

SOME INEQUALITIES RELATED TO DIFFERENTIAL MONOMIALS

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ABSTRACT. The aim of this paper is to consider the value distribution of a differential monomial generated by a transcendental meromorphic function.

1. INTRODUCTION

In this article, we use the standard notations of value distribution theory (see, Hayman's Monograph ([1])). It will be convenient to let E denote any set of positive real numbers of finite linear (Lebesgue) measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f , we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \text{ as } r \rightarrow \infty, r \notin E.$$

In addition, in this paper, we also use another type of notation $S^*(r, f)$ which is defined as

$$S^*(r, f) = o(T(r, f)) \text{ as } r \rightarrow \infty, r \notin E^*,$$

where E^* is a set of logarithmic density 0.

By small function with respect to a non-constant meromorphic function f , we mean a meromorphic function $b = b(z) (\neq 0, \infty)$ which satisfies that $T(r, b) = S(r, f)$ as $r \rightarrow \infty, r \notin E$.

Throughout this paper, we always assume that f is a transcendental meromorphic function in the complex plane \mathbb{C} .

In 1979, Mues ([6]) proved that for a transcendental meromorphic function $f(z)$ in \mathbb{C} , $f^2 f' - 1$ has infinitely many zeros. In 1992, Q. Zhang ([10]) proved the quantitative version of Mues's Result as follows:

Theorem A. For a transcendental meromorphic function f , the following inequality holds :

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2 f' - 1}\right) + S(r, f).$$

In this direction Huang and Gu ([2]) obtained the following result:

Theorem B. Let f be a transcendental meromorphic function and k be a positive integer. Then

$$T(r, f) \leq 6N\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r, f).$$

In this connection, one can easily see that the following result is an immediate corollary of *Theorem 3.2* of Lahiri and Dewan ([4]).

Theorem C. Let f be a transcendental meromorphic function and a be a non zero complex constant. Let $l \geq 3, n \geq 1, k \geq 1$ be positive integers. Then

$$T(r, f) \leq \frac{1}{l-2} \bar{N}\left(r, \frac{1}{f^l (f^{(k)})^n - a}\right) + S(r, f).$$

Next we introduce the following definition:

2010 Mathematics Subject Classification: 30D30, 30D20, 30D35.

Key words and phrases : Transcendental Meromorphic function, Differential Monomials.

Definition 1.1. Let q_1, q_2, \dots, q_k be $k (\geq 1)$ non-negative integers and a be a non zero complex constant. Then the expression defined by

$$M[f] = a(f)^{q_0}(f')^{q_1} \dots (f^{(k)})^{q_k}$$

is known as differential monomial generated by f . Next we define $\mu = q_0 + q_1 + \dots + q_k$ and $\mu_* = q_1 + 2q_2 + \dots + kq_k$. In literature, the terms μ and $\mu + \mu_*$ are known as the degree and weight of the differential monomial respectively.

Here, in our paper, we always take $q_0 \geq 1$, $q_k \geq 1$.

Since differential monomial $M[f]$ is the general form of $(f)^{q_0}(f^{(k)})^{q_k}$, so from the above discussion it is natural to ask the following questions:

Question 1.1. Are there any positive constants $B_1, B_2 > 0$ such that following hold?

- i) $T(r, f) \leq B_1 N\left(r, \frac{1}{M[f]-c}\right) + S(r, f)$,
- ii) $T(r, f) \leq B_2 \bar{N}\left(r, \frac{1}{M[f]-c}\right) + S(r, f)$,

where $M[f]$ is a differential monomial generated by a non constant transcendental meromorphic function f and c is any non zero constant.

To answer the above questions are the motivations of this paper. Before going to our main results we first explain some notations and definitions:

Definition 1.2. Let k be a positive integer, for any constant a in the complex plane. We denote

- i) by $N_{(k)}(r, \frac{1}{(f-a)})$ the counting function of a -points of f with multiplicity $\leq k$,
- ii) by $N_{(k)}(r, \frac{1}{(f-a)})$ the counting function of a -points of f with multiplicity $\geq k$.

Similarly, the reduced counting functions $\bar{N}_{(k)}(r, \frac{1}{(f-a)})$ and $\bar{N}_{(k)}(r, \frac{1}{(f-a)})$ are defined.

2. MAIN RESULTS

Theorem 2.1. Let f be a transcendental meromorphic function and $k \geq 2$, $q_0 \geq 2$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 2$ be integers. Then

$$(2.1) \quad T(r, f) \leq \frac{1}{q_0 - 1} N\left(r, \frac{1}{M[f] - 1}\right) + S^*(r, f),$$

where $S^*(r, f) = o(T(r, f))$ as $r \rightarrow \infty$, $r \notin E$, E is a set of logarithmic density 0.

Corollary 2.1. Let f be a transcendental meromorphic function and $k \geq 2$, $q_0 \geq 2$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 2$ be integers. For a no zero complex constant α , we have

$$(2.2) \quad T(r, f) \leq \frac{1}{q_0 - 1} N\left(r, \frac{1}{(f)^{q_0}(f')^{q_1} \dots (f^{(k)})^{q_k} - \alpha}\right) + S^*(r, f).$$

Corollary 2.2. Let f be a transcendental meromorphic function and $k \geq 2$, $q_0 \geq 2$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 2$ be integers. Then $(f)^{q_0}(f')^{q_1} \dots (f^{(k)})^{q_k}$ assumes every non-zero finite value infinitely often.

Theorem 2.2. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 3$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. Then

$$(2.3) \quad T(r, f) \leq \frac{1}{\mu - \mu_* - 2} \bar{N}\left(r, \frac{1}{M[f] - 1}\right) + S(r, f),$$

where $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$, $r \notin E$, E is a set of finite linear measure.

Corollary 2.3. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 3$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. For a no zero complex constant α , we have

$$(2.4) \quad T(r, f) \leq \frac{1}{\mu - \mu_* - 2} \overline{N} \left(r, \frac{1}{(f)^{q_0} (f')^{q_1} \dots (f^{(k)})^{q_k} - \alpha} \right) + S(r, f).$$

Corollary 2.4. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 3$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. Then $(f)^{q_0} (f')^{q_1} \dots (f^{(k)})^{q_k}$ assumes every non-zero finite value infinitely often.

Theorem 2.3. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 5 - q_0$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. Then

$$(2.5) \quad T(r, f) \leq \frac{1}{\mu - \mu_* - 4 + q_0} \overline{N} \left(r, \frac{1}{M[f] - 1} \right) + S(r, f).$$

Corollary 2.5. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 5 - q_0$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. For a no zero complex constant α , we have

$$(2.6) \quad T(r, f) \leq \frac{1}{\mu - \mu_* - 4 + q_0} \overline{N} \left(r, \frac{1}{(f)^{q_0} (f')^{q_1} \dots (f^{(k)})^{q_k} - \alpha} \right) + S(r, f).$$

Corollary 2.6. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 5 - q_0$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. Then $(f)^{q_0} (f')^{q_1} \dots (f^{(k)})^{q_k}$ assumes every non-zero finite value infinitely often.

3. LEMMAS

Let a be a non zero complex constant and q_1, q_2, \dots, q_k be $k(\geq 1)$ non-negative integers. Define $\mu = q_0 + q_1 + \dots + q_k$ and $\mu_* = q_1 + 2q_2 + \dots + kq_k$.

Let $M[f] = a(f)^{q_0} (f')^{q_1} \dots (f^{(k)})^{q_k}$ be a differential monomial generated by a non constant transcendental meromorphic function f where we take $q_0 \geq 1$, $q_k \geq 1$.

Lemma 3.1. For a non constant meromorphic function g ,

$$N(r, \frac{g'}{g}) - N(r, \frac{g}{g'}) = \overline{N}(r, g) + N(r, \frac{1}{g}) - N(r, \frac{1}{g'}).$$

Proof. For the proof, one go through the technique of formula (12) of ([3]). \square

Now the following Lemma which plays the major role to prove Theorem2.1 is a immediate corollary of Yamanoi's Celebrated Theorem([8]). Yamanoi's Theorem is a correspondent result to the famous Gol'dberg Conjecture.

Lemma 3.2. ([8]) Let f be a transcendental meromorphic function in \mathbb{C} and let $k \geq 2$ be an integer. Then

$$(k-1) \overline{N}(r, f) \leq N(r, \frac{1}{f^{(k)}}) + S^*(r, f),$$

where $S^*(r, f) = o(T(r, f))$ as $r \rightarrow \infty$, $r \notin E$, E is a set of logarithmic density 0.

Lemma 3.3. For any small function $b = b(z) (\neq 0, \infty)$ of f , $b(z)M[f]$ can not be a constant.

Proof. On contrary, let us assume

$$(3.1) \quad b(z)M[f] \equiv C,$$

for some constant C .

As f is non constant transcendental meromorphic function, so $C \neq 0$.

Thus from (3.1) and Lemma of logarithmic derivative, it is clear that

$$(3.2) \quad m(r, \frac{1}{f}) = S(r, f).$$

Also it is clear from that (3.1) that

$$(3.3) \quad N(r, 0; f) \leq N(r, 0; M[f]) = S(r, f).$$

Thus $T(r, f) = S(r, f)$, which is absurd as f is transcendental. \square

Lemma 3.4. Let f be a transcendental meromorphic function, then $T\left(r, b(z)M[f]\right) = O(T(r, f))$ and $S\left(r, b(z)M[f]\right) = S(r, f)$.

Proof. The proof is similar to the proof of the Lemma 2.4 of ([5]). \square

Lemma 3.5. Let f be a transcendental meromorphic function. Then

$$(3.4) \quad \mu T(r, f) \leq \mu N(r, \frac{1}{f}) + \overline{N}(r, \infty; f) + N(r, \frac{1}{bM[f] - 1}) - N(r, \frac{1}{(bM[f])'}) + S(r, f).$$

Proof. As Lemma 3.3 yields that $b(z)M[f] \not\equiv \text{constant}$, so we can write

$$\frac{1}{f^\mu} = \frac{bM[f]}{f^\mu} - \frac{(bM[f])'(bM[f] - 1)}{f^\mu (bM[f])'}.$$

Thus in view of Lemma 3.4, First Fundamental Theorem and Lemma 3.1 we have

$$(3.5) \quad \begin{aligned} \mu m(r, \frac{1}{f}) &\leq m(r, \frac{bM[f]}{f^\mu}) + m(r, \frac{(bM[f])'}{f^\mu}) + m(r, \frac{bM[f] - 1}{(bM[f])'}) + O(1) \\ &\leq 2m(r, \frac{bM[f]}{f^\mu}) + m(r, \frac{(bM[f])'}{bM[f]}) + m(r, \frac{bM[f] - 1}{(bM[f])'}) + O(1) \\ &\leq T(r, \frac{bM[f] - 1}{(bM[f])'}) - N(r, \frac{bM[f] - 1}{(bM[f])'}) + S(r, f) \\ &\leq \overline{N}(r, \infty; f) + N(r, \frac{1}{bM[f] - 1}) - N(r, \frac{1}{(bM[f])'}) + S(r, f) \end{aligned}$$

Thus

$$\mu T(r, f) \leq \mu N(r, \frac{1}{f}) + \overline{N}(r, \infty; f) + N(r, \frac{1}{bM[f] - 1}) - N(r, \frac{1}{(bM[f])'}) + S(r, f).$$

\square

Lemma 3.6. Let f be a transcendental meromorphic function. If $q_0 \geq 1$, $q_k \geq 1$, then

$$(3.6) \quad \begin{aligned} \mu T(r, f) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \mu_* \overline{N}_{(k+1)}(r, 0; f) + (\mu - q_0) N_k(r, 0; f) \\ &\quad + \overline{N}(r, \frac{1}{M[f] - 1}) - N_0(r, \frac{1}{M[f]'}) + S(r, f), \end{aligned}$$

where $N_0(r, \frac{1}{(M[f])'})$ is the counting function of the zeros of $(M[f])'$ but not the zeros of $f(M[f] - 1)$.

Proof. Clearly

$$(3.7) \quad \begin{aligned} &\mu N(r, \frac{1}{f}) + N(r, \frac{1}{M[f] - 1}) - N(r, \frac{1}{(M[f])'}) \\ &\leq \mu N(r, \frac{1}{f}) - N_*(r, \frac{1}{(M[f])'}) + \overline{N}(r, \frac{1}{M[f] - 1}) - N_0(r, \frac{1}{(M[f])'}), \end{aligned}$$

where $N_*(r, \frac{1}{(M[f])'})$ is the counting function of the zeros of $(M[f])'$ which comes from the zeros of f .

Let z_0 be a zero of f with multiplicity q .

Case-1 $q \geq k + 1$.

It is easy to observe that

$$\begin{aligned} q\mu - \mu_* &\geq k\mu - \mu_* + \mu \\ &\geq \sum_{i=0}^{k-1} (k-i)q_i + \mu \\ &\geq (k+1)q_0 + q_k \\ &\geq 3. \end{aligned}$$

Then z_0 is the zero of $(M[f])'$ of order atleast $q\mu - \mu_* - 1$.

Case-2 $q \leq k$.

Then z_0 is the zero of $(M[f])'$ of order atleast $qq_0 - 1$. Thus

$$(3.8) \quad \begin{aligned} &\mu N(r, \frac{1}{f}) - N_*(r, \frac{1}{(M[f])'}) \\ &\leq (\mu - q_0)N_k(r, \frac{1}{f}) + \overline{N}_k(r, \frac{1}{f}) + (\mu_* + 1)\overline{N}_{(k+1)}(r, \frac{1}{f}) \end{aligned}$$

Now the proof follows from the Lemma 3.5 and the inequalities (3.7),(3.8). \square

4. PROOF OF THE THEOREMS

Proof of Theorem 2.1 . It is given that, f is a transcendental meromorphic function and $k \geq 2$, $q_0 \geq 2$, $q_k \geq 2$. It is clear that

$$(q_0 - 1)N(r, 0; f) + (q_k - 1)N(r, 0; f^{(k)}) \leq N(r, 0; (M[f])').$$

Now in view of Lemma 3.2 and Lemma 3.5, we have

$$\begin{aligned} &\mu T(r, f) \\ &\leq (\mu - q_0 + 1)N(r, \frac{1}{f}) + (1 - (k-1)(q_k - 1))\overline{N}(r, \infty; f) + N(r, \frac{1}{M[f] - 1}) \\ &\quad + S(r, f) + S^*(r, f) \\ &\leq (\mu - q_0 + 1)N(r, \frac{1}{f}) + N(r, \frac{1}{M[f] - 1}) + S^*(r, f). \end{aligned}$$

Thus

$$(q_0 - 1)T(r, f) \leq N(r, \frac{1}{M[f] - 1}) + S^*(r, f),$$

This completes the proof. \square

Proof of Theorem 2.2 . In view of Lemma 3.6, we can write

$$(4.1) \quad \begin{aligned} &\mu T(r, f) \\ &\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + (\mu - q_0)\{N_k(r, 0; f) + \overline{N}_{(k+1)}(r, 0; f)\} \\ &\quad + (\mu_* - \mu + q_0)\overline{N}_{(k+1)}(r, 0; f) + \overline{N}(r, \frac{1}{M[f] - 1}) - N_0(r, \frac{1}{(M[f])'}) + S(r, f). \end{aligned}$$

Thus

$$(\mu - \mu_* - 2)T(r, f) \leq \overline{N}(r, \frac{1}{M[f] - 1}) + S(r, f),$$

This completes the proof. \square

Proof of Theorem 2.3 . Since $k \geq 1$ and $2\overline{N}_{(k+1)}(r, 0; f) \leq N(r, 0; f)$, so from inequality (4.1), we can write

$$(q_0 - 2)T(r, f) \leq \frac{(\mu_* - \mu + q_0)}{2}N(r, 0; f) + \overline{N}(r, \frac{1}{M[f] - 1}) - N_0(r, \frac{1}{(M[f])'}) + S(r, f).$$

Thus

$$(\mu - \mu_* - 4 + q_0)T(r, f) \leq \overline{N}\left(r, \frac{1}{M[f]-1}\right) + S(r, f)$$

This completes the proof. \square

5. APPLICATIONS

If there exists positive constants $B_1, B_2 > 0$ such that

$$(1) \quad T(r, f) \leq B_1 N\left(r, \frac{1}{M[f]-c}\right) + S(r, f),$$

$$(2) \quad T(r, f) \leq B_2 \overline{N}\left(r, \frac{1}{M[f]-c}\right) + S(r, f),$$

holds, then we can write

$$(1) \quad T(r, M[f]) \leq (\mu + \mu_*)T(r, f) + S(r, f) \leq B_1(\mu + \mu_*) N\left(r, \frac{1}{M[f]-c}\right) + S(r, f),$$

$$(2) \quad T(r, M[f]) \leq (\mu + \mu_*)T(r, f) + S(r, f) \leq B_2(\mu + \mu_*) \overline{N}\left(r, \frac{1}{M[f]-c}\right) + S(r, f),$$

where $M[f]$ is a differential monomial generated by a non constant transcendental meromorphic function f and c is any non zero constant.

Let $\psi = (f)^{q_0}(f')^{q_1}\dots(f^{(k)})^{q_k}$ and a be a non zero finite value. Then

$$(5.1) \quad \begin{aligned} \delta(a; \psi) &= 1 - \limsup_{r \rightarrow \infty} \frac{N(r, a; \psi)}{T(r, \psi)} \\ &\leq 1 - \frac{1}{B_1(\mu + \mu_*)}. \end{aligned}$$

and

$$(5.2) \quad \begin{aligned} \Theta(a; \psi) &= 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, a; \psi)}{T(r, \psi)} \\ &\leq 1 - \frac{1}{B_2(\mu + \mu_*)}. \end{aligned}$$

Thus the following theorems are immediate in view of Theorems 2.1, 2.2, 2.3.

Theorem 5.1. Let f be a transcendental meromorphic function and $k \geq 2$, $q_0 \geq 2$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 2$ be integers. Then

$$(5.3) \quad \delta(a; \psi) \leq 1 - \frac{q_0 - 1}{(\mu + \mu_*)}.$$

Remark 5.1. Thus the Theorem 5.1 improves, extends and generalizes the result of Lahiri and Dewan ([4]).

Theorem 5.2. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 3$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. Then

$$(5.4) \quad \Theta(a; \psi) \leq 1 - \frac{\mu - \mu_* - 2}{(\mu + \mu_*)}.$$

Theorem 5.3. Let f be a transcendental meromorphic function and $k \geq 1$, $\mu - \mu_* \geq 5 - q_0$, $q_0 \geq 1$, $q_i \geq 0$ ($i = 1, 2, \dots, k-1$), $q_k \geq 1$ be integers. Then

$$(5.5) \quad \Theta(a; \psi) \leq 1 - \frac{\mu - \mu_* - 4 + q_0}{(\mu + \mu_*)}.$$

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