j \equiv \sum j c_j \equiv 0 \mod p$.

Therefore, by Lemma 1.4, we have $F(X,Y) \in V_r^{**}$, but $F(X,Y) \notin V_r^{***}$ because the coefficient c_{r-2} of X^2Y^{r-2} is $\binom{r-2}{2} \not\equiv 0 \mod p$ by hypothesis. Thus, $X_{r-2}^{**}/X_{r-2}^{***} \neq 0$. ◻

Lemma 3.13. *If $r \geq 3p+2$ and $r \equiv p+3 \mod p-1$ and $r \equiv 2 \mod p$, then*

$$X_{r-2}^{**}/X_{r-2}^{***} \neq 0.$$

Proof: Let

$$F(X,Y) := \sum_{k \in \mathbb{F}_p} k^{p-3}(kX+Y)^r + 3X^2Y^{r-2} + 3X^{r-2}Y^2 \in X_{r-2}.$$

Working $\mod p$:

$$\begin{aligned} F(X,Y) &\equiv - \sum_{\substack{0<j\leq r-2 \\ j\equiv a-2\equiv 2 \mod p-1}} \binom{r}{j} X^{r-j}Y^j + 3X^2Y^{r-2} + 3X^{r-2}Y^2 \\ &\equiv - \sum_{\substack{0<j<r-2 \\ j\equiv 2 \mod p-1}} \binom{r}{j} X^{r-j}Y^j - \binom{r}{r-2} X^2Y^{r-2} + 3X^2Y^{r-2} + 3X^{r-2}Y^2. \end{aligned}$$

As $r \equiv 2 \mod p$, we see that $\binom{r}{r-2} \equiv 1 \mod p$. Thus,

$$F(X,Y) \equiv - \sum_{\substack{0<j<r-2 \\ j\equiv 2 \mod p-1}} \binom{r}{j} X^{r-j}Y^j + 2X^2Y^{r-2} + 3X^{r-2}Y^2.$$

Let c_j denote the coefficients of F.

By Lemma 1.6 for $a = 4$ and $i = 2$, using $r \equiv 2 \mod p$,

$$\sum c_j \equiv -\frac{(4-2)(4+2-1)}{2} + 2 + 3 \equiv 0 \mod p$$

and

$$\begin{aligned}\sum jc_j &\equiv -\sum_{\substack{0<j<r-2\\ j\equiv 2 \bmod p-1}} j\binom{r}{j} + 2(r-2) + 3\cdot 2\\ &\equiv -r\sum_{\substack{0<j'<r-2\\ j'\equiv 1 \bmod p-1}} \binom{r'}{j'} + 2(r-2) + 3\cdot 2\\ &\equiv -\frac{r((a-1)-(r-1))(a-1+r-1-1)}{2} + 0 + 6\\ &\equiv -\frac{2(3-1)(3+1-1)}{2} + 6 \equiv -6+6 \equiv 0 \bmod p.\end{aligned}$$

Therefore, by Lemma 1.4, we have $F(X, Y) \in V_r^{**}$, but $F(X, Y) \notin V_r^{***}$ because the coefficient c_{r-2} of X^2Y^{r-2} is $2 \not\equiv 0 \bmod p$. Thus, $X_{r-2}^{**}/X_{r-2}^{***} \neq 0$. □

Lemma 3.14. *If $r \geq 3p+2$ and $r \equiv p+3 \bmod p-1$ and $r \equiv 4 \bmod p$, then*

$$X_{r-2}^{**}/X_{r-2}^{***} = V_{p-1} \otimes D^2.$$

Proof: We follow [BG15, Lemma 5.5]: We have the short exact sequence

$$0 \to X_{r''}^{*} \otimes V_2 \to X_{r''} \otimes V_2 \to X_{r''}/X_{r''}^{*} \otimes V_2.$$

Let $a = 4$. Because $X_{r''}/X_{r''}^{*} = V_2$ and $X_{r''}^{*} = V_{p-3} \otimes D^2$, its Jordan-Hölder factors are

$$\begin{aligned}0 &\to (V_{p-a+3} \otimes D^{a-2}) \oplus (V_{p-a+1} \otimes D^{a-1}) \oplus (V_{p-a-1} \otimes D^{a})\\ &\to X_{r''} \otimes V_2\\ &\to (V_{a-4} \otimes D^2) \oplus (V_{a-2} \otimes D) \oplus V_a \to 0.\end{aligned}$$

Because $r \equiv 4 \bmod p$, we have $\Sigma(r), \Sigma(r')$ and $\Sigma(r'') \geq p$; thus $\phi\colon X_{r''} \otimes V_2 \xrightarrow{\sim} X_{r-2}$. Therefore

$$\begin{aligned}0 &\to (V_{p-a+3} \otimes D^{a-2}) \oplus (V_{p-a+1} \otimes D^{a-1}) \oplus (V_{p-a-1} \otimes D^{a})\\ &\to X_{r-2}^{*}\\ &\to (V_{a-4} \otimes D^2) \oplus (V_{a-2} \otimes D) \to 0.\end{aligned}$$

By Lemma 3.1, we have $X_{r-2}^{*} = X_{r-2}^{**}$. In particular, $X_{r-2}^{**} \twoheadrightarrow V_0 \otimes D^2$. We will show that $X_{r-2}^{***} \twoheadrightarrow V_0 \otimes D^2$ as well.

Let $F(X, Y) := X^{r-2}Y^2 - Y^{r-2}X^2$ in X_{r-2}^{**}.

Claim: $F(X,Y)$ has nonzero image under the map $X^{**}_{r-2} \twoheadrightarrow V_0 \otimes D^2$.

Proof of our Claim: Because $X^{r''}$ and $Y^{r''}$ in $X_{r''}$ map to X^2 and Y^2 under $X_{r''} \to X_{r''}/X^*_{r''} \xrightarrow{\sim} V_2$, we find that $F(X,Y)$ maps to $X^2 \otimes Y^2 - Y^2 \otimes X^2$ in $V_2 \otimes V_2$.

More exactly,

$$0 \to V_1 \otimes V_1 \otimes D \to V_2 \otimes V_2 \to V_4 \to 0$$

be the short exact sequence whose left-hand map is given by $u \otimes v \mapsto ux \otimes vy - uy \otimes vx$ and whose right-hand map m is given by multiplication. Because $m \colon X^2 \otimes Y^2 - X^2 \otimes Y^2 \mapsto 0$, we find $X^2 \otimes Y^2 - X^2 \otimes Y^2$ to lie in the left-hand side of the above short exact sequence; its preimage under the left-hand map is $X \otimes Y - Y \otimes X$ in $V_1 \otimes V_1$.

Even more exactly, we have again the (split) short exact sequence

$$0 \to V_0 \otimes D \to V_1 \otimes V_1 \to V_2 \to 0,$$

whose left-hand map is $u \otimes v \mapsto ux \otimes vy - uy \otimes vx$ and whose right-hand map m is multiplication. Because $m \colon x \otimes y - y \otimes x \mapsto 0$, we find $x \otimes y - y \otimes x$ to lie in the left-hand side of the above short exact sequence; its preimage under the left-hand map is the base element e of V_0.

Therefore, as claimed F maps onto a nonzero element in $V_0 \otimes D^2$.

Claim: $F(X,Y) \mapsto 0$ under the map $X^{**}_{r-2} \to X^{**}_{r-2}/X^{***}_{r-2} \twoheadrightarrow V_0 \otimes D^2$.

Proof of our Claim: We follow [BG15, Lemma 5.1]: We have the following composition of maps:

$$X^{**}_{r-2}/X^{***}_{r-2} \hookrightarrow V^{**}_r/V^{***}_r \cong V_{r-2p-2}/V^*_{r-2p-2} \otimes D^2 \xrightarrow{\psi^{-1}} V_{2p-2}/V^*_{2p-2} \otimes D^2 \overset{\beta}{\twoheadrightarrow} V_0 \otimes D^2,$$

where the map ψ^{-1} is from [Glo78, (4.2)] and the map β from [Bre03, 5.3(ii)]. By Lemma 1.4, we have $F(X,Y) = X^{r-2}Y^2 - X^2Y^{r-2} \in X^{**}_{r-2}$ as $r \equiv 4 \bmod p$. Thus, $F(X,Y) \mapsto f(X,Y)$ in $V_{r-2p-2}/V^*_{r-2p-2} \otimes D^2$, where

$$f(X,Y) = \sum_{i=0}^{\frac{r-2p-2}{p-1}} (i+1) X^{r-2p-2-i(p-1)} Y^{i(p-1)}.$$

As in [BG15, Lemma 5.1], under $\beta \circ \psi^{-1}$, we have $X^{r-2p-2-i(p+1)}Y^{i(p-1)} \mapsto X^0Y^0 = e$ for $i = 1, \ldots, \frac{r-2p-2}{p-1} - 1$, while the initial and last term of the sum X^{r-2p-2} and

Y^{r-2p-2} vanish. Thus, under this projection, the coefficient of the basis vector e of $V_0 \otimes D^2$ is given by

$$\sum_{i=1,\ldots,\frac{r-2p-2}{p-1}-1} (i+1) = 2 + \ldots + \frac{r-2p-2}{p-1}$$

$$= \left(\frac{r-2p-2}{p-1}\right)\left(\frac{r-2p-2}{p-1}+1\right)/2 - 1 \equiv (-2)(-1)/2 - 1 = 0$$

mod p because $r \equiv 4 \bmod p$. That is, as claimed, $f \mapsto 0$ in $V_0 \otimes D^2$.

We conclude by both claims that $X^{***}_{r-2} \twoheadrightarrow V_0 \otimes D^2$, because the Jordan-Hölder factor $V_0 \otimes D^2$ occurs only once in X^{**}_{r-2}.

Let

$$W := \phi(X^{*}_{r''} \otimes V_2) = (V_{p-a+3} \otimes D^{a-2}) \oplus (V_{p-a+1} \otimes D^{a-1}) \oplus (V_{p-a-1} \otimes D^{a})$$

where the last equality uses that ϕ is injective. Because, by Lemma 2.8, we have $X^{*}_{r''} = X^{**}_{r''}$, it follows

$$X^{**}_{r-2} \supseteq \phi(X^{**}_{r''} \otimes V_2) = W$$

Because, by Lemma 2.8, we have $X^{**}_{r''} \neq X^{***}_{r''}$, and because ϕ is injective,

$$W = \phi(X^{**}_{r''} \otimes V_2) \neq \phi(X^{***}_{r''} \otimes V_2).$$

In particular, $X^{**}_{r-2} \neq X^{***}_{r-2}$. Because $V_0 \otimes D^2$ is a Jordan-Hölder factor of X^{***}_{r-2}, by Lemma 1.3

$$V^{**}_r / V^{***}_r \supset X^{**}_{r-2}/X^{***}_{r-2} = V_{p-1} \otimes D^2. \qquad \square$$

4 The Jordan-Hölder series of Q

Let $r \geq 3p+2$. To study the Jordan-Hölder series of $Q := V_r/(V_r^{***} + X_{r-2})$, we consider the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{X_{r-2}^{*}}{X_{r-2}^{***}} & \longrightarrow & \dfrac{X_{r-2}}{X_{r-2}^{***}} & \longrightarrow & \dfrac{X_{r-2}}{X_{r-2}^{*}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{V_r^{*}}{V_r^{***}} & \longrightarrow & \dfrac{V_r}{V_r^{***}} & \longrightarrow & \dfrac{V_r}{V_r^{*}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{V_r^{*}}{X_{r-2}^{*} + V_r^{***}} & \longrightarrow & Q & \longrightarrow & \dfrac{V_r}{X_{r-2} + V_r^{*}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{4.1}
$$

By Proposition 2.6 and Lemma 1.3 the two factors of V_r/V_r^* and (one or two) factors X_{r-2}/X_{r-2}^* are known, so we can determine the factor on the right-hand side of the bottom line:

$$
U := \frac{V_r}{X_{r-2} + V_r^*} = \begin{cases} 0, & \text{for } a = 1, 2, \\ V_{p-a-1} \otimes D^a, & \text{for } a = 3, \ldots, p-1, \end{cases} \tag{4.2}
$$

where a in $\{1, \ldots, p-1\}$ such that $r \equiv a \bmod p-1$. Therefore, we are left with determining the factor of the left-hand side of the bottom line,

$$
W := \frac{V_r^*}{X_{r-2}^* + V_r^{***}}.
$$

By Lemma 1.3 the four factors of V_r^*/V_r^{***} are known, so by looking at the short exact sequence of the left column of Diagram (4.1), we are reduced to determining the Jordan-Hölder factors of

$$
X_{r-2}^*/X_{r-2}^{***},
$$

that is, of

$$
X_{r-2}^*/X_{r-2}^{**} \quad \text{and} \quad X_{r-2}^{**}/X_{r-2}^{***},
$$

where we computed in Section 3 whether the quotient X^*_{r-2}/X^{**}_{r-2} respectively $X^{**}_{r-2}/X^{***}_{r-2}$ is nonzero or not.

By Section 2, we have the exact sequence:

$$0 \to \phi(X^*_{r''} \otimes V_2) \to X_{r-2} \to X_{r-2}/\phi(X^*_{r''} \otimes V_2) \to 0. \tag{4.3}$$

Let a in $\{3, \ldots, p+1\}$ such that $r \equiv a \bmod p-1$. By Lemma 2.8,

- for $a = 3$ and $p \mid r-2$,
- for $a = 4$ and $r-2 \equiv 0, 1 \bmod p$, and
- for $a = 5, \ldots, p+1$,

we have $X^*_{r''} = X^{**}_{r''} = X^{***}_{r''}$, so $\phi(X^*_{r''} \otimes V_2) \subseteq X^{***}_{r-2}$. Thus, the Jordan-Hölder series of X^*_{r-2}/X^{***}_{r-2} is included in the largest non-singular submodule of the right-hand side $X_{r-2}/\phi(X^*_{r''} \otimes V_2)$ of (4.3).

The following observation is informed by [BG15, Sections 3,5,6]: These show that, given the congruence class of $r \bmod p-1$, the Jordan-Hölder series of Q only indirectly depends on that of X_{r-1} which is determined by the minimality of $\Sigma(r')$ and $\Sigma(r)$; instead, given the congruence class of $r \bmod p-1$, it is directly determined by the congruence class of $r \bmod p$:

Observation 4.1. The Jordan-Hölder factors of X^*_{r-2}/X^{***}_{r-2}, thus of Q, are reduced from the conditions on $\Sigma(r'')$, $\Sigma(r')$ or $\Sigma(r) < p$ to those on r'', r' or $r \bmod p$ given in Lemma 2.8:

If the conditions given in Lemma 2.8 on $r'' \bmod p$ are satisfied, then $X^*_{r''} = X^{***}_{r''}$, thus $\phi(X^*_{r''} \otimes V_2) \subseteq X^{***}_{r-2}$. Therefore only if the conditions on r'' given in Lemma 2.8 are not satisfied, then X^*_{r-2}/X^{***}_{r-2} depends on whether $X^*_{r''} \neq 0$, equivalently (by Lemma 2.7), whether $\Sigma(r'') \geq p$. If they are not satisfied, then in particular $X^*_{r''} \neq 0$, equivalently, $\Sigma(r'') \geq p$.

Similarly, only if the conditions on r' respectively $r \bmod p$ given in Lemma 2.8 are not satisfied, then X^*_{r-1}/X^{***}_{r-1} respectively X^*_r/X^{***}_r depends on whether $X^*_{r'} \neq 0$ respectively $X^*_r \neq 0$, equivalently, whether $\Sigma(r') \geq p$ respectively $\Sigma(r) \geq p$. If they are not satisfied, then in particular $X^*_{r'} \neq 0$ respectively $X^*_r \neq 0$, equivalently, $\Sigma(r') \geq p$ respectively $\Sigma(r) \geq p$.

By Proposition 2.6, the Jordan-Hölder factor V_a (and $V_{p-a-1} \otimes D^a$ for $a = p, p+1$) in (4.3) vanishes when we reduce X_{r-2} in (4.3) to its largest singular subspace X^*_{r-2}. Thus, by Proposition 2.9 there is a single Jordan-Hölder factor for $a = 3$, two Jordan-Hölder factors in $X^*_{r-2}/\phi(X^*_{r''} \otimes V_2)$ for $a = 4, \ldots, p$, but three for $a = p+1$. In particular, under the conditions of Lemma 2.8,

- if $a = 3$ and one of the quotients X^*_{r-2}/X^{**}_{r-2} and $X^{**}_{r-2}/X^{***}_{r-2}$ is nonzero,
- or $a = 4, \dots, p$ and both quotients X^*_{r-2}/X^{**}_{r-2} and $X^{**}_{r-2}/X^{***}_{r-2}$ are non-zero,

then we know all Jordan-Hölder factors of X^*_{r-2}/X^{***}_{r-2}. The remaining cases when, the conditions of Lemma 2.8 are not satisfied, that is,

- $a = 3$ and $r \not\equiv 2$, or
- $a = 4$ and $r \not\equiv 2, 3 \mod p$

or there are more than two Jordan-Hölder factors in $X^*_{r-2}/\phi(X^*_{r''} \otimes V_2)$,

- $a = p + 1$

were handled separately in Section 3.

4.1 $a = 3$

Proposition 4.2. *If $r \geq 3p + 2$ and $r \equiv 3 \mod p - 1$ then the structure of Q is:*

$$0 \to W \to Q \to U \to 0$$

where $U = V_{p-4} \otimes D^3$ and W has Jordan-Hölder factors given by:

(i) *None, if $r \not\equiv 0, 1, 2 \mod p$.*

(ii) $V_1 \otimes D$*, if $r \equiv 0 \mod p$.*

(iii) $V_{p-2} \otimes D^2, V_{p-2} \otimes D^2$ *and* $V_1 \otimes D$ *if $r \equiv 1, 2 \mod p$.*

Proof: By (4.2), we have $U = V_{p-4} \otimes D^3$. We now use the results of the previous section.

(i) By Lemma 3.5 and by Lemma 3.9 none of the factors in W appear as $X^*_{r-2}/X^{***}_{r-2} = V^*_r/V^{***}_r$.

(ii) By Lemma 3.5 we see that $X^*_{r-2}/X^{**}_{r-2} = V^*_r/V^{**}_r$ while by Lemma 3.10 we have that $X^{**}_{r-2}/X^{***}_{r-2} = V_{p-2} \otimes D^2$, hence the only factor that appears in W is $V_1 \otimes D$.

(iii) If $r \equiv 1, 2 \mod p$, then by Lemma 3.5, we know that $X^*_{r-2}/X^{**}_{r-2} = V_1 \otimes D$ while by Lemma 3.11 we know that $X^{**}_{r-2}/X^{***}_{r-2} = 0$ hence both factors of V^{**}_r/V^{***}_r appear in W. □

4.2 $a = 4$

Proposition 4.3. *If $r \geq 3p+2$ and $r \equiv 4 \bmod p-1$, then the structure of* Q *is:*

$$0 \to \mathrm{W} \to \mathrm{Q} \to \mathrm{U} \to 0$$

where $\mathrm{U} = \mathrm{V}_{p-5} \otimes \mathrm{D}^4$ *and:*

(i) *If $r \equiv 2 \bmod p$ then* W *has Jordan-Hölder factors* $\mathrm{V}_{p-3} \otimes \mathrm{D}^3$ *and* $\mathrm{V}_{p-1} \otimes \mathrm{D}^2$.

(ii) *If $r \equiv 3 \bmod p$ then* W *has Jordan-Hölder factors* $\mathrm{V}_{p-3} \otimes \mathrm{D}^3$ *and at least one of* $\mathrm{V}_0 \otimes \mathrm{D}^2$ *and* $\mathrm{V}_{p-1} \otimes \mathrm{D}^2$.

(iii) *If $r \equiv 4 \bmod p$ then* W *has Jordan-Hölder factors* $\mathrm{V}_{p-3} \otimes \mathrm{D}^3$, $\mathrm{V}_2 \otimes \mathrm{D}$ *and* $\mathrm{V}_0 \otimes \mathrm{D}^2$.

(iv) *If $r \not\equiv 2,3,4 \bmod p$ then* W *has Jordan-Hölder factors* $\mathrm{V}_{p-3} \otimes \mathrm{D}^3$ *and at most one of* $\mathrm{V}_0 \otimes \mathrm{D}^2$ *and* $\mathrm{V}_{p-1} \otimes \mathrm{D}^2$.

Proof: By (4.2), we have $\mathrm{U} = \mathrm{V}_{p-5} \otimes \mathrm{D}^4$.

By Lemma 2.8, we always have $\phi(\mathrm{X}^*_{r''} \otimes \mathrm{V}_2) \subseteq \mathrm{X}^{**}_{r-2}$ and if and only if $r \equiv 2,3 \bmod p$ we have $\phi(\mathrm{X}^*_{r''} \otimes \mathrm{V}_2) \subseteq \mathrm{X}^{***}_{r-2}$. Therefore if $r \equiv 2,3 \bmod p$, then, by looking at (4.3), we find that $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2}$ has at most one Jordan-Hölder factor, that is, $\mathrm{V}_0 \otimes \mathrm{D}^2$ while $\mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2}$ has at most one Jordan-Hölder factor, that is, $\mathrm{V}_2 \otimes \mathrm{D}^3$.

(i) If $r \equiv 2 \bmod p$, then by Lemma 3.13 we have $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2} \neq 0$ and by Lemma 3.2 we know $\mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2} \neq 0$. As $\phi(\mathrm{X}_{r''} \otimes \mathrm{V}_2) \subseteq \mathrm{X}^{**}_{r-2}$ we see that $\mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2} = \mathrm{V}_2 \otimes \mathrm{D}$ and $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2} = \mathrm{V}_0 \otimes \mathrm{D}^2$. Hence, W has Jordan-Hölder factors $\mathrm{V}_{p-3} \otimes \mathrm{D}^3$ and $\mathrm{V}_{p-1} \otimes \mathrm{D}^2$.

(ii) If $r \equiv 3 \bmod p$, then by Lemma 3.2 we know $\mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2} \neq 0$, so $\mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2} = \mathrm{V}_2 \otimes \mathrm{D}$. Since $\phi(\mathrm{X}_{r''} \otimes \mathrm{V}_2) \subseteq \mathrm{X}^{***}_{r-2}$, so $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2}$ is included in $\mathrm{V}_0 \otimes \mathrm{D}$. (In fact, the recent preprint [GR19b, Lemma 4.20] shows $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2} = 0$.) We cannot determine whether $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2} = 0$ is zero or not, but in Section 5 we can eliminate both factors coming from $\mathrm{V}^{**}_r/\mathrm{V}^{***}_r$. Hence, we may assume that W contains both Jordan-Hölder factors from $\mathrm{V}^{**}_r/\mathrm{V}^{***}_r$.

(iii) If $r \equiv 4 \bmod p$, then by Lemma 3.1 we know $\mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2} = 0$ and by Lemma 3.14 we have $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2} = \mathrm{V}_{p-1} \otimes \mathrm{D}^2$. Hence, W has Jordan-Hölder factors $\mathrm{V}_{p-3} \otimes \mathrm{D}^3, \mathrm{V}_2 \otimes \mathrm{D}$ and $\mathrm{V}_0 \otimes \mathrm{D}$.

(iv) If $r \not\equiv 2,3,4 \bmod p$, then by Lemma 3.2 we know $0 \neq \mathrm{X}^{*}_{r-2}/\mathrm{X}^{**}_{r-2} = \mathrm{V}_2 \otimes \mathrm{D}$. By Lemma 3.12 we see that $\mathrm{X}^{**}_{r-2}/\mathrm{X}^{***}_{r-2} \neq 0$. We cannot determine whether

$X^{**}_{r-2}/X^{***}_{r-2}$ contains only one or both factors from V^{**}_r/V^{***}_r, but in Section 5 we can eliminate all factors except the one coming from V^{**}_r/V^{***}_r. Both give the same induced representations so we may assume that W contains one of the Jordan-Hölder factors from V^{**}_r/V^{***}_r. □

4.3 $a = p$

Proposition 4.4. *If* $r \geq 3p+2$ *and* $r \equiv p \bmod p-1$ *then the structure of* Q *is:*

$$0 \to W \to Q \to U \to 0$$

where $U = 0$ *and:*

(i) *If* $r \equiv p \bmod p$ *then* $W = V^{*}_r/V^{***}_r$.

(ii) *If* $r \not\equiv p, p-1 \bmod p$ *then the Jordan-Hölder factors of* W *are* V_1 *and* $V_3 \otimes D^{p-2}$.

(iii) *If* $r \equiv p-1 \bmod p$ *the Jordan-Hölder factors of* W *are* V_1 *and possibly* $V_{p-4} \otimes D^2$ *and* $V_3 \otimes D^{p-2}$.

Proof: By (4.2), we have $U = 0$.

(i) When $r \equiv p \bmod p$, then by Lemma 3.1 we have $X^{*}_{r-2}/X^{***}_{r-2} = 0$, hence $W = V^{*}_r/V^{***}_r$.

(ii) If $r \not\equiv p, p-1 \bmod p$, then by Lemma 3.2 and Lemma 3.6 we have $X^{*}_{r-2}/X^{**}_{r-2} \neq 0$ and $X^{**}_{r-2}/X^{***}_{r-2} \neq 0$. By Lemma 2.8, we have $X^{*}_{r''} = X^{***}_{r''}$, thus $\phi(X^{*}_{r''} \otimes V_2) \subseteq X^{***}_{r-2}$. By comparing with (the non-singular part of) the right-hand side of the short exact sequence of Proposition 2.9, therefore W contains one Jordan-Hölder factor of V^{*}_r/V^{**}_r and one of V^{**}_r/V^{***}_r.

(iii) If $r \equiv p-1 \bmod p$, then by Lemma 3.2 we have $0 \neq X^{*}_{r-2}/X^{**}_{r-2}$. By Lemma 2.8, we have in particular $X^{*}_{r''} = X^{**}_{r''}$, thus $\phi(X^{*}_{r''} \otimes V_2) \subseteq X^{**}_{r-2}$. By comparing with (the non-singular part of) the right-hand side of the short exact sequence of Proposition 2.9, therefore $X^{*}_{r-2}/X^{**}_{r-2} = V_{p-2} \otimes D$.

Therefore W contains only one Jordan-Hölder factor of V^{*}_r/V^{**}_r and possibly both of V^{**}_r/V^{***}_r. □

4.4 $a = p + 1$

Proposition 4.5. *If $r \geq 3p + 2$ and $r \equiv p + 1 \mod p - 1$ then the structure of Q is:*

$$0 \to W \to Q \to U \to 0$$

where $U = 0$ and:

(i) *If $r \not\equiv 0, 1 \mod p$ then W has only one Jordan-Hölder factor V_2.*

(ii) *If $r \equiv 1 \mod p$ then W has Jordan-Hölder factors $V_{p-3} \otimes D^2, V_2$ and $V_{p-1} \otimes D$.*

(iii) *If $r \equiv 0 \mod p$, then W has Jordan-Hölder factors $V_{p-3} \otimes D^2, V_2$ and $V_0 \otimes D$.*

Proof: By (4.2), we have $U = 0$.

(i) If $r \not\equiv 0, 1 \mod p$, then by Lemma 3.3 we have $X^*_{r-2}/X^{**}_{r-2} = V^*_r/V^{**}_r$ while by Lemma 3.7 $X^{**}_{r-2}/X^{***}_{r-2} = V_{p-3} \otimes D^2$. Hence the structure for Q follows.

(ii) If $r \equiv 1 \mod p$ then by Lemma 3.8 we have $X^{**}_{r-2}/X^{***}_{r-2} = 0$ and by Lemma 3.3 we have $X^*_{r-2}/X^{**}_{r-2} = V_0 \otimes D$. Thus, we conclude the Jordan-Hölder series of Q.

(iii) If $r \equiv 0 \mod p$, then by Lemma 3.8 we have $X^{**}_{r-2}/X^{***}_{r-2} = 0$ and by Lemma 3.3 we have $X^*_{r-2}/X^{**}_{r-2} = V_{p-1} \otimes D$. Thus, we conclude the Jordan-Hölder series of Q. □

4.5 r has the same representative mod $p - 1$ and p

Proposition 4.6. *Let a in $\{5, \ldots, p - 1\}$ such that $r \equiv a \mod p - 1$. If $r \equiv a \mod p$, then the structure of Q is:*

$$0 \to W \to Q \to U \to 0$$

where $W = V^*_r/V^{***}_r$ and $U = V_{p-a-1} \otimes D^a$.

Proof: By (4.2), we have $U = V_{p-a-1} \otimes D^a$. By Lemma 3.1, we know $X^*_{r-2}/X^{***}_{r-2} = 0$. Hence, $W = V^*_r/V^{***}_r$. □

4.6 r does *not* have the same representative mod $p-1$ and p

Proposition 4.7. *Let a in $\{5,\ldots,p-1\}$ be such that $r \equiv a \bmod p-1$. If $r \geq 3p+2$ and $r \not\equiv a, a-1 \bmod p$, then the structure of Q is:*

$$0 \to W \to Q \to U \to 0$$

where W has the two Jordan-Hölder factors $V_{p-a+1} \otimes D^{a-1}$ and $V_{p-a+3} \otimes D^{a-2}$ and $U = V_{p-a-1} \otimes D^{a}$.

Proof: By (4.2), we have $U = V_{p-a-1} \otimes D^{a}$.

To compute the left-hand side W, we compare X^{*}_{r-2}/X^{**}_{r-2} and $X^{**}_{r-2}/X^{***}_{r-2}$ with the Jordan-Hölder series of V^{*}_{r}/V^{**}_{r} and V^{**}_{r}/V^{***}_{r} in Lemma 1.3: By Lemma 3.2 and Lemma 3.6 we have $X^{*}_{r-2}/X^{**}_{r-2} \neq 0$ and $X^{**}_{r-2}/X^{***}_{r-2} \neq 0$. By Lemma 2.8, we have $X^{*}_{r''} = X^{***}_{r''}$, thus $\phi(X^{*}_{r''} \otimes V_2) \subseteq X^{***}_{r-2} \subseteq X^{**}_{r-2}$. By comparing with (the non-singular part of) the right-hand side of Proposition 2.9, we find that W contains exactly one Jordan-Hölder factor each of V^{*}_{r}/V^{**}_{r} and of V^{**}_{r}/V^{***}_{r}. □

Proposition 4.8. *Let a in $\{5,\ldots,p-1\}$ such that $r \equiv a \bmod p-1$. If $r \geq 3p+2$ and $r \equiv a-1 \bmod p$, then the Γ-module structure of Q is given by:*

$$0 \to W \to Q \to U \to 0$$

where the Jordan-Hölder factors of W are $V_{p-a+1} \otimes D^{a-1}$ and possibly $V_{a-4} \otimes D^{2}$ and $V_{p-a+3} \otimes D^{a-2}$, and $U = V_{p-a-1} \otimes D^{a}$.

Proof: By (4.2), we have $U = V_{p-a-1} \otimes D^{a}$.

By Lemma 3.2 we have $X^{*}_{r-2}/X^{**}_{r-2} \neq 0$. By Lemma 2.8, we have $X^{*}_{r''} = X^{***}_{r''}$, thus $\phi(X^{*}_{r''} \otimes V_2) \subseteq X^{***}_{r-2} \subseteq X^{**}_{r-2}$. By comparing with (the non-singular part of) the right-hand side of Proposition 2.9, we find that $X^{*}_{r-2}/X^{**}_{r-2} = V_{a-2} \otimes D$. Therefore W contains only one Jordan-Hölder factor of V^{*}_{r}/V^{**}_{r} and possibly both of V^{**}_{r}/V^{***}_{r}. □

5 Eliminating Jordan-Hölder factors

Throughout this section we assume that $p \geq 5$. We refer the reader to [BG15] and [Bre03] for details but summarize the formulae needed.

For $m = 0$ we set $I_0 = \{0\}$ and for $m > 0$ we let $I_m = \{[\lambda_0] + [\lambda_1]p + \ldots. + [\lambda_{m-1}]p^{m-1} : \lambda_i \in \mathbb{F}_p\}$, where $[\cdot]$ denotes the Teichmüller representative. For $m \geq 1$, there is a truncation map $[\cdot]_{m-1} : I_m \to I_{m-1}$ given by taking the first $m-1$ terms in the p-adic expansion above. For $m = 1$, the truncation map is the 0-map. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. For $m \geq 0$ and $\lambda \in I_m$, let

$$g^0_{m,\lambda} = \begin{pmatrix} p^m & \lambda \\ 0 & 1 \end{pmatrix} \text{ and } g^1_{m,\lambda} = \begin{pmatrix} 1 & 0 \\ p\lambda & p^{m+1} \end{pmatrix},$$

where $g^0_{0,0} = \mathrm{id}$ and $g^1_{0,0} = \alpha$. We have the decomposition $G = \amalg_{i=0,1} KZ(g^i_{m,\lambda})^{-1}$.

An element in $\mathrm{ind}^G_{KZ} V$ is a finite sum of functions of the form $[g, v]$ where $g = g^0_{m,\lambda}$ or $g^1_{m,\lambda}$ for some $\lambda \in I_m$ and $v = \sum_{i=0}^r c_i X^{r-i} Y^i \in V = \mathrm{Sym}^r R^2 \otimes D^s$.

The Hecke operator T that acts on $\mathrm{ind}^G_{KZ} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$ can be written as $T = T^+ + T^-$, where:

$$T^+([g^0_{n,\mu}, v]) = \sum_{\lambda \in I_1} \left[g^0_{n+1,\mu+p^n\lambda}, \sum_{j=0}^r \left(p^j \Big(\sum_{i=j}^r c_i \binom{i}{j} \Big) (-\lambda)^{i-j} \right) X^{r-j} Y^j \right]$$

and

$$T^-([g^0_{n,\mu}, v]) = \left[g^0_{n-1,[\mu]_{n-1}}, \sum_{j=0}^r \left(\sum_{i=j}^r p^{r-i} c_i \binom{i}{j} \left(\frac{\mu - [\mu]_{n-1}}{p^{n-1}} \right)^{i-j} \right) X^{r-j} Y^j \right], (n > 0)$$

and

$$T^-([g^0_{n,\mu}, v]) = [\alpha, \sum_{j=0}^r p^{r-j} c_j X^{r-j} Y^j], (n = 0).$$

We will use these explicit formulae for T to eliminate all but one of the Jordan-Hölder factors from Section 4 to be able to apply [BG09, Proposition 3.3].

To explain the calculations using the T^+ and T^- operators, we use the following heuristic:

- For T^+, we note that the terms with p^j appear depending on the valuation of c_i. For example if $c_i = \frac{1}{pa_p}$, then $v(c_i) < -4$, so we need to consider only the first 4 values of j, while the terms for $j \geq 4$ vanish as p^j kills c_i.
- For T^- we typically consider the highest index i for which $c_i \not\equiv 0$ as p^{r-i} usually kills the other c_i terms. For example, if $c_{r-1} \not\equiv 0$, then the terms in T^-, which we consider are $pc_{r-1}\binom{r-1}{j}(-\lambda)^{r-1-j}$.

Lemma 5.1. *Let $5 \leq a \leq p+3$. We have the short exact sequence of Γ-modules:*

$$0 \to J_0 := V_{a-4} \otimes D^2 \to V_r^{**}/V_r^{***} \to J_1 := V_{p-a+3} \otimes D^{a-2} \to 0,$$

where

- *The monomials* $X^{a-4}, Y^{a-4} \in J_0$ *map to* $\theta^2 X^{r-2p-2}$, $\theta^2 Y^{r-2p-2}$, *respectively, in* V_r^{**}/V_r^{***}.
- *The polynomials* $\theta^2 X^{r-2p-2}$, $\theta^2 Y^{r-2p-2}$ *map to* $0 \in J_1$ *and* $\theta^2 X^{r-2p-a+2} Y^{a-4}$, $\theta^2 X^{r-3p-1} Y^{p-1}$ *map to* X^{p-a+3}, Y^{p-a+3}, *respectively in* J_1.

Proof: Following [BG15, Lemma 8.5], we have the following exact sequence:

$$0 \to V_{a-4} \otimes D^2 \to V_r^{**}/V_r^{***} \to V_{r-2p-2}/V^*_{r-2p-2} \otimes D^2$$
$$\xrightarrow{\psi^{-1}} V_{p+a-5}/V^*_{p+a-5} \xrightarrow{\beta} V_{p-a+3} \otimes D^{a-2} \to 0.$$

where the map ψ^{-1} is from [Glo78, (4.2)] and β from [Bre03, Lemma 5.3]. Under these maps $\psi^{-1} : X^{r-2p-a+2} Y^{a-4} \mapsto X^{p-1} Y^{a-4}$ and $\beta : X^{p-1} Y^{a-4} \mapsto X^{p-a+3}$. Similarly $\psi^{-1} : X^{r-3p-1} Y^{p-1} \mapsto X^{a-4} Y^{p-1}$ and $\beta : X^{a-4} Y^{p-1} \mapsto Y^{p-a+3}$. □

5.1 r has the same representative mod $p-1$ and p

Proposition 5.2. *Let* $a = 6, \ldots, p-1$. *If* $r \equiv a \bmod p-1$ *and* $r \equiv a \bmod p^2$, *then:*

$$\mathrm{ind}_{KZ}^{G}(V_{p-a-1} \otimes D^a) \twoheadrightarrow \overline{\Theta}_{k,a_p}.$$

Proof: By Proposition 4.6, we have the following structure for Q:

$$0 \to V_r^{*}/V_r^{***} \to Q \to V_{p-a-1} \otimes D^a \to 0.$$

To eliminate the factors coming from V_r^{**}/V_r^{***} we consider $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g^0_{1,[\lambda]}, \frac{p^2}{a_p} [\lambda]^{p-3} (Y^r - X^{r-a} Y^a) \right] + \left[g^0_{1,0}, \frac{\binom{r}{2}(1-p)}{a_p} (X^2 Y^{r-2} - X^{r-a+2} Y^{a-2}) \right],$$

and

$$f_0 = \left[\mathrm{id}, \frac{p^2(p-1)}{a_p^2} \sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} \gamma_j X^{r-j} Y^j \right],$$

where the γ_j are integers as in Lemma 1.10.

In f_1, for the first part we observe that $v(p^2/a_p) < -1$, so we consider only the term with $j = 0$ for the first part of $T^+ f_1$. For $j = 0$, we observe $\binom{r}{0} - \binom{a}{0} = 0$. Regarding the second part, we note that $v(1/a_p) < -3$, so we consider the terms with $j = 0, 1, 2$ for the second part of $T^+ f_1$. For $j = 0$, we see that $\binom{r-2}{0} - \binom{a-2}{0} = 0$.

For $j = 1, 2$ we obtain $\frac{p^j}{a_p}(\binom{r-2}{j} - \binom{a-2}{j}) \equiv 0 \bmod p$ as $r \equiv a \bmod p^2$. Thus $T^+ f_1 \equiv 0 \bmod p$.

In f_0 we see that $v(p^2/a_p^2) < -4$. Due to the properties of γ_j from Lemma 1.10, we have $\sum \binom{j}{n}\gamma_j \equiv 0 \bmod p^{4-n}$ and $j \equiv a - 2 \geq 4$, so the terms in $T^+ f_0$ vanish mod p. In f_0 the highest index i for which $c_i \not\equiv 0 \bmod p$ is $i = r - p - 1$. So we have $p^{r-i} = p^{p+1}$, which kills p^2/a_p^2 as $p \geq 5$. Thus $T^- f_0 \equiv 0 \bmod p$.

For $T^- f_1$, we note that the highest terms for which $c_i \not\equiv 0$ are $i = r$ and $i = r-2$. In the case $i = r - 2$ we note that it forces $j = r - 2$ (as $\lambda = 0$), so the non-zero term is $\dfrac{p^2(1-p)}{a_p}\binom{r}{2}X^2Y^{r-2}$. If $i = r$, then

$$T^- f_1 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p} \sum_{\substack{0<j\leq r-2 \\ j\equiv a-2 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right]$$

The last term in the above expansion (when $j = r - 2$) is $\dfrac{p^2\binom{r}{2}(p-1)}{a_p}X^2Y^{r-2}$, which is cancelled out by the term for $i = r - 2$. Thus:

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p} \sum_{\substack{0<j<r-2 \\ j\equiv a-2 \bmod p-1}} \left(\binom{r}{j} - \gamma_j\right) X^{r-j}Y^j\right]$$

where the $\gamma_j \equiv \binom{r}{j} \bmod p$ due to Lemma 1.10, so $T^- f_1 - a_p f_0 \equiv 0 \bmod p$.

So $(T - a_p)f = -a_p f_1 \bmod p$ and as $r \equiv a \bmod p$ we have

$$\begin{aligned}
&(T - a_p)f \\
\equiv\ & -\left[g_{1,0}^0, \binom{r}{2}(1-p)(X^2Y^{r-2} - X^{r-a+2}Y^{a-2})\right] \\
\equiv\ & -\left[g_{1,0}^0, \binom{a}{2}\theta^2\left(\sum_{i=0}^{\frac{r-2p-a+2}{p-1}} (i+1)\binom{r-2p-a+2}{p-1} X^{r-2p-a+2}Y^{a-4} + Y^{r-2p-2}\right)\right] \bmod V_r^{***} \\
\equiv\ & \left[g_{1,0}^0, -2\binom{a}{2}\theta^2(X^{r-2p-a+2}Y^{a-4} - Y^{r-2p-2})\right] \bmod p.
\end{aligned}$$

Let v be the image of $-2\binom{a}{2}\theta^2(X^{r-2p-a+2}Y^{a-4} - Y^{r-2p-2})$ in V_r^{**}/V_r^{***}. By Lemma 5.1 the reduction $\overline{(T - a_p)f}$ maps to $[g_{1,0}^0, -2\binom{a}{2}X^{p-a+3}] \neq 0$. Because the short exact sequence for the structure of V_r^{**}/V_r^{***} is non-split, the element $[g_{1,0}^0, -2\binom{a}{2}X^{p-a+3}]$ generates $\mathrm{ind}_{KZ}^{G}(V_r^{**}/V_r^{***})$ over G.

To eliminate the factors coming from V_r^*/V_r^{**} we consider $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g^0_{1,[\lambda]}, \frac{p}{a_p}[\lambda]^{p-2}(Y^r - X^{r-a}Y^a) \right] + \left[g^0_{1,0}, \frac{r(1-p)}{a_p}(XY^{r-1} - X^{r-a+1}Y^{a-1}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{p(p-1)}{a_p^2} \sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} \beta_j X^{r-j}Y^j \right],$$

where the β_j are the integers from Lemma 1.9, but due to the condition $r \equiv a \mod p^2$ we have $\beta_j \equiv \binom{r}{j} \mod p^2$.

In f_1 for the first part we have $v(p/a_p) < -2$, so we consider the terms with $j = 0, 1$ for the first part of $T^+ f_1$. For $j = 0$, we see that $\binom{r}{0} - \binom{a}{0} = 0$ while for $j = 1$, we see that $\frac{p}{a_p}(\binom{r}{1} - \binom{a}{1}) \equiv 0 \mod p$ as $r \equiv a \mod p^2$. Regarding the second part, we note that $v(1/a_p) < -3$, so we consider the terms in $T^+ f_1$ for $j = 0, 1, 2$. For $j = 0$ we see that $\binom{r}{0} - \binom{a}{0} = 0$ while for $j = 1, 2$, we see that $\frac{p^j}{a_p}(\binom{r}{j} - \binom{a}{j}) \equiv 0 \mod p^2$ as $r \equiv a \mod p^2$. Thus $T^+ f_1 \equiv 0 \mod p$.

In f_0 we see that $v(p/a_p^2) < -5$. Due to the properties of β_j, we have $\sum \binom{j}{n} \beta_j \equiv 0 \mod p^{5-n}$ (as $r \equiv a \mod p^2$) and $j \equiv a - 1 \geq 5$, so the terms in $T^+ f_0$ vanish mod p. In f_0 the highest index i for which $c_i \not\equiv 0 \mod p$ is $i = r - p$. Thus, $p^{r-i} = p^p$ but $p \geq 5$, so $T^- f_0 \equiv 0 \mod p$.

For $T^- f_1$, we note that the highest terms for which $c_i \not\equiv 0$ are $i = r$ and $i = r-1$. In case $i = r - 1$, we note that it forces $j = r - 1$ (as $\lambda = 0$), so the nonzero term is $\frac{pr(1-p)}{a_p} XY^{r-1}$. If $i = r$, then

$$T^- f_1 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0<j\leq r-1 \\ j \equiv a-1 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j \right].$$

The last term in the above expansion ($j = r - 1$) is $\dfrac{p\binom{r}{1}(p-1)}{a_p} XY^{r-1}$, which is cancelled out by the term for $i = r - 1$. Thus:

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} \left(\binom{r}{j} - \beta_j \right) X^{r-j}Y^j \right]$$

where the $\beta_j \equiv \binom{r}{j} \bmod p^2$, so $T^- f_1 - a_p f_0 \equiv 0 \bmod p$. Thus $(T - a_p)f = -a_p f_1 \bmod p$, and

$$(T - a_p)f \equiv -\left[g^0_{1,0}, r(1-p)(XY^{r-1} - X^{r-a+1}Y^{a-1})\right]$$

The rest follows as in the proof of [BG15, Lemma 8.6], so we can eliminate the factors from V^*_r/V^{**}_r.

Thus, the only remaining factor is $V_{p-a-1} \otimes D^a$. □

Proposition 5.3. *If $r \equiv 5 \bmod p-1$ and $r \equiv 5 \bmod p^2$, and, when $v(a_p) = \frac{5}{2}$, assume that $v(a_p^2 - p^5) = 5$. Then*

$$\mathrm{ind}^G_{KZ}(V_{p-6} \otimes D^5) \twoheadrightarrow \overline{\Theta}_{k,a_p}$$

Proof: The structure of Q is the same as in Proposition 5.2. We will eliminate the factors from V^*_r/V^{**}_r and V^{**}_r/V^{***}_r leaving us with $V_{p-a-1} \otimes D^a$ as in Proposition 5.2.

To eliminate the terms from V^*_r/V^{**}_r, we distinguish two cases:

- If $v(a_p) \leq 5/2$ we use the functions from Proposition 5.2, but note that $T^+ f_0$ has the term $\frac{p^5(p-1)}{a_p^2}\beta_4 X^{r-4}Y^4$, which is integral as $v(a_p) \leq 5/2$. Noting that $\beta_4 \equiv 5 \bmod p$, we can write $(T - a_p)f = T^+ f_0 - a_p f_1$

$$\equiv \left[g^0_{1,0}, \frac{5p^5(p-1)}{a_p^2}X^{r-4}Y^4 - 5(1-p)(XY^{r-1} - X^{r-4}Y^4)\right]$$

  and the follow the argument of Theorem 8.7 of [BG15].

- If $v(a_p) > 5/2$, then consider $f' = \frac{a_p^2}{p^5}f$. All terms are zero except $T^+ f_0 = \left[g^0_{1,0}, \beta_4 X^{r-4}Y^4\right]$ where $\beta_4 \equiv 5 \bmod p$. By adding an appropriate term of XY^{r-1}, we can follow the argument as in the previous case to eliminate the factors from V^*_r/V^{**}_r.

To eliminate the terms from V^{**}_r/V^{***}_r we distinguish two cases:

- If $v(a_p) \leq 5/2$ we use the functions from Proposition 5.2 but note that $T^+ f_0$ has the term $\frac{p^5(p-1)}{a_p^2}\beta_3 X^{r-3}Y^3$, which is integral as $v(a_p) \leq 5/2$. As $\beta_3 \equiv 10 \bmod p$, so we can write $(T - a_p)f = T^+ f_0 - a_p f_1$

$$\equiv \left[g^0_{1,0}, \frac{10p^5(p-1)}{a_p^2}X^{r-3}Y^3 - \binom{5}{2}(1-p)(X^2Y^{r-2} - X^{r-3}Y^3)\right]$$

and follow the argument as in the previous case.

- If $v(a_p) > 5/2$, then consider $f' = \frac{a_p^2}{p^5} f$. All terms are zero except $T^+ f_0 = \left[g_{1,0}^0, \beta_3 X^{r-3} Y^3\right]$. By adding an appropriate term of X^2Y^{r-2}, we can follow the argument as in the previous case to eliminate the factors from V_r^{**}/V_r^{***}.

□

Proposition 5.4. *Let $a = 5, \dots, p-1$. If $r \equiv a \bmod p(p-1)$ but $r \not\equiv a \bmod p^2$, then:*

$$\mathrm{ind}_{KZ}^{G}(V_{p-a+1} \otimes D^{a-1}) \twoheadrightarrow \overline{\Theta}_{k,a_p}.$$

Proof: By Proposition 4.6, we have the following structure for Q:

$$0 \to V_r^{*}/V_r^{***} \to Q \to V_{p-a-1} \otimes D^{a} \to 0.$$

To eliminate the factors coming from V_r^{**}/V_r^{***} we consider $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g_{1,[\lambda]}^0, \frac{(p-1)}{p^2}[\lambda]^{p-3}(XY^{r-1} - 2X^pY^{r-p} + X^{2p-1}Y^{r-2p+1})\right]$$
$$+ \frac{2(p-1)}{(a+r-1)p}\left[g_{1,0}^0, (Y^r - 2X^{p-1}Y^{r-p+1} + X^{2p-2}Y^{r-2p+2})\right]$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{pa_p}\left(C_1\theta^2(X^{4p}Y^{r-4p} - Y^{r-2p-2}) + \sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} D_j X^{r-j} Y^j\right)\right],$$

where

$$D_j = \binom{r-1}{j} - \left(\frac{2}{a+r-1} + O(p)\right)\binom{r}{j}$$

and $O(p)$ is chosen such that

$$\sum_{\substack{0<j<r-2 \\ j \equiv a-2 \bmod p-1}} D_j = 0.$$

We let $C_1 = -\sum jD_j$ and by Lemma 1.6 we see $\sum jD_j = \dfrac{(r-a)(r-a+1)}{(a+r-1)}$.

In the first part of f_1 we see $v(1/p^2) = -2$, so we consider $j = 0, 1, 2$ for $T^+ f_1$. For $j = 0$ we obtain $\binom{r-1}{0} - 2\binom{r-p}{0} + \binom{r-2p+1}{0} = 0$ while for $j = 1$, we see that

$\binom{r-1}{1}-2\binom{r-p}{1}+\binom{r-2p+1}{1}=0$, too. For $j=2$, the term $X^{r-2}Y^2$ has integral coefficients, so it maps to zero in Q. In the second part of f_1 we see $v(-2/(a+r-1)p)=-1$ and for $j=0$ we have $\binom{r}{0}-2\binom{r-p+1}{0}+\binom{r-2p+2}{0}=0$. For $j=1$ the coefficient of the term $X^{r-1}Y$ is integral and maps to zero in A. Hence, $T^+f_1 \equiv 0 \bmod p$. Because $v(a_p)>2$, we see that $a_pf_1 \equiv 0 \bmod p$.

In f_0 we see that $v(1/pa_p)<-4$, so we need to consider $j=0,1,2,3$. For $j=0$ we have $\sum D_j=0$, for $j=1$ we have $C_1=-\sum jD_j$. For $j=2$, observing that $v(C_1)=v(\sum jD_j)\geq 1$ yields that the term with $X^{r-2}Y^2$ is integral, which vanishes in Q. Finally, for $j=3$, we see that $v(C_1)=v(\sum jD_j)\geq 1$, so $T^+f_0\equiv 0 \bmod p$ as well. Since the highest index is $i=r-p$, we obtain $p^{r-i}=p^p$. Since $p\geq 5$, we have $T^-f_0\equiv 0 \bmod p$. For T^-f_1 we consider $i=r$ and $i=r-1$. For $i=r-1$, we obtain:

$$\left[\mathrm{id}, \frac{(p-1)}{p} \sum_{\substack{0<j<r-2 \\ j\equiv a-2 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j\right],$$

and for $i=r$:

$$\left[\mathrm{id}, \frac{-2(p-1)}{a+r-1} \sum_{\substack{0<j<r-2 \\ j\equiv a-2 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right].$$

Thus:

$$T^-f_1 \equiv \left[\mathrm{id}, \sum_{\substack{0<j<r-2 \\ j\equiv a-2 \bmod p-1}} \frac{1}{p} D_j X^{r-j}Y^j\right] \bmod p$$

Hence, we compute that

$$(T-a_p)f = T^-f_1 - a_pf_0 \equiv \left[\mathrm{id}, \frac{C_1(p-1)}{p}\left(\theta^2(X^{4p}Y^{r-4p}-Y^{r-2p-2})\right)\right],$$

which maps to $\dfrac{C_1(p-1)}{p}X^{p-a+3}$ by Lemma 5.1. Because $p \parallel r-a$, we have

$$\frac{C_1}{p} \equiv \frac{(r-a)(r-a+1)}{p(a+r-1)} \neq 0 \bmod p,$$

so we can eliminate the factors from V_r^{**}/V_r^{***}.

To eliminate the factor $V_{p-a-1}\otimes D^a$ we consider $f=f_0+f_1 \in \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda\in\mathbb{F}_p^*}\left[g^0_{1,[\lambda]}, \frac{1}{p^2}(Y^r - X^{r-a}Y^a)\right] + \left[g^0_{1,0}, \frac{(p-1)}{p}(Y^r-X^{r-a}Y^a)\right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{p^2 a_p} \sum_{\substack{0<j<r \\ j\equiv a \bmod p-1}} \alpha_j X^{r-j} Y^j\right],$$

where the α_j are the integers from Lemma 1.8 with the added conditions that $\alpha_j \equiv \binom{r}{j} \bmod p^2$ and $\sum \binom{j}{n}\alpha_j \equiv 0 \bmod p^{5-n}$ as $r \equiv a \bmod p$.

For $T^+ f_1$ in the first part of f_1 we note that $v(1/p^2) = -2$, so we need to consider $j = 0, 1, 2$. For $j = 0$, we see that $\binom{r}{0} - \binom{a}{0} = 0$ while for $j = 1$, we see that $\frac{p}{p^2}(\binom{r}{1} - \binom{a}{1}) = \frac{r-a}{p}$, which is integral as $p \mid r - a$, so the term involving $X^{r-1}Y$ maps to zero in Q. The term for $j = 2$ is zero $\bmod p$ as $r \equiv a \bmod p$. For the second part, we note that $v(1/p) = -1$. The term with $j = 0$ is identically zero while the coefficient of $X^{r-1}Y$ with $j = 1$ is integral, which vanishes in Q. Thus $T^+ f_1 \equiv 0 \bmod p$.

In f_0 we see that $v(1/p^2 a_p) < -5$. Due to the properties of α_j, we have $\sum \binom{j}{n}\alpha_j \equiv 0 \bmod p^{5-n}$ and $j \equiv a \geq 5$, so the terms in $T^+ f_0$ vanish $\bmod p$. Because the highest index i for which $c_i \not\equiv 0 \bmod p$ is $i = r - p + 1$, we have $p^{r-i} = p^{p-1}$. Thus $T^- f_0 \equiv 0 \bmod p$ for $p > 5$. Note that $6 \leq a \leq p - 1$ means that $p \geq 7$, so we do not need to worry about the case $p = 5$.

For $T^- f_1$ we note that the highest index of a nonzero coefficient is $i = r$, hence

$$T^- f_1 = \left[\mathrm{id}, \frac{(p-1)}{p^2} \sum_{\substack{0<j\leq r \\ j\equiv a \bmod p-1}} \binom{r}{j} X^{r-j} Y^j\right]$$

The last term in the above expansion (when $j = r$) is $\dfrac{p(p-1)\binom{r}{r}}{p^2} Y^r$, which is cancelled out by the term for $i = r$ from the second part (where $\lambda = 0$) which is $\dfrac{(1-p)}{p} Y^r$. We compute

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)}{p^2} \sum_{\substack{0<j<r \\ j\equiv a \bmod p-1}} \left(\binom{r}{j} - \alpha_j\right) X^{r-j} Y^j\right]$$

where the $\alpha_j \equiv \binom{r}{j} \bmod p^2$, so $T^- f_1 - a_p f_0$ is integral. Now we follow the argument as in the proof of [BG15, Theorem 8.3] and see that this maps to $\dfrac{a-r}{pa} X^{p-a-1}$, which is nonzero as $p^2 \nmid a - r$.

To eliminate the factor $V_{a-2} \otimes D$, we consider $f = f_0 + f_1 \in \operatorname{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g_{1,[\lambda]}^0, \frac{[\lambda]^{p-2}}{p} (Y^r - X^{r-a} Y^a) \right]$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{p a_p} \sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} \beta_j X^{r-j} Y^j \right],$$

where the β_j are the integers from Lemma 1.9.

In f_1 we see that $v(1/p) = -1$, so we consider $j = 0, 1$. For $j = 0$ we see $\binom{r}{0} - \binom{a}{0} = 0$ while for $j = 1$ we obtain $\frac{p}{p}(\binom{r}{1} - \binom{a}{1}) \equiv 0 \bmod p$ as $r - a \equiv 0 \bmod p$. Thus, $T^+ f_1 \equiv 0 \bmod p$. As $v(a_p) > 2$, we see that $a_p f_1 \equiv 0 \bmod p$.

For f_0, we note that $v(1/pa_p) < -4$ while $\sum \binom{j}{n} \beta_j \equiv 0 \bmod p^{4-n}$ and $j \equiv a-1 \geq 4$, hence $T^+ f_0 \equiv 0 \bmod p$. For $T^- f_0$ we note that the highest index is $i = r - p$, hence $p^{r-i} = p^p$, which kills $1/pa_p$ for $p \geq 5$.

For $T^- f_1$ the highest index of a non-zero coefficient is $i = r$, hence $T^- f_1$ is equivalent to:

$$\equiv \left[\mathrm{id}, \frac{(p-1)}{p} \sum_{\substack{0<j\leq r-1 \\ j \equiv a-1 \bmod p-1}} \binom{r}{j} X^{r-j} Y^j \right]$$

$$\equiv \left[\mathrm{id}, \frac{(p-1)}{p} \sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} \left(\binom{r}{j} X^{r-j} Y^j + r X Y^{r-1} \right) \right]$$

We compute that

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)}{p} \left(\sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} \left(\binom{r}{j} - \beta_j \right) X^{r-j} Y^j + r X Y^{r-1} \right) \right]$$

As in Theorem 8.9(i) of [BG15] we change the above polynomial by a suitable XY^{r-1} term and see that this has the same image in Q as

$$\left[\mathrm{id}, (p-1)\big(F(X,Y) + \frac{(a-r)}{p} \theta Y^{r-p-1}\big) \right],$$

where:

$$F(X,Y) = \left[\mathrm{id}, \frac{1}{p} \sum_{\substack{0<j<r-1 \\ j \equiv a-1 \bmod p-1}} \left(\binom{r}{j} - \beta_j \right) X^{r-j} Y^j - (a-r) X^p Y^{r-p} \right) \Bigg].$$

This function is integral as $\beta_j \equiv \binom{r}{j} \bmod p$ and $r \equiv a \bmod p$. By the conditions in Lemma 1.4 and recalling that $\sum_j \beta_j = 0$, $\sum j\beta_j \equiv 0 \bmod p^3$ and $r \equiv a \bmod p$ we see that $F(X,Y) \in V_r^{**}$. Thus, $(T - a_p)f$ is equivalent to $\frac{a-r}{p}\theta Y^{r-p-1}$, which, by Lemma 8.5 of [BG15], maps to $\frac{a-r}{p}Y^{a-2}$. This term is not zero as $r \not\equiv a \bmod p^2$.

Hence, the only surviving factor is $V_{p-a+1} \otimes D^{a-1}$.

□

Proposition 5.5. *If $r \equiv p \bmod p-1$ and $r \equiv p \bmod p$ (where in the case $p = 5$ and $v(a_p) = 5/2$ we assume $v(a_p^2 - p^5) = 5$), then:*

(i) *If $p^2 \nmid p - r$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_1) \to \overline{\Theta}_{k,a_p}.$$

(ii) *If $p^2 \mid p - r$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_{p-2} \otimes D) \to \overline{\Theta}_{k,a_p}.$$

Proof: We follow the proof of [BG15, Theorem 8.9]. By Proposition 4.4,

$$0 \to V_r^*/V_r^{***} \to Q \to 0,$$

that is, $Q \simeq V_r^*/V_r^{***}$.

(i) We have $p^2 \nmid p - r$:

To eliminate the factors from V_r^{**}/V_r^{***} we choose the functions as in Proposition 5.4 putting $a = p$ and seeing that $p^2 \nmid r - p$.

To eliminate the factor $V_{p-2} \otimes D$ we choose the functions $f = f_0 + f_1 + f_2 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_2 = \sum_{\lambda \in \mathbb{F}_p} \left[g_{2,p[\lambda]}^0, \frac{[\lambda]^{p-2}}{p}(Y^r - X^{r-p}Y^p) \right],$$

and

$$f_1 = \left[g_{1,0}^0, \frac{(p-1)}{pa_p} \sum_{\substack{0<j<r-1 \\ j \equiv 0 \bmod p-1}} \beta_j X^{r-j}Y^j \right],$$

and
$$f_0 = \left[\mathrm{id}, \frac{(1-p)}{p}(X^r - X^p Y^{r-p})\right]$$
where the integers β_j are those given in Lemma 1.9.

In f_2 we see that $v(1/p) = -1$, so we only consider $j = 0, 1$. For $j = 0$ we see that $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1$ we obtain $\frac{p}{p}(\binom{r}{1} - \binom{p}{1}) \equiv 0 \bmod p$ as $r \equiv p \bmod p$. Thus $T^+ f_2 \equiv 0 \bmod p$. Since $v(a_p) > 2$, we see that $a_p f_2 \equiv 0 \bmod p$.

In f_1 we see $v(1/pa_p) < -4$. Because $\sum \binom{j}{n}\beta_j \equiv 0 \bmod p^{4-n}$, we have $T^+ f_1 \equiv 0 \bmod p$. Since the highest index is $i = r - p$, we see that $p^{r-i} = p^p$ kills $1/pa_p$ for $p \geq 5$, which means $T^- f_1 \equiv 0 \bmod p$.

In f_0 we have $v(1/p) = -1$, so we only consider $j = 0, 1$. For $j = 0$ we see that $\frac{(1-p)}{p}(\binom{0}{0} - \binom{r-p}{0})X^r = \frac{(1-p)}{p}X^r$ while for $j = 1$ $\frac{p(1-p)}{p}(\binom{0}{1} - \binom{r-p}{1}) \equiv 0 \bmod p$ as $r \equiv p \bmod p$. Hence, $T^+ f_0 \equiv \left[g^0_{1,0}, \frac{(1-p)}{p}X^r\right]$. Since $v(a_p) > 2$, we see that $a_p f_0 \equiv 0 \bmod p$.

For $T^- f_2$, for $i = r$ we see that:
$$T^- f_2 = \left[g^0_{1,0}, \frac{(p-1)}{p} \sum_{\substack{0 \leq j \leq r-1 \\ j \equiv 0 \bmod p-1}} \binom{r}{j} X^{r-j} Y^j\right].$$

The last term above (when $j = r-1$) is $\frac{(p-1)r}{p}XY^{r-1}$ while the first term (when $j = 0$) is cancelled out by $T^+ f_0 = \left[g^0_{1,0}, \frac{(1-p)}{p}X^r\right]$.

This yields
$$T^- f_2 - a_p f_1 + T^+ f_0 = \left[g^0_{1,0}, \frac{(p-1)}{p}\left(\sum_{\substack{0 < j < r-1 \\ j \equiv 0 \bmod p-1}} \left(\binom{r}{j} - \beta_j\right) X^{r-j} Y^j + rXY^{r-1}\right)\right]$$

which is integral as $\beta_j \equiv \binom{r}{j} \bmod p$ and $p \mid r$.

Now, we follow the same argument as in the proof of [BG15, Theorem 8.9(i)] to eliminate the factor $V_{p-2} \otimes D$. Thus, we are left with the factor V_1.

(ii) We have $p^2 \mid p - r$:

We first assume that $v(a_p^2) < 5$ if $p = 5$. To eliminate the factors from V_r^{**}/V_r^{***} we consider $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g_{1,[\lambda]}^0, \frac{p}{a_p} [\lambda]^{p-3} (Y^r - X^{r-p} Y^p) \right] + \left[g_{1,0}^0, \frac{\binom{r}{2}(1-p)}{p a_p} (X^2 Y^{r-2} - X^{r-p+2} Y^{p-2}) \right],$$

and

$$f_0 = \left[\mathrm{id}, \frac{p(p-1)}{a_p^2} \sum_{\substack{0<j<r-2 \\ j \equiv p-2 \bmod p-1}} \gamma_j X^{r-j} Y^j \right],$$

where the integers γ_j are those given in Lemma 1.10 that satisfy $\gamma_j \equiv \binom{r}{j} \bmod p^2$ due to the condition that $p^2 \mid p - r$.

In f_1, we note that in the first part $v(p/a_p) < -2$, so for $T^+ f_1$ we consider $j = 0, 1$. For $j = 0$ we see that $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1$ we see $\frac{p^2}{a_p}\left(\binom{r}{1} - \binom{p}{1}\right) \equiv 0 \bmod p$ as $p^2 \mid r - p$. In the second part of f_1 we have $v(\binom{r}{2}/p a_p) < -3$, so we consider $j = 0, 1, 2$. For $j = 0$ we see that $\binom{r-2}{0} - \binom{p-2}{0} = 0$ while for $j = 1, 2$ we see $\frac{p^j \binom{r}{2}(1-p)}{p a_p}\left(\binom{r-2}{j} - \binom{p-2}{j}\right) \equiv 0 \bmod p$ as $p^2 \mid r - p$. Thus, $T^+ f_1 \equiv 0 \bmod p$.

In f_0 we have $v(p/a_p^2) < -5$. Because $\sum \binom{j}{n} \gamma_j \equiv 0 \bmod p^{5-n}$, we have $T^+ f_0 \equiv 0 \bmod p$. Note that for $p = 5$, $T^+ f_0 = [g_{1,0}, \frac{p^4}{a_p^2} \gamma_3 \binom{3}{3} X^{r-3} Y^3]$. Because $v(a_p^2) < 5$ and $\gamma_3 \equiv \binom{r}{3} \equiv 0 \bmod p$, we obtain $T^+ f_0 \equiv 0 \bmod p$. Because the highest index is $i = r - p - 1$, we see that $p^{r-i} = p^{p+1}$ kills p/a_p^2. Hence $T^- f_0 \equiv 0 \bmod p$.

For $T^- f_1$, for the first part $(i = r)$ we see that:

$$T^- f_1 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0<j \le r-2 \\ j \equiv p-2 \bmod p-1}} \binom{r}{j} X^{r-j} Y^j \right].$$

The last term when $j = r - 2$ is $\frac{\binom{r}{2} p}{a_p} X^2 Y^{r-2}$, which is cancelled out by the second part of $T^- f_1$ $(i = r - 2)$. This yields

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0<j<r-2 \\ j \equiv p-2 \bmod p-1}} \left(\binom{r}{j} - \gamma_j \right) X^{r-j} Y^j \right]$$

which is zero $\mod p$ as $\gamma_j \equiv \binom{r}{j} \mod p^2$ while $v(p/a_p) < -2$.

Hence $(T - a_p)f \equiv -a_p f_1$, which is equivalent to:

$$-\left[g^0_{1,0}, \frac{\binom{r}{2}(1-p)}{p}(X^2Y^{r-2} - X^{r-p+2}Y^{p-2})\right]$$

By the hypothesis $\frac{r}{p} \equiv 1 \mod p$ and $r - 1 \equiv p - 1 \mod p$, so the above function is congruent to

$$-\left[g^0_{1,0}, \frac{(p-1)(1-p)}{2}(X^2Y^{r-2} - X^{r-p+2}Y^{p-2})\right].$$

Therefore

$$\begin{aligned}
&(X^2Y^{r-2} - X^{r-p+2}Y^{p-2})\\
=&\theta^2\Big(\sum_{i=0}^{\frac{r-3p+2}{p-1}} (i+1)\big(\frac{r-3p+2}{p-1}\big)(X^{r-3p+2}Y^{p-4} + Y^{r-2p-2}\Big)\\
\equiv&\theta^2(X^{r-3p+2}Y^{p-4} - Y^{r-2p-2}) \mod V_r^{***}.
\end{aligned}$$

Thus, $\overline{(T - a_p)f}$ maps to $[g^0_{1,0}, X^3]$ by Lemma 5.1. Following previous arguments, this shows that we can eliminate the factors from V_r^{**}/V_r^{***}.

In the case $p = 5$ and $v(a_p^2) \geq 5$ we assume $v(a_p^2 - p^5) =$ if $v(a_p) = 5/2$. Now, we consider the function $f' = \frac{a_p^2}{p^5} f$ where f is the function above, obtaining:

$$\begin{aligned}
f_1' = &\sum_{\lambda \in \mathbb{F}_p^*} \left[g^0_{1,[\lambda]}, \frac{a_p}{p^4}[\lambda]^{p-3}(Y^r - X^{r-p}Y^p)\right]\\
&+\left[g^0_{1,0}, \frac{\binom{r}{2}(1-p)a_p}{p^6}(X^2Y^{r-2} - X^{r-p+2}Y^{p-2})\right],
\end{aligned}$$

and

$$f_0' = \left[\mathrm{id}, \sum_{\substack{0<j<r-2 \\ j \equiv p-2 \mod p-1}} \frac{(p-1)}{p^4}\gamma_j X^{r-j}Y^j\right],$$

where the integers γ_j are those given in Lemma 1.10 that satisfy $\gamma_j \equiv \binom{r}{j} \mod p^2$ due to the condition that $p^2 \mid p - r$.

In f_1' we have $v(a_p/p^4) < -2$ in the first part of f_1', so we consider $j = 0, 1$. For $j = 0$ we see that $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1$ we see $\frac{pa_p}{p^4}\left(\binom{r}{1} - \binom{p}{1}\right) \equiv 0 \mod p$ as $p^2 \mid r - p$. In the second part of f_1' we see that $v(\binom{r}{2}a_p/p^6) < -3$, so we consider $j = 0, 1, 2$. For $j = 0$ we see that $\binom{r-2}{0} - \binom{p-2}{0} = 0$ while for $j = 1, 2$ we see $\frac{p^j\binom{r}{2}a_p}{p^6}\left(\binom{r-2}{j} - \binom{p-2}{j}\right) \equiv 0 \mod p$ as $p^2 \mid r - p$. Thus, the second part of $T^+f_1' \equiv 0 \mod p$ as well.

In f_0' we have $v(1/p^4) = -4$. The highest index in f_0' is $i = r - p - 1$, so $p^{r-i} = p^{p+1}$, which kills $1/p^4$. Hence, $T^-f_0' \equiv 0 \mod p$. We obtain $T^+f_0' = [\mathrm{id}, \frac{\gamma_3}{p}X^{r-3}Y^3]$ (observing that $p - 2 = 3$), which is integral as $\gamma_3 \equiv \binom{r}{3} \equiv 0 \mod p$.

For T^-f_1', for the first part (when $i = r$) we that:

$$T^-f_1' = \left[\mathrm{id}, \frac{(p-1)a_p}{p^4} \sum_{\substack{0<j\le r-2 \\ j\equiv p-2 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right].$$

The last term above (when $j = r - 2$) is $\frac{\binom{r}{2}a_p}{p^4}X^2Y^{r-2}$, which is cancelled out by the second part of T^-f_1' (when $i = r - 2$). This yields

$$T^-f_1' - a_pf_0' = \left[\mathrm{id}, \frac{(p-1)a_p}{p^4} \sum_{\substack{0<j<r-2 \\ j\equiv p-2 \bmod p-1}} \left(\binom{r}{j} - \gamma_j\right) X^{r-j}Y^j\right]$$

which is zero $\mod p$ as $\gamma_j \equiv \binom{r}{j} \mod p^2$ while $v(a_p/p^4) < -2$.

Hence $(T - a_p)f' \equiv -a_pf_1' + T^+f_0'$, which is equivalent to:

$$-\left[g_{1,0}^0, \frac{\binom{r}{2}(1-p)}{p}(X^2Y^{r-2} - X^{r-p+2}Y^{p-2}) + \frac{\gamma_3}{p}X^{r-3}Y^3\right]$$

Like in the previous case, this is equivalent to:

$$-\left[g_{1,0}^0, \frac{(p-1)(1-p)}{2}(X^2Y^{r-2} - X^{r-p+2}Y^{p-2}) + \frac{\gamma_3}{p}X^{r-3}Y^3\right].$$

We follow the argument as in the previous case and, noting that $p - 2 = 3$, this maps to $c[g_{1,0}^0, X^3]$ where $c = 1 - \frac{a_p^2}{p^5}$ is not zero $\mod p$ due to the hypothesis.

To eliminate the factor V_1 we choose the functions $f = f_0 + f_1 + f_2$ in $\mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_2 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g_{2,p[\lambda]}^0, \frac{[\lambda]^{p-2}p}{a_p}(Y^r - X^{r-p}Y^p) \right] + \left[g_{2,0}^0, \frac{r(1-p)}{pa_p}(XY^{r-1} - X^{r-p+1}Y^{p-1}) \right],$$

$$f_1 = \left[g_{1,0}^0, \frac{(p-1)p}{a_p^2} \sum_{\substack{0<j<r-1 \\ j \equiv 0 \bmod p-1}} \gamma_j X^{r-j}Y^j \right],$$

and

$$f_0 = \left[\mathrm{id}, \frac{(1-p)p}{a_p}(X^r - X^pY^{r-p}) \right]$$

where the integers γ_j are those given in Lemma 1.11 that satisfy $\gamma_j \equiv \binom{r}{j} \bmod p^2$ due to the condition that $p^2 \mid p - r$.

In f_2 we see that $v(p/a_p) < -2$ in the first part of f_2, so we consider $j = 0, 1$. For $j = 0$ we see that $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1$ we see $\frac{p^2}{a_p}(\binom{r}{1} - \binom{p}{1}) \equiv 0 \bmod p$ as $p^2 \mid r - p$. In the second part of f_2 we see that $v(r/pa_p) < -3$, so we consider $j = 0, 1, 2$. For $j = 0$ we see that $\binom{r-1}{0} - \binom{p-1}{0} = 0$ while for $j = 1, 2$ we see $\frac{p^j r}{pa_p}(\binom{r-1}{j} - \binom{p-1}{j}) \equiv 0 \bmod p$ as $p^2 \mid r - p$. Thus $T^+ f_2 \equiv 0 \bmod p$.

In f_1 we have $v(p/a_p^2) < -5$. Since $\sum \binom{j}{n}\gamma_j \equiv 0 \bmod p^{5-n}$, we see that $T^+ f_1 \equiv 0 \bmod p$. Note that for $p = 5$, $T^+ f_1 = [g_{1,0}, \frac{p^5}{a_p^2}\gamma_4\binom{3}{3}X^{r-4}Y^4] \bmod p$ but $\gamma_4 \equiv \binom{r}{4} \equiv 0$, so $T^+ f_1 \equiv 0 \bmod p$. Since the highest index is $i = r - p - 1$, we see that $p^{r-i} = p^{p+1}$ kills p/a_p^2 hence $T^- f_0 \equiv 0 \bmod p$.

In f_0 we have $v(p/a_p) < -2$, so we only consider $j = 0, 1$. For $j = 0$ we see that $\frac{(1-p)p}{a_p}(\binom{0}{0} - \binom{r-p}{0})X^r = \frac{-(1-p)p}{a_p}X^r$ while for $j = 1$ we obtain $\frac{p^2(1-p)}{a_p}(\binom{0}{1} - \binom{r-p}{1}) \equiv 0 \bmod p$ as $r \equiv p \bmod p$. Thus,

$$T^+ f_0 = [g_{1,0}^0, \frac{-(1-p)p}{a_p}X^r].$$

For T^-f_2, for the first part $(i = r)$ we that:

$$T^-f_2 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0\le j\le r-1\\ j\equiv 0 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right].$$

The last term (when $j = r-1$) is $\dfrac{(p-1)r}{a_p}XY^{r-1}$, which is cancelled out by the second part of T^-f_2 (when $i = r-1$). The first term (when $j = 0$) is cancelled out by T^+f_0. This yields

$$T^-f_2 - a_pf_1 + T^+f_0 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0<j<r-1\\ j\equiv 0 \bmod p-1}} \left(\binom{r}{j} - \gamma_j\right) X^{r-j}Y^j\right]$$

which is zero $\bmod p$ as $\gamma_j \equiv \binom{r}{j} \bmod p^2$ while $v(p/a_p) < -2$.

Hence $(T - a_p)f \equiv -a_pf_2$, which is equivalent to:

$$-\left[g^0_{2,0}, \frac{r(1-p)}{p}(XY^{r-1} - X^{r-p+1}Y^{p-1})\right].$$

By assumption, $\frac{r}{p} \equiv 1 \bmod p$. We then follow the same argument as in the proof of [BG15, Thm 8.9(ii)] to eliminate the factor V_1. Thus, the only factor left is $V_{p-2} \otimes D$. □

5.2 r does *not* have the same representative mod $p-1$ and p

Proposition 5.6. *If $r \equiv a \bmod p-1$ and $r \not\equiv a, a-1 \bmod p$ for $5 \le a \le p$, then there is a surjection*

$$\mathrm{ind}^G_{KZ}(V_{p-a+3} \otimes D^{a-2}) \twoheadrightarrow \overline{\Theta}_{k,a_p}$$

Proof: By Proposition 4.7, we have the following structure for Q:

$$0 \to W \to Q \to V_{p-a-1} \otimes D^a \to 0$$

where W has $V_{p-a+1} \otimes D^{a-1}$ and $V_{p-a+3} \otimes D^{a-2}$ as factors.

To eliminate the factor $V_{p-a-1} \otimes D^a$, we consider $f = f_0 + f_1 \in \mathrm{ind}^G_{KZ} \mathrm{Sym}^r \overline{\mathbb{Q}}^2_p$, where:

$$f_1 = \sum_{\lambda\in\mathbb{F}_p} \left[g^0_{1,[\lambda]}, \frac{1}{p}(Y^r - X^{r-a}Y^a)\right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{pa_p} \sum_{\substack{0<j<r \\ j\equiv a \bmod p-1}} \alpha_j X^{r-j}Y^j\right],$$

where the α_j are chosen as in Lemma 1.8.

In f_1 we have $v(1/p) = -1$, so we consider only $j = 0, 1$ in $T^+ f_1$. For $j = 0$, we obtain $\frac{1}{p}(\binom{r}{0} - \binom{a}{0}) = 0$ while for $j = 1$ we obtain $\frac{p}{p}(\binom{r}{1} - \binom{a}{1})X^{r-1}Y$, which is integral and goes to zero in Q. Because $v(a_p) > 2$, we have $a_p f_1 \equiv 0 \bmod p$.

In f_0 we note that $v(1/pa_p) < -4$. As $\binom{j}{n}\alpha_j \equiv 0 \bmod p^{4-n}$ and $j \equiv a \geq 4$, hence $T^+ f_0 \equiv 0 \bmod p$. For $T^- f_0$ the highest index $i = r-(p-1)$ and $p^{r-i} = p^{p-1}$, which kills $1/pa_p$ for $p \geq 5$. For $T^- f_1$ we consider $i = r$, obtaining:

$$T^- f_1 = \left[\mathrm{id}, \frac{(p-1)}{p} \sum_{\substack{0<j<r \\ j\equiv a \bmod p-1}} \binom{r}{j} X^{r-j}Y^j + pY^r\right].$$

Because Y^r is sent to zero in Q:

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)}{p} \sum_{\substack{0<j<r \\ j\equiv a \bmod p-1}} \left(\binom{r}{j} - \alpha_j\right) X^{r-j}Y^j\right]$$

which is integral as $\binom{r}{j} \equiv \alpha_j \bmod p$. Following the same argument as in the proof of [BG15, Theorem 8.3], we see that $(T - a_p)f$ maps to $[\mathrm{id}, \frac{r-a}{a}X^{p-a-1}]$, which is nonzero as $r \not\equiv a \bmod p$.

To eliminate the factor $V_{p-a+1} \otimes D^{a-1}$, we consider $f = f_1 + f_0$, where

$$f_1 = \sum_{\lambda\in\mathbb{F}_p^*} \left[g^0_{1,[\lambda]}, \frac{-1}{a-1}[\lambda]^{p-2}(Y^r - 2X^{p-1}Y^{r-p+1} + X^{2p-2}Y^{r-2p+2})\right]$$
$$+ \left[g^0_{1,0} \frac{1}{p^2}(XY^{r-1} - 2X^pY^{r-p} + X^{2p-1}Y^{r-2p+1})\right]$$

and

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{pa_p}\left(\frac{C_1}{p-1}(X^pY^{r-p} - X^{2p-1}Y^{r-2p+1}) + \sum_{\substack{0<j<r-1 \\ j\equiv a-1 \bmod p-1}} D_j\right)X^{r-j}Y^j\right],$$

where

$$D_j = \binom{r-1}{j} - \left(\frac{p}{a-1} + O(p^2)\right)\binom{r}{j}$$

and $O(p^2)$ is chosen so that

$$\sum_{\substack{0<j<r-1 \\ j\equiv a-1 \bmod p-1}} D_j = 0.$$

We let $C_1 = -\sum jD_j$. By Lemma 1.6:

$$\sum_{\substack{0<j<r-1 \\ j\equiv a-1 \bmod p-1}} jD_j = \frac{p(r-a)(r-a+1)}{(a-1)(a-2)}.$$

In the second part of f_1 we have $v(1/p^2) = -2$, so we consider $j = 0, 1, 2$ for $T^+ f_1$. For $j = 0$ we obtain $\binom{r-1}{0} - 2\binom{r-p}{0} + \binom{r-2p+1}{0} = 0$ while for $j = 1$, we see that $\binom{r-1}{1} - 2\binom{r-p}{1} + \binom{r-2p+1}{1} = 0$, too. For $j = 2$ the term $X^{r-2}Y^2$ has integral coefficients, so it maps to zero in Q. In the first part, we only consider $j = 0$ and see that $\binom{r}{0} - 2\binom{r-p+1}{0} + \binom{r-2p+2}{0} = 0$. Hence, $T^+ f_1 \equiv 0 \bmod p$. As $v(a_p) > 2$, we see that $a_p f_1 \equiv 0 \bmod p$.

In f_0 we see that $v(1/pa_p) < -4$, so we need to consider $j = 0, 1, 2, 3$ for $T^+ f_0$. For $j = 0$ we have $\sum D_j = 0$, for $j = 1$ we have $C_1 = -\sum jD_j$, so $T^+ f_0 \equiv 0$ for $j = 0, 1$. For $j = 2$, the term with $X^{r-2}Y^2$ is integral, which vanishes in Q. Finally, for $j = 3$, we see that $v(C_1) = v(\sum jD_j) \geq 1$, so $T^+ f_0 \equiv 0 \bmod p$ as well.. As the highest $i = r - p$, we see that $T^- f_0 \equiv 0 \bmod p$.

For $T^- f_1$ we consider $i = r$ and $i = r - 1$. For $i = r - 1$ we obtain:

$$\left[\mathrm{id}, \frac{(p-1)}{p} \sum_{\substack{0<j<r-1 \\ j\equiv a-1 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j\right],$$

and for $i = r$:

$$\left[\mathrm{id}, \frac{-(p-1)}{a-1} \sum_{\substack{0<j<r-1 \\ j\equiv a-1 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right].$$

Thus:

$$T^- f_1 \equiv \sum_{\substack{0<j<r-1 \\ j\equiv a-1 \bmod p-1}} \left[\mathrm{id}, \frac{(p-1)}{p} D_j X^{r-j}Y^j\right].$$

We hence compute that $(T - a_p)f = T^- f_1 - a_p f_0 \equiv [\mathrm{id}, \frac{C_1}{p}\theta X^{p-1}Y^{r-2p}]$, which maps to $[\mathrm{id}, \frac{C_1}{p} X^{p-a+1}]$. Because $r \not\equiv a, a-1 \bmod p$,

$$\frac{C_1}{p} \equiv \frac{(r-a)(r-a+1)}{(a-1)(a-2)} \not\equiv 0 \bmod p.$$

Hence, the only remaining factor is $V_{p-a+3} \otimes D^{a-2}$. □

Proposition 5.7. *If $r \equiv a \bmod p-1$ and $p \parallel r-a+1$ for $5 \le a \le p$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_{p-a+1} \otimes D^{a-1}) \twoheadrightarrow \overline{\Theta}_{k,a_p}$$

Proof: By Proposition 4.8, we have the following structure for Q:

$$0 \to W \to Q \to V_{p-a-1} \otimes D^{a} \to 0$$

where W has $V_{p-a+1} \otimes D^{a-1}, V_{a-4} \otimes D^2$ and V_{p-a+3} as factors.

We can eliminate the factor $V_{p-a-1} \otimes D^a$ by the functions in the proof of Proposition 5.6 as $r \not\equiv a \bmod p$.

As $p \parallel r-a+1$, we can eliminate both factors $V_{a-4} \otimes D^2$ and $V_{p-a+3} \otimes D^{a-2}$ by the functions in the proof of Proposition 5.4 used to this end.

Hence, the only remaining factor is $V_{p-a+1} \otimes D^{a-1}$. □

Proposition 5.8. *If $r \equiv a \bmod p-1$ and $r \equiv a-1 \bmod p^2$ for $5 \le a \le p$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_{a-4} \otimes D^{2}) \twoheadrightarrow \overline{\Theta}_{k,a_p}$$

Proof: By Proposition 4.8, we have the following structure for Q:

$$0 \to W \to Q \to V_{p-a-1} \otimes D^{a} \to 0$$

where W has $V_{p-a+1} \otimes D^{a-1}, V_{a-4} \otimes D^2$ and $V_{p-a+3} \otimes D^{a-2}$ as factors.

We can eliminate the factor $V_{p-a-1} \otimes D^a$ by the functions in the proof of Proposition 5.6 as $r \not\equiv a \bmod p$.

To eliminate the factor $V_{p-a+1} \otimes D^{a-1}$ we consider the functions $f = f_1 + f_0 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, where:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g_{1,[\lambda]}^0, \frac{p^2}{a_p}(Y^r - X^{r-a}Y^a) \right] + \left[g_{1,0}^0, \frac{1}{a_p}(X^{r-1}Y - X^{a-2+p^2}Y^{r-a+2-p^2}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p^2} \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \alpha_j X^{r-j}Y^j \right],$$

where the α_j are chosen as in Lemma 1.8. Note that as $r \equiv a-1 \bmod p^2$, we have $r - a + 2 - p^2 > 0$.

In the first part of f_1, we have $v(p^2/a_p) < -1$, so we consider $j = 0$ for the first part of $T^+ f_1$. Because $\binom{r}{0} - \binom{a}{0} = 0$, the first part of $T^+ f_1$ vanishes. For the second part, $v(1/a_p) < -3$, so we consider $j = 0, 1, 2$. Now see that $p^j/a_p(\binom{1}{j} - \binom{r-a+2-p^2}{j}) \equiv 0 \bmod p$ as $r \equiv a-1 \bmod p^2$ for $j = 0, 1, 2$. Thus, $T^+ f_1 \equiv 0 \bmod p$. For $T^- f_1$ we consider $i = r$ and obtain

$$T^- f_1 \equiv \left[\mathrm{id}, \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \frac{p^2(p-1)}{a_p} \binom{r}{j} X^{r-j} Y^j \right].$$

Thus:

$$T^- f_1 - a_p f_0 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \left(\binom{r}{j} - \alpha_j \right) X^{r-j} Y^j \right]$$

which vanishes as $\binom{r}{j} \equiv \alpha_j \bmod p$ and $v(p^2/a_p) < -1$.

In f_0, we have $v(p^2/a_p^2) < -4$. For $T^- f_0$, the highest index $i = r - (p-1)$, so $p^{r-i} = p^{p-1}$ kills p^2/a_p^2 for $p \geq 5$. Finally, for $T^+ f_0$, we see that $\sum \binom{j}{n} \alpha_j \equiv 0 \bmod p^{4-n}$ and $j \equiv a \geq 4$. Thus, $T^+ f_0 \equiv 0 \bmod p$.

Hence $(T - a_p) f \equiv -a_p f_1 \equiv [g_{1,0}^0, X^{r-1} Y - X^{a-2+p^2} Y^{r-a+2-p^2}]$

$$\equiv \left[g_{1,0}^0, \theta \sum_{i=0}^{\frac{r-a-(p-1)(p+2)}{(p-1)}} X^{r-i(p-1)} Y^{i(p-1)} \right] \bmod V_r^{**}.$$

By an argument similar to Lemma 5.1 $\psi^{-1} : X^{r-2p} Y^{p-1} \mapsto X^{a-2} Y^{p-1}$ and $\beta : X^{a-2} Y^{p-1} \mapsto Y^{p-a+1}$. Thus, $[g_{1,0}^0, \theta \sum_{i=0}^{\frac{r-a-(p-1)(p+2)}{(p-1)}} X^{r-i(p-1)} Y^{i(p-1)}]$ maps to $[g_{1,0}^0, \frac{2}{p-1} Y^{p-a+1}]$ and eliminates the factor $V_{p-a+1} \otimes D^{a-1}$.

To eliminate the factor $V_{p-a+3} \otimes D^{a-2}$ we consider the functions $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$, where:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g_{1,[\lambda]}^0, \frac{p^2}{a_p} (Y^r - X^{r-a} Y^a) \right] + \left[g_{1,0}^0, \frac{1}{a_p} (X^{r-2} Y^2 - X^{a-3+p^2} Y^{r-a+3-p^2}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p^2} \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \alpha_j X^{r-j} Y^j \right],$$

where the α_j are chosen as in Lemma 1.8.

Note that as $r \equiv a - 1 \bmod p^2$, we have $r - a + 3 - p^2 > 0$.

In the first part of f_1, we have $v(p^2/a_p) < -1$, so we consider $j = 0$ for the first part of $T^+ f_1$. Because $\binom{r}{0} - \binom{a}{0} = 0$, the first part of $T^+ f_1$ vanishes. For the second part, $v(1/a_p) < -3$, so we consider $j = 0, 1, 2$. Now see that $p^j/a_p\left(\binom{2}{j} - \binom{r-a+3-p^2}{j}\right) \equiv 0 \bmod p$ as $r \equiv a - 1 \bmod p^2$ for $j = 0, 1, 2$. Thus, $T^+ f_1 \equiv 0 \bmod p$. For $T^- f_1$ we consider $i = r$ and obtain

$$T^- f_1 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \binom{r}{j} X^{r-j} Y^j\right].$$

Thus:

$$T^- f_1 - a_p f_0 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \left(\binom{r}{j} - \alpha_j\right) X^{r-j} Y^j\right]$$

which vanishes as $\binom{r}{j} \equiv \alpha_j \bmod p$ and $v(p^2/a_p) < -1$.

In f_0, we have $v(p^2/a_p^2) < -4$. For $T^- f_0$, the highest index $i = r - (p-1)$, so $p^{r-i} = p^{p-1}$ kills p^2/a_p^2 for $p \geq 5$. Finally, for $T^+ f_0$, we see that $\sum \binom{j}{n} \alpha_j \equiv 0 \bmod p^{4-n}$ and $j \equiv a \geq 4$. Thus, $T^+ f_0 \equiv 0 \bmod p$.

Thus, $(T - a_p) f \equiv -a_p f_1 \equiv \left[g^0_{1,0}, (X^{r-2}Y^2 - X^{a-3+p^2} Y^{r-a+3-p^2})\right]$

$$\equiv \left[g^0_{1,0}, \theta^2 \sum_{i=0}^{\frac{r-a-(p-1)(p+3)}{(p-1)}} (i+1)(X^{r-i(p-1)} Y^{i(p-1)})\right].$$

By Lemma 5.1, each $X^{r-i(p-1)}Y^{i(p-1)}$ term (for $i \neq 0$) maps to Y^{p-a+3}. Thus, $\theta^2 \sum_{i=0}^{\frac{r-a-(p-1)(p+3)}{(p-1)}} (i+1) X^{r-i(p-1)} Y^{i(p-1)}$ maps to $\frac{-1}{p-1} Y^{p-a+3}$ and eliminates the factor $V_{p-a+3} \otimes D^{a-2}$.

Therefore, the only remaining factor is $V_{a-4} \otimes D^2$. □

5.3 $r \equiv 3 \bmod p - 1$

In the following proposition we eliminate all but one Jordan-Hölder factor. We note that while eliminating the factors from V_r^*/V_r^{**} we consider $a = 3$ but while eliminating the factors from V_r^{**}/V_r^{***}, we consider $a = p + 2$, following the convention set in the beginning of the paper in Lemma 1.4.

Proposition 5.9. *If $r \equiv 3 \bmod p - 1$, and:*

(i) *If* $r \not\equiv 0, 1, 2 \bmod p$, *then* $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}(\mathrm{V}_{p-4} \otimes \mathrm{D}^3) \twoheadrightarrow \overline{\Theta}_{k,a_p}$.

(ii) *If* $r \equiv 0 \bmod p$ *then* $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}(\mathrm{V}_1 \otimes \mathrm{D}) \twoheadrightarrow \overline{\Theta}_{k,a_p}$.

(iii) *If* $p \parallel r-p-1$ *then* $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}(\mathrm{V}_{p-2} \otimes \mathrm{D}^2) \twoheadrightarrow \overline{\Theta}_{k,a_p}$ *where it corresponds to* $\mathrm{V}_{p-a+1} \otimes \mathrm{D}^2$ *(for* $a = 3$*).*

(iv) *If* $r \equiv p+1 \bmod p^2$ *then* $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}(\mathrm{V}_{p-2} \otimes \mathrm{D}^2) \twoheadrightarrow \overline{\Theta}_{k,a_p}$ *where it corresponds to* $\mathrm{V}_{a-4} \otimes \mathrm{D}^2$ *(for* $a = p+2$*).*

(v) *If* $r \equiv 2 \bmod p$ *then* $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}(\mathrm{V}_{p-2} \otimes \mathrm{D}^2) \twoheadrightarrow \overline{\Theta}_{k,a_p}$ *where it corresponds to* $\mathrm{V}_{p-a+1} \otimes \mathrm{D}^2$ *(for* $a = 3$*).*

Proof:

(i) If $r \not\equiv 0, 1, 2 \bmod p$, then by Proposition 4.2 we already have the result.

(ii) If $r \equiv 0 \bmod p$, then to eliminate the factor $\mathrm{V}_{p-4} \otimes \mathrm{D}^3$ we use the functions as in Proposition 5.6 used to eliminate $\mathrm{V}_{p-a-1} \otimes \mathrm{D}^a$ (for $a = 3$) but note that $\mathrm{T}^+ f_0$ has the term $\frac{p-1}{pa_p} p^3 \alpha_3 \mathrm{X}^{r-3}\mathrm{Y}^3 = \frac{p-1}{pa_p} p^3 \binom{r}{3} \mathrm{X}^{r-3}\mathrm{Y}^3$ by Lemma 1.8. As $p \mid r$ we see that $\binom{r}{3} = 0$, so $\mathrm{T}^+ f_1 = 0$. The rest follows as in Proposition 5.6. Hence, the only remaining factor is $\mathrm{V}_1 \otimes \mathrm{D}$.

(iii) If $r \equiv 1 \bmod p$, then to eliminate the factor $\mathrm{V}_{p-4} \otimes \mathrm{D}^3$ we use the functions as in Proposition 5.6 used to eliminate $\mathrm{V}_{p-a-1} \otimes \mathrm{D}^a$ (for $a = 3$) but note that $\mathrm{T}^+ f_0$ has the term $\frac{p-1}{pa_p} p^3 \alpha_3 \mathrm{X}^{r-3}\mathrm{Y}^3 = \frac{p-1}{pa_p} p^3 \binom{r}{3} \mathrm{X}^{r-3}\mathrm{Y}^3$ by Lemma 1.8. If $p \mid r-1$ we see that $\binom{r}{3} = 0$, so $\mathrm{T}^+ f_1 = 0$. The rest follows as in Proposition 5.6.

If $p \parallel r-p-1$, we can eliminate the factors from $\mathrm{V}_r^{**}/\mathrm{V}_r^{***}$ by using $a = p+2$ in Proposition 5.4. Hence, we are left with only $\mathrm{V}_{p-2} \otimes \mathrm{D}^2$ (which corresponds to the factor $\mathrm{V}_{p-a+1} \otimes \mathrm{D}^{a-1}$ for $a = 3$).

(iv) If $p^2 \mid r-p-1$, first we note that as in the previous case, we can eliminate the term $\mathrm{V}_{p-4} \otimes \mathrm{D}^3$. To eliminate the factor $\mathrm{V}_{p-2} \otimes \mathrm{D}^2$ (i.e the factor corresponding to $\mathrm{V}_{p-a+1} \otimes \mathrm{D}^{a-1}$ for $a = 3$) we consider the functions $f = f_1 + f_0 \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, where:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g_{1,[\lambda]}^0, \frac{p^2}{a_p}(\mathrm{Y}^r - \mathrm{X}^{r-p+2}\mathrm{Y}^{p+2}) \right] + \left[g_{1,0}^0, \frac{1}{a_p}(\mathrm{X}^{r-1}\mathrm{Y} - \mathrm{X}^{p+p^2}\mathrm{Y}^{r-p-p^2}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p^2} \sum_{\substack{0<j<r\\ j\equiv 3 \bmod p-1}} \alpha_j X^{r-j}Y^j\right],$$

where the α_j are chosen as in Lemma 1.8.

Note that as $r \equiv p+1 \bmod p^2$, we have $r-p-p^2 > 0$.

In the first part of f_1, we have $v(p^2/a_p) < -1$, so we consider $j=0$ for the first part of T^+f_1. Because $\binom{r}{0} - \binom{p+2}{0} = 0$, the first part of T^+f_1 vanishes. For the second part, $v(1/a_p) < -3$, so we consider $j = 0,1,2$. Now see that $p^j/a_p(\binom{1}{j} - \binom{r-p-p^2}{j}) \equiv 0 \bmod p$ as $r \equiv p+1 \bmod p^2$ for $j=0,1,2$. Thus, $T^+f_1 \equiv 0 \bmod p$. For T^-f_1 we consider $i = r$ and obtain

$$T^-f_1 \equiv \left[\mathrm{id}, \sum_{\substack{0<j<r\\ j\equiv 3 \bmod p-1}} \frac{p^2(p-1)}{a_p}\binom{r}{j} X^{r-j}Y^j\right].$$

Thus:

$$T^-f_1 - a_p f_0 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r\\ j\equiv 3 \bmod p-1}} \left(\binom{r}{j} - \alpha_j\right) X^{r-j}Y^j\right]$$

which vanishes as $\binom{r}{j} \equiv \alpha_j \bmod p$ and $v(p^2/a_p) < -1$.

In f_0, we have $v(p^2/a_p^2) < -4$. For T^-f_0, the highest index $i = r-(p-1)$, so $p^{r-i} = p^{p-1}$ kills p^2/a_p^2 for $p \geq 5$. Finally, for T^+f_0, we see that $T^+f_0 \equiv \frac{p^5(p-1)}{a_p^2}\alpha_3 Y^{r-3}Y^3 \equiv 0 \bmod p$ as $\alpha_3 \equiv \binom{r}{3} \equiv 0 \bmod p$ as $r-1 \equiv 0 \bmod p$.

Hence $(T-a_p)f \equiv -a_p f_1 \equiv [g_{1,0}^0, X^{r-1}Y - X^{p+p^2}Y^{r-p-p^2}]$

$$\equiv \left[g_{1,0}^0, \theta \sum_{i=0}^{\frac{r-p-2-(p-1)(p+2)}{(p-1)}} X^{r-i(p-1)}Y^{i(p-1)}\right] \bmod V_r^{**}.$$

By an argument similar to Lemma 5.1 $\psi^{-1} : X^{r-2p}Y^{p-1} \mapsto Y^{p-1}$ and $\beta : Y^{p-1} \mapsto Y^{p-2}$. Thus, $[g_{1,0}^0, \theta \sum_{i=0}^{\frac{r-p-2-(p-1)(p+2)}{(p-1)}} X^{r-i(p-1)}Y^{i(p-1)}]$ maps to $[g_{1,0}^0, \frac{2}{p-1}Y^{p-2}]$ and eliminates the factor $V_{p-2} \otimes D^2$.

To eliminate the factor $V_1 \otimes D$ (which corresponds to $V_{p-a+3} \otimes D^{a-2}$ for $a = p+2$) we consider the functions $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, where:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g^0_{1,[\lambda]}, \frac{p^2}{a_p}(Y^r - X^{r-p-2}Y^{p+2}) \right] + \left[g^0_{1,0}, \frac{1}{a_p}(X^{r-2}Y^2 - X^{p-1+p^2}Y^{r-p+1-p^2}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p^2} \sum_{\substack{0<j<r \\ j \equiv 3 \bmod p-1}} \alpha_j X^{r-j}Y^j \right],$$

where the α_j are chosen as in Lemma 1.8.

Note that as $r \equiv p+1 \bmod p^2$, we have $r - p + 1 - p^2 > 0$.

In the first part of f_1, we have $v(p^2/a_p) < -1$, so we consider $j = 0$ for the first part of $T^+ f_1$. Because $\binom{r}{0} - \binom{p+2}{0} = 0$, the first part of $T^+ f_1$ vanishes. For the second part, $v(1/a_p) < -3$, so we consider $j = 0, 1, 2$. Now see that $p^j/a_p(\binom{2}{j} - \binom{r-p+1-p^2}{j}) \equiv 0 \bmod p$ as $r \equiv a - 1 \bmod p^2$ for $j = 0, 1, 2$. Thus, $T^+ f_1 \equiv 0 \bmod p$. For $T^- f_1$ we consider $i = r$ and obtain

$$T^- f_1 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j \equiv 3 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j \right].$$

Thus:

$$T^- f_1 - a_p f_0 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j \equiv a \bmod p-1}} \left(\binom{r}{j} - \alpha_j \right) X^{r-j}Y^j \right]$$

which vanishes as $\binom{r}{j} \equiv \alpha_j \bmod p$ and $v(p^2/a_p) < -1$.

In f_0, we have $v(p^2/a_p^2) < -4$. For $T^- f_0$, the highest index $i = r - (p-1)$, so $p^{r-i} = p^{p-1}$ kills p^2/a_p^2 for $p \geq 5$. Finally, for $T^+ f_0$, we see that $T^+ f_0 \equiv \frac{p^5(p-1)}{a_p^2} \alpha_3 Y^{r-3}Y^3 \equiv 0 \bmod p$ as $\alpha_3 \equiv \binom{r}{3} \equiv 0 \bmod p$ as $r - 1 \equiv 0 \bmod p$.

$$\text{Thus, } (T - a_p)f \equiv -a_p f_1 \equiv \left[g^0_{1,0}, (X^{r-2}Y^2 - X^{p-1+p^2}Y^{r-p+1-p^2}) \right]$$

$$\equiv \left[g^0_{1,0}, \theta^2 \sum_{i=0}^{\frac{r-p-2-(p-1)(p+3)}{(p-1)}} (i+1)(X^{r-i(p-1)}Y^{i(p-1)}) \right].$$

By Lemma 5.1, each $X^{r-i(p-1)}Y^{i(p-1)}$ term (for $i \neq 0$) maps to Y. Thus, $\theta^2 \sum_{i=0}^{\frac{r-p-2-(p-1)(p+3)}{(p-1)}} (i+1)X^{r-i(p-1)}Y^{i(p-1)}$ maps to $\frac{-1}{p-1}Y$ and eliminates the factor $V_1 \otimes D$.

Hence, we are left with only $V_{p-2} \otimes D^2$ (which corresponds to the term $V_{a-4} \otimes D^2$ for $a = p+2$).

(v) If $r \equiv 2 \bmod p$, then to eliminate the factor $V_{p-4} \otimes D^3$ we use the functions as in Proposition 5.6 used to eliminate $V_{p-a-1} \otimes D^a$ (for $a = 3$) but note that $T^+ f_0$ has the term $\frac{p-1}{pa_p} p^3 \alpha_3 X^{r-3}Y^3 = \frac{p-1}{pa_p} p^3 \binom{r}{3} X^{r-3}Y^3$ by Lemma 1.8. As $p \mid r-2$ we see that $\binom{r}{3} = 0$, so $T^+ f_1 = 0$. The rest follows as in Proposition 5.6.

If $r \equiv p+2 \bmod p^2$, then to eliminate the factors coming from V_r^{**}/V_r^{***} we consider $f = f_0 + f_1 + f_2 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_2 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g^0_{2,p[\lambda]}, \frac{p^2}{a_p}[\lambda]^{p-3}(Y^r - X^{r-p-2}Y^{p+2}) \right] + \left[g^0_{2,0}, \frac{(1-p)}{a_p}(X^2Y^{r-2} - X^{r-p}Y^p) \right],$$

$$f_1 = \left[g^0_{1,0}, \frac{p^2(p-1)}{a_p^2} \sum_{\substack{0<j<r-2 \\ j \equiv p \bmod p-1}} \gamma_j X^{r-j}Y^j \right],$$

and

$$f_0 = \left[\mathrm{id}, \frac{pr}{a_p}(X^{r-1}Y - X^{r-p}Y^p) \right],$$

where the $\gamma_j \equiv \binom{r-2}{j} \bmod p^2$ are integers as in Lemma 1.11.

In f_2, for the first part we observe that $v(p^2/a_p) < -1$, so we consider only the term with $j = 0$ for the first part of $T^+ f_2$. For $j = 0$, we observe $\binom{r}{0} - \binom{p+2}{0} = 0$. Regarding the second part, we note that $v(1/a_p) < -3$, so we consider the terms with $j = 0, 1, 2$ for the second part of $T^+ f_2$. For $j = 0$, we see that $\binom{r-2}{0} - \binom{p}{0} = 0$. For $j = 1, 2$ we obtain $\frac{p^j}{a_p}(\binom{r-2}{j} - \binom{p}{j}) \equiv 0 \bmod p$ as $r \equiv p+2 \bmod p^2$. Thus $T^+ f_2 \equiv 0 \bmod p$.

In f_1 we see that $v(p^2/a_p^2) < -4$. Due to the properties of γ_j from Lemma 1.10, we have $\sum \binom{j}{n}\gamma_j \equiv 0 \bmod p^{5-n}$ and $j \equiv p \geq 5$, so the terms in $T^+ f_1$ vanish mod p. In f_1 the highest index i for which $c_i \not\equiv 0 \bmod p$ is $i = r-p-1$. So we have $p^{r-i} = p^{p+1}$, which kills p^2/a_p^2 as $p \geq 5$. Thus $T^- f_1 \equiv 0 \bmod p$.

For T^-f_2, we note that the highest terms for which $c_i \not\equiv 0$ are $i = r$ and $i = r-2$. In the case $i = r-2$ we note that it forces $j = r-2$ (as $\lambda = 0$), so the non-zero term is $\frac{p^2(1-p)}{a_p}\binom{r}{2}X^2Y^{r-2}$. If $i = r$, then

$$T^-f_2 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p} \sum_{\substack{p \le j \le r-2 \\ j \equiv a-2 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right]$$

The last term in the above expansion (when $j = r-2$) is $\frac{p^2\binom{r}{2}(p-1)}{a_p}X^2Y^{r-2}$, which is cancelled out by the term for $i = r-2$. The first term in the above expansion (for $j = 1$) is $\frac{p^2 r}{a_p}X^{r-1}Y$, which gets cancelled by the term from T^+f_0.

Thus:

$$T^-f_2 - a_pf_1 + T^+f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p} \sum_{\substack{0<j<r-2 \\ j \equiv p \bmod p-1}} \left(\binom{r}{j} - \gamma_j\right) X^{r-j}Y^j\right]$$

where the $\gamma_j \equiv \binom{r-2}{j} \bmod p$, so $T^-f_2 - a_pf_1 + T^+f_0 \equiv 0 \bmod p$ as $j \ge p$.

So $(T - a_p)f = -a_pf_2 \bmod p$, and we follow the argument as in Proposition 5.2.

If $p \parallel r-2$, then we consider $a = p+2$ in Proposition 5.4 to eliminate the factors from V_r^{**}/V_r^{***}. Hence, we are left with only $V_{p-2} \otimes D^2$ where it corresponds to $V_{p-a+1} \otimes D^2$ (for $a = 3$).. □

5.4 $r \equiv 4 \bmod p-1$

In the following proposition we eliminate all but one Jordan-Hölder factor. We note that while eliminating the factors from V_r^*/V_r^{**} we consider $a = 4$ but while eliminating the factors from V_r^{**}/V_r^{***}, we consider $a = p+3$, following the convention set in the beginning of the paper in Lemma 1.4.

Proposition 5.10. *If $r \equiv 4 \bmod p-1$ and:*

(i) *If $r \equiv 4 \bmod p$ then* $\mathrm{ind}_{KZ}^{G}(V_{p-5} \otimes D^4) \twoheadrightarrow \overline{\Theta}_{k,a_p}$.

(ii) *If $r \equiv 3 \bmod p$ then* $\mathrm{ind}_{KZ}^{G}(V_{p-3} \otimes D^3) \twoheadrightarrow \overline{\Theta}_{k,a_p}$.

(iii) *If* $r \equiv 2 \bmod p$ *then* $\operatorname{ind}_{KZ}^{G}(V_{p-1} \otimes D^2) \twoheadrightarrow \overline{\Theta}_{k,a_p}$.

(iv) *If* $r \not\equiv 2, 3, 4 \bmod p$ *then* $\overline{V}_{k,a_p} \cong \operatorname{ind}(\omega_2^{a+2p-1})$

Proof: First, we note that in the structure of Q in some cases we could not determine which factors from V_r^{**}/V_r^{***} appear in W, but we can eliminate those factors if $r \not\equiv 1, 2 \bmod p$, by considering the following function:

$$f = \left[\mathrm{id}, \frac{\theta^2}{a_p}(X^{r-3p-1}Y^{p-1} - X^{r-2p-2})\right].$$

Expanding yields $f = \dfrac{1}{a_p}(-X^{r-2}Y^2 + 3X^{r-p-1}Y^{p+1} - 3X^{r-2p}Y^{2p} + X^{r-3p+1}Y^{3p-1})$. For T^+f we see that $v(1/a_p) < -3)$, so we consider $j = 0, 1, 2$. For $j = 0$, we obtain $1/a_p(-1+3-3+1) = 0$. For $j = 1, 2$, we see that $p^j/a_p(-\binom{2}{j}+3\binom{p+1}{j}-3\binom{2p}{j}+\binom{3p-1}{j} = 0)$. Hence $T^+f \equiv 0 \bmod p$. For T^-f, we consider the highest $i = 3p-1$ and note that if $r \not\equiv 1, 2 \bmod p$, then $r - i > 2p$, which means that p^{r-i} kills $1/a_p$, so $T^-f \equiv 0 \bmod p$. Thus $(T - a_p)f \equiv a_p f \equiv \left[\mathrm{id}, \theta^2(X^{r-3p-1}Y^{p-1} - X^{r-2p-2})\right]$, which eliminates the factors from V_r^{**}/V_r^{***} by Lemma 5.1.

We also note that if $r \not\equiv 4 \bmod p$, then the functions from Proposition 5.6 can be used to eliminate the factor $V_{p-5} \otimes D^4$.

(i) Let $r \equiv 4 \bmod p$. To eliminate the factors from V_r^{*}/V_r^{**} we consider $f = f_0 + f_1 \in \operatorname{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g_{1,[\lambda]}^0, \frac{a_p[\lambda]^{p-2}}{p^3}(Y^r - X^{r-4}Y^4)\right] + \left[g_{1,0}^0, \frac{r a_p(1-p)}{p^4}(XY^{r-1} - X^{r-3}Y^3)\right],$$

and

$$f_0 = \left[\mathrm{id}, \frac{1}{p^3} \sum_{\substack{0<j<r-1 \\ j \equiv 3 \bmod p-1}} \beta_j X^{r-j} Y^j\right],$$

where the β_j are chosen as in Lemma 1.9.

In f_1 we have $v(a_p/p^3) < -1$ in the first part, so we consider $j = 0$. As $\binom{r}{0} - \binom{4}{0} = 0$, the first part of $T^+f_1 = 0$. In the second part we observe that $v(ra_p/p^4) < -2$, so we consider $j = 0, 1$. For $j = 0$ we obtain $\binom{r-1}{0} - \binom{3}{0} = 0$ while for $j = 1$ we see $\frac{rpa_p}{p^4}(\binom{r-1}{1} - \binom{3}{1}) \equiv 0 \bmod p$ due to $r \equiv 4 \bmod p$. Hence $T^+f_1 \equiv 0 \bmod p$. As $v(a_p) > 2$, we obtain $a_p f_1 \equiv 0 \bmod p$.

For T^-f_1, we consider $i = r$ (in the first part of f_1) and $i = r-1$ (in the second part of f_1). For the first part, we observe

$$T^- f_1 = \left[\mathrm{id}, \frac{a_p(p-1)}{p^3} \sum_{\substack{0<j\le r-1 \\ j\equiv 3 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right].$$

The last term (when $j = r-1$) is $\frac{r a_p}{p^3} XY^{r-1}$, which is cancelled out by the second part of T^-f_1. This yields

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{a_p(p-1)}{p^3} \sum_{\substack{0<j<r-1 \\ j\equiv 3 \bmod p-1}} \left(\binom{r}{j} - \beta_j\right) X^{r-j}Y^j\right]$$

which is zero $\bmod p$ as $\beta_j \equiv \binom{r}{j} \bmod p$ and $v(a_p/p^3) < -1$.

In f_0 as the highest $i = r - p$, we see that $T^- f_0 \equiv 0 \bmod p$. However, for $j = 3$, we obtain $T^+ f_0 = [g^0_{1,0}, \beta_3 X^{r-3}Y^3] \equiv [g^0_{1,0}, 4X^{r-3}Y^3] \bmod p$.

Hence, $(T - a_p)f = T^+ f_0 = [g^0_{1,0}, 4X^{r-3}Y^3]$. Since XY^{r-1} maps to zero in Q, we see that $(T - a_p)f = T^+ f_0 \equiv [g^0_{1,0}, 4(X^{r-3}Y^3 - XY^{r-1})]$. Now, we follow the argument as in Theorem 8.6 of [BG15] (for $a = 4$) and see that we can eliminate the factors from V_r^*/V_r^{**}.

To eliminate the factor $V_0 \otimes D$, we consider the function in the beginning of the proof above as $r \equiv 4 \not\equiv 1, 2 \bmod p$.

Thus, $V_{p-5} \otimes D^4$ is the only remaining factor.

(ii) If $r \equiv 3 \bmod p$ then the factors from V_r^{**}/V_r^{***} and $V_{p-5} \otimes D^4$ can be eliminated as explained in the beginning of the proof. Thus, $V_{p-3} \otimes D^3$ is the only surviving factor.

(iii) If $r \equiv 2 \bmod p$ we see that $r \not\equiv 3, 4 \bmod p$, so we can use the functions from Proposition 5.6 to eliminate the factors $V_{p-3} \otimes D^3$ and $V_{p-5} \otimes D^4$. Hence, we are left with $V_{p-1} \otimes D^2$.

(iv) If $r \not\equiv 2, 3, 4 \bmod p$ then the functions from Proposition 5.6 can be used to eliminate the factor $V_{p-5} \otimes D^4$. As $r \not\equiv 3, 4 \bmod p$ we can use the functions from Proposition 5.6 to eliminate the factor $V_{p-3} \otimes D^3$. Hence, the only surviving factor is $V_{p-1} \otimes D^2$ or $V_0 \otimes D^2$. As in [BG15] Theorem 8.4, both give the same induced representation. □

5.5 $r \equiv p + 1 \bmod p - 1$

In the following proposition we eliminate all but one Jordan-Hölder factor. We note that while eliminating the factors from V_r^*/V_r^{**} and V_r^{**}/V_r^{***}, we consider $a = p+1$, following the convention set in the beginning of the paper in Lemma 1.4.

Proposition 5.11. *If $r \equiv p + 1 \bmod p - 1$ then*

(i) *If $r \not\equiv 0, 1 \bmod p$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_2) \to \overline{\Theta}_{k,a_p}.$$

(ii) *If $r \equiv 1 \bmod p$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_{p-1} \otimes D) \to \overline{\Theta}_{k,a_p}.$$

(iii) *If $p \parallel r - p$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_0 \otimes D) \to \overline{\Theta}_{k,a_p}.$$

(iv) *If $p^2 \mid r - p$, then there is a surjection*

$$\mathrm{ind}_{KZ}^{G}(V_{p-3} \otimes D^2) \to \overline{\Theta}_{k,a_p}.$$

Proof: We consider the latter two cases and further separate the congruence conditions modulo p^2.

(i) If $r \not\equiv 0, 1 \bmod p$, then by Proposition 4.5 we know that V_2 is the only factor.

(ii) If $p^2 \nmid r - p - 1$, then to eliminate the factors from V_r^{**}/V_r^{***} we use the functions in Proposition 5.4 with $a = p + 1$. We see that $(T - a_p)f$ maps to a non-zero element in V_r^{**}/V_r^{***} as $p^2 \nmid r$. Hence, the only remaining factor is $V_{p-1} \otimes D$.

If $p^2 \mid r-p-1$, then to eliminate the factors $V_{p-3} \otimes D^2$ and V_2 from V_r^{**}/V_r^{***}

we consider $f = f_0 + f_1 + f_2 \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_2 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g^0_{2,p[\lambda]}, \frac{p}{a_p}[\lambda]^{p-3}(\mathrm{Y}^r - \mathrm{X}^{r-p-1}\mathrm{Y}^{p+1}) \right]$$
$$+ \left[g^0_{2,0}, \frac{\binom{r}{2}(1-p)}{pa_p}(\mathrm{X}^2\mathrm{Y}^{r-2} - \mathrm{X}^{r-p+1}\mathrm{Y}^{p-1}) \right]$$
$$f_1 = \left[g^0_{1,0}, \frac{p(p-1)}{a_p^2} \sum_{\substack{0<j<r-2 \\ j \equiv p-1 \bmod p-1}} \gamma_j \mathrm{X}^{r-j}\mathrm{Y}^j \right], \quad \text{and}$$
$$f_0 = \left[\mathrm{id}, \frac{(p-1)p}{a_p}(\mathrm{X}^r - \mathrm{X}^p\mathrm{Y}^{r-p}) \right],$$

where the γ_j are chosen as in Lemma 1.10 with the extra condition that $\gamma_j \equiv \binom{r}{j} \bmod p^2$ and $\sum \binom{j}{n}\gamma_j \equiv 0 \bmod p^{5-n}$.

In f_2, we note that in the first part $v(p/a_p) < -2$, so for $\mathrm{T}^+ f_2$ we consider $j = 0, 1$. For $j = 0$ we see that $\binom{r}{0} - \binom{p+1}{0} = 0$ while for $j = 1$ we see $\frac{p^2}{a_p}(\binom{r}{1} - \binom{p+1}{1}) \equiv 0 \bmod p$ as $p^2 \mid r - p - 1$. In the second part of f_2 we have $v(\binom{r}{2}/pa_p) < -3$, so we consider $j = 0, 1, 2$. For $j = 0$ we see that $\binom{r-2}{0} - \binom{p-1}{0} = 0$ while for $j = 1, 2$ we see $\frac{p^j\binom{r}{2}}{pa_p}(\binom{r-2}{j} - \binom{p-1}{j}) \equiv 0 \bmod p$ as $p^2 \mid r - p - 1$. Thus, $\mathrm{T}^+ f_2 \equiv 0 \bmod p$.

In f_1 we see that $v(p/a_p^2) < -5$. Since $\sum \binom{j}{n}\gamma_j \equiv 0 \bmod p^{5-n}$, we obtain $\mathrm{T}^+ f_1 \equiv 0 \bmod p$. Note that for $p = 5$, $\mathrm{T}^+ f_1 = \frac{p^5(p-1)}{a_p^2}\gamma_4\binom{4}{4}\mathrm{X}^{r-4}\mathrm{Y}^4 \equiv 0 \bmod p$ as $\gamma_4 \equiv \binom{r}{4} \equiv 0 \bmod p$. Since the highest index $i = r - p - 1$ we see that $p^{r-i} = p^{p+1}$ kills p/a_p^2 hence $\mathrm{T}^- f_1 \equiv 0 \bmod p$.

For $\mathrm{T}^- f_2$, for the first part $(i = r)$ we that:

$$\mathrm{T}^- f_2 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0 \le j \le r-2 \\ j \equiv p-1 \bmod p-1}} \binom{r}{j} \mathrm{X}^{r-j}\mathrm{Y}^j \right].$$

The last term (when $j = r-2$) is $\frac{\binom{r}{2}p}{a_p}\mathrm{X}^2\mathrm{Y}^{r-2}$, which is cancelled out by the second part of $\mathrm{T}^- f_1$ $(i = r-2)$. The first term (when $j = 0$) gets cancelled by $\mathrm{T}^- f_0 \equiv 0 \bmod p$ but $\mathrm{T}^+ f_0 = [g^0_{1,0} \frac{(1-p)p}{a_p}\mathrm{X}^r]$

This tells us:

$$T^- f_2 - a_p f_1 + T^+ f_0 = \left[\mathrm{id}, \frac{(p-1)p}{a_p} \sum_{\substack{0<j<r-2 \\ j\equiv p-1 \bmod p-1}} \left(\binom{r}{j} - \gamma_j\right) X^{r-j} Y^j\right]$$

which is zero mod p as $\gamma_j \equiv \binom{r}{j} \bmod p^2$ while $v(p/a_p) < -2$.

Thus, noting that $\dfrac{r-1}{p} \equiv 1 \bmod p$, we see that $(T - a_p)f \equiv -a_p f_2$, which is equivalent to:

$$\begin{aligned} &\left[g^0_{2,0}, \frac{\binom{r}{2}}{p}(X^2Y^{r-2} - X^{r-p+1}Y^{p-1})\right] \\ \equiv &\left[g^0_{2,0}, \frac{-1}{2}\theta^2 \Big(\sum_{i=0}^{(r-3p+1)/(p-1)} (X^{r-p+3}Y^{p-3} + Y^{r-2p-2})\right] \end{aligned}$$

The rest follows as in previous cases to eliminate the factors from V_r^{**}/V_r^{***} leaving $V_0 \otimes D$ as the only remaining factor.

(iii) $p \parallel r - p$.

To eliminate the factor $V_0 \otimes D$, we use the functions from Proposition 5.6 to eliminate the factor $V_{p-a+1} \otimes D^{a-1}$ with $a = p + 1$ and noting that $p \parallel r - p$.

To eliminate the factor $V_{p-3} \otimes D^2$, we consider the function $f = f_0 + f_1 + f_2 \in \mathrm{ind}_{KZ}^{G} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$ for

$$f_2 = \sum_{\lambda\in\mathbb{F}_p} \left[g^0_{2,p[\lambda]}, \frac{1}{p}(Y^{r-2} - X^{r-p+1}Y^{p-1})\right],$$

$$f_1 = \left[g^0_{1,0}, \frac{(p-1)}{pa_p} \sum_{\substack{0<j<r-2 \\ j\equiv p-1 \bmod p-1}} \alpha_j X^{r-j}Y^j\right],$$

where the $\alpha_j \equiv \binom{r-2}{j} \bmod p$ are chosen as in Lemma 1.8 and

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{p}(X^r - X^pY^{r-p})\right].$$

For $T^+ f_2$ we have $v(1/p) = -1$, so we consider $j = 0, 1$. For $j = 0$, we have $\binom{r-2}{0} - \binom{p-1}{0} = 0$ while for $j = 1$ the coefficient of $X^{r-1}Y$ is integral and hence zero in Q. Thus $T^+ f_2 \equiv 0 \bmod p$. As $v(a_p) > 2$, we see that $a_p f_2 \equiv 0 \bmod p$.

For f_1, we have $v(1/pa_p) < -4$. We have that $\sum \binom{j}{n} \gamma_j \equiv 0 \bmod p^{4-n}$ and the smallest $j = p-1 \geq 4$, so $T^+ f_1 = 0 \bmod p$. Since the highest index i for which $c_i \not\equiv 0 \bmod p$ is $i = r-p-1$, we see that $p^{r-i} = p^{p+1}$, which kills $1/pa_p$. Hence, $T^- f_1 \equiv 0 \bmod p$

For $T^- f_2$, the highest $i = r-2$, so

$$T^- f_2 = \left[g_{1,0}^0, \frac{(p-1)}{p} \sum_{\substack{0 \leq j \leq r-2 \\ j \equiv p-1 \bmod p-1}} \binom{r-2}{j} X^{r-j} Y^j \right].$$

We see that $T^- f_0 \equiv 0 \bmod p$ but $T^+ f_0 = [g_{1,0}^0 \frac{(1-p)}{p} X^r]$ Thus $(T - a_p)f$ is equivalent to:

$$T^- f_2 - a_p f_1 + T^+ f_0 = \left[g_{1,0}^0, \frac{(p-1)}{p} \left(\sum_{\substack{0 < j < r-2 \\ j \equiv p-1 \bmod p-1}} \left(\binom{r-2}{j} - \alpha_j \right) X^{r-j} Y^j + p X^2 Y^{r-2} \right) \right]$$

As $\alpha_j \equiv \binom{r-2}{j} \bmod p$, the above sum is integral. By changing the above polynomial by a suitable $X^2 Y^{r-2}$ term we can rewrite it as

$$T^- f_2 - a_p f_1 + T^+ f_0 = \left[g_{1,0}^0, (p-1) F(X, Y) + \frac{r-p}{p} \theta^2 Y^{r-2p-2} \right]$$

where

$$F(X, Y) = \frac{1}{p} \sum_{\substack{0 < j < r-2 \\ j \equiv p-1 \bmod p-1}} \left(\binom{r-2}{j} - \alpha_j \right) X^{r-j} Y^j + (p-r)(X^{2p} Y^{r-2p} - 2X^{p+1} Y^{r-p-1}).$$

By Lemma 1.6 and Lemma 1.4 we see that $F(X, Y) \in V_r^{***}$, so by Lemma 5.1 $(T - a_p)f$ maps to $\left[g_{1,0}^0, \frac{p-r}{p} Y^{p-3} \right]$, which is not zero as $r \not\equiv p \bmod p^2$. Hence, the only remaining factor is V_2.

(iv) $p^2 \mid r - p$.

To eliminate the factor $V_0 \otimes D$ we consider the functions $f = f_1 + f_0 \in \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, where:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g_{1,[\lambda]}^0, \frac{p^2}{a_p} (Y^r - X^{r-p-1} Y^{p+1}) \right] + \left[g_{1,0}^0, \frac{1}{a_p} (X^{r-1} Y - X^{p-1+p^2} Y^{r-p+1-p^2}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p^2} \sum_{\substack{0<j<r \\ j\equiv 2 \bmod p-1}} \alpha_j X^{r-j}Y^j\right],$$

where the α_j are chosen as in Lemma 1.8.

In the first part of f_1, we have $v(p^2/a_p) < -1$, so we consider $j = 0$ for the first part of $T^+ f_1$. Because $\binom{r}{0} - \binom{p+1}{0} = 0$, the first part of $T^+ f_1$ vanishes. For the second part, $v(1/a_p) < -3$, so we consider $j = 0, 1, 2$. Now see that $p^j/a_p(\binom{1}{j}) - \binom{r-p+1-p^2}{j}) \equiv 0 \bmod p$ as $r \equiv 0 \bmod p^2$ for $j = 0, 1, 2$. Thus, $T^+ f_1 \equiv 0 \bmod p$. For $T^- f_1$ we consider $i = r$ and obtain

$$T^- f_1 \equiv \left[\mathrm{id}, \sum_{\substack{0<j<r \\ j\equiv 2 \bmod p-1}} \frac{p^2(p-1)}{a_p}\binom{r}{j} X^{r-j}Y^j\right].$$

Thus:

$$T^- f_1 - a_p f_0 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j\equiv 2 \bmod p-1}} \left(\binom{r}{j} - \alpha_j\right) X^{r-j}Y^j\right]$$

which vanishes as $\binom{r}{j} \equiv \alpha_j \bmod p$ and $v(p^2/a_p) < -1$.

In f_0, we have $v(p^2/a_p^2) < -4$. For $T^- f_0$, the highest index $i = r - (p-1)$, so $p^{r-i} = p^{p-1}$ kills p^2/a_p^2 for $p \geq 5$. Finally, for $T^+ f_0$, we see that $\sum \binom{j}{n}\alpha_j \equiv 0 \bmod p^{4-n}$ and $j \equiv 2 \geq 4$, so $T^+ f_0 = \frac{p^4}{a_p^2}\alpha_2 \equiv 0 \bmod p$ as $\alpha_2 \equiv \binom{r}{2} \equiv 0 \bmod p^2$.

Hence $(T - a_p)f \equiv -a_p f_1 \equiv [g_{1,0}^0, X^{r-1}Y - X^{p-1+p^2}Y^{r-p+1-p^2}]$

$$\equiv \left[g_{1,0}^0, \theta \sum_{i=0}^{\frac{r-a-(p-1)(p+2)}{(p-1)}} X^{r-i(p-1)}Y^{i(p-1)}\right] \bmod V_r^{**}.$$

By an argument similar to Lemma 5.1 $\psi^{-1}\colon X^{r-2p}Y^{p-1} \mapsto X^{a-2}Y^{p-1}$ and $\beta\colon X^{p-1}Y^{p-1} \mapsto Y^0$. Thus, $[g_{1,0}^0, \theta \sum_{i=0}^{\frac{r-p-1-(p-1)(p+2)}{(p-1)}} X^{r-i(p-1)}Y^{i(p-1)}]$ maps to $[g_{1,0}^0, \frac{2}{p-1}Y^0]$ and eliminates the factor $V_0 \otimes D$.

To eliminate the factor V_2 we consider the functions $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, where:

$$f_1 = \sum_{\lambda\in\mathbb{F}_p}\left[g_{1,[\lambda]}^0, \frac{p^2}{a_p}(Y^r - X^{r-p-1}Y^{p+1})\right] + \left[g_{1,0}^0, \frac{1}{a_p}(X^{r-2}Y^2 - X^{p-2+p^2}Y^{r-p+2-p^2})\right],$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p^2} \sum_{\substack{0<j<r \\ j\equiv 2 \bmod p-1}} \alpha_j X^{r-j}Y^j\right],$$

and where the α_j are chosen as in Lemma 1.8.

In the first part of f_1, we have $v(p^2/a_p) < -1$, so we consider $j = 0$ for the first part of T^+f_1. Because $\binom{r}{0} - \binom{p+1}{0} = 0$, the first part of T^+f_1 vanishes. For the second part, $v(1/a_p) < -3$, so we consider $j = 0, 1, 2$. Now see that $p^j/a_p(\binom{2}{j} - \binom{r-p+2-p^2}{j}) \equiv 0 \bmod p$ as $r \equiv 0 \bmod p^2$ for $j = 0, 1, 2$. Thus, $T^+f_1 \equiv 0 \bmod p$. For T^-f_1 we consider $i = r$ and obtain

$$T^-f_1 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j\equiv 2 \bmod p-1}} \binom{r}{j} X^{r-j}Y^j\right].$$

Thus:

$$T^-f_1 - a_pf_0 \equiv \left[\mathrm{id}, \frac{p^2(p-1)}{a_p} \sum_{\substack{0<j<r \\ j\equiv 2 \bmod p-1}} \left(\binom{r}{j} - \alpha_j\right) X^{r-j}Y^j\right],$$

which vanishes as $\binom{r}{j} \equiv \alpha_j \bmod p$ and $v(p^2/a_p) < -1$.

In f_0, we have $v(p^2/a_p^2) < -4$. For T^-f_0, the highest index $i = r - (p-1)$, so $p^{r-i} = p^{p-1}$ kills p^2/a_p^2 for $p \geq 5$. Finally, for T^+f_0, we see that $\sum \binom{j}{n}\alpha_j \equiv 0 \bmod p^{4-n}$ and $j \equiv 2 \geq 4$, so $T^+f_0 = \frac{p^4}{a_p^2}\alpha_2 \equiv 0 \bmod p$ as $\alpha_2 \equiv \binom{r}{2} \equiv 0 \bmod p^2$.

Thus,

$$\begin{aligned} (T - a_p)f &\equiv - a_pf_1 \equiv \left[g_{1,0}^0, (X^{r-2}Y^2 - X^{p-2+p^2}Y^{r-p+2-p^2})\right] \\ &\equiv \left[g_{1,0}^0, \theta^2 \sum_{i=0}^{\frac{r-p-1-(p-1)(p+3)}{(p-1)}} (i+1)(X^{r-i(p-1)}Y^{i(p-1)})\right]. \end{aligned}$$

By Lemma 5.1, each $X^{r-i(p-1)}Y^{i(p-1)}$ term (for $i \neq 0$) maps to Y^2. Thus, $\theta^2 \sum_{i=0}^{\frac{r-p-1-(p-1)(p+3)}{(p-1)}} (i+1)X^{r-i(p-1)}Y^{i(p-1)}$ maps to $\frac{-1}{p-1}Y^2$ and eliminates the factor V_2. Hence, the only remaining factor is $V_{p-3} \otimes D^2$. □

6 Separating Reducible and Irreducible cases

We follow the methods of [BG15, Section 9] to separate the reducible and irreducible cases when $\overline{\Theta}_{k,a_p}$ is a quotient of $\mathrm{ind}(V_{p-2} \otimes D^n)$. This happens in Proposition 5.9 (for $a = 3$), Proposition 5.6 (for $a = 5$) and Proposition 5.5 (for $a = p$ and $p^2 \mid p - r$). By [BG13, Lemma 3.2], we need to check whether $\overline{\Theta}$ is a quotient

- of $(\mathrm{ind}_{KZ}^{G} V_{p-2})/T$ (in which case we obtain irreducibility), or
- of $(\mathrm{ind}_{KZ}^{G} V_{p-2})/(T^2 - cT + 1)$ for some c in $\overline{\mathbb{F}}_p$ (in which case we obtain reducibility).

The following two theorems are based on [BG15, Theorem 9.1]:

Theorem 6.1. *Let $r \equiv 3 \bmod p-1$ and $r \equiv 1, 2 \bmod p$. Then $\overline{V}_{k,a_p}$ is irreducible.*

Proof: First, we note that the factor $V_{p-2} \otimes D^2$ appears in two different forms. When $r \equiv 2 \bmod p$, then the factor $V_{p-2} \otimes D^2 = V_{p-a+1} \otimes D^2$ (for $a = 3$). If $r \equiv p+1 \bmod p^2$ then it appears as $V_{a-4} \otimes D^2$ (for $a = p+2$). If $p \parallel r-p-1$ then it appears as $V_{p-a+1} \otimes D^2$ (for $a = 3$). Hence, we need to treat them separately.

- $r \equiv 2 \bmod p$ then have two further considerations.

  If $p^2 \mid r-2$, we consider $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G} \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by:

$$f_1 = \sum_{\lambda \in \mathbb{F}_p^*} \left[g_{1,[\lambda]}^0, \frac{1}{a_p} \theta(X^{r-2p-2}Y - Y^{r-p-1}) \right] + \left[g_{1,0}^0, \frac{p(1-p)}{a_p} (XY^{r-1} - X^{r-p-1}Y^{p+1}) \right],$$

$$f_0 = \left[\mathrm{id}, \frac{p^2(p-1)}{a_p^2} \sum_{\substack{0<j<r-1 \\ j \equiv 2 \bmod p-1}} \alpha_j X^{r-j} Y^j \right],$$

  where the integers α_j are those given in Lemma 1.8 that satisfy $\alpha_j \equiv \binom{r-1}{j} \bmod p$.

  In the first part of f_1 we have $v(1/a_p) < -3$, so we consider $j = 0, 1, 2$. Expanding f_1 in the first part yields $\theta(X^{r-2p-2}Y - Y^{r-p-1}) = X^{r-2}Y^2 - X^pY^{r-p} - Y^{p+1} + Y^{r-1}$. For $j = 0$ we obtain $p^0/a_p(\binom{2}{0} - \binom{r-p}{0} - \binom{p+1}{0} + \binom{r-1}{0}) = 0$. For $j = 1$ we obtain $p/a_p(\binom{2}{1} - \binom{r-p}{1} - \binom{p+1}{1} + \binom{r-1}{1}) = 0$, while for $j = 2$, we obtain $p^2/a_p(\binom{2}{2} - \binom{r-p}{2} - \binom{p+1}{2} + \binom{r-1}{2}) \equiv 0 \bmod p$ as $p \mid r-2$. Regarding the second part, $v(p/a_p) < -2$, so we consider $j = 0, 1$. For $j = 0$, we see that

T^+f_1 is identically zero, while for $j=1$ we obtain $p^2/a_p(\binom{r-1}{1}-\binom{p+1}{1})\equiv 0 \bmod p$ as $p \mid r-2$. Thus, we obtain $T^+f_1 \equiv 0 \bmod p$.

In f_0 we have $v(p^2/a_p^2) < -4$. By the properties of the α_j we have $\sum \binom{j}{n}\alpha_j \equiv 0 \bmod p^{4-n}$, so $T^+f_0 \equiv \frac{p^4}{a_p^2}\alpha_2 X^{r-2}Y^2$. But $\alpha_2 \equiv \binom{r-1}{2} \equiv 0 \bmod p^2$ as $r-2\equiv 0 \bmod p^2$, so $T^+f_0 \equiv 0 \bmod p$. The highest index of a nonzero coefficient in f_0 is $i = r-p$. Since $p \geq 5$, we see that T^-f_0 vanishes as well.

For T^-f_1 we consider $i = r-1$ and see that:

$$T^-f_1 \equiv \left[\mathrm{id}, \frac{p(p-1)}{a_p}\sum_{\substack{0<j<r-1\\ j\equiv 2 \bmod p-1}}\binom{r-1}{j}X^{r-j}Y^j + \frac{p^2(1-p)}{a_p}XY^{r-1}\right]$$

The last term above is cancelled by the second part (when $i=r-1$) of $T^-f_1 \equiv [\mathrm{id}, \frac{p^2}{a_p}XY^{r-1}]$.

Thus, we compute that:

$$T^-f_1 - a_pf_0 \equiv \left[\mathrm{id}, \frac{p(p-1)}{a_p}\sum_{\substack{0<j<r-1\\ j\equiv 2 \bmod p-1}}\left(\binom{r-1}{j} - p\alpha_j\right)X^{r-j}Y^j\right]$$

which vanishes as $v(p/a_p) < 2$ and $\binom{r-1}{j} - p\alpha_j \equiv 0 \bmod p^2$.

Thus, $(T-a_p)f = -a_pf_1 = \sum_{\lambda\in\mathbb{F}_p}[g^0_{1,[\lambda]}, \theta(X^{r-2p-2}Y - Y^{r-p-1})]$, which maps to:

$$\sum_{\lambda\in\mathbb{F}_p}[g^0_{1,[\lambda]}, -X^{p-2}]$$

by [BG15, Lemma 8.5]. This equals $-T([\mathrm{id}, X^{p-2}])$. Then, as in [BG15, Theorem 9.1], the reducible case cannot occur.

If $p \parallel r-2$, then we consider $f = f_0 + f_1 \in \mathrm{ind}_{KZ}^{G}\mathrm{Sym}^r\,\overline{\mathbb{Q}}_p^2$, given by::

$$f_1 = \sum_{\lambda\in\mathbb{F}_p}\left[g^0_{1,[\lambda]}, \frac{a_p}{p^3}\theta(X^{r-2p-2}Y - Y^{r-p-1})\right]$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{p^3}\sum_{\substack{0<j<r-1\\ j\equiv 2 \bmod p-1}}\alpha_jX^{r-j}Y^j\right],$$

where the integers α_j are those given in Lemma 1.8 that satisfy $\alpha_j \equiv \binom{r-1}{j} \bmod p$.

In f_1 we have $v(a_p/p^3) < -1$, so we consider $j = 0$. Expanding f_1 in the first part yields $\theta(X^{r-2p-2}Y - Y^{r-p-1}) = X^{r-2}Y^2 - X^pY^{r-p} - Y^{p+1} + Y^{r-1}$. For $j = 0$ we obtain $p^0/a_p(\binom{2}{0} - \binom{r-p}{0} - \binom{p+1}{0} + \binom{r-1}{0}) = 0$. Thus, we obtain $T^+f_1 \equiv 0 \mod p$.

For T^-f_1 we consider $i = r-1$. Because $v(a_p) > 2$,

$$T^-f_1 \equiv \left[\mathrm{id}, \frac{a_p(p-1)}{p^2} \sum_{\substack{0<j<r-1 \\ j\equiv 2 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j\right] \equiv 0 \mod p.$$

The highest index of a nonzero coefficient in f_0 is $i = r-p$. Since $p \geq 5$, we see that T^-f_0 vanishes as well. In f_0 we have $v(1/p^2) = -2$. By the properties of the α_j we have $\sum \binom{j}{n}\alpha_j \equiv 0 \mod p^{4-n}$, so (for $j = 2$) $T^+f_0 \equiv \frac{p^2}{p^3}\alpha_2 X^{r-3}Y^3$ but as $\alpha_2 \equiv \binom{r-1}{2} \mod p$, we see that $T^+f_0 \equiv \frac{\binom{r-1}{2}}{p}X^{r-2}Y^2 \mod p$. As $v(a_p) > 2$ and each $\binom{r-1}{j} \equiv 0 \mod p$ for $0 < j < r-1$ we see that $a_pf_0 \equiv 0 \mod p$. Thus, $(T - a_p)f = T^+f_0 \equiv \sum_{\lambda\in\mathbb{F}_p}[g^0_{1,[\lambda]}, \frac{\binom{r-1}{2}}{p}(X^{r-2}Y^2 - XY^{r-1})]$, which as in [BG15, Theorem 9.1] equals $-T([\mathrm{id}, X^{p-2}])$. Thus, the reducible case cannot occur.

- If $p^2 \mid r-p-1$, we consider $f = f_0 + f_1 \in \mathrm{ind}^G_{KZ} \mathrm{Sym}^r\, \overline{\mathbb{Q}}_p^2$, given by: :

$$f_1 = \sum_{\lambda\in\mathbb{F}_p}\left[g^0_{1,[\lambda]}, \frac{a_p}{p^3}\theta(X^{r-2p-2}Y - Y^{r-p-1})\right]$$

$$f_0 = \left[\mathrm{id}, \frac{(p-1)}{p^2} \sum_{\substack{0<j<r-2 \\ j\equiv 1 \bmod p-1}} \beta_j X^{r-j}Y^j\right],$$

where the integers β_j are those given in Lemma 1.9 that satisfy $\beta_j \equiv \binom{r-1}{j} \mod p$. In f_1 we have $v(a_p/p^3) < -1$, so we consider $j = 0$. Expanding f_1 in the first part yields $\theta(X^{r-2p-2}Y - Y^{r-p-1}) = X^{r-2}Y^2 - X^pY^{r-p} - X^{r-p-1}Y^{p+1} + XY^{r-1}$. For $j = 0$ we obtain $p^0/a_p(\binom{2}{0} - \binom{r-p}{0} - \binom{p+1}{0} + \binom{r-1}{0}) = 0$. Thus, we obtain $T^+f_1 \equiv 0 \mod p$.

For T^-f_1 we consider $i = r-1$. As $v(a_p) > 2$,

$$T^-f_1 \equiv \left[\mathrm{id}, \frac{a_p(p-1)}{p^2} \sum_{\substack{0<j<r-2 \\ j\equiv 1 \bmod p-1}} \binom{r-1}{j} X^{r-j}Y^j\right] \equiv 0 \mod p$$

The highest index of a nonzero coefficient in f_0 is $i = r - p$. Since $p \geq 5$, we see that $T^- f_0$ vanishes as well. In f_0 we have $v(1/p^2) = -2$. By the properties of the β_j we have $\sum \binom{j}{n}\beta_j \equiv 0 \bmod p^{4-n}$, so (for $j = 1$) $T^+ f_0 \equiv \frac{p}{p^2}\beta_1 X^{r-1}Y^1$ but as $\beta_1 \equiv \binom{r-1}{1} \bmod p$, we see that $T^+ f_0 \equiv \frac{r-1}{p} X^{r-1}Y \bmod p$. As $v(a_p) > 2$, we see that $a_p f_0 \equiv 0 \bmod p$.

Thus, $(T - a_p)f = T^+ f_0 \equiv \sum_{\lambda \in \mathbb{F}_p}[g^0_{1,[\lambda]}, \frac{r-1}{p}(X^{r-1}Y - X^2Y^{r-2})]$, which, as in [BG15, Theorem 9.1], equals $-T([\mathrm{id}, X^{p-2}])$ (where we note that $\frac{r-1}{p} \equiv 1 \bmod p$). Thus, the reducible case cannot occur.

- If $p \parallel r - p - 1$, then we consider the same functions as in the case $p \parallel r - 2$ and observe that $\alpha_2 \equiv \binom{r-1}{2} \not\equiv 0 \bmod p$ as $p \parallel r - 2$. The rest of the argument is the same. □

Theorem 6.2. *Let $r \equiv 5 \bmod p - 1$ and $r \equiv 2, 3 \bmod p$. Then $\overline{V}_{k,a_p}$ is irreducible.*

Proof: We consider $f = f_1 + f_0 \in \mathrm{ind}^G_{KZ} \operatorname{Sym}^r \overline{\mathbb{Q}}_p^2$, where

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g^0_{1,[\lambda]}, \frac{\theta^2}{a_p}(X^{r-2p-3}Y - Y^{r-2p-2}) \right]$$

and

$$f_0 = \left[\mathrm{id}, \frac{p^2(p-1)}{a_p^2} \sum_{\substack{0<j<r-2 \\ j \equiv 3 \bmod p-1}} \alpha_j X^{r-j}Y^j \right],$$

where the α_j are chosen similar to Lemma 1.8 with the condition that $\alpha_j \equiv \binom{r-2}{j} \bmod p$.

In the first part of f_1 as $v(1/a_p) < -3$, we consider $j = 0, 1, 2$ for $T^+ f_1$. We see that $\theta^2(X^{r-2p-3}Y - Y^{r-2p-2}) = X^{r-2p-1}Y^{2p+1} - 2X^{r-p-2}Y^{p+2} + X^{r-3}Y^3 + X^2Y^{r-2} - 2X^{p+1}Y^{r-p-1} + X^{2p}Y^{r-2p}$. For $j = 0, 1$ we obtain that $T^+ f_1$ is identically zero. For $j = 2$ we see that $\sum a_i \binom{i}{2} \equiv 0 \bmod p$ where a_i is the coefficient of $X^{r-i}Y^i$ in $\theta^2(X^{r-2p-3}Y - Y^{r-2p-2})$, so $T^+ f_1 \equiv 0 \bmod p$.

In f_0 we have $v(p^2/a_p^2) < -4$. As $\sum_j \binom{j}{n}\alpha_j \equiv 0 \bmod p^{4-n}$ we obtain $T^+ f_0 \equiv \frac{p^5(p-1)}{a_p^2}\binom{r-2}{3} X^{r-3}Y^3 \equiv 0 \bmod p$ since $r \equiv 2, 3 \bmod p$. Finally, in f_0 the highest $i = r - p - 1$, so $p^{r-i} = p^{p+1}$ kills p^2/a_p^2 for $p \geq 5$. Thus, $T^- f_0 \equiv 0 \bmod p$.

For $T^- f_1$ we consider $i = r - 2$ and obtain:

$$T^- f_1 = \left[\mathrm{id}, \sum_{\substack{0<j<r-2 \\ j \equiv 3 \bmod p-1}} \frac{(p-1)p^2}{a_p} \binom{r-2}{j} X^{r-j}Y^j + \frac{p^3}{a_p} X^2Y^{r-2} \right].$$

As $v(a_p) < 3$ we obtain:

$$T^- f_1 - a_p f_0 = \left[\mathrm{id}, \frac{(p-1)p^2}{a_p} \sum_{\substack{0<j<r-2 \\ j \equiv 3 \bmod p-1}} \left(\binom{r-2}{j} - \alpha_j\right) X^{r-j} Y^j\right],$$

which dies mod p as $\alpha_j \equiv \binom{r-2}{j}$ mod p.

Hence, $(T - a_p)f = -a_p f_1 = \sum_{\lambda \in \mathbb{F}_p} [g^0_{1,[\lambda]}, \theta^2(X^{r-2p-3}Y - Y^{r-2p-2})]$.

By Lemma 5.1 this maps to $\sum_{\lambda \in \mathbb{F}_p} [g^0_{1,[\lambda]}, -X^{p-2}]$, which equals $-T[\mathrm{id}, X^{p-2}]$. Thus, the reducible case cannot occur. □

Remark. In the above theorem we imposed the additional condition that $p \mid (r-2)(r-3)$ instead of just $p \nmid r-5, r-4$ as in Proposition 5.6. This is also equivalent to saying $v(a_p^2) < v(\binom{r-2}{3}p^5)$ as in the remark after the main theorem. Without these conditions $T^+ f_0 \equiv \frac{p^5(p-1)}{a_p^2}\binom{r-2}{3}X^{r-3}Y^3$ but, unlike in the proof of [BG15, Theorem 9.1], we cannot subtract the term X^2Y^{r-2}, so that the value is in V_r^{**}/V_r^{***}, which would then map to X^{p-2}.

Theorem 6.3 (Extension of [BG15, Theorem 9.2]). *Let $r \equiv p \bmod p-1$ and $p^2 \mid p - r$. If $p = 5$ and $v(a_p^2) = 5$ then assume that $v(a_p^2 - p^5) = 5$. Then:*

(i) *If $p^3 \nmid p - r$, then $\overline{V}_{k,a_p}$ is irreducible.*

(ii) *If $p^3 \mid p - r$, then $\overline{V}_{k,a_p} \cong u(\sqrt{-1})\omega \oplus u(-\sqrt{-1})\omega$ is reducible.*

Proof.

(i) We have $p^3 \nmid p - r$:

Consider the function $f = f_0 + f_1 + f_2 \in \mathrm{ind}_{KZ}^G \mathrm{Sym}^r \overline{\mathbb{Q}}_p^2$, given by::

$$f_2 = \sum_{\lambda \in \mathbb{F}_p, \mu \in \mathbb{F}_p^*} \left[g^0_{2,p[\mu]+[\lambda]}, \frac{1}{p^2}(Y^r - X^{r-p}Y^p)\right] + \sum_{\lambda \in \mathbb{F}_p} \left[g^0_{2,[\lambda]}, \frac{(1-p)}{p}(Y^r - X^{r-p}Y^p)\right],$$

$$f_1 = \sum_{\lambda \in \mathbb{F}_p} \left[g^0_{1,\lambda}, \sum_{\substack{1<j<r \\ j \equiv 1 \bmod p-1}} \frac{(p-1)}{p^2 a_p} \gamma_j X^{r-j} Y^j\right],$$

and

$$f_0 = \left[\mathrm{id}, \frac{r}{p^3}(X^{r-1}Y - X^{r-p}Y^p)\right],$$

where the integers γ_j are given in Lemma 1.11.

In the first part of f_2 we have $v(1/p^2) = -2$, so we consider $j = 0, 1, 2$ for $T^+ f_2$. For $j = 0$ we have $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1, 2$ we see that $\frac{p^j}{p^2}(\binom{r}{j} - \binom{p}{j}) \equiv 0 \bmod p^2$ as $p^2 \mid r - p$. In the second part of f_2 we have $v(1/p) = -1$, so we consider $j = 0, 1$ for $T^+ f_2$. For $j = 0$ we have $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1$ we see that $\frac{p}{p}(\binom{r}{1} - \binom{p}{1}) \equiv 0 \bmod p^2$ as $p^2 \mid r - p$. Thus $T^+ f_2 \equiv 0 \bmod p$.

In f_1 we have $v(1/p^2 a_p) < -5$. By the properties of the γ_j we have $\sum \binom{j}{n} \gamma_j \equiv 0 \bmod p^{5-n}$, so $T^+ f_1 \equiv 0 \bmod p$. We see that $a_p f_2$ and $a_p f_0$ die mod p as $v(a_p) > 2$.

In f_0, we have $v(r/p^3) = -2$. For $T^+ f_0$ we consider $j = 0, 1, 2$. For $j = 0$ we obtain $\frac{r}{p^3}(\binom{1}{0} - \binom{p}{0}) = 0$. For $j = 1$, we obtain $\frac{pr}{p^3}(\binom{1}{1} - \binom{p}{1}) X^{r-1} Y = \frac{r(1-p)}{p^2} X^{r-1} Y$. For $j = 2$, we obtain $\frac{p^2 r(1-p)}{p^3}(\binom{1}{2} - \binom{p}{2}) X^{r-2} Y^2$, which is integral, hence vanishes in Q.

In $T^- f_1$, the highest index of a nonzero coefficient is $i = r - p + 1$. Therefore $p^{r-i} = p^{p-1}$ kills $1/p^2 a_p$ for $p \geq 7$. If $p = 5$, we note that $T^- f_1$ has the term $\frac{(p-1)p^4}{p^2 a_p} \gamma_4$. As $r \equiv p \bmod p^2$, we see that $\gamma_4 \equiv \binom{r}{4} \equiv 0 \bmod p$ and hence $T^- f_1 \equiv 0$.

For $T^- f_2$ we consider $i = r$ in the first part of f_2, obtaining:

$$\sum_{\lambda \in \mathbb{F}_p} \left[g^0_{1,\lambda}, \frac{(p-1)}{p^2} \sum_{\substack{1<j<r \\ j \equiv 1 \bmod p-1}} \binom{r}{j} X^{r-j} Y^j + \frac{1}{p} Y^r + \frac{r(p-1)}{p^2} X^{r-1} Y \right].$$

The term $\frac{1}{p} Y^r$ is cancelled out by the second part of $T^- f_2$, while the term $\frac{r(p-1)}{p^2} X^{r-1} Y$ is cancelled out by $T^+ f_0$. Thus $(T - a_p) f \equiv T^- f_2 - a_p f_1 + T^+ f_0$ is equivalent to:

$$\sum_{\lambda \in \mathbb{F}_p} \left[g^0_{1,\lambda}, \frac{(p-1)}{p^2} \sum_{\substack{1<j<r \\ j \equiv 1 \bmod p-1}} \left(\binom{r}{j} - \gamma_j \right) X^{r-j} Y^j \right].$$

As $\binom{r}{j} \equiv \gamma_j \bmod p^2$ the above function is integral. Because each of the monomials $X^{r-j} Y^j$ maps to X^{p-2}, by the properties of $\sum_j \gamma_j$ the expression above maps to $c X^{p-2}$, where $c = \frac{(p-1)(p-r)}{p^2}$ due to Lemma 1.7. As $p^2 \mid p -$

r this sum is integral, but is nonzero as $p^3 \nmid p-r$. Thus $(T-a_p)f = \sum_{\lambda\in\mathbb{F}_p}[g^0_{1,\lambda}, cX^{p-2}] = cT[\mathrm{id}, X^{p-2}]$, which means that $\overline{V}_{k,a_p}$ is irreducible.

(ii) We have $p^3 \mid p-r$:

If $v(a_p) < 5/2$ we consider the function $f = f_0 + f_1 + f_2$, where:

$$f_2 = \sum_{\lambda\in\mathbb{F}_p,\mu\in\mathbb{F}_p^*}\left[g^0_{2,p[\mu]+\lambda}, \frac{1}{a_p}(Y^r - X^{r-p}Y^p)\right] + \sum_{\lambda\in\mathbb{F}_p}\left[g^0_{2,[\lambda]}, \frac{(1-p)p}{a_p}(Y^r - X^{r-p}Y^p)\right]$$

and

$$f_1 = \sum_{\lambda\in\mathbb{F}_p}\left[g^0_{1,\lambda}, \frac{(p-1)}{a_p^2}\sum_{\substack{1<j<r\\ j\equiv 1 \bmod p-1}}\gamma_j X^{r-j}Y^j\right],$$

where the $\gamma_j \equiv \binom{r}{j} \bmod p^3$ are chosen as in Lemma 1.11 and

$$f_0 = \left[\mathrm{id}, \frac{r}{pa_p}(X^{r-1}Y - X^{r-p}Y^p)\right].$$

In the first part of f_2 we have $v(1/a_p) < -3$, so we consider $j = 0,1,2$ for T^+f_2. For $j=0$ we have $\binom{r}{0} - \binom{p}{0} = 0$ while for $j = 1,2$ we see that $\frac{p^j}{a_p}(\binom{r}{j} - \binom{p}{j}) \equiv 0 \bmod p$ as $p^3 \mid r-p$. In the second part of f_2 we have $v(p/a_p) < -2$, so we consider $j = 0,1$ for T^+f_2. For $j=0$ we have $\binom{r}{0}-\binom{p}{0} = 0$ while for $j=1$ we see that $\frac{p^2}{a_p}(\binom{r}{1} - \binom{p}{1}) \equiv 0 \bmod p$ as $p^3 \mid r-p$ Thus, we see that $T^+f_2 \equiv 0 \bmod p$.

In f_1 we see that $v(1/a_p^2) < -6$. By the properties of the γ_j we have $\sum\binom{\gamma_j}{n} \equiv 0 \bmod p^{6-n}$, so $T^+f_1 \equiv 0 \bmod p$. In T^-f_1 the highest index of a non-zero coefficient is $i = r-p+1$, and $p^{r-i} = p^{p-1}$ kills $1/a_p^2$ for $p \geq 7$. For $p=5$ we see that T^-f_1 has the terms $\frac{p^4}{a_p^2}\binom{r-4}{j} \equiv 0 \bmod$ as $p^3 \mid r-p$ and $v(a_p^2) < 5$, so $T^-f_1 \equiv 0 \bmod p$.

In f_0, we have $v(r/pa_p) < -3$. For T^+f_0 we consider $j = 0,1,2$. For $j=0$ we obtain $\frac{r}{pa_p}(\binom{1}{0} - \binom{p}{0}) = 0$. For $j=1$, we obtain $\frac{pr}{pa_p}(\binom{1}{1} - \binom{p}{1})X^{r-1}Y = \frac{r(1-p)}{a_p}X^{r-1}Y$. For $j=2$, we obtain $\frac{p^2r}{pa_p}(\binom{1}{2} - \binom{p}{2}) \equiv 0 \bmod p$. Hence, $T^+f_0 = [g^0_{1,0}, \frac{r(1-p)}{a_p}X^{r-1}Y]$.

For T^-f_2 we consider $i = r$ in the first part and obtain:

$$\sum_{\lambda\in\mathbb{F}_p}\left[g^0_{1,\lambda}, \frac{(p-1)}{a_p}\sum_{\substack{1<j<r\\ j\equiv 1 \bmod p-1}}\binom{r}{j}X^{r-j}Y^j + \frac{p}{a_p}Y^r + \frac{r(p-1)}{a_p}X^{r-1}Y\right].$$

The term $\dfrac{p}{a_p}\mathrm{Y}^r$ is cancelled out by the second part of $\mathrm{T}^- f_2$, while the term $\dfrac{r(p-1)}{a_p}\mathrm{X}^{r-1}\mathrm{Y}$ is cancelled out by $\mathrm{T}^+ f_0$.

Thus, $(\mathrm{T}-a_p)f \equiv \mathrm{T}^- f_2 - a_p f_1 + \mathrm{T}^+ f_0$ is equivalent to:

$$\sum_{\lambda\in\mathbb{F}_p}\left[g^0_{1,\lambda}, \frac{(p-1)}{a_p}\sum_{\substack{1<j<r\\ j\equiv 1 \bmod p-1}}\left(\binom{r}{j}-\gamma_j\right)\mathrm{X}^{r-j}\mathrm{Y}^j\right],$$

which is zero as $\gamma_j \equiv \binom{r}{j} \bmod p^3$.

Thus, $(\mathrm{T}-a_p)f \equiv -a_p f_2 - a_p f_0 \bmod p$. Following the argument given in the proof of [BG15, Theorem 9.2], this turns out to be the same as $(\mathrm{T}^2+1)[\mathrm{id}, -\mathrm{X}^{p-2}]$. Therefore the representation is reducible.

If $p = 5$ and $v(a_p) \geq 5/2$, then we are in a situation similar to [BG15] Theorem 9.2 for $p = 3$ and $v(a_p) \geq 3/2$. We consider the function $f' = \dfrac{a_p^2}{p^5}$. Then $(\mathrm{T}-a_p)f'$ is integral and has reduction equal to the image of $c(\mathrm{T}^2+1)[\mathrm{id}, \mathrm{X}]$ where $c = \overline{1 - a_p^2/p^5}$, which by the extra hypothesis is not zero. Thus, the representation is reducible. □

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