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Localizing Weak Convergence in L_∞

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Abstract

In a general measure space $(X, \mathcal{L}, \lambda)$, a characterization of weakly null sequences in $L_\infty(X, \mathcal{L}, \lambda)$ ($u_k \rightharpoonup 0$) in terms of their pointwise behaviour almost everywhere is derived from the Yosida-Hewitt identification of $L_\infty(X, \mathcal{L}, \lambda)^*$ with finitely additive measures, and extreme points of the unit ball in $L_\infty(X, \mathcal{L}, \lambda)^*$ with $\pm\mathfrak{G}$, where $\mathfrak{G}$ denotes the set of finitely additive measures that take only values 0 or 1. When (X, τ) is a locally compact Hausdorff space with Borel σ-algebra $\mathcal{B}$, the well-known identification of $\mathfrak{G}$ with ultrafilters means that this criterion for nullity is equivalent to localized behaviour on open neighbourhoods of points x_0 in the one-point compactification of X. Notions of weak convergence at x_0 and the essential range of u at x_0 are natural consequences. When a finitely additive measure ν represents $f \in L_\infty(X, \mathcal{B}, \lambda)^*$ and $\hat{\nu}$ is the Borel measure representing f restricted to $C_0(X, \tau)$, a minimax formula for $\hat{\nu}$ in terms ν is derived and those ν for which $\hat{\nu}$ is singular with respect to λ are characterized.

Keywords: Dual of L_∞; finitely additive measures; weak convergence; ultrafilters; pointwise convergence; extreme points

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1 Introduction

In the usual Banach space $C(Z)$ of real-valued continuous functions on a compact metric space Z with the maximum norm, it is well-known [3] that v_k converges weakly to v ($v_k \rightharpoonup v$) if and only if $\{\|v_k\|\}$ is bounded and $v_k(z) \to v(z)$ for all $z \in Z$. This observation amounts to a simple test for weak convergence in $C(Z)$ from which follows, for example, the weak sequential continuity [2] of composition maps $u \mapsto f \circ u,\ u \in C(Z)$, when $f : \mathbb{R} \to \mathbb{R}$ is continuous. However $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{L}, \lambda)$ implies that $\{\|u_k\|_\infty\}$ is bounded and often that $u_k(x) \to u(x)$ almost everywhere (Lemma 3.3), but the converse is false (Remark 3.5) and, despite the identification of $L_\infty(X, \mathcal{L}, \lambda)$ with $C(Z)$ for some compact Z [5, VIII 2.1], it can be difficult to decide whether or not a given sequence is weakly convergent in $L_\infty(X, \mathcal{L}, \lambda)$. To address this issue Theorem 3.6 characterises sequences that are weakly convergent to 0 in $L_\infty(X, \mathcal{L}, \lambda)$ (hereafter referred to as weakly null) purely in terms of their pointwise behaviour almost everywhere, and a practical test for weak nullity ensues (Corollary 3.7 and Section 3.1). When (X, τ) is a locally compact Hausdorff topological space, localization in terms of opens sets, as opposed to pointwise, follows from the identification of ultrafilters in the corresponding Borel measure space $(X, \mathcal{B}, \lambda)$ with extreme points in the unit ball of $L_\infty(X, \mathcal{B}, \lambda)^*$. When ν is the finitely additive measure corresponding to $f \in L_\infty(X, \mathcal{B}, \lambda)^*$ we give a formula for the Borel measure $\hat{\nu}$ that represent the restriction $\hat{f}$ of f to

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$C_0(X,\tau)$, defined in Section 5, and use it to characterize those ν for which $\hat{\nu}$ is singular relative to λ. These observations are motivated by examples [8, 13] of singular finitely additive measures that do not yield singular Borel measures when restricted to continuous functions, contrary to a claim by Yosida & Hewitt [15, Thm. 3.4]. The material is organized as follows.

Section 2 is a brief survey of finitely additive measures on σ-algebras and of weak convergence in $L_\infty(X,\mathcal{L},\lambda)$ in terms of the Yosida-Hewitt representation of the dual space $L_\infty(X,\mathcal{L},\lambda)^*$ as a space $L_\infty^*(X,\mathcal{L},\lambda)$ of finitely additive measures. When $\mathfrak{G}$ denotes elements of $L_\infty^*(X,\mathcal{L},\lambda)$ that take only values $\{0,1\}$, it follows that $u_k \rightharpoonup u$ in $L_\infty(X,\mathcal{L},\lambda)$ if and only if $f(u_k) \to f(u)$ for all f represented by elements of $\mathfrak{G}$. Although obtained independently, this is a special case of Rainwater's Theorem, see Appendix and the Closing Remarks at the end of the paper. The section ends with a brief account of weak sequential continuity of composition operators.

Section 3 begins by remarking that $u_k \rightharpoonup u$ if and only if $|u_k| \rightharpoonup |u|$, noting aspects of the pointwise behaviour of weakly convergent sequences in $L_\infty(X,\mathcal{L},\lambda)$, and observing that a necessary condition, which turns out to be sufficient, is given by Mazur's theorem. The characterization of null sequences in terms of their pointwise behaviour in Theorem 3.6 follows from Yosida-Hewitt theory and the fact that any $u \in L_\infty(X,\mathcal{L},\lambda)$ is a constant ω-almost everywhere in the sense of finitely additive measures when $\omega \in \mathfrak{G}$ (see Remark following Theorem 2.8). An $L_\infty(X,\mathcal{L},\lambda)$ analogue of Dini's theorem on the uniform convergence of sequences of continuous functions that are monotonically convergent pointwise is a corollary, and Theorem 3.6 is illustrated by several examples.

In Section 4, when (X,τ) is a locally compact Hausdorff space and $(X,\mathcal{B},\lambda)$ is the corresponding Borel measure space, the well-known one-to-one correspondence (2.8) between $\mathfrak{G}$ and a set $\mathfrak{F}$ of ultrafilters (Definition 2.10) leads to a local description of weak convergence: a sequence is weakly convergent in $L_\infty(X,\mathcal{B},\lambda)$ if and only if it is weakly convergent at each $x_0 \in X_\infty$, the one-point compactification of X. This notion of weak convergence at a point leads naturally to a definition of the essential range $\mathcal{R}(u)(x_0)$ of u at $x_0 \in X_\infty$. For the relation between weak convergence and the pointwise essential range, see Remark 4.5.

For $\nu \in L_\infty^*(X,\mathcal{B},\lambda)$, let $\hat{\nu}$ denote the Borel measure that, by the Riesz Representation Theorem [12, Thm. 6.19]), corresponds to the restriction to $C_0(X,\tau)$ of the functional defined on ν on $L_\infty(X,\mathcal{B},\lambda)$ by (2.1). Section 5 develops a minimax formula (Theorem 5.7) for $\hat{\nu}$ in terms of ν. It follows that if (X,τ) is not compact, $\hat{\nu}$ may be zero when $\nu \geqslant 0$ is non-zero. In particular when $\omega \in \mathfrak{G}$, either $\hat{\omega} = 0$ or $\hat{\omega} \in \mathfrak{D}$ (a Dirac measure on X) and if (X,τ) is compact $\hat{\omega} \in \mathfrak{D}$. An arbitrary Hahn-Banach extension to $L_\infty(X,\mathcal{B},\lambda)$ of a δ-function on $C_0(X,\tau)$ need not be in $\mathfrak{G}$, but from Section 4.1 there may be infinitely many extensions that are in $\mathfrak{G}$. Those ν for which $\hat{\nu}$ is singular with respect to λ are characterised in Corollary 5.6.

2 L_∞ and its Dual

Let λ be a non-negative, complete, countably additive measure on a σ-algebra $\mathcal{L}$ in a set X and let $\mathcal{N} = \{E \in \mathcal{L} : \lambda(E) = 0\}$. So $(X,\mathcal{L},\lambda)$ is a measure space and $\mathcal{N}$ denotes its null sets. As usual $(L_\infty(X,\mathcal{L},\lambda), \|\cdot\|_\infty)$ denotes the corresponding Banach space of (equivalence classes of) essentially bounded functions. In notation summarised in Section 2.1, the analogue of the Riesz Representation Theorem [12, Thm. 6.19] for functionals in $L_\infty(X,\mathcal{L},\lambda)^*$ is the following.

Theorem 2.1. (Yosida & Hewitt [15], see also [6, Theorem IV.8.16]). *For every bounded linear func-*

tional on $L_\infty(X,\mathcal{L},\lambda)$ there exists a finitely additive measure (Definition 2.2) ν on $\mathcal{L}$ such that

$$f(w) = \int_X w\,d\nu \text{ for all } w \in L_\infty(X,\mathcal{L},\lambda), \tag{2.1}$$
$$\nu(N) = 0 \text{ for all } N \in \mathcal{N} \text{ and } |\nu|(X) = \|f\|_\infty < \infty.$$

Conversely if ν is a finitely additive measure on X with $\nu(N) = 0$ for all $N \in \mathcal{N}$, then f defined by (2.1) *is in $L_\infty(X,\mathcal{L},\lambda)^*$. We write $\nu \in L^*_\infty(X,\mathcal{L},\lambda)$ if* (2.1) *holds for some $f \in L_\infty(X,\mathcal{L},\lambda)^*$.* □

Because ν is finitely additive, but not necessarily σ-additive, integrals in (2.1) should be treated with care. For example, the Monotone Convergence Theorem and Fatou's Lemma do not hold, and the Dominated Convergence Theorem holds only in a restricted form. The next section is a review of notation and standard theory; for a comprehensive account see [15], [6, Ch. III] or [4, Ch. 4]. When combined with the Hahn-Banach theorem, Theorem 2.1 yields the existence of a variety of finitely additive measures.

2.1 Finitely Additive Measures: Notation and Definitions

Although finitely additive measures are defined on algebras (closed under complementation and finite unions), here they are considered only on σ-algebras, where their theory is somewhat more satisfactory, because $\mathcal{L}$ in Theorem 2.1 is a σ-algebra.

Definition 2.2. [15, §1.2-§1.7] *A finitely additive measure ν on $\mathcal{L}$ is a mapping from $\mathcal{L}$ into $\mathbb{R}$ with*

$$\nu(\emptyset) = 0 \text{ and } \sup_{A\in\mathcal{L}} |\nu(A)| < \infty;$$
$$\nu(A \cup B) = \nu(A) + \nu(B) \text{ for all } A,\ B \in \mathcal{L} \text{ with } A \cap B = \emptyset.$$

A finitely additive measure is σ-additive if and only if

$$\nu\left(\cup_{k\in\mathbb{N}} E_k\right) = \sum_{k\in\mathbb{N}} \nu(E_k) \text{ for all } \{E_k\} \subset \mathcal{L} \text{ with } E_j \cap E_k = \emptyset, j \neq k.$$

Let $\Upsilon(\mathcal{L})$ and $\Sigma(\mathcal{L})$ denote, respectively, the families of finitely additive and σ-additive measures on $\mathcal{L}$.

Since finitely-additive measures are not one-signed, the hypothesis that $\sup_{A\in\mathcal{L}} |\nu(A)| < \infty$ does not follow from the fact that $\nu(X) < \infty$. The following results are from [15, §1.9-§1.12].

For $\nu_1,\ \nu_2 \in \Upsilon(\mathcal{L}),\ E \in \mathcal{L}$, let

$$\begin{aligned}(\nu_1 \vee \nu_2)(E) &= \sup_{E\supset F\in\mathcal{L}}\{\nu_1(F) + \nu_2(E \setminus F)\},\\ (\nu_1 \wedge \nu_2)(E) &= -\big((-\nu_1) \vee (-\nu_2)\big)(E).\end{aligned} \tag{2.2a}$$

Then $\nu_1 \vee \nu_2,\ \nu_1 \wedge \nu_2 \in \Upsilon(\mathcal{L})$, which is a lattice, and $\nu \in \Upsilon(\mathcal{L})$ can be written

$$\nu = \nu^+ - \nu^- \text{ where } \nu^+ = \nu \vee 0,\ \ \nu^- = (-\nu) \vee 0 \text{ and } \nu^+ \wedge \nu^- = 0. \tag{2.2b}$$

$\nu^\pm$ are the positive and negative parts of ν and $|\nu| := \nu^+ + \nu_-$ is its total variation (see Theorem 2.1). For $\nu_1,\ \nu_2 \in \Upsilon(\mathcal{L})$ write $\nu_1 \ll \nu_2$ (ν_1 is absolutely continuous with respect to ν_2), if for all $\epsilon > 0$ there exists δ such that $|\nu_1(E)| < \epsilon$ when $|\nu_2|(E) < \delta$, and write $\nu_1 \perp \nu_2$ if for every $\epsilon > 0$ there exists $E \in \mathcal{L}$ such that $|\nu_1|(E) + |\nu_2|(X \setminus E) < \epsilon$.

Remark 2.3. When $\nu_1, \nu_2 \in \Sigma(\mathcal{L}) \subset \Upsilon(\mathcal{L})$ the above definitions imply:

$$\nu_1 \ll \nu_2 \text{ if and only if } |\nu_2|(E) = 0 \text{ implies } \nu_1(E) = 0 \text{ for all } E \in \mathcal{L};$$
$$\nu_1 \perp \nu_2 \text{ if and only if } |\nu_1|(E) + |\nu_2|(X \setminus E) = 0 \text{ for some } E \in \mathcal{L}.$$

However it is important that a non-negative finitely additive measure ν which vanishes on $\mathcal{N}$ (see Theorems 2.1 and 2.9) need not satisfy $\nu \ll \lambda$ if $\nu \notin \Sigma(\mathcal{L})$. □

Definition 2.4. [15, §1.13] *A non-negative $\nu \in \Upsilon(\mathcal{L})$ is purely finitely additive (written $\nu \in \Pi(\mathcal{L})$) if*

$$\{\gamma \in \Sigma(\mathcal{L}) : 0 \leqslant \gamma \leqslant \nu\} = \{0\}.$$

Equivalently, $0 \leqslant \nu \in \Pi(\mathcal{L})$ if and only if $\nu \wedge \gamma = 0$ for all $0 \leqslant \gamma \in \Sigma(\mathcal{L})$. In general, $\nu \in \Upsilon(\mathcal{L})$ is purely finitely additive if ν^+ and ν^- are purely finitely additive.

Note that $\Pi(\mathcal{L}) \cap \Sigma(\mathcal{L}) = \{0\}$ and if $\alpha \in \mathbb{R}$ and $\nu \in \Pi(\mathcal{L})$ then $\alpha\nu \in \Pi(\mathcal{L})$. Moreover $\Pi(\mathcal{L})$ is a lattice [15, Thm. 1.17]: if $\nu_i \in \Pi(\mathcal{L})$, $i = 1, 2$, then $\nu_1 + \nu_2$, $\nu_1 \wedge \nu_2$, $\nu_1 \vee \nu_2 \in \Pi(\mathcal{L})$. The sense in which a purely finitely additive measure *on a σ-algebra* is singular with respect to any σ-additive measure is captured by the following observation which is not true if $\mathcal{L}$ is only an algebra.

Theorem 2.5. [15, Thm 1.22] *For $0 \leqslant \gamma \in \Sigma(\mathcal{L})$ and $0 \leqslant \mu \in \Pi(\mathcal{L})$ there exists $\{E_k\} \subset \mathcal{L}$ with*

$$E_{k+1} \subset E_k, \quad \mu(E_k) = \mu(X) \text{ for all } k \text{ and } \gamma(E_k) \to 0 \text{ as } k \to \infty.$$

Conversely if $0 \leqslant \mu \in \Upsilon(\mathcal{L})$ and for all $0 \leqslant \gamma \in \Sigma(\mathcal{L})$ a sequence $\{E_k\}$ with these properties exists, then $\mu \in \Pi(\mathcal{L})$.

The significance of purely finitely additive measures is evident from the following.

Theorem 2.6. [15, Thms 1.23 & 1.24] *Any $\nu \in \Upsilon(\mathcal{L})$ can be written uniquely as $\nu = \mu + \gamma$ where $\mu \in \Pi(\mathcal{L})$ and $\gamma \in \Sigma(\mathcal{L})$. Any $\nu \in L^*_\infty(X, \mathcal{L}, \lambda)$ can be written uniquely as*

$$\nu = \mu + \gamma \in (L^*_\infty(X, \mathcal{L}, \lambda) \cap \Pi(\mathcal{L})) \oplus (L^*_\infty(X, \mathcal{L}, \lambda) \cap \Sigma(\mathcal{L})). \tag{2.3}$$

If $\nu \geqslant 0$ the elements of the decomposition are non-negative. This is the Yosida-Hewitt Decomposition of finitely additive measures.

By (2.3), $\nu = \mu + \gamma$ where $\mu \in (L^*_\infty(X, \mathcal{L}, \lambda) \cap \Pi(\mathcal{L}))$ and $\lambda \gg \gamma \in \Sigma(\mathcal{L})$. If $(X, \mathcal{L}, \lambda)$ is σ-finite, by the Lebesgue-Radon-Nikodym Theorem [7, Ch. 3.8] there exists $g \in L_1(X, \mathcal{L}, \lambda)$ with

$$\int_X u\, d\gamma = \int_X ug\, d\lambda \text{ for all } u \in L_\infty(X, \mathcal{L}, \lambda). \tag{2.4}$$

In this case (2.3) can be re-written

$$\nu = \mu + g\lambda, \quad \mu \in \Pi(\mathcal{L}) \cap L^*_\infty(X, \mathcal{L}, \lambda), \quad g \in L_1(X, \mathcal{L}, \lambda). \tag{2.5}$$

The relation between this and the Lebesgue decomposition of Borel measures is the topic of Section 5.

2.2 $\mathfrak{G}$: 0-1 Measures

Recall that $L^*_\infty(X, \mathcal{L}, \lambda)$ is the set of finitely additive measures on $\mathcal{L}$ that are zero on $\mathcal{N}$. Let

$$\mathfrak{G} = \{\omega \in L^*_\infty(X, \mathcal{L}, \lambda) : \ \omega(X) = 1 \text{ and } \omega(A) \in \{0, 1\} \text{ for all } A \in \mathcal{L}\}. \tag{2.6}$$

$A \in \mathcal{L}$ is called a λ-atom if $\lambda(A) > 0$ and if $A \supset E \in \mathcal{L}$ implies $\lambda(E) \in \{0, \lambda(A)\}$.

Theorem 2.7. *Suppose $\omega \in \mathfrak{G}$. (a) Either $\omega \in \Pi(\mathcal{L})$ or $\omega \in \Sigma(\mathcal{L})$. (b) If $(X, \mathcal{L}, \lambda)$ is σ-finite and $\omega \in \Sigma(\mathcal{L})$, there exists a λ-atom E_ω such that $\omega(E) = \lambda(E \cap E_\omega)/\lambda(E_\omega)$ for all $E \in \mathcal{L}$.*

Remark. Hence $\mathfrak{G} \subset \Pi(\mathcal{L})$ if $(X, \mathcal{L}, \lambda)$ is σ-finite and $\mathcal{L}$ has no λ-atoms. A stronger statement, Lemma 4.1, can be made when $\mathcal{L}$ is the Borel σ-algebra of a locally compact Hausdorff space. □

Proof. (a) For $\omega \in \mathfrak{G}$, by Theorem 2.6, $\omega = \mu + \gamma$ where $\mu \in \Pi(\mathcal{L})$ and $\gamma \ll \lambda$, $\gamma \in \Sigma(\mathcal{L})$ are non-negative. By Theorem 2.5 there exists $\{E_k\} \subset \mathcal{L}$ with $\mu(E_k) = \mu(X)$ for all k and $\gamma(E_k) \to 0$ as $k \to \infty$. If $\omega(E_k) = 0$ for some k then $0 = \omega(E_k) \geqslant \mu(E_k) = \mu(X)$ and so $\omega = \gamma \in \Sigma(\mathcal{L})$. If $\omega(E_k) = 1$ for all k, then

$$1 = \omega(E_k) = \mu(E_k) + \gamma(E_k) = \mu(X) + \gamma(E_k) \to \mu(X) \text{ as } k \to \infty.$$

Hence $\omega(X) = 1 = \mu(X)$ and consequently $\gamma(X) = 0$. Thus $\omega = \mu \in \Pi(\mathcal{L})$.

(b) Since $\omega \ll \lambda$ where $\omega \in \Sigma(\mathcal{L})$ is finite and λ is σ-additive, by (2.4) there exists $g \in L_1(X, \mathcal{L}, \lambda)$ with $\omega(E) = \int_E g\, d\lambda$ for all $E \in \mathcal{L}$. So $g \geqslant 0$ λ-almost everywhere on X. Since $g \in L_1(X, \mathcal{L}, \lambda)$, $\lambda(\{x \in X : g(x) \geqslant n\}) \to 0$ as $n \to \infty$, and hence, by [7, Cor. 3.6],

$$\omega(\{x \in X : g(x) \geqslant n\}) = \int_{\{x \in X : g(x) \geqslant n\}} g\, d\lambda \to 0 \text{ as } n \to \infty.$$

Since $\omega \in \mathfrak{G}$ it follows that $\omega(\{x \in X : g(x) \geqslant N\}) = 0$ for some $N \in \mathbb{N}$. Now, by finite additivity, $\omega(X) = 1$ and $\omega(E) \in \{0, 1\}$ implies that for every $K \in \mathbb{N}$ there exists a unique $k_{_K} \in \{1, \cdots, N2^K\}$ such that

$$1 = \omega(X) = \sum_{k=1}^{N2^K} \omega(E_k) = \omega(E_{k_{_K}}) \text{ where } E_k = \left\{x \in X : \frac{k-1}{2^K} \leqslant g(x) < \frac{k}{2^K}\right\}.$$

Hence $E_{k_{_{K+1}}} \subset E_{k_{_K}}$ and since ω is σ-additive it follows that $\omega(E_\omega) = 1$ where $E_\omega = \{x \in X : g(x) = \alpha\}$ for some $\alpha \in [0, N]$. Then $\lambda(E_\omega) > 0$ because $\omega(E_\omega) = 1$ and, for all $E \in \mathcal{L}$,

$$\omega(E) = \omega(E \cap E_\omega) = \int_{E \cap E_\omega} \alpha\, d\lambda = \alpha\lambda(E \cap E_\omega).$$

Hence $\alpha = 1/\lambda(E_\omega)$, and E_ω is a λ-atom with the required properties because $\omega \in \mathfrak{G}$. □

Theorem 2.8. *For $u \in L_\infty(X, \mathcal{L}, \lambda)$ and $\omega \in \mathfrak{G}$ there is a unique $\alpha \in I := [-\|u\|_\infty, \|u\|_\infty]$ such that*

$$\omega(\{x \in X : |u(x) - \alpha| < \epsilon\}) = 1 \textit{ for all } \epsilon > 0, \tag{2.7a}$$

$$\int_X u\, d\omega = \alpha \textit{ and } \int_X |u|\, d\omega = |\alpha|. \tag{2.7b}$$

Remark. Thus, on $L_\infty(X, \mathcal{L}, \lambda)$ elements of $\mathfrak{G}$ are analogous to Dirac measures $\mathfrak{D}$ in the theory of continuous functions on topological spaces. When (2.7a) holds we say that $u = \alpha$ on X ω-almost everywhere even though it does not imply that $\omega(\{x \in X : u(x) = \alpha\}) = \omega(X)$ if $\omega \notin \Sigma(\mathcal{L})$. □

Proof. Since ω is zero on $\mathcal{N}$, it is clear that $\alpha \in I$ if (2.7a) holds. Now (2.7a) cannot hold for distinct $\alpha_1 < \alpha_2$ because, with $\epsilon = (\alpha_2 - \alpha_1)/4$ the sets $\omega(\{x \in X : |u(x) - \alpha_i| < \epsilon\})$, $i = 1, 2$, are disjoint and by finite additivity the ω-measure of their union would be 2. Since $\omega \in \mathfrak{G}$, there is at most one α for which (2.7a) holds.

Now suppose that there is no α for which (2.7a) holds. Then for each $\alpha \in I$ there is an $\epsilon_\alpha > 0$ such that $\omega(\{x \in X : |u(x) - \alpha| < \epsilon_\alpha\}) = 0$. By compactness there exists $\{\alpha_1, \cdots, \alpha_K\} \subset I$ such that $I \subset \cup_{k=1}^K (\alpha_k - \epsilon_{\alpha_k}, \alpha_k + \epsilon_{\alpha_k})$ and consequently

$$\begin{aligned} 1 = \omega(X) &= \omega\left(\{x : u(x) \in \cup_{k=1}^K (\alpha_k - \epsilon_{\alpha_k}, \alpha_k + \epsilon_{\alpha_k})\right) \\ &\leqslant \sum_{k=1}^K \omega(\{x : u(x) \in (\alpha_k - \epsilon_{\alpha_k}, \alpha_k + \epsilon_{\alpha_k})\}) = 0. \end{aligned}$$

Hence (2.7a) holds for a unique α. The first part of (2.7b) follows because, by (2.7a), $u = \alpha$ ω-almost everywhere on X and $\omega(X) = 1$. Finally, $\omega\left(\{x \in X : \big||u(x)| - |\alpha|\big| < \epsilon\}\right) = 1$ for all $\epsilon > 0$, and the second part of (2.7b) follows. □

The next result give the existence elements of $\mathfrak{G}$.

Theorem 2.9. [15, Thm. 4.1] *Let $\mathcal{E} \subset \mathcal{L} \setminus \mathcal{N}$ have the property that $E_\ell \in \mathcal{E}$, $1 \leqslant \ell \leqslant L$ implies that $\cap_{\ell=1}^{L} E_\ell \notin \mathcal{N}$. Then there exists $\omega \in \mathfrak{G}$ with $\omega(E) = 1$ for all $E \in \mathcal{E}$.*

The proof is by Zorn's lemma and for given $\mathcal{E}$ there can be uncountably many ω. The same argument underlies the correspondence between elements of $\mathfrak{G}$ and ultrafilters.

Definition 2.10. *Given $(X, \mathcal{L}, \lambda)$, a filter is a family $\mathcal{F}$ of subsets of X satisfying: (i) $X \in \mathcal{F}$ and $\mathcal{N} \cap \mathcal{F} = \emptyset$; (ii) $E_1,\ E_2 \in \mathcal{F}$, implies that $E_1 \cap E_2 \in \mathcal{F}$; (iii) $E_2 \supset E_1 \in \mathcal{F}$ implies that $E_2 \in \mathcal{F}$. A maximal filter $\mathcal{F}$, one which satisfies (iv) $\mathcal{F} \subset \hat{\mathcal{F}}$ implies $\mathcal{F} = \hat{\mathcal{F}}$, is called an ultrafilter. Let $\mathfrak{F}$ denote the family of ultrafilters.*

It is obvious that when $\omega \in \mathfrak{G}$

$$\mathcal{F}(\omega) := \{E \in \mathcal{L} :\ \omega(E) = 1\} \in \mathfrak{F}. \tag{2.8a}$$

Conversely, when $\mathcal{F} \in \mathfrak{F}$,

$$\omega(E) := \left\{ \begin{array}{l} 1 \text{ if } E \in \mathcal{F} \\ 0 \text{ otherwise} \end{array} \right\} \in \mathfrak{G}. \tag{2.8b}$$

This holds because, exactly as in the proof of [15, Thm. 4.1], the maximality of $\mathcal{F} \in \mathfrak{F}$ implies that for $E \in \mathcal{L}$ precisely one of E and $X \setminus E$ is in $\mathcal{F}$. Thus (2.8b) defines $\omega \in \mathfrak{G}$ with $\mathcal{F} = \mathcal{F}(\omega)$ and hence $\omega \leftrightarrow \mathcal{F}(\omega)$ is a one-to-one correspondence between $\mathfrak{G}$ and $\mathfrak{F}$.

By the essential range of u is meant the set

$$\mathcal{R}(u) := \big\{\alpha \in \mathbb{R} : \lambda\big(\{x : |u(x) - \alpha| < \epsilon\}\big) > 0 \text{ for all } \epsilon > 0\big\}. \tag{2.9}$$

Corollary 2.11. *For $u \in L_\infty(X, \mathcal{L}, \lambda)$,*

$$\left\{\int_X u\, d\omega : \quad \omega \in \mathfrak{G}\right\} = \mathcal{R}(u).$$

Proof. It follows from Theorems 2.8 and 2.9 that the right side is a subset of the left. Since $\omega(E) = 1$, $E \in \mathcal{L}$, implies $\lambda(E) > 0$, it is immediate from Theorem 2.8 that the right side contains the left. □

In a topological space (2.8), (2.9) and Corollary 2.11 can be localized to points, (4.1), (4.2) and (4.3).

For $A \in \mathcal{L}$, let $\Delta_A = \{\omega \in \mathfrak{G} :\ \omega(A) = 1\}$ and let $\{\Delta_A : A \in \mathcal{L}\}$ be a base for the topology $\mathfrak{t}$ on $\mathfrak{G}$. Note from Theorem 2.9 that Δ_A is empty if and only if $A \in \mathcal{N}$ and Δ_A is both open and closed because $\mathfrak{G} \setminus \Delta_A = \Delta_{X\setminus A}$. For $u \in L_\infty(X, \mathcal{L}, \lambda)$ let $L[u] : \mathfrak{G} \to \mathbb{R}$ be defined by

$$L[u](\omega) = \int_X u\, d\omega \text{ for all } \omega \in \mathfrak{G}. \tag{2.10}$$

Theorem 2.12. [15, Thms. 4.2 & 4.3] *(a) $(\mathfrak{G}, \mathfrak{t})$ is a compact Hausdorff topological space.*
(b) For $u \in L_\infty(X, \mathcal{L}, \lambda)$, $L[u]$ is continuous on $(\mathfrak{G}, \mathfrak{t})$ with

$$\|u\|_\infty = \|L[u]\|_{C(\mathfrak{G},\mathfrak{t})} :\ = \sup\nolimits_{\omega \in \mathfrak{G}} |L[u](\omega)|,$$

and $u \mapsto L[u]$ *is linear from* $L_\infty(X, \mathcal{L}, \lambda)$ *to* $C(\mathfrak{G}, \mathfrak{t})$*. Moreover, for* $u, v \in L_\infty(X, \mathcal{L}, \lambda)$*,*

$$L[u](\omega)L[v](\omega) = L[uv](\omega) \text{ for all } \omega \in \mathfrak{G}. \tag{2.11}$$

Conversely, for every real-valued continuous function U *on* $(\mathfrak{G}, \mathfrak{t})$ *there exists* $u \in L_\infty(X, \mathcal{L}, \lambda)$ *with* $U = L[u]$*. So* L *is an isometric isomorphism between Banach algebras* $L_\infty(X, \mathcal{L}, \lambda)$ *and* $C(\mathfrak{G}, \mathfrak{t})$*.*

Since $L_\infty(X, \mathcal{L}, \lambda)$ and $C(\mathfrak{G}, \mathfrak{t})$ are isometrically isomorphic, $u_k \rightharpoonup u_0$ in $L_\infty(X, \mathcal{L}, \lambda)$ if and only if $L[u_k] \rightharpoonup L[u_0]$ in $C(\mathfrak{G}, \mathfrak{t})$. Since $(\mathfrak{G}, \mathfrak{t})$ is a compact Hausdorff topological space, it follows from the opening remarks of the Introduction that $L[u_k] \rightharpoonup L[u_0]$ in $C(\mathfrak{G}, \mathfrak{t})$ if and only if $\{\|L[u_k]\|_{C(\mathfrak{G},\mathfrak{t})}\}$ is bounded and $L[u_k] \to L[u_0]$ pointwise on $\mathfrak{G}$. Hence $u_k \rightharpoonup u_0$ in $L_\infty(X, \mathcal{L}, \lambda)$ if and only if

$$\|u_k\|_\infty \leqslant M \text{ and } \int_X u_k \, d\omega \to \int_X u_0 \, d\omega \text{ as } k \to \infty \text{ for all } \omega \in \mathfrak{G}. \tag{2.12}$$

Sequential weak continuity of composition operators is an obvious consequence.

Theorem 2.13. *If* $u_k^n \rightharpoonup u_0^n$ *in* $L_\infty(X, \mathcal{L}, \lambda)$ *as* $k \to \infty$*,* $n \in \{1, \cdots, N\}$*, and* $F : \mathbb{R}^N \to \mathbb{R}$ *is continuous, then* $F(u_k^1, \cdots, u_k^N) \rightharpoonup F(u_0^1, \cdots, u_0^N)$ *in* $L_\infty(X, \mathcal{L}, \lambda)$*.*

Proof. When $u_k^n \rightharpoonup u_0^n$ in $L_\infty(X, \mathcal{L}, \lambda)$, $L[u_k^n] \rightharpoonup L[u_0^n]$ in $C(\mathfrak{G}, \mathfrak{t})$ and consequently $L[u_k^n](\omega) \to L[u_0^n](\omega)$ pointwise in $\mathfrak{G}$ as $k \to \infty$. Therefore, for continuous F,

$$F\big(L[u_k^1](\omega), \cdots, L[u_k^N](\omega)\big) \to F\big(L[u_0^1](\omega), \cdots, L[u_0^N](\omega)\big), \ \omega \in \mathfrak{G}.$$

If F is a polynomial it follows from (2.11) that

$$L[F\big(u_k^1, \cdots, u_k^N\big)](\omega) \to L[F\big(u_0^1, \cdots, u_0^N\big)](\omega), \ \omega \in \mathfrak{G},$$

and this holds for continuous F, by approximation. Consequently, for continuous F,

$$L[F\big(u_k^1, \cdots, u_k^N\big)] \rightharpoonup L[F\big(u_0^1, \cdots, u_0^N\big)] \text{ in } C(\mathfrak{G}, \mathfrak{t})$$

and so $F\big(u_k^1, \cdots, u_k^N\big) \rightharpoonup F\big(u_0^1, \cdots, u_0^N\big)]$ in $L_\infty(X, \mathcal{L}, \lambda)$. □

3 Pointwise and Weak Convergence in $L_\infty(X, \mathcal{L}, \lambda)$

The goal is to characterise weakly null sequences in $L_\infty(X, \mathcal{L}, \lambda)$ in terms of their pointwise behaviour, but we begin with some observations on the pointwise behaviour of weakly convergent sequences.

Lemma 3.1. *In* $L_\infty(X, \mathcal{L}, \lambda)$*,* $u_k \rightharpoonup 0$ *if and only if* $|u_k| \rightharpoonup 0$*.*

Proof. 'Only if' follows from Theorem 2.13 and 'if' is a consequence of (2.12) since $u_k = u_k^+ - u_k^-$, $0 \leqslant u_k \pm \leqslant |u_k|$ and $\omega \geqslant 0$. □

Lemma 3.2. *If* $(X, \mathcal{L}, \lambda)$ *is* σ*-finite and* $\{u_k\}$ *is weakly null, there is a subsequence* $\{u_{k_j}\}$ *with* $u_{k_j}(x) \to 0$ λ*-almost everywhere on* X*.*

Proof. Since $(X, \mathcal{L}, \lambda)$ is σ-finite there exists $f \in L_1(X, \mathcal{L}, \lambda)$ which is positive almost everywhere. Since $|u_k|f \to 0$ in $L_1(X, \mathcal{L}, \lambda)$, there is a subsequence with $|u_{k_j}(x)| \to 0$ for λ-almost all $x \in X$. □

Lemma 3.3. *Suppose that (X,ρ) is a metric space on which λ is a regular Borel measure with the property that for all locally integrable functions f and balls $B(x,r)$ centred at x and radius r,*

$$\lim_{0<r\to 0} \fint_{B(x,r)} f d\lambda = f(x) \text{ for } \lambda\text{-almost all } x \in X \text{ where } \fint_{B(x,r)} f d\lambda := \frac{1}{\lambda(B(x,r))}\int f\,d\lambda. \quad (3.1)$$

Then $u_k \rightharpoonup u_0$ in $L_\infty(X,\mathcal{L},\lambda)$ implies that $u_k(x) \to u_0(x)$ pointwise λ-almost everywhere.

Remark 3.4. From [9, Ch. 1], (3.1) holds in particular when λ is a doubling measure on (X,ρ) (i.e. there exists a constant C such that $\lambda(B(x,2r)) \leqslant C\lambda(B(x,r))$, or on $\mathbb{R}^n$ with the standard metric when λ is any Radon measure (i.e. λ is finite on compact sets). □

Proof. By hypothesis, for $u \in L_\infty(X,\mathcal{L},\lambda)$ there exists a set $E(u) \in \mathcal{B}$ with $\lambda(X \setminus E(u)) = 0$ and

$$u(x) := \lim_{0<r\to 0} \fint_{B(x,r)} u\,d\lambda \text{ for all } x \in E(u). \quad (3.2)$$

Now for $u_k \rightharpoonup u_0$ the set $E = \cap_0^\infty E(u_k)$ has full measure. Let V denote the subspace of $L_\infty(X,\mathcal{L},\lambda)$ spanned by $\{u_k : k \geqslant 0\}$ and for fixed $x \in E$ define a linear functional ℓ_x on V by $\ell_x(u) = u(x)$. Then

$$|\ell_x(u)| = |u(x)| = \left|\lim_{0<r\to 0} \fint_{B(x,r)} u\,d\lambda\right| \leqslant \|u\|_\infty,\ u \in V,$$

and, by the Hahn-Banach Theorem, there exists $L_x \in L_\infty(X,\mathcal{L},\lambda)^*$ with $L_x(u) = \ell_x(u)$ for all $u \in V$. Therefore since $u_k \rightharpoonup u_0$,

$$u_k(x) = \ell_x(u_k) = L_x(u_k) \to L_x(u_0) = \ell(u_0) = u_0(x) \text{ for all } x \in E.$$

Hence $u_k \rightharpoonup u_0$ in $L_\infty(X,\mathcal{L},\lambda)$ implies $u_k(x) \to u_0(x)$ for almost all $x \in X$. □

Remark 3.5. By contrast there follows an example where $\{\|u_k\|_\infty\}$ is bounded, u_k is continuous except at one point and $u_k(x) \to 0$ everywhere as $k \to \infty$, but $u_k \not\rightharpoonup 0$ in $L_\infty(X,\mathcal{L},\lambda)$. Let $X = (-1,1)$, for each $k > 2$ let $u_k(0) = 0$, $u_k(x) = 0$ when $|x| \geqslant 2/k$, $u_k(x) = 1$ if $0 < |x| \leqslant 1/k$, and linear elsewhere. Now in Theorem 2.9 let $E_\ell = (-1/2\ell, 0) \cup (0, 1/2\ell)$ for each ℓ and let ω be a finitely additive measure that takes the value 1 on E_ℓ for all ℓ. Then $\omega \in \mathfrak{G}$ and, by Theorem 2.8, $\int_X u_k\,d\omega = 1$ for all k. So $u_k \not\rightharpoonup 0$, yet it is clear that $u_k(x) \to 0$ for all $x \in X$. □

By a well-known result of Mazur, $y_k \rightharpoonup y$ in a normed linear space implies, for any strictly increasing sequence $\{k_j\}$ in $\mathbb{N}$, that some $\{\overline{y}_i\}$ in the convex hull of $\{y_{k_j} : j \in \mathbb{N}\}$ converges strongly to y. Hence if $u_k \rightharpoonup 0$ in $L_\infty(X,\mathcal{L},\lambda)$, by Lemma 3.1 there exists $\{\overline{u}_i\}$ in the convex hull of $\{|u_{k_j}| : j \in \mathbb{N}\}$ with

$$\overline{u}_i \to 0 \text{ as } i \to \infty \text{ and, for all } i,\ \overline{u}_i = \sum_{j=1}^{m_i} \gamma_j^i |u_{k_j}|, \quad \gamma_j^i \in [0,1] \text{ and } \sum_{j=1}^{m_i} \gamma_j^i = 1, \text{ for some } m_i \in \mathbb{N}.$$

Since γ_j^i may be zero there is no loss in assuming that $\{m_i\}$ is increasing. Therefore, for a strictly increasing sequence $\{k_j\}$ in $\mathbb{N}$,

$$0 \leqslant w_i(x) := \inf\left\{|u_{k_j}(x)| : j \in \{1,\cdots,m_i\}\right\} \leqslant \overline{u}_i(x),\ x \in X,$$

defines a non-increasing sequence in $L_\infty(X,\mathcal{L},\lambda)$ with $\|w_i\|_\infty \to 0$. It follows that if $u_k \rightharpoonup 0$

$$v_J(x) = \inf\left\{|u_{k_j}(x)| : j \in \{1,\cdots,J\}\right\} \quad (3.3)$$

is a non-increasing sequence in $L_\infty(X,\mathcal{L},\lambda)$ with $\|v_J\|_\infty \to 0$ as $J \to \infty$. We now show that a sequence is weakly null in $L_\infty(X,\mathcal{L},\lambda)$ if and only if every sequence $\{v_J\}$, defined as above in terms of a strictly increasing $\{k_j\}$, converges strongly to 0 in $L_\infty(X,\mathcal{L},\lambda)$. To do so, for $u \in L_\infty(X,\mathcal{L},\lambda)$ and $\alpha > 0$, let

$$A_\alpha(u) = \{x \in X : |u(x)| > \alpha\}.$$

Theorem 3.6. *A bounded sequence $\{u_k\}$ in $L_\infty(X,\mathcal{L},\lambda)$ converges weakly to zero if and only if for every $\alpha > 0$ and every strictly increasing sequence $\{k_j\}$ in $\mathbb{N}$ there exists $J \in \mathbb{N}$ with the property that*

$$\lambda\left\{\cap_{j=1}^{J} A_\alpha(u_{k_j})\right\} = 0. \tag{3.4}$$

This criterion is equivalent to saying that for every strictly increasing sequence $\{k_j\}$ in $\mathbb{N}$ the corresponding sequence $\{v_J\}$ in (3.3) *converges strongly to zero in $L_\infty(X,\mathcal{L},\lambda)$.*

Proof. Suppose, for a strictly increasing sequence $\{k_j\}$ and $\alpha > 0$, that (3.4) is false for all $J \in \mathbb{N}$. Then $\mathcal{E} = \{A_\alpha(u_{k_j}) : j \in \mathbb{N}\}$ satisfies the hypothesis of Theorem 2.9. Hence there exists $\omega \in \mathfrak{G}$ such that $\omega(A_\alpha(u_{k_j})) = 1$ for all j. It follows that

$$\int_X |u_{k_j}|d\omega \geqslant \int_{A_\alpha(u_{k_j})} |u_{k_j}|d\omega \geqslant \alpha > 0 \text{ for all } j.$$

Hence $|u_k| \not\rightharpoonup 0$ by (2.12) and so, by Lemma 3.1, $u_k \not\rightharpoonup 0$.

Conversely suppose $u_k \not\rightharpoonup 0$. Then by Lemma 3.1 and (2.12), there exists $\alpha > 0$, a strictly increasing sequence $\{k_j\} \subset \mathbb{N}$ and $\omega \in \mathfrak{G}$ such that

$$\int_X |u_{k_j}|d\omega =: \alpha_j > \alpha > 0 \text{ for all } j \in \mathbb{N}.$$

Since $\alpha_j - \alpha > 0$, by Theorem 2.8,

$$\omega\big(\{x : \big||u_{k_j}| - \alpha_j\big| < \alpha_j - \alpha\}\big) = 1 \text{ for all } j.$$

Therefore, since $|u_{k_j}| - \alpha = |u_{k_j}| - \alpha_j + \alpha_j - \alpha$, it follows that $\omega(A_\alpha(u_{k_j})) = 1$ for all j. Hence, since ω is a 0-1 measure, by finite additivity $\omega\Big(\cap_{j=1}^{J} A_\alpha(u_{k_j})\Big) = 1$ for all J. Since $\omega \in \mathfrak{G} \subset L^*_\infty(X,\mathcal{L},\lambda)$, it follows that (3.4) is false for all J. Finally note that for a strictly increasing sequence $\{k_j\}$ and $\alpha > 0$,

$$\lambda\{x : v_J(x) > \alpha\} = \lambda\big\{x : |u_{k_j}(x)| > \alpha \text{ for all } j \in \{1,\cdots,J\}\big\} = \lambda\big\{\cap_{j=1}^{J} A_\alpha(u_{k_j})\big\}.$$

Since $v_J(x) \geqslant v_{J+1}(x) \geqslant 0$ it follows that $v_J \to 0$ in $L_\infty(X,\mathcal{L},\lambda)$ if and only if (3.4) holds for every $\alpha > 0$. This completes the proof. □

There follows an analogue of Dini's theorem that on compact topological spaces monotone, pointwise convergence of sequences of continuous functions to a continuous function is uniform; equivalently, for bounded monotone sequences weak and strong convergence coincide.

Corollary 3.7. *Suppose $\{u_k\}$ is bounded in $L_\infty(X,\mathcal{L},\lambda)$ and $|u_k(x)| \geqslant |u_{k+1}(x)|$, $k \in \mathbb{N}$, for λ-almost all $x \in X$. Then $u_k \rightharpoonup 0$ if and only if $u_k \to 0$ in $L_\infty(X,\mathcal{L},\lambda)$.*

Proof. The monotonicity of $\{|u_k|\}$ implies that v_J coincides with $|u_J|$ in Theorem 3.6 and so that $|u_J| \to 0$ in $L_\infty(X,\mathcal{L},\lambda)$ as $J \to \infty$ when $u_k \rightharpoonup 0$ if in $L_\infty(X,\mathcal{L},\lambda)$. The converse is obvious. □

3.1 Illustrations of Theorem 3.6

(1) In this example $X = (-1,1)$ with Lebesgue measure, u_k is supported in $[-1/2,1/2]$, $\|u_k\|_\infty = 1$ and u_k, $u_k^- \rightharpoonup 0$, but $u_k^+ \not\rightharpoonup 0$ where $u_k^\pm(x) = u_k(x \pm 1/2^{k+1})$. To see this, let $A_k = [1/2^{k+1}, 1/2^k)$, $A_k^\pm = A_k \mp 1/2^{k+1}$, $u_k = \chi_{A_k}$ and $u_k^\pm = \chi_{A_k^\pm}$. Clearly $u_k^\pm(x) = u_k(x \pm 1/2^{k+1})$ and $u_k^+ \not\rightharpoonup 0$ because v_J, defined in (3.3) by u_k^+, is 1 on $(0, 1/2^{J+1})$. But since $\{A_k\}$ and $\{A_k^-\}$ are two mutually disjoint families,

in (3.3) v_J, defined for any $\{k_j\} \subset \mathbb{N}$ by u_{k_j} or $u^-_{k_j}$, is zero for $J \geqslant 2$. Hence $u_k \rightharpoonup 0$ and $u^-_k \rightharpoonup 0$. That $\chi_{A_k} \rightharpoonup 0$ for a disjoint family of sets is used in Remark 4.5. □

(2) In $L_\infty(X, \mathcal{L}, \lambda)$ let $u_k(x) = \sum_{i=1}^\infty \alpha_i \chi_{A^i_k}, x \in X$, where $\Sigma_{i=1}^\infty |\alpha_i| < \infty$ and, for each $i \in \mathbb{N}$, $\{A^i_k\}_{k\in\mathbb{N}}$ is a family of mutually disjoint non-null measurable sets. Then $u_k \rightharpoonup 0$ in $L_\infty(X, \mathcal{L}, \lambda)$.

To see this, note that for each $x \in X$ and $i \in \mathbb{N}$ there exists at most one $k \in \mathbb{N}$, denoted, if it exists, by $\kappa(x, i)$, such that $x \in A^i_k$ if and only if $k = \kappa(x, i)$. Note also that for $\epsilon > 0$ there exists $I_\epsilon \in \mathbb{N}$ such that $\Sigma_{I_\epsilon+1}^\infty |\alpha_i| < \epsilon$. Hence, for any given $k \in \mathbb{N}$ and $x \in X$,

$$|u_k(x)| \leqslant \sum_{i=1}^{I_\epsilon} |\alpha_i| \chi_{A^i_k}(x) + \epsilon = \sum_{\substack{i\in\{1,\cdots,I_\epsilon\}\\ \kappa(x,i)=k,}} |\alpha_i| + \epsilon.$$

Since $\{\kappa(x, i) : i \in \{1, \cdots , I_\epsilon\}\}$ has at most I_ϵ elements, there exists $k \in \{1, \cdots , I_\epsilon + 1\}$ such that $k \neq \kappa(x, i)$ for any $i \in \{1, \cdots , I_\epsilon\}$. Consequently $\inf\{|u_k(x)| : 1 \leqslant k \leqslant I_\epsilon + 1\} \leqslant \epsilon$, independent of $x \in X$. Since this argument can be repeated with $k \in \mathbb{N}$ replaced by any strictly increasing subsequence $\{k_j\}$, it follows that $\{v_J\}$ defined in terms of any subsequence in (3.3) has $\|v_J\|_\infty \to 0$ in $L_\infty(X, \mathcal{L}, \lambda)$. The weak convergence of $\{u_k\}$ follows. For the special case, take $\alpha_1 = 1$ and $\alpha_i = 0$, $i \geqslant 2$. □

(3) Let $u : \mathbb{R} \to \mathbb{R}$ be essentially bounded and measurable with $|u(x)| \to 0$ as $|x| \to \infty$ and let $u_k(x) = u(x + k)$. Then $u_k \rightharpoonup 0$ in $L_\infty(X, \mathcal{L}, \lambda)$ where λ is Lebesgue measure on $\mathbb{R}$. To see this, for $\epsilon > 0$ suppose that $|u(x)| < \epsilon$ if $|x| > K_\epsilon$. The for any $\{k_j\} \subset \mathbb{N}$, $\|v_J\|_\infty < \epsilon$ for all $J \geqslant K_\epsilon$ where $\{v_J\}$ is defined in terms of $\{u_{k_j}\}$ by (3.3), and the result follows. □

(4) Let $u : \mathbb{R} \to \mathbb{R}$ be essentially bounded and measurable with $u(x) \to 0$ as $x \to \infty$ and $u(x) \to 1$ as $x \to -\infty$. Let $u_k(x) = u(x + k)$. Then $u_k(x) \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}$, but u_k is not weakly convergent to 0 because of Theorem 3.6. However, in the notation of Definition 4.3, $u_k \rightharpoonup 0$ at every point of $\mathbb{R}$, but not at the point at infinity in its one-point compactification. □

(5) Define $\{u_k\}_{k\in\mathbb{N}} \subset L_\infty(X, \mathcal{L}, \lambda)$ by $u_k(x) = \sin(1/(kx))$, $x \in X = (0, 2\pi)$, with the standard measure λ on the Lebesgue σ-algebra on X. Clearly $|u_k(x)| \to 0$ as $k \to \infty$ uniformly on $(\epsilon, 2\pi)$ for any $\epsilon \in (0, 2\pi)$. Therefore if a subsequence $\{u_{k_j}\}$ is weakly convergent, its weak limit must be zero.

To see that no subsequence of $\{u_k\}$ is weakly convergent to 0, consider first a strictly increasing sequence $\{k_j\}$ of natural numbers for which there exists a prime power p^m which does not divide k_j for all j. Then, for $J \in \mathbb{N}$ sufficiently large, let

$$x_J = \left\{\frac{\pi}{p^m}\text{lcm}\,\{k_1, \cdots , k_J\}\right\}^{-1} \in (0, 2\pi),$$

where lcm denotes the least common multiple. Then, since $p^m \nmid k_j$ and p is prime,

$$\frac{1}{k_j x_J} = \frac{\text{lcm}\,\{k_1, \cdots , k_J\}}{p^m k_j}\,\pi \quad \text{where} \quad \frac{\text{lcm}\,\{k_1, \cdots , k_J\}}{k_j} = r \bmod p^m, \quad r \in \{1, \cdots , p^m - 1\},$$

from which it follows that $|u_{k_j}(x_J)| \geqslant |\sin(\pi/p^m)| > 0$, independent of J. Since, for all $j \in \{1, \cdots , J\}$, u_{k_j} is continuous at x_J, it follows that $\|v_J\|_{L_\infty(X,\mathcal{L},\lambda)} \geqslant |\sin(\pi/p^m)| > 0$ for all J sufficiently large. By Theorem 3.6 this shows that $u_{k_j} \not\rightharpoonup 0$ if $\{k_j\}$ has a subsequence $\{k'_j\}$for which $p^m \nmid k'_j$ for all $j \in \mathbb{N}$. Note that if this hypothesis is not satisfied by $\{k_j\}$ for any prime p and $m \in \mathbb{N}$, then every $K \in \mathbb{N}$ is a divisor of k_j for all j sufficiently large, how large depending on K. Consequently, if $u_{k_j} \rightharpoonup 0$, $\{k_j\}$ has subsequence $\{k'_j\}$ with the property that $2^{j+2}k'_j$ divides k'_{j+1} for all j. In other words $2^{j+2}n_j k'_j = k'_{j+1}$, $n_j \in \mathbb{N}$, and $[0, k'_{j+1})$ is a union of $2^{j+2}n_j$ disjoint intervals of length k'_j.

Now fixed $J \in \mathbb{N}$, let m_J denote the mid-point of $I_J := [0, k'_J)$, and let $I_{J-1} := [m_J - k'_{J-1}, m_J)$, which is a half open interval of length k'_{J-1} to the left of m_J. Then

$$x = r \bmod k'_J \text{ where } r \in \left[\frac{k'_J}{2}\left(1 - \frac{1}{2^J n_{J-1}}\right), \frac{k'_J}{2}\right] \text{ for all } x \in I_{J-1},$$

since $2^{J+1} n_{J-1} k'_{J-1} = k'_J$. Note that m_J, and hence the end-points of I_{J-1}, are integer multiples of k'_{J-1}. Now denote the mid-point of I_{J-1} by m_{J-1} and let $I_{J-2} = [m_{J-1} - k'_{J-2}, m_{J-1})$, the interval of length k'_{J-2} to the left of m_{J-1}. Then $I_{J-2} \subset I_{J-1} \subset I_J$ and

$$x = r \bmod k'_{J-1} \text{ where } r \in \left[\frac{k'_{J-1}}{2}\left(1 - \frac{1}{2^{J-1} n_{J-2}}\right), \frac{k'_{J-1}}{2}\right] \text{ for all } x \in I_{J-2},$$

since $2^J n_{J-2} k'_{J-2} = k'_{J-1}$. Repeating this construction leads to a nested sequence of intervals, $I_0 \subset I_1 \subset \cdots \subset I_{J-1}$ with the property that

$$x = r \bmod k'_{i+1} \text{ where } r \in \left[\frac{k'_{i+1}}{2}\left(1 - \frac{1}{2^{i+1} n_i}\right), \frac{k'_{i+1}}{2}\right] \text{ for all } x \in I_i.$$

Now let $x_J = \{m_0 \pi\}^{-1}$ where $m_0 \in I_0$. Since $I_0 = \cap_{i=0}^{J-1} I_i$ and $\frac{1}{2}\left(1 - \frac{1}{2^{i+1} n_i}\right) \geqslant \frac{1}{4}$ for $i \geqslant 0$,

$$|u_{k'_j}(x_J)| = \sin\left(\frac{m_0}{k'_j}\pi\right) \geqslant \sin(\pi/4) = \frac{1}{\sqrt{2}},\ j \in \{1, \cdots, J\},$$

independent of $J \in \mathbb{N}$. Hence $\{v_J\}$ defined by (3.3) using the subsequence $\{k'_j\}$ does not converge to 0 in $L_\infty(X, \mathcal{L}, \lambda)$. It follows that $u_{k_j} \not\to 0$ as $j \to \infty$ for any $\{k_j\} \subset \mathbb{N}$. □

4 $L^*_\infty(X, \mathcal{B}, \lambda)$ when (X, τ) is a Topological Space

This section deals with $L^*_\infty(X, \mathcal{B}, \lambda)$ when (X, τ) is a locally compact Hausdorff topological space, $\mathcal{B}$ is the corresponding Borel σ-algebra and $\lambda \geqslant 0$ is a measure on $\mathcal{B}$ as described in Section 2. In addition here λ is assumed regular and finite on compact sets. In that setting a regular Borel measure that takes only values 0 or 1 is a Dirac measure concentrated at a point $x_0 \in X$. As before, $\mathfrak{G}$ is defined by (2.6).

Lemma 4.1. *For $\omega \in \mathfrak{G}$ there exists a compact set $K \in \mathcal{B}$ with $\omega(K) = 1$ if and only if there exists $x_0 \in X$ such that $\omega(G) = 1$ for all open sets G with $x_0 \in G$. For all $\omega \in \mathfrak{G}$ there is at most one such x_0 and when (X, τ) is compact there is exactly one such x_0.*

Proof. Suppose that $\omega(K) = 1$, K compact, and the result is false. Then for $x \in K$ there is an open G_x with $x \in G_x$ and $\omega(G_x) = 0$. By compactness, $K \subset \cup_{i=1}^K G_{x_i}$ where $\omega(G_{x_i}) = 0$, $1 \leqslant i \leqslant K$, which implies $\omega(K) = 0$. Since this is false, $\omega(K) = 1$ for compact K implies the existence of $x_0 \in K$ with the required property. Since X is Hausdorff, if there is another $x_1 \in X$ with this property there are open sets with $x_0 \in G_{x_0}$, $x_1 \in G_{x_1}$ and $G_{x_0} \cap G_{x_1} = \emptyset$. But this is impossible because by finite additivity $\omega\big(G_{x_0} \cup G_{x_1}\big) = 2$. Now suppose that $\omega(K) = 0$ for all compact sets K. By local compactness, for $x \in X$ there is an open set G_x with $x \in G_x$ and its closure $\overline{G_x}$ is compact. Since $\omega(G_x) \leqslant \omega(\overline{G_x}) = 0$, there is no $x \in X$ with the required property. Finally, the existence of x_0 when X is compact follows because $\omega(X) = 1$. This completes the proof. □

Let (X_∞, τ_∞) denote the one-point compactification [10] of (X, τ). Then $X_\infty = X \cup \{x_\infty\}$, $x_\infty \notin X$ (the "point at infinity"), and a subset G of X_∞ is open if either $G \subset X$ is open in X, or $G = \{x_\infty\} \cup (X \setminus K)$ for some compact $K \subset X$. Then (X_∞, τ_∞) is a compact Hausdorff topological space because (X, τ) is locally compact Hausdorff, and (X, τ) is compact if and only if $\{x_\infty\}$ is an isolated point (open and closed) in (X_∞, τ_∞). For $\omega \in \mathfrak{G}$, let ω_∞ be defined on Borel subsets E of X_∞ by $\omega_\infty(E) = \omega(E \cap X)$. Then ω_∞ is the unique finitely additive measure on X_∞ which takes only values 0 and 1 and coincides with ω on Borel sets in X. In this setting Lemma 4.1 can be re-stated:

Lemma 4.2. *Let (X, τ) be a locally compact Hausdorff space and $\omega \in \mathfrak{G}$. Then there exists a unique $x_0 \in X_\infty$ such that $\omega_\infty(G) = 1$ for all open sets G in X_∞ with $x_0 \in G$; $x_0 = x_\infty$ if and only if $\omega(K) = 0$ for all compact $K \subset X$ and $x_0 \in X$ if (X, τ) is compact.*

4.1 Localization of Weak Convergence in $L_\infty(X, \mathcal{B}, \lambda)$

By (2.8) there is a one-to-one correspondence between $\mathfrak{G}$ and $\mathfrak{F}$. For $x_0 \in X_\infty$, let $\mathfrak{G}(x_0) \subset \mathfrak{G}$ denote the set of $\omega \in \mathfrak{G}$ for which the conclusions of Lemma 4.1 holds, and let $\mathfrak{F}(x_0) \subset \mathfrak{F}$ be the corresponding family of ultrafilters. Then, by Lemma 4.1,

$$\mathfrak{G} = \cup_{x_0 \in X_\infty} \mathfrak{G}(x_0), \qquad \mathfrak{F} = \cup_{x_0 \in X_\infty} \mathfrak{F}(x_0), \tag{4.1}$$

which leads to the following definition of weak pointwise convergence.

Definition 4.3. *u_k converges weakly to u at $x_0 \in X_\infty$ if*

$$\int_X u_k \, d\omega \to \int_X u \, d\omega \textit{ for all } \omega \in \mathfrak{G}(x_0).$$

The localized version of Theorem 3.6 is immediate. For $u \in L_\infty(X, \mathcal{L}, \lambda)$, $\alpha > 0$ and $E \in \mathcal{L}$ let $A_\alpha(u|_E) = \{x \in E : |u(x)| > \alpha\}$.

Theorem 4.4. *A bounded sequence $\{u_k\}$ in $L_\infty(X, \mathcal{B}, \lambda)$ converges weakly to zero at $x_0 \in X_\infty$ if and only if for every $\alpha > 0$, every strictly increasing sequence $\{k_j\}$ in $\mathbb{N}$ and every open $G \subset X_\infty$ with $x_0 \in G$ there exists J with $\lambda\big\{ \cap_{j=1}^J A_\alpha(u_{k_j}|_G)\big\} = 0$. Equivalently, in* (3.3), *$v_J \to 0$ in $L_\infty(G, \mathcal{B}, \lambda)$.*

By analogy with (2.9), for $x_0 \in X_\infty$ the essential range of u at $x_0 \in X_\infty$ is defined by

$$\mathcal{R}(u)(x_0) = \left\{ \int_X u \, d\omega : \omega \in \mathfrak{G}(x_0) \right\}. \tag{4.2}$$

As in Corollary 2.11, for open G with $x_0 \in G$,

$$\mathcal{R}(u)(x_0) = \left\{ \int_G u \, d\omega : \omega \in \mathfrak{G}(x_0) \right\} = \{\alpha : \lambda\{x \in G : |\alpha - u(x)| < \epsilon\} > 0 \text{ for all } \epsilon > 0\}. \tag{4.3}$$

Note that $\mathcal{R}(u)(x_0)$ is closed in $\mathbb{R}$ because, by (4.3), for any $x_0 \in X$ its complement is open. It is immediate from (2.12), Lemmas 4.1 and 4.2 that $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if for all $x_0 \in X_\infty$

$$\alpha_k := \int_X u_k \, d\omega \to \int_X u \, d\omega =: \alpha \text{ as } k \to \infty \text{ for all } \omega \in \mathfrak{G}(x_0),$$

which is not equivalent to $\alpha_k \to \alpha$ when $\alpha_k \in \mathcal{R}(u_k)(x_0)$ and $\alpha \in \mathcal{R}(u)(x_0)$ because, possibly,

$$\alpha_k = \int_X u_k \, d\omega_k \text{ and } \alpha = \int_X u \, d\omega, \text{ but } \omega_k \neq \omega.$$

However, $\alpha = \int_X u \, d\omega \in \mathcal{R}(u)(x_0)$, $\omega \in \mathfrak{G}(x_0)$, may be thought of as a directional limit of u at x_0, the "direction" being determined by $\mathcal{F}(\omega) \in \mathfrak{F}(x_0)$. Then weak convergence in $L_\infty(X, \mathcal{B}, \lambda)$ is equivalent to convergence, for each $\mathcal{F} \in \mathfrak{F}(x_0)$, of the directional limits of u_k at x_0 to corresponding directional limits of u at x_0, for each $x_0 \in X_\infty$. Therefore, by Theorem 2.8, $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if for all $x_0 \in X_\infty$ and all $\omega \in \mathfrak{G}(x_0)$

$$|\alpha_k - \alpha| \to 0 \text{ and } \omega\{x \in G : |u_k(x) - \alpha_k| + |u(x) - \alpha| < \epsilon\} = 1$$
$$\text{for all } \epsilon > 0 \text{ and all open sets } G \subset X_\infty \text{ with } x_0 \in G, \tag{4.4a}$$

equivalently $u_k \rightharpoonup u$ in $L_\infty(X, \mathcal{B}, \lambda)$ if and only if for all $x_0 \in X_\infty$ and all $\mathcal{F} \in \mathfrak{F}(x_0)$,

$$|\alpha_k - \alpha| \to 0 \text{ and } \{x \in G : |u_k(x) - \alpha_k| + |u(x) - \alpha| < \epsilon\} \in \mathcal{F}$$
$$\text{for all } \epsilon > 0 \text{ and all open sets } G \subset X_\infty \text{ with } x_0 \in G. \tag{4.4b}$$

Remark 4.5. It follows that for $u_k \rightharpoonup u$ it is *necessary* that for every $x_0 \in X_\infty$ and every $\alpha \in \mathcal{R}(u)(x_0)$ there exist $\alpha_k \in \mathcal{R}(u_k)(x_0)$ such that $\alpha_k \to \alpha$ as $k \to \infty$ and *sufficient* that for every $x_0 \in X_\infty$

$$\sup\{|\gamma| : \gamma \in \mathcal{R}(u_k - u)(x_0)\} \to 0 \text{ as } k \to \infty.$$

As noted earlier, the necessary condition is not sufficient. To see that the sufficient condition is not necessary, let $u_k = \chi_{A_k}$ where $\{A_k\}$ is a sequence of disjoint segments centred on the origin 0 of the unit disc X in $\mathbb{R}^2$. Then $\mathcal{R}(u_k)(0) = \{0, 1\}$ but $u_k \rightharpoonup 0$ by the last remark in Section 3.1 (1) or, equivalently, by Section 3.1 (2) with $\alpha_1 = 1$ and $\alpha_i = 0,\ i \geqslant 2$. In this example $\int_X u_k d\omega \to 0$, but not uniformly, for every $\omega \in \mathfrak{G}(0)$.

5 Restriction to $C_0(X, \tau)$ of Elements of $L_\infty^*(X, \mathcal{B}, \lambda)$

Throughout this section (X, τ) is a locally compact Hausdorff topological space and $C_0(X, \tau)$ is the space of real-valued continuous functions v on X with the property that for all $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|v(x)| < \epsilon$ for all $x \in X \setminus K$. When endowed with the maximum norm

$$\|v\|_\infty = \max_{x \in X} |v(x)|, \quad v \in C_0(X, \tau), \tag{5.1}$$

$C_0(X, \tau)$ is a Banach space which if X is compact consists of all real-valued continuous functions on X. Let $\nu \in L_\infty^*(X, \mathcal{B}, \lambda)$, as in Theorem 2.1 define $f \in L_\infty(X, \mathcal{B}, \lambda)^*$ by

$$f(u) = \int_X u\, d\nu,\ u \in L_\infty(X, \mathcal{B}, \lambda),$$

and let $\hat{f}$ denote the restriction of f to $C_0(X, \tau)$. By the Riesz Representation theorem [12, Thm. 6.19] there is a unique bounded regular Borel measure $\hat{\nu} \in \Sigma(\mathcal{B})$ corresponding to $\hat{f}$, and consequently

$$\int_X v\, d\nu = \int_X v\, d\hat{\nu} \text{ for all } v \in C_0(X, \tau). \tag{5.2}$$

The goal is to understand how $\hat{\nu}$ depends on ν and, since $\hat{\nu}^\pm = \widehat{\nu^\pm}$ (see (2.2b)), there is no loss of generality in restricting attention to non-negative $\nu \in L_\infty^*(X, \mathcal{B}, \lambda)$. Recall

(i) from the Yosida-Hewitt decomposition (2.5), $\nu = \mu + g\lambda$ where $\mu \in L_\infty^*(X, \mathcal{B}, \lambda)$ is purely finitely additive and $g\lambda$, $g \in L_1(X, \mathcal{B}, \lambda)$, is σ-additive.

(ii) from the Lebesgue-Radon-Nikodym Theorem [7, Thm. 3.8], [12, Thm. 6.10], $\hat{\nu} = \rho + k\lambda$ where ρ and $k\lambda$ are σ-additive, $k \in L_1(X, \mathcal{B}, \lambda)$, and ρ is singular with respect to λ. Thus $\hat{\nu}$ has a singularity with respect to λ if $\hat{\nu}(E) \neq 0$ (equivalently $\rho(E) \neq 0$) for some $E \in \mathcal{N}$, and $\hat{\nu}$ is singular if in addition $k = 0$, where

$$\int_X v\, d\mu + \int_X vg\, d\lambda = \int_X v\, d\nu = \int_X v\, d\hat{\nu} = \int_X v\, d\rho + \int_X vk\, d\lambda \text{ for all } v \in C_0(X, \tau), \tag{5.3}$$

where $\rho \perp \lambda$ in $\Sigma(\mathcal{B})$, $\mu \perp \lambda$ in $\Upsilon(\mathcal{B})$ (see Remark 2.3 for the distinction), and $g, k \in L_1(X, \mathcal{B}, \lambda)$. Valadier was first to note that the relation between μ and ρ, and g and k is not straightforward.

Theorem (Valadier [13]). *When λ is Lebesgue measure on $[0, 1]$ there is a non-negative $\nu \in \Pi(\mathcal{B})$ with*

$$\int_0^1 v\, d\nu = \int_0^1 v\, d\lambda \textit{ for all } v \in C[0, 1].$$

Thus in (i), (ii) and (5.3)*, $0 \neq \mu \in \Pi(\mathcal{B})$ and $g = 0$ but $\rho = 0$ and $k \equiv 1$, and $\hat{\nu}$ has no singularity.*

Hensgen independently observed that the last claim in [15, Theorem 3.4] is false.

Theorem (Hensgen [8])**.** *With $X = (0,1)$ there exists $\nu \in L_\infty^*(X,\mathcal{B},\lambda)$ which is non-zero and not purely finitely additive but $\int_0^1 v\,d\nu = 0$ for all $v \in C(0,1)$.*

Subsequently Abramovich & Wickstead [1] provided wide ranging generalizations and recently Wrobel [14] gave a sufficient condition on ν for $\hat\nu$ to be singular with respect to Lebesgue measure on $[0,1]$. To find a formula for $\hat\nu$ satisfying (5.2) for a given non-negative $\nu \in L_\infty^*(X,\mathcal{B},\lambda)$, and to characterise those ν for which $\hat\nu$ has a singularity, recall the following version of Urysohn's Lemma.

Lemma 5.1. [12, §2.12] *If $K \subset G \subset X$ where K is compact and G is open, there exists a continuous function $f : X \to [0,1]$ such that $f(K) = 1$, $\overline{\{x : f(x) > 0\}} \subset G$ is compact, and hence $f \in C_0(X,\tau)$.*

Lemma 5.2. *Suppose $0 \leqslant \nu \in L_\infty^*(X,\mathcal{B},\lambda)$ and $B \in \mathcal{B}$. Then $\nu(K) \leqslant \hat\nu(B) \leqslant \nu(G)$ for compact K and open G with $K \subset B \subset G$. Moreover*

$$\nu(K) \leqslant \hat\nu(K) \text{ and } \hat\nu(G) \leqslant \nu(G) \text{ for all compact } K \text{ and open } G \text{ in } X$$

and $\nu(F) \leqslant \hat\nu(F) + \nu(X) - \hat\nu(X)$ if F is closed. Thus $\hat\nu(X) = \nu(X)$ implies that $\nu(F) \leqslant \hat\nu(F)$ for all closed sets $F \subset X$. (That $\nu(X) = \hat\nu(X)$ when (X,τ) is compact was noted following (5.2).*)*

Proof. For a given Borel set B and $K \subset B \subset G$ as in the statement, let f be the continuous function determined in Lemma 5.1 by K and G. Then

$$\nu(K) \leqslant \int_X f\,d\nu \leqslant \nu(G) \text{ and } \hat\nu(K) \leqslant \int_X f\,d\hat\nu \leqslant \hat\nu(G).$$

It follows from (5.2) that $\nu(K) \leqslant \hat\nu(G)$ and $\hat\nu(K) \leqslant \nu(G)$ whence, since $\hat\nu$ is regular [12, Thm. 6.19], $\nu(K) \leqslant \hat\nu(B) \leqslant \nu(G)$. In particular, if $B = K$ is compact, $\nu(K) \leqslant \hat\nu(K)$, and if $B = G$ is open, $\hat\nu(G) \leqslant \nu(G)$. That $\nu(F) \leqslant \hat\nu(F) + \nu(X) - \hat\nu(X)$ when F is closed follows by finite additivity since $0 \leqslant \nu(X), \hat\nu(X) < \infty$. □

Remark 5.3. A non-negative finitely additive set function ν on $\mathcal{B}$ is said to be regular [6, III.5.11] if for all $E \in \mathcal{B}$ and $\epsilon > 0$ there are sets $F \subset E \subset G$ with F closed, G open and $\nu(G \setminus F) < \epsilon$. If X is compact and ν is regular, by a theorem of Alexandroff [6, III.5.13] ν is σ-additive and hence $\hat\nu = \nu$. By Lemma 5.2, if $\nu(X) = \hat\nu(X)$ and $F \subset E \subset G$, where F is closed and G is open,

$$\nu(F) \leqslant \hat\nu(F) \leqslant \hat\nu(E) \leqslant \hat\nu(G) \leqslant \nu(G).$$

Hence if $\nu(X) = \hat\nu(X)$ and $\nu \geqslant 0$ regular implies that $\nu = \hat\nu$ is σ-additive on $\mathcal{B}$. □

Theorem 5.4. *Suppose K is compact, G is open, $K \subset G$ and $0 \leqslant \nu \in L_\infty^*(X,\mathcal{B},\lambda)$. Then for $n \in \mathbb{N}$ there exist compact K_n and open G_n with*

$$K \subset G_n \subset K_n \subset G, \quad G_n \subset G_{n-1},\ K_n \subset K_{n-1},$$
$$\hat\nu(K) \leqslant \nu(K_n), \quad \hat\nu(G) \geqslant \nu(G_n) \text{ and } \lambda(K_n) < \lambda(K) + 1/n.$$

Proof. Since λ is a regular Borel measure that is finite on compact sets there exist open sets G^k with $K \subset G^k \subset G$ and $\lambda(G^k) < \lambda(K) + 1/k$ for $k \in \mathbb{N}$. By Lemma 5.1 there exists a continuous function $f_k : X \to [0,1]$ such that $f_k(K) = 1$ and $\overline{\{x : f_k(x) > 0\}}$ is a compact subset of G^k. For $x \in X$, let $g_n(x) = \min\{f_k(x) : k \leqslant n\}$ so that $g_n \leqslant g_{n-1}$, g_n is continuous on X, $g_n(K) = 1$ and $\overline{\{x : g_n(x) > 0\}} \subset G^n$ is compact.

Let $G_n = \{x : g_n(x) > 0\}$ and $K_n = \overline{\{x : g_n(x) > 0\}}$. Then $K \subset G_n \subset K_n \subset G^n \subset G$ and, by Lemma 5.2,

$$\hat{\nu}(K) \leqslant \hat{\nu}(G_n) \leqslant \nu(G_n) \leqslant \nu(K_n), \quad \hat{\nu}(G) \geqslant \hat{\nu}(K_n) \geqslant \nu(K_n) \geqslant \nu(G_n),$$

and $\lambda(K_n) < \lambda(K) + 1/n$ because $K_n \subset G^n$. Now $\{G_n\}$ and $\{K_n\}$ are nested sequences of open and compact sets, respectively, because $g_n(x)$ is decreasing in n, with the required properties. □

Corollary 5.5. *For G open, K compact and $\nu \in L^*_\infty(X, \mathcal{B}, \lambda)$ non-negative,*

$$\hat{\nu}(G) = \sup\{\nu(K) : K \subset G,\ K \text{ compact}\}, \quad \hat{\nu}(K) = \inf\{\nu(G) : K \subset G,\ G \text{ open}\}.$$

Proof. Let G be open. Then for any $\epsilon > 0$ there exists compact $K_\epsilon \subset G$ with $\hat{\nu}(K_\epsilon) > \hat{\nu}(G) - \epsilon$, since $\hat{\nu}$ is regular, and $\nu(K_\epsilon) \leqslant \hat{\nu}(K_\epsilon) \leqslant \hat{\nu}(G)$ by Lemma 5.2. Now by Theorem 5.4 there exists compact K_1 with $K_\epsilon \subset K_1 \subset G$ and $\hat{\nu}(G) \geqslant \hat{\nu}(K_1) \geqslant \nu(K_1) \geqslant \hat{\nu}(K_\epsilon) > \hat{\nu}(G) - \epsilon$. This establishes the first identity. Similarly for compact K and $\epsilon > 0$ there exists open G_ϵ with $K \subset G_\epsilon$ and $\hat{\nu}(G_\epsilon) < \hat{\nu}(K) + \epsilon$, and an open G_1 with $\hat{\nu}(G_\epsilon) \geqslant \nu(G_1)$, $K \subset G_1 \subset G_\epsilon$, whence $\hat{\nu}(K) + \epsilon > \hat{\nu}(G_\epsilon) \geqslant \nu(G_1)$. □

Corollary 5.6. *For $0 \leqslant \nu \in L^*_\infty(X, \mathcal{B}, \lambda)$, $\hat{\nu} \in \Sigma(\mathcal{B})$ has a singularity if and only if there exists $\alpha > 0$ and a sequence of compact sets with $\nu(K_n) \geqslant \alpha$, $K_{n+1} \subset K_n$ for all n, and $\lambda(K_n) \to 0$ as $n \to \infty$.*

Proof. If $\alpha > 0$ and such a sequence exists, by Lemma 5.2, $\hat{\nu}(K_n) \geqslant \alpha$ for all n. Since $\{K_n\}$ is nested and $\hat{\nu}$ is σ-additive it follows that $\hat{\nu}(K) \geqslant \alpha$ where $K = \cap_n K_n$. Since $K \in \mathcal{N}$, because $\lim_{n\to\infty} \lambda(K_n) = 0$ and λ is σ-additive, ν has a singularity. Conversely if $\hat{\nu} \geqslant 0$ has a singularity there exists $E \in \mathcal{N}$ and $\alpha > 0$ with $\hat{\nu}(E) = 2\alpha$. Since $\hat{\nu}$ is regular, there exists a compact $K \subset E$ with $\hat{\nu}(K) \geqslant \alpha > 0$. Now since $\lambda(K) = 0$ because $K \subset E \in \mathcal{N}$, the existence of compact sets with $\nu(K_n) \geqslant \hat{\nu}(K) \geqslant \alpha$, $K_{n+1} \subset K_n$ for all n, and $\lambda(K_n) \to 0$ as $n \to \infty$ follows from Theorem 5.4. □

Theorem 5.7. *For $B \in \mathcal{B}$ and $0 \leqslant \nu \in L^*_\infty(X, \mathcal{B}, \lambda)$,*

$$\hat{\nu}(B) = \inf_{\substack{G \text{ open} \\ B \subset G}} \left\{ \sup_{\substack{K \text{ compact} \\ K \subset G}} \nu(K) \right\} = \sup_{\substack{K \text{ compact} \\ K \subset B}} \left\{ \inf_{\substack{G \text{ open} \\ K \subset G}} \nu(G) \right\}. \tag{5.4}$$

Proof. This follows from Corollary 5.5 since $\hat{\nu}$ is a regular Borel measure. □

Corollary 5.8. *(a) For $\omega \in \mathfrak{G}$, either $\hat{\omega}$ is zero or $\hat{\omega}$ is a Dirac measure. (b) Both possibilities in (a) may occur when (X, τ) is not compact. (c) If $\hat{\omega} = \delta_{x_0} \in \mathfrak{D}$, then $\omega \in \mathfrak{G}(x_0)$.*

Proof. (a) If $\omega(K) = 0$ for all compact K, the first formula of (5.4) implies that $\hat{\omega} = 0$. If $\omega(K) = 1$ for some compact K, by Lemma 4.1 there is a unique $x_0 \in X$ for which $\omega(G) = 1$ if $x_0 \in G$ and G is open. From the second part of (5.4) it is immediate that $\hat{\omega}(B) = 1$ if and only if $x_0 \in B$. Hence $\hat{\omega} \in \mathfrak{D}$.

(b) For an example of both possibilities let $X = (0, 1)$ with the standard locally compact topology and Lebesgue measure. Let $\omega \in \mathfrak{G}$ be defined by Theorem 2.9 with $E_\ell = (0, 1/\ell)$, $\ell \in \mathbb{N}$. Then $\omega(K) = 0$ for all compact $K \subset (0, 1)$ and hence $\hat{\omega} = 0$. On the other hand if $E_\ell = (1/2 + 1/\ell, 1/2)$ in Theorem 2.9, $\omega \in \mathfrak{G}$ with $\omega([1/2 + 1/\ell, 1/2]) = 1$ for all ℓ and hence $\hat{\omega} = \delta_{1/2} \in \mathfrak{D}$.

(c) When $\hat{\omega} = \delta_{x_0}$ let $x_0 \in G$, open. Since $\{x_0\}$ is compact by Lemma 5.1 there exists $v \in C_0(X, \tau)$ with $v(X) \subset [0, 1]$, $v(x_0) = 1$, $v(X \setminus G) = 0$. Now $\omega(G) = 1$ for every open set with $x_0 \in G$ since

$$1 \geqslant \omega(G) \geqslant \int_G v \, d\omega = \int_X v \, d\omega = \int_X v \, d\hat{\omega} = v(x_0) = 1.$$

□

A Appendix: $\mathfrak{G}$ and Extreme Points of the Unit Ball in $L_\infty^*(X, \mathcal{L}, \lambda)$

Theorem A.1 (Rainwater [11]). *In a Banach space B, $x_k \rightharpoonup x$ as $k \to \infty$ if and only if $f(x_k) \to f(x)$ in $\mathbb{R}$ for all extreme points f of the closed unit ball in B^*.* $\square$

Recall that $L_\infty^*(X, \mathcal{L}, \lambda)$ is the set of finitely additive measures on $\mathcal{L}$ that are zero on $\mathcal{N}$. Let $U_\infty^* = \{\nu \in L_\infty^*(X, \mathcal{L}, \lambda) : |\nu|(X) \leqslant 1\}$, the closed unit ball in $L_\infty^*(X, \mathcal{L}, \lambda)$. Then $\nu \in U_\infty^*$ is an extreme point of U_∞^* if for $\nu_1, \nu_2 \in U_\infty^*$ and $\alpha \in (0,1)$

$$\nu(E) = \alpha\nu_1(E) + (1-\alpha)\nu_2(E) \text{ for all } E \in \mathcal{L} \text{ implies that } \nu = \nu_1 = \nu_2.$$

Clearly extreme points have $|\nu|(X) = 1$ and, by Theorem A.1, $u_k \rightharpoonup u_0$ in $L_\infty(X, \mathcal{L}, \lambda)$ if and only if for some M

$$\|u_k\|_\infty \leqslant M \text{ and } \int_X u_k \, d\nu \to \int_X u_0 \, d\nu \text{ as } k \to \infty \text{ for all extreme points } \nu \text{ of } U_\infty^*.$$

Thus (2.12) is a consequence of the following result.

Lemma A.2. *ν is an extreme point of U_∞^* if and only if either ν or $-\nu \in \mathfrak{G}$, see* (2.6).

Proof. If $|\nu|(X) = 1$ but ν is not one signed, then $|\nu| = \nu^+ + \nu^-$ where $\nu^+ \wedge \nu^- = 0$ and $\nu^+(X) \in (0,1)$. Let $0 < \epsilon_0 = \frac{1}{2}\min\{\nu^+(X), 1-\nu^+(X)\}$ and, by (2.2b), choose $A \in \mathcal{L}$ such that $\nu^+(X \setminus A) + \nu^-(A) = \epsilon < \epsilon_0$. If $\nu(A) = 0$ then $\nu^+(X) = \nu^+(X \setminus A) + \nu^+(A) = \nu^+(X \setminus A) + \nu^-(A) = \epsilon < \epsilon_0$, which is false. So $\nu(A) \neq 0$ and hence $|\nu|(A) > 0$. If $|\nu|(A) = 1$ then $\nu^+(X) = 1 + \epsilon - 2\nu^-(A) \geqslant 1 - \epsilon$, and hence $1 - \nu^+(X) < \epsilon < \epsilon_0$, which is false. So $|\nu|(A) \in (0,1)$. Let

$$\nu_1(E) = \frac{\nu(A \cap E)}{|\nu|(A)}, \quad \nu_2(E) = \frac{\nu((X \setminus A) \cap E)}{|\nu|(X \setminus A)} \text{ for all } E \in \mathcal{L}.$$

Then ν_1, $\nu_2 \in U_\infty^*$ and, for all $E \in \mathcal{L}$,

$$\nu(E) = \alpha\nu_1(E) + (1-\alpha)\nu_2(E), \text{ where } \alpha = |\nu|(A),\ (1-\alpha) = |\nu|(X \setminus A).$$

Since $\alpha \in (0,1)$, $\nu_1(A) = \nu(A)/|\nu|(A) \neq 0$ and $\nu_2(A) = 0$, this shows that ν is not an extreme element of U_∞^* if ν is not one-signed.

Now suppose $0 \leqslant \nu \in U_\infty^*$ (for $\nu \leqslant 0$ replace ν with $-\nu$). If $\nu \notin \mathfrak{G}$ there exists $A \in \mathcal{L}$ with $\nu(A) \in (0,1)$. Let

$$\nu_1(E) = \frac{\nu(A \cap E)}{\nu(A)}, \quad \nu_2(E) = \frac{\nu((X \setminus A) \cap E)}{\nu(X \setminus A)} \text{ for all } E \in \mathcal{L}.$$

Then ν_1, $\nu_2 \in U_\infty^*$,

$$\nu(E) = \alpha\nu_1(E) + (1-\alpha)\nu_2(E) \text{ for all } E \in \mathcal{L}, \text{ where } \alpha = \nu(A),\ (1-\alpha) = \nu(X \setminus A).$$

Since $\nu_1(A) = 1 \neq 0 = \nu_2(A)$, ν is not extreme. Hence ν extreme implies that $\pm\nu \in \mathfrak{G}$.

Now suppose that $\nu \in \mathfrak{G}$ and for all $E \in \mathcal{L}$,

$$\nu(E) = \alpha\nu_1(E) + (1-\alpha)\nu_2(E), \quad \alpha \in (0,1), \quad \nu_1, \nu_2 \in U_\infty^*.$$

Then $\nu \geqslant 0$ and if $\nu(E) = 1$,

$$1 = \nu(E) = \alpha\nu_1(E) + (1-\alpha)\nu_2(E) \leqslant \alpha|\nu_1|(X) + (1-\alpha)|\nu_2|(X) \leqslant 1$$

which implies that $\nu_1(E) = \nu_2(E) = \nu(E) = 1$. In particular $\nu_1(X) = \nu_2(X) = 1$. If $\nu(E) = 0$ then $\nu(X \setminus E) = 1$ and so $\nu_1(X \setminus E) = \nu_2(X \setminus E) = 1$, whence $\nu_1(E) = \nu_2(E) = \nu(E) = 0$. Thus $\nu = \nu_1 = \nu_2$ and ν is extreme if $\nu \in \mathfrak{G}$. $\square$

Closing Remark. Although the main result, Theorem 3.6, is derived above from Yosida-Hewitt theory [15] without reference to other sources, Theorem A.1 and Lemma A.2 lead to (2.12), and Lemma 2.8 yields Corollary 3.1, thus dispensing with any need for Theorems 2.12 and 2.13. □

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References

[1] Y. A. Abramovich and A. W. Wickstead. Singular extensions and restrictions of order continuous functionals. *Hokkaido Math. J.* **21** (1992), 475–482.

[2] J. M. Ball. Weak continuity properties of mappings and semigroups. *Proc. Roy. Soc. Edin. Sect.* A **72** (4) (1975), 275–280.

[3] S. Banach. *Théorie des Opérations Linéaires.* Monografje Matematyczne, Tom I, Warszawa 1932; Chelsea Publishing Co. New York 1955; English translation by F. Jellett, Dover Books, 1987.

[4] K. P. S. Bhaskara Rao and M. Bhaskara Rao. *Theory of Charges*. Academic Press, London 1983.

[5] J. B. Conway. *A Course in Functional Analysis.* Second Edition, Springer, New York 2007,

[6] N. Dunford and J. T. Schwartz. *Linear Operators.* Volume I, Wiley Interscience, New York 1988.

[7] G. B. Folland. *Real Analysis.* Second Edition, John Wiley, New York, 1999.

[8] W. Hensgen. An example concerning the Yosida-Hewitt decomposition of finitely additive measures. *Proc. Amer. Math. Soc.* **121** (1994), 641–642.

[9] J. Heinonen. *Lectures on Analysis on Metric Spaces.* Springer, New York, 2001.

[10] J. L. Kelley. *General Topology.* Van Nostrand Reinhold, Toronto, 1955.

[11] J. Rainwater. Weak convergence of bounded sequences. *Proc. Amer. Math. Soc.* **14** (6), (1963), 999.

[12] W. Rudin. *Real and Complex Analysis*. Third Edition, McGraw-Hill, New York, 1986.

[13] M. Valadier. Une singulière forme linéaire sur L^∞. *Sém. d'Analyse Convexe Montpelier*, **4**, (1987).

[14] A. J. Wrobel. A sufficient condition for a singular functional on $L^\infty[0,1]$ to be represented on $C[0,1]$ by a singular measure. *Indag. Math.* **29** (2), (2018), 746 – 751.

[15] K. Yosida and E. Hewitt. Finitely additive measures. *Trans. Amer. Math. Soc.* **72**, (1952), 46-66.

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