# SOME SPECTRUM PROPERTY OF PERIODIC COUPLING AMO OPERATOR

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Abstract. We study spectrum of the periodic coupling AMO model. Meantime there establish the continuity of Lyapunov exponent about the the periodic coupling of AMO model. Through the dynamical method can find a interval the AMO model only have absolutely continuous spectrum. At the same time, some condition make the periodic coupling of AMO model is singular continuous.

## 1. Introduction

In this paper, we construct examples of ergodic Schrödinger operators $H_\omega$, whose Lyapunov exponent $L(E)$ is continuous and have some intrigue property. Since Anderson [1]introduce that absence of diffusion in certain random lattices, random can make the model localization, and AMO model is determine in some sense, so if want to get more random in the AMO model is consider more demension.

Since E. Dinaburg and Y. G. Sinai[2] use KAM theory to prove the reducible of cocycle, many absolutely continuous spectrum's existence have be proved. Eliasson[3] developed the method and prove that full measure of the reducibility. So when get the Lyapunov exponent vanishes in somewhere, through the KAM theory can prove purely absolutely continuous spectrum's existence in a internal about zero.

Through the Herman[4] method can prove Lyapunov exponent is positive, so can get a condition about purely singular continuous spectrum under some condition.

The method is Gordon[5] method, many place use it to prove can not exist point spectrum. The power of Gordon method is very strong, when talk about continuous potential, can get the generic singular continuous spectrum.

Some author such as Simon, Jitomirskaya consider the transition of pure point spectrum and absolutely continuous spectrum of AMO model. At the last forty years, many people consider the coupling is constant and study the famous conjecture 'Ten Martini Problem ' and 'Dry Ten Martini Problem '. Recetently the 'Ten Martini Problem' was solved by Avila and Jitomirskaya[6]. And recently the "Dry Ten Martini Problem " was partly solved by Avila, You and Zhou [7]. The above authors always consider the coupling is constant. What property about the

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coupling is not a constant in the AMO model? There is only a few results. In this paper we consider the coupling is not a constant in the AMO model, concrete to study the periodic couplings in the AMO model.

In physical the coupling and potential function determine present the magnetic, like in the Anderson model. If consider it from the coupling, we can think the coupling is iid and the potential function is identity, and no trival iid make the spectrum is Anderson localization then the material is not conductive. If in the AMO model the coupling is constant but if the absolutely of coupling is large than one, then under some condition the material is not conductive. And when the absolutely of coupling is large than one is little than one but not zero, the material will can not be conductive. This show that the coupling is very important in the conductive of the material. But the magnetic have many form, so the coupling maybe not is a constant. There we will consider the non-constant coupling, concrete we study the periodic of the AMO model.

Since 1980 many author study schödinger operator through a family of ergodic opertor, such as Simon [8], Sinai[9], Avron and Simon[10]. The famous model AMO is define in a irrational rotation on circle. Throgh the frame of strict ergodicity and transformation of the circle can get many information about the spectrum. So in order to study the spectrum of the periodic coupling of AMO model we need to establish the dynamical system. we get it in the second section.

Through the dynamical method and Lyapunov exponent there will be get some interesting spectrum property of the periodic coupling AMO model. Find a property of that periodic coupling AMO model has purely absolutely continuous spectrum on the neighborhood of zero (Theorem5.5). Meantime give some singular continuous spectrum of periodic coupling AMO model.

In section 2 show the motivation to consider the periodic coupling AMO model, and give the definition about the Lyapunov exponent. In section 3 give the continuity of Lyapunov exponent of periodic coupling AMO model. It is important about to use the Kotani theory. In section 4 give the spectral property of a class of periodic coupling AMO model. In section 5 Find a property of that periodic coupling AMO model has purely absolutely continuous spectrum on the neighborhood of zero. In section 6 studying a special case of periodic coupling AMO model, which give the connection with the classical AMO model. In section 7 give some singular continuous spectrum of periodic coupling AMO model.

## 2. PRELIMINARIES

First we consider the periodic coupling AMO model is not want to get a extensions about classical AMO model, but want get some spectrum property of the 2-dimension AMO model. In addition to being a result of interest of its own, we were motivated by the following observation. Studying how to get the Anderson-location in the 2-dimension, this is the case when I consider the schodinger operator:

$$H : L^2(Z) \to L^2(Z),$$

satisfy

$$(H\phi)_n = \phi_{n+1} + \phi_{n-1} + \lambda V(n\omega + \theta)\phi_n$$
$$\omega = (\omega_1, \omega_2) \in T^1 \times T^1$$
$$\forall \phi \in L^2(Z).$$

We want to search the spectral property in some sense 2-demention, chose

$$V(n\omega+\theta)=\lambda(\cos(\theta+n\omega_1))+\beta(\cos(\theta+n\omega_2)),$$

the spectrum of model equation becomes

$$\phi_{n+1}+\phi_{n-1}+\lambda(\cos(\theta+n\omega_1))+\beta(\cos(\theta+n\omega_2))\phi_n=E\phi_n.$$

For simple chose

$$\lambda=\beta,$$

then

$$\lambda(\cos(\theta+n\omega_1))+\lambda(\cos(\theta+n\omega_2))=2\lambda\cos(\theta+\frac{n}{2}(\omega_1+\omega_2))\cos(\frac{n}{2}(\omega_1-\omega_2)).$$

For simple there chose

$$\frac{n}{2}(\omega_1-\omega_2)=\frac{p}{q}2\pi,$$

$p,q$ is co-prime and $p\in N,q\in Z$. So we reduce the problem to the potential $v$ is a

$$qusiperiodic\times periodic.$$

So for generally chose

$$V(n\omega+\theta)=\lambda\cos(n\omega+\theta)\times T(n),$$

where exist $k\in N$

$$T(n+k)=T(n)$$
$$\forall n\in Z.$$

So, if we know

$$T(0),T(1),...,T(k-1),$$

can get the potential. Through the Herman method there will can get some information about the positive of Lyapunov exponent.

We can get the transfer matrix through the above argument

$$\begin{aligned}A^n(\theta)&=\begin{pmatrix}E-V(n\omega+\theta) & -1\\ 1 & 0\end{pmatrix}\\ &=\begin{pmatrix}E-\lambda\cos(n\omega+\theta)\times T(n) & -1\\ 1 & 0\end{pmatrix},\end{aligned}\tag{2.1}$$

when

$$T=constant.$$

Then the model become AMO model so we really consider bigger than AMO model.

Define

$$A_n(\theta)=\prod_{k=0}^{n-1}A^k(\theta);$$

$$L(E)=\liminf_{n\to\infty}\int_0^{2\pi}\frac{\log|A_n(\theta)|}{n}d\theta.$$

Use the Herman method achieve that

$$\begin{aligned}A^n(\theta)&=\begin{pmatrix}E-\lambda\cos(n\omega+\theta)\times T(n) & -1\\ 1 & 0\end{pmatrix}\\ &=\begin{pmatrix}E-\frac{\lambda}{2}(e^i(n\omega+\theta)\times T(n)+e^{-i(n\omega+\theta)\times T(n)}) & -1\\ 1 & 0\end{pmatrix}.\end{aligned}\tag{2.2}$$

Let

$$e^{i(n\omega+\theta)} = z,$$

Then

$$\begin{aligned} A^n(z) &= \begin{pmatrix} E - \frac{\lambda}{2}(z \times T(n) + \frac{1}{z} \times T(n)) & -1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{z}\begin{pmatrix} Ez - \frac{\lambda}{2}(z^2 \times T(n) + 1 \times T(n)) & -z \\ z & 0 \end{pmatrix}; \end{aligned} \tag{2.3}$$

$$\begin{aligned} L(E) &= \liminf_{n\to\infty} \int_0^{2\pi} \frac{\log |A_n(\theta)|}{n} d\theta \\ &= \liminf_{n\to\infty} \int_{|z|=1} \frac{\log |A_n(z)|}{n} \frac{dz}{z} \\ &\geq \int_{|z|=0} \frac{\log |A_n(0)|}{n} |dz| \\ &= \log \| \frac{\lambda}{2} | \sqrt[k]{T(1) \times T(2) ... \times T(k)}. \end{aligned}$$

Establish

$$L(E) > 0,$$

when

$$\| \frac{\lambda}{2} | \sqrt[k]{T(1) \times T(2) ... \times T(k)} > 1.$$

so can get that the Lyapunov exponent is positive in some place. It is interesting to search the

$$\{T(0), T(1), ..., T(k-1)\},$$

have some one is zero, what happended about spectrum. Consider a special situation, the number of set

$$\{T(0), T(1), ..., T(k-1)\},$$

is 2. And

$$\{T(0), T(1)\},$$

is

$$\{0, T(1)\},$$

then $E = 0$ is the Lyapunov exponent vanishes, no matter how large the value of $T(1)$.

We will begin to discuss some basics dynamical systems about the periodic coupling of AMO model.

Given a bounded sequence $V : \mathbb{Z} \to \mathbb{R}$, we denote by $\Omega_V$ the hull of its translates. That is

$$\Omega_V = \overline{\{V_m, \quad m \in \mathbb{Z}\}}^{\ell^\infty(\mathbb{Z})}, \tag{2.4}$$

where $V_m(n) = V(n-m)$. If $\Omega_V$ is compact in the $\ell^\infty(\mathbb{Z})$ topology, then $V$ is called *almost-periodic*. The shift map on $\ell^\infty(\mathbb{Z})$ becomes a translation on the group $\Omega_V$ and it is uniquely ergodic with respect to the Haar measure of $\Omega_V$. Embedding a schödinger operator in a suit family is the main sutdy method. This method get a great progress in study *limit-periodic* by Avila[11],he give a suit dynamical system on a contour group. This re $V$ is called *limit-periodic*, if there exists a sequence of periodic potentials $V^k$ such that

$$V = \lim_{k\to\infty} V^k \tag{2.5}$$

in the $\ell^\infty(\mathbb{Z})$ topology. It should be remarked that limit-periodic $V$ are almost-periodic. In fact, then $\Omega_V$ has the extra structure of being a Cantor group. Then Avila confirm that limit-periodic one-to-one correspond a minimal transform Cantor group.

So use the similar method to study the periodic coupling AMO model. The dyanmical system in study the periodic coupling AMO model is define in a compact metric space.

Let

$$M = S^1 \times \{0, 1, ..., k-1\}.$$

The hemeomorphism is

$$T : M \to M,$$

$$T(\theta, h) = (\theta + \omega, h + 1),$$

and when

$$h = k - 1,$$

then

$$h + 1 = 0.$$

eg: the function in

$$\{0, 1, ..., k-1\}$$

is cyclic group.

The topology in $M$ is product topology, and we chose the topology in $\{0, 1, ..., k-1\}$ is discrete topology, the topology of $S^1$ is the general topology induce by the metric.

The metric in $S^1$ is

$$d(\theta_1, \theta_2) = min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|).$$

Then we can get the dynamical system $(M, T)$ is strict ergodicity and transformation of the space$M$ when the number $\omega/2\pi$ is irrational. This result is prove by many author, such as a book of F. R. Hertz, J. R. Hertz and R. Ures[12], and the unique probability mesaure is that:

$$d\mu = \frac{\frac{d\theta}{2\pi} \times \{\delta_0 + \delta_1 + ... + \delta_{k-1}\}}{k}.$$

So can study the periodic coupling AMO model through the base dynamical system.

Since the base dynamical system is strict ergodicity, so the Lyapunov exponent can give many information about the periodic coupling AMO model.

First give the Lyapunov exponent in the model let

$$\forall \theta \in S^1, n \in \{0, 1, ..., k-1\}, A(\theta, n) = \begin{pmatrix} E - \lambda\cos(\theta) \times T(n) & -1 \\ 1 & 0 \end{pmatrix},$$

$$A^m(\theta,n)=\begin{pmatrix} E-\lambda\cos(\theta+(m-1)\omega)\times T(n+m-1) & -1\\ 1 & 0\end{pmatrix}\cdots$$
$$\cdots\begin{pmatrix} E-\lambda\cos(\theta+\omega)\times T(n+1) & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} E-\lambda\cos(\theta)\times T(n) & -1\\ 1 & 0\end{pmatrix}.$$

**Definition 2.1.** *Let*

$$L_m(E)=\int_M A^m(\theta,n)d\mu,$$

*then through the subadditive ergodic theorem*[13] *know that*

$$\lim_{m\to\infty} L_m(E),$$

*is exist and*

$$\lim_{m\to\infty} L_m(E)=\inf_{m\geq 1} L_m(E),$$

*then*

$$L(E)=\lim_{m\to\infty} L_m(E)=\inf_{m\geq 1} L_m(E).$$

The study continuity of Lyapunov exponent of periodic coupling AMO model is very important, through the Kotani theory[14], if know the continuity of Lyapunov exponent of periodic coupling AMO model, then the absolute continuous spectrum is in the essential closed of the set $\{E|L(E)=0\}$. But if Lyapunov exponent of periodic coupling AMO model is continuous, then set $\{E|L(E)=0\}$ is closed. So the essential closed of the set $\{E|L(E)=0\}$ in $\{E|L(E)=0\}$. The continuity of Lyapunov exponent is not always can be established, so the absolute continuous spectrum may have some support in the set$\{E|L(E)>0\}$. This argument maybe construct many example to get discontinuous Lyapunov exponent, such as Z. Gan and H. Krueger[15].

But the most important is that maybe get some $E$ in the absolute continuous spectrum meantime in the set $\{E|L(E)\geq 0\}$, this maybe give a negative answer about the "schödinger conjecture". Avial[16] give a counterexample about the "schödinger conjecture", his method is subtle, I think this argument can give a simple example. This problem will be done in the future.

## 3. Continuity of Lyapunov exponent of periodic coupling AMO model

The study continuity of Lyapunov exponent AMO model has long history, the first important result in this line is that J Bourgain's theorem:

**Theorem 3.1.** [17] *The Lyapunov exponent*

$$L(\beta+.,.):S^1\times B^\omega(S^1,SL(2,R))\to R$$

*is jointly continuous at every irrational $\beta$.*

In [17], Theorem 3.1 was stated and proven for the Schrdinger, SL(2,R) case; strictly speaking, the extension to SL(2,R)follows from [18]. Many generalization to non-singular and singular cocycles has been carried out explicitly by[19][20].Many other line about the continuity of Lyapunov exponent of different model, such as Duarte and Klein[21] who prove many continuity of Lyapunov exponent about general cocycle. But for prove the continuity of Lyapunov exponent of periodic coupling AMO model, the theorem3.1 of J Bourgain is enough. Meantime the method of J Bourgain can be use to prove the positive in many place. But in this paper only use the Herman and Avila's method to prove the positive.

**Theorem 3.2.** *Given a periodic coupling AMO model $(M, T)$, if the frequency $\omega$ is irrational, then the Lyapunov exponent is continuous in the real number.*

*Proof.* Since the dynamical system $(M, T)$ is strict ergodicity and transformation of the space$M$ when the number $\omega/2\pi$ is irrational. the unique probability mesaure is that:

$$d\mu = \frac{\frac{d\theta}{2\pi} \times \{\delta_0 + \delta_1 + ... + \delta_{k-1}\}}{k},$$

and

$$L(E) = \lim_{m\to\infty} L_m(E) = \inf_{m\geq 1} L_m(E).$$

Set $B(\theta) = A^k(\theta, 0)$, then let

$$\bar{L}_m(E) = \int_M \frac{||B^m(\theta)||}{m} \frac{\frac{d\theta}{2\pi}}{k},$$

$$L(E) = \lim_{m\to\infty} \bar{L}_m(E) = \inf_{m\geq 1} \bar{L}_m(E).$$

Since the dynamical system $(M, T)$ is strict ergodicity and transformation of the space $M$ when the number $\omega/2\pi$ is irrational, the equation Set up. If the Lyapunov exponent is defined by

$$L(E) = \lim_{m\to\infty} \bar{L}_m(E) = \inf_{m\geq 1} \bar{L}_m(E).$$

Then the Lyapunov exponent can be reduced to a simple dynamical systems. The new dynamical system is :

$$\bar{T} : S^1 \to S^1;$$
$$\bar{T}(\theta) = \theta + k\omega;$$
$$B(\theta) = A^k(\theta, 0).$$

Then this is a coycle on circle. Thorough the definition of

$$B(\theta) = A^k(\theta, 0),$$

find that

$$\begin{aligned} B(\theta) &= A^k(\theta, 0) \\ &= \begin{pmatrix} E - \lambda\cos(\theta + (m-1)\omega) \times T(k-1) & -1 \\ 1 & 0 \end{pmatrix} \cdots \\ &\cdots \begin{pmatrix} E - \lambda\cos(\theta + \omega) \times T(1) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda\cos(\theta) \times T(0) & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

So

$$B(\theta) = A^k(\theta, 0).$$

is analytical about the variant $\theta \in S^1$. and Then continuity of Lyapunov exponent of periodic coupling AMO model reduced to the continuity of Lyapunov exponent of a simple dynamical systems$(S^1, \bar{T})$. So $(B, T, S^1) \in S^1 \times B^\omega(S^1, SL(2, R))$, so use the theorem 3.1the Lyapunov exponent is continuous in the model $(B, T, S^1)$, and because periodic coupling AMO model reduced to the continuity of Lyapunov exponent of a simple dynamical systems$(S^1, \bar{T})$, so Lyapunov exponent of periodic coupling AMO model is continuous.

□

## 4. SPECTRAL PROPERTY OF A CLASS OF PERIODIC COUPLING AMO MODEL

If we let the number of set

$$\{T(0), T(1), ..., T(k-1)\}$$

is 2.

That is to say

$$\{T(0), T(1), ..., T(k-1)\} = \{T(0), T(1)\}.$$

Then the model $(B, T, S^1)$ have some simple form:

(4.1)
$$B(\theta) = \begin{pmatrix} (\cos(\theta)T(0) - E)(\cos(\theta+\omega)T(1) - E) - 1 & E - \cos(\theta+\omega)T(1) \\ \cos(\theta)T(0) - E & -1 \end{pmatrix}.$$

**Theorem 4.1.** *Lyapunov exponent of periodic coupling AMO model,* $L(0) = 0$*,when* $T(0) = 0$*, no matter what the value of* $T(1)$*.*

*Proof.*
(4.2)
$$B(\theta) = \begin{pmatrix} (\cos(\theta)T(0) - E)(\cos(\theta+\omega)T(1) - E) - 1 & E - \cos(\theta+\omega)T(1) \\ \cos(\theta)T(0) - E & -1 \end{pmatrix}.$$

When $T(0) = 0$, then

$$B(\theta) = \begin{pmatrix} -E(\cos(\theta+\omega)T(1) - E) - 1 & E - \cos(\theta+\omega)T(1) \\ -E & -1 \end{pmatrix}. \tag{4.3}$$

When $E = 0$,

$$B(\theta) = \begin{pmatrix} -1 & -\cos(\theta+\omega)T(1) \\ 0 & -1 \end{pmatrix}. \tag{4.4}$$

Since $\det B(\theta) = 1$ so $L(E) \geq 0$. For calculate the $L(0)$, use the Schmidt norm to done it. When $T(0) = 0, E = 0$, then

$$\begin{aligned} ||B(\theta)|| &= \sqrt{T(1)^2\cos^2(\theta+\omega) + 2} \\ &\leq \sqrt{T(1)^2 + 2}, \end{aligned}$$

$$\begin{aligned} ||B^m(\theta)|| &\leq \left(T(1)^2\cos^2(\theta+\omega)m + 2\right) \\ &\leq \left(T(1)^2 m + 2\right). \end{aligned}$$

So, can get the lower bound

$$\begin{aligned} L(E) &= \lim_{m\to\infty} \int_{S^1} \frac{\ln ||B^m(\theta)||}{m} \frac{d\theta}{2\pi} \\ &\leq \lim_{m\to\infty} \int_{S^1} \frac{\ln\left(T(1)^2 m + 2\right)}{m} \frac{d\theta}{2\pi}. \end{aligned}$$

Since

$$\lim_{m\to\infty} \frac{\ln\left(T(1)^2 m + 2\right)}{m} = 0,$$

so

$$\lim_{m\to\infty} \int_{S^1} \frac{\ln\left(T(1)^2 m + 2\right)}{m} \frac{d\theta}{2\pi} = 0.$$

Since $L(E) \geq 0$, can get $L(E) = 0$, under the condition $T(0) = 0, E = 0$. $\square$

Some interesting property can get in this periodic coupling AMO model. The classic AMO model is that:

$$H : L^2(Z) \to L^2(Z),$$

satisfy

$$(H\phi)_n = \phi_{n+1} + \phi_{n-1} + \lambda V(n\omega + \theta)\phi_n.$$

Let $V(x) = \cos(x)$,the spectrum of model equation becomes

$$\phi_{n+1} + \phi_{n-1} + \lambda \cos(n\omega + \theta)\phi_n = E\phi_n.$$

And through the transform matrix to study the spectrum. We know that Bourgain and Jitomirskaya result[17] when the frequency $\omega$ is irrational, then $L(E) > 0$ in the spectrum, if $|\lambda| > 2$ and $L(E) = 0$, if $|\lambda| \leq 2$, in the spectrum.

And through many people's work get the complete spectral picture, when $|\lambda| > 2$ if the irrational frequency $\omega$ is well approximate by rational number, such as Liouville number ,then there only have the singular continuous spectrum. when $|\lambda| > 2$ if the irrational frequency $\omega$ is diophantine number, then there only have the pure point spectrum. when $|\lambda| < 2$ if the frequency $\omega$ is irrational, then there only have absolutely continuous spectrum. There are many gap in the above example, use quantity of irrational frequency $\omega$. Let $\frac{p_n}{q_n}$ be the continued fraction approximation to $\omega$ and let

$$\beta = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}.$$

In the paper [22] have prove the conjecture that for alomst Mathieu family,under the condition that localization for almost $a.e.x$ has been $e^{\beta} < \lambda$ where $\beta$ is the upper rate of exponential growth of denominators of the continued fractions approximation to $\alpha$.

So many people wonder if every analytic potential have these property. Then Avila study the neighborhood of $cos(\theta)$, and Bjerklöv [23] give explicit examples of arbitrarily large analytic ergodic potentials for which the Schrdinger equation has zero Lyapunov exponent for certain energies on matter what the value of the coupling.

In the two periodic of the periodic coupling AMO model, if one of coupling is zero, then have get the similar property.

We can se some reason about the property that zero Lyapunov exponent for certain energies if the first coupling is zero no matter what the value of the second coupling.

$$\begin{aligned} A(\theta, 0) &= \begin{pmatrix} \cos(\theta)T(0) & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{4.5}$$

$$A(\theta+\omega, 1) = \begin{pmatrix} \cos(\theta+\omega)T(1) - E & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.6}$$

$$B(\theta) = A(\theta+\omega, 1)A(\theta, 0), \tag{4.7}$$

then can see that

$$A(\theta,0) = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{pmatrix} = R_{\frac{\pi}{2}}. \tag{4.8}$$

So $R_{\frac{\pi}{2}}$ can exchange the contraction direction and expand direction, so the zero Lyapunov exponent for certain energies if the first coupling is zero no matter what the value of the second coupling. Such mechanism have give by Furstenberg and Kesten[24] to and use it to establish the example to explain his theorem is optimism. Bocker-Neto and Viana[25] use it to give the example to emphasize the positive is important in the condition.

The mechanism of the product the $SL(2,R)$ matrix if one is $R_{\frac{\pi}{2}}$,then the contraction direction and expand direction can not be distinguished.So the Lyapunov exponent become zero.

So in the next section, we study the absolutely continuous spectrum on the neighborhood of zero.

## 5. ABSOLUTELY CONTINUOUS SPECTRUM ON THE NEIGHBORHOOD OF ZERO

Sinai[2] use KAM theory to prove the reducible of cocycle, many absolutely continuous spectrum's existence. But the notion reducible is not a open condition.So many time use reducible only can prove the exist the absolutely continuous spectrum, but can not prove the pure absolutely continuous spectrum in some place.

Avila and Jitomirskaya[26] introduce a new notion almost reducible about the reducible of cocycle. Use this notion they prove the pure absolutely continuous spectrum in AMO model, if the absolutely value of coupling is little than 2, and frequency is irrational.

Resently X Hou, J You use the notion of almost reducibility and the equality of continuous dynamical system and discrete cocycle they prove the almost reducible of AMO model if frequency is Liouville number.

In the progress in reduce a cocycle we always incounter a small denominators problem, this produce in look for a answer of coboundary problem. This always be called the rigidity of irrational rotation.This paper not to get a sharp condition the interval of the exitance of pure absolutely continuous spectrum, the usual answer is enough to get a interval of zero.

If we let the number of set

$$T(0), T(1), ..., T(k-1)$$

is 2.

That is to say

$$T(0), T(1), ..., T(k-1) = T(0), T(1)$$

then the model $(B, T, S^1)$ have some simple form:

$$B(\theta) = \begin{pmatrix} (\cos(\theta)T(0) - E)(\cos(\theta+\omega)T(1) - E) - 1 & E - \cos(\theta+\omega)T(1) \\ \cos(\theta)T(0) - E & -1 \end{pmatrix}. \tag{5.1}$$

**Theorem 5.1.** *There is a positive number $\epsilon > 0$, the schödinger operator is almost reducible in the interval $[-\epsilon, \epsilon]$, if the frequency $\frac{\omega}{2\pi}$ is a diophantine number.*

*Proof.*
(5.2)
$$B(\theta) = \begin{pmatrix} (\cos(\theta)T(0)-E)(\cos(\theta+\omega)T(1)-E)-1 & E-\cos(\theta+\omega)T(1) \\ \cos(\theta)T(0)-E & -1 \end{pmatrix}.$$
When $T(0)=0$, then
$$B(\theta) = \begin{pmatrix} -E(\cos(\theta+\omega)T(1)-E)-1 & E-\cos(\theta+\omega)T(1) \\ -E & -1 \end{pmatrix}. \tag{5.3}$$
When $E=0$,
$$B(\theta) = \begin{pmatrix} -1 & -\cos(\theta+\omega)T(1) \\ 0 & -1 \end{pmatrix}. \tag{5.4}$$
Look for :
$$\cos(\theta+\omega)T(1) = h(\theta+3\omega)-h(\theta+\omega).$$
Since $\int_{S^1}\cos(\theta)d\theta = 0$,and the frequency $\frac{\omega}{2\pi}$is a diophantine number. the equation
$$\cos(\theta+\omega)T(1) = h(\theta+3\omega)-h(\theta+\omega),$$
can be solve.

This problem of Small denominators about the mapping the circle onto itself deal with by the KAM method used by Arnol'd[27] and Herman[28]. And the result extended by Yoccoz[29], Sinai and Khanin[30]. The frequency has basic importance in the Small denominators. If the frequency is diophantine number then $h$ of the equation :
$$\cos(\theta+\omega)T(1) = h(\theta+3\omega)-h(\theta+\omega),$$
is exist and analytic.

An resent the problem of
$$\cos(\theta+\omega)T(1) = h(\theta+3\omega)-h(\theta+\omega),$$
has been extend that the analytic radius $\rho$ of the function
$$\cos(\theta+\omega)T(1).$$

The irrational frequency $\omega$ have: Let $\frac{p_n}{q_n}$ be the continued fraction approximation to $\omega$ and let
$$\beta = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}.$$
and there is a inequality in the two number:
$$\beta < \rho.$$
Then there exist the analytic $h$ of the equation:
$$\cos(\theta+\omega)T(1) = h(\theta+3\omega)-h(\theta+\omega).$$

So there can get more general result in the problem, but only consider the diophantine number in this paper.

Then
(5.5)
$$\begin{aligned}\bar{B}(\theta) &= \begin{pmatrix} 1 & -h(\theta+3\omega) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & T(1)(-\cos(\theta+\omega)) \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & h(\theta+\omega) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -h(\theta+\omega)+h(\theta+3\omega)-\cos(\theta+\omega)T(1) \\ 0 & -1 \end{pmatrix}.\end{aligned}$$

Because the equation

$$\cos(\theta+\omega)T(1) = h(\theta+3\omega) - h(\theta+\omega),$$

$$\bar{B}(\theta) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = R_\pi,$$

is a constant rotation, so it should in absolutely spectrum. Next talk about the neighborhood of zero.

When $E \neq 0$,

$$\begin{aligned} B(\theta) &= \begin{pmatrix} -E(\cos(\theta+\omega)T(1)-E)-1 & E-\cos(\theta+\omega)T(1) \\ -E & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\cos(\theta+\omega)T(1) \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -E(\cos(\theta+\omega)T(1)-E) & E \\ -E & 0 \end{pmatrix}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \bar{B}(\theta) = & \begin{pmatrix} 1 & -h(\theta+3\omega) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -E(\cos(\theta+\omega)T(1)-E) & E \\ -E & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & h(\theta+\omega) \\ 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} 1 & -h(\theta+3\omega) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & -\cos(\theta+\omega)T(1) \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & h(\theta+\omega) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} Eh(\theta+3\omega) - E(\cos(\theta+\omega)T(1)-E) & G \\ -E & -Eh(\theta+\omega) \end{pmatrix} \\ &+ \begin{pmatrix} -1 & -h(\theta+\omega)+h(\theta+3\omega)-\cos(\theta+\omega)T(1) \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} Eh(\theta+3\omega) - E(\cos(\theta+\omega)T(1)-E) & G \\ -E & -Eh(\theta+\omega) \end{pmatrix} \\ &+ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

where $G = h(\theta+\omega)h(\theta+3\omega)E - h(\theta+\omega)(\cos(\theta+\omega)T(1)-E)E + E$. The every entry of the matrix

$$C(\theta) = \begin{pmatrix} Eh(\theta+3\omega) - E(\cos(\theta+\omega)T(1)-E) & G \\ -E & -Eh(\theta+\omega) \end{pmatrix},$$

is polynomial about $E$ and $h$, and the constant item is zero, and $h$ is analytic in the condition, so the matrix $C(\theta)$ is approximate $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is $o(E)$.

So

$$\bar{B}(\theta) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + o(E).$$

The new dynamical system is :

$$\bar{T} : S^1 \to S^1,$$

$$\bar{T}(\theta) = \theta + 2\omega,$$

$$\bar{B}(\theta) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + o(E).$$

Then this is a coycle on circle.

And $\bar{B}(\theta)$ is a approximate the constant matrix,but not arbitrary approximate the constant matrix, through the theorem of Hou and You [31], there exist a $\epsilon > 0$, if

$$o(E) < \epsilon,$$

$\bar{B}$ is almost reducible in the interval $[-\epsilon, \epsilon]$.

There is a theorem of Hou and You[31] in the continuous case, the local almost reducibility result is completely established recently, while there is no result for global reducibility.

In the discrete case, various global reducibility results [32][33][33] [34]were obtained, local almost reducibility results are not enough. Since they not deal with the Liouville frequency.

Since the result of Hou and You[31], if we have a connection about the quasi-periodic cocycle close to constant and quasi-periodic linear system close to constant, they can get the almost reducibility result about Liouville frequency.

So,they prove a Embedding theorem to connection the quasi-periodic cocycle close to constant and quasi-periodic linear system close to constant.

**Theorem 5.2.** [31] *Any analytic quasi-periodic cocycle close to constant is the Poincar map of an analytic quasi-periodic linear system close to constant.*

So the prove that

**Theorem 5.3.** [35] *Any analytic quasi-periodic cocycle close to constant is the if the frequency is irrational then the analytic quasi-periodic cocycle have almost reducibility .*

The precise result can be ref [35].

Continuous the proof of theorem 5.1, through the theorem5.3, there is a positive number $\epsilon > 0$, then $o(E) < \epsilon$,that cocycle close to constant. Then the schödinger operator is almost reducible in the interval $[-\epsilon, \epsilon]$.

□

The almost reducibility of periodic coupling AMO model on the neighborhood of zero has been establish in the theorem 5.1. So use the property of the almost reducibility can get some spectral property of of periodic coupling AMO model on the neighborhood of zero.

Avila have get a result in his paper :

**Theorem 5.4.** *If the transform matrix have almost reducibility in some interval,then the schödinger operator is absolutely in that interval.*

Then use this theorem5.4, get the theorem:

**Theorem 5.5.** *There is a positive number $\epsilon > 0$, the schödinger operator is purely absolutely continuous in the interval $[-\epsilon, \epsilon]$,if the frequency $\frac{\omega}{2\pi}$ is a diophantine number.*

*Proof.* Through the the theorem5.1,There is a positive number $\epsilon > 0$, the schödinger operator is almost reducible in the interval $[-\epsilon, \epsilon]$.

So, by the theorem5.5,schödinger operator is purely absolutely continuous in the interval $[-\epsilon, \epsilon]$. □

Although there only give the simplest result about the periodic coupling AMO model have the purely absolutely continuous in a interval. Since the result of You and Zhou[35] is generally include some Liouville frequency. That is:

$$\beta = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}.$$

When $\beta > 0$,can control the analytic radius $\rho$ of the potential $V$ have the condition:

$$\rho > 5\beta > 0,$$

then can get a $\epsilon$ such that the schödinger operator is purely absolutely continuous in the interval $[-\epsilon, \epsilon]$.

In this time only consider the diophantine number. But can easily extended to large number by the above argument.

## 6. A special case of periodic coupling AMO model

If we let the number of set

$$\{T(0), T(1), ..., T(k-1)\},$$

is 2.

That is to say

$$\{T(0), T(1), ..., T(k-1)\} = \{T(0), T(1)\},$$

and the case that one of

$$\{T(0), T(1)\},$$

is zero has been consider in the previous section. There consider the case:

$$\{T(0), T(1)\} = \{1, -1\}.$$

The potetial is

$$\begin{aligned} V(n\omega + \theta) &= \lambda \cos(n\omega + \theta) \times T(n) \\ &= \lambda \cos(n\omega + \theta) \times (-1)^n \\ &= \lambda \cos(n\omega + \theta + n\pi) \\ &= \lambda \cos(n(\omega + \pi) + \theta). \end{aligned}$$

The periodic coupling AMO model become a classic AMO medel:

$$\bar{T} : S^1 \to S^1,$$

$$\bar{T}(\theta) = \theta + \omega + \pi,$$

$$D(\theta) = \begin{pmatrix} E - \lambda\cos(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Then this is a coycle on circle. But have a variant in frequency in the classic AMO medel. So in the condition in the above, has a obvious theorem.

**Theorem 6.1.** *If consider the value of $\lambda$, there has:*

- *If $|\lambda| > 2$ , the Lyapunov exponent $L(E)$of the cocycle $(D, T)$ is positive an get the value $\ln(\frac{|\lambda|}{2})$,if $E$ in the spectrum of schödinger operator,and the measure of the spectrum is $4|1 - \frac{2}{|\lambda|}|$.*

- *If $|\lambda| = 2$, the Lyapunov exponent $L(E)$ of the cocycle $(D, T)$ is positive an get the value 0, if $E$ in the spectrum of schödinger operator, and the measure of the spectrum is 0.*
- *If $|\lambda| < 2$,the Lyapunov exponent $L(E)$ of the cocycle $(D, T)$ is positive an get the value 0, if $E$ in the spectrum of schödinger operator, and the measure of the spectrum is $4|1 - \frac{|\lambda|}{2}|$.*

Only in this special case can get the complete information about the Lyapunov exponent in the spectrum. mean time there is a theorem about the spectrum of the special example. The reason is that the special AMO model can be reduced to the classic model. So the spectrum of the special example only translate the spectrum of the classic AMO to the the special example.

## 7. SINGULAR CONTINUOUS SPECTRUM OF PERIODIC COUPLING AMO MODEL

The explanation about the positive of Lyapunov exponent of periodic coupling AMO model has been given in the introduction. In this section give a deep result about the positive of Lyapunov exponent of periodic coupling AMO model.

The Gordon [5] use the periodic approximate the model can exclude the point spectrum under some condition. This method has been used in ubiquitous about schödinger operator.

Avron and Simon [10] use the positive of Lyapunov exponent of AMO model exclude the absolutely continuous spectrum. And they use the method of Gorodon they prove that: If frequency is a Liouville number and the coupling$\lambda > 2$, we prove that for a.e. phase$\theta$, the operator's spectral measures are all singular continuous.

In the case of the potential is substitution Hamiltonians, Damanik [36] consider discrete one-dimensional Schrdinger operators with potentials generated by primitive substitutions. A purely singular continuous spectrum with probability one is established provided that the potentials have a local four-block structure. The local four-block structure is the main tool of Gorodon method. So, the Gorodon method is strongly in prove the spectral type of Schrdinger operators.

The detail of Gorodon method can be find in Damanik[37]. Damanik and Stolz use it prove a fully general result[38].

Although there is a method to exclude the point spectrum and absolutely spectrum, I wonder if there is a method prove singular continuous spectrum directly, and will be search it in the future.

The periodic coupling AMO model is similar classic AMO model, so use the Gorodon method get a result about the spectrum type of periodic coupling AMO model.

For simplify there consider the number of set

$$\{T(0), T(1), ..., T(k-1)\},$$

is 2.

That is to say

$$\{T(0), T(1), ..., T(k-1)\} = \{T(0), T(1)\}.$$

**Definition 7.1.** *A number $\alpha \in \mathbb{R}/\mathbb{Q}$ is called a Liouville number, if for any $k \in \mathbb{N}$, there exist $p_k, q_k \in \mathbb{N}$ such that:*

$$|\alpha - \frac{p_k}{q_k}| \leq k^{-q_k}.$$

The Liouville numbers have many interesting property in topology and measure.

The set of Liouville numbers is small form an analyst's point of view : It has Lebesgue measure zero .

However, from a topologist's point of view,it is rather big : It is a $G_\delta-$set, so it is generic in the topology sense. For prove the theorem,there is a lemma in Cycon, Froese, Kirsch and Simon's book[39].

**Lemma 7.2.** [39] *Let $A$ be an invertible $2\times 2$ matrix,and $v$ ia a vector of norm 1. Then*

$$\max(||Av||, ||A^2v||, ||A^{-1}v||, ||A^{-2}v||) \geq \frac{1}{2}.$$

**Theorem 7.3.** *If the frequency $\frac{\omega}{2\pi}$ is a Liouville number , and the periodic coupling is $T(0), T(1)$ and $|T(0)T(1)| > 4$, then the periodic coupling AMO model is pure singular measure in the spectrum.*

*Proof.* In this case there only consider the one dimension, for fix $E$ consider the one-dimensional difference schödinger operator :

$$u(n+1) + u(n-1) + [cos(\theta + n\omega) - E]u(n) = 0.$$

For the strict ergodic of periodic coupling AMO model, only need to consider the transform matrix:

$$A(\theta) = \begin{pmatrix} E - T(0)\cos(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

$$A(\theta+\omega) = \begin{pmatrix} E - T(1)\cos(\theta+\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{aligned} B(\theta) &= A(\theta)A(\theta+\omega) \\ &= \begin{pmatrix} E - T(1)\cos(\theta+\omega) & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} E - T(0)\cos(\theta) & -1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

$$A^{-1}(\theta) = \begin{pmatrix} 0 & 1 \\ -1 & E - T(0)\cos(\theta) \end{pmatrix},$$

$$A^{-1}(\theta+\omega) = \begin{pmatrix} 0 & 1 \\ -1 & E - T(1)\cos(\theta+\omega) \end{pmatrix},$$

$$\begin{aligned} B^{-1}(\theta) &= [A(\theta)A(\theta+\omega)]^{-1} \\ &= A^{-1}(\theta+\omega)\cdot A^{-1}(\theta) \\ &= A^{-1}(\theta+\omega) \\ &= \begin{pmatrix} 0 & 1 \\ -1 & e - T(1)\cos(\theta+\omega) \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & E - \cos(\theta)T(0) \end{pmatrix}. \end{aligned}$$

So there give the explicit expression of the transform matrix, next talk about the local block structure. Since have know the Lyapunov exponent is positive upon the condition $|T(0)T(1)| > 4$, and the continuity of Lyapunov exponent of the periodic coupling AMO model has been proved. So the absolutely continuous spectrum of the periodic coupling AMO model is empty. Then look for the like-Gorodon block structure.

Assume that $\frac{\omega}{2\pi}$ is well approximated by $\frac{p_k}{q_k}$ in the sense of above definition. By choosing a subsequence $\frac{p_{k'}}{q_{k'}}$ of $\frac{p_k}{q_k}$, assumme

$$|\frac{\omega}{2\pi}-\frac{p_{k'}}{q_{k'}}|\leq q_{k'}^{-1}k^{-q_{k'}}.$$

Then set

$$V_k(n)=T(n)\cos(2\pi\frac{p_{k'}}{q_{k'}}n+\theta).$$

Then $T_k=2q_{k'}$ is a periodic for $V_k$. estamate:

$$\begin{aligned}\sup_{|n|\leq 4q_{k'}}|V_k(n)-V(n)|&=\sup_{|n|\leq 4q_{k'}}|T(n)\cos(2\pi\frac{p_{k'}}{q_{k'}}n+\theta)-T(n)\cos(\omega n+\theta)|\\&\leq C\sup_{|n|\leq 4q_{k'}}2\pi|n||\frac{p_{k'}}{q_{k'}}-\omega|\\&\leq 4C\pi k^{-T_k}.\end{aligned}\tag{7.1}$$

So this give the accurate error of periodic approximate. Then use the $V_k(n)$ to approximate the periodic coupling AMO model. Then use the the follwing periodic cocycle to approximate the periodic coupling AMO model:

$$A_k(\theta)=\begin{pmatrix}E-T(0)\cos(\theta)&-1\\1&0\end{pmatrix},$$

$$A_k(\theta+2\pi\frac{p_{k'}}{q_{k'}})=\begin{pmatrix}E-T(1)\cos(\theta+2\pi\frac{p_{k'}}{q_{k'}})&-1\\1&0\end{pmatrix},$$

$$\begin{aligned}B_k(\theta)&=A_k(\theta)A_k(\theta+2\pi\frac{p_{k'}}{q_{k'}})\\&=\begin{pmatrix}E-T(1)\cos(\theta+2\pi\frac{p_{k'}}{q_{k'}})&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}E-T(0)\cos(\theta)&-1\\1&0\end{pmatrix},\end{aligned}$$

$$A_k^{-1}(\theta)=\begin{pmatrix}0&1\\-1&E-T(0)\cos(\theta)\end{pmatrix},$$

$$A_k^{-1}(\theta+2\pi\frac{p_{k'}}{q_{k'}})=\begin{pmatrix}0&1\\-1&E-T(1)\cos(\theta+2\pi\frac{p_{k'}}{q_{k'}})\end{pmatrix},$$

$$\begin{aligned}B_k^{-1}(\theta)&=[A_k(\theta)A_k(\theta+2\pi\frac{p_{k'}}{q_{k'}})]^{-1}\\&=A_k^{-1}(\theta+2\pi\frac{p_{k'}}{q_{k'}})\cdot A_k^{-1}(\theta)\\&=A_k^{-1}(\theta+2\pi\frac{p_{k'}}{q_{k'}})\\&=\begin{pmatrix}0&1\\-1&E-T(1)\cos(\theta+2\pi\frac{p_{k'}}{q_{k'}})\end{pmatrix}\cdot\begin{pmatrix}0&1\\-1&E-\cos(\theta)T(0)\end{pmatrix}.\end{aligned}$$

Through the lemma7.2, if $v$ ia a arbitrary vector of norm 1, $\forall\theta\in S^1$ can get :

$$\max(||B_k^{T_k}(\theta)v||,||B_k^{2T_k}(\theta)v||,||B_k^{-T_k}(\theta)v||,||B_k^{-2T_k}(\theta)v||)\geq\frac{1}{2}.$$

Let $M$ is :

$$M=\max\{|T(0)|,|T(1)|\}.$$

Then for $\forall\theta \in S^1$:

$$A(\theta, n) = \begin{pmatrix} E - T(n)\cos(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Since there only consider the $E$ in the spectrum, so

$$|E| \leq M + 2,$$

$$\begin{aligned} ||A||^2_{Schmidt} &= (E - T(n)\cos(\theta))^2 + (-1)^2 + (1)^2 + (0)^2 \\ &= E^2 - 2T(n)\cos(\theta)E + T(n)^2\cos^2(\theta) + 2 \\ &\leq (M+2)^2 + 2M(M+2) + M^2 + 2. \end{aligned}$$

so

$$\begin{aligned} ||A||_{Schmidt} &\leq \sqrt{(M+2)^2 + 2M(M+2) + M^2 + 2)} \\ &\leq 2(M+2). \end{aligned}$$

Let

$$\bar{M} = \max(2(M+2), M).$$

The difference of the two cocycle is :

$$\begin{aligned}
&(7.2)\\
&||B_k^{T_k}(\theta) - B^{T_k}(\theta)||\\
&= ||A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) \cdots A(\theta + 2\pi\frac{p_{k'}}{q_{k'}}, 1)A(\theta, 0)\\
&- A(\theta + 2(T_k-1)\omega, T(2(T_k-1)) \cdots A(\theta+\omega, 1)A(\theta, 0)||\\
&= ||A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) \cdots A(\theta + 2\pi\frac{p_{k'}}{q_{k'}}, 1)A(\theta, 0)\\
&- A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) \cdots A(\theta+\omega, 1)A(\theta, 0)\\
&+ A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) \cdots A(\theta+\omega, 1)A(\theta, 0)\\
&\cdots\\
&- A(\theta + 2(T_k-1)\omega, T(2(T_k-1)) \cdots A(\theta+\omega, 1)A(\theta, 0)||\\
&= ||A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) \cdots (A(\theta + (2\pi\frac{p_{k'}}{q_{k'}}), 1) - A(\theta+\omega, 1))A(\theta, 0)\\
&\cdots\\
&(A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) - A(\theta + 2(T_k-1)\omega, T(2(T_k-1))) \cdots A(\theta+\omega, 1)A(\theta, 0)||.
\end{aligned}$$

The main difference of equation of 7.2 is

$$\begin{aligned}
&(A(\theta + (2\pi\frac{p_{k'}}{q_{k'}}), 1) - A(\theta+\omega, 1))\\
&\cdots\\
&(A(\theta + 4(T_k - 1)\pi\frac{p_{k'}}{q_{k'}}, T(2(T_k-1)) - A(\theta + 2(T_k-1)\omega, T(2(T_k-1))).
\end{aligned}$$

This can be control by 7.1, and the other term can be control by $\bar{M}$

So,there has:

$$||B_k^{T_k}(\theta) - B^{T_k}(\theta)||$$
$$\leq 2T_k\bar{M}^{2T_k-1}4C\pi k^{-T_k},$$

and there are the similar difference of the other term:

$$||B_k^{2T_k}(\theta) - B^{2T_k}(\theta)||,$$
$$||B_k^{-T_k}(\theta) - B^{-T_k}(\theta)||,$$
$$||B_k^{-2T_k}(\theta) - B^{-2T_k}(\theta)||.$$

Thus,

$$\max_{a=\pm1,\pm2}\{||B(aT_k) - B_k(aT_k)||\} \to 0,$$

as

$$k \to 0.$$

Then for arbitrary norm 1 vector $v$,

$$\max_{a=\pm1,\pm2}\{||B(aT_k)||\} \geq \frac{1}{2}||v|| \geq \frac{1}{2},$$

as

$$\limsup_n \frac{||B(n)v||}{||v||} \geq \limsup_n \frac{\max_{a=\pm1,\pm2}\{||B(aT_k)||\}}{||v||} \geq \frac{1}{4}.$$

So in the condition there can not have point spectrum. And through the Lyapunove exponent is positive and continuous know that there can not have absolute continuous spectrum.

And the spectrum is not empty, so the spectrum of periodic coupling AMO model in the condition is purely singular continuous. □

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