EQUIDIMENSIONALITY AND REGULARITY

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Theorem. — *Let S be a locally noetherian normal algebraic space of residue characteristics zero with regular formal fibers. Assume that there is a morphism $f : X \to S$, which is locally of finite type, equidimensional and étale locally on S admits sections, from a regular algebraic space X. Then S is regular and f is flat.*

Proof. The hypothesis on S says that every affine scheme étale over S is a noetherian $\mathbf{Q}$-scheme with normal excellent local rings.

The hypothesis on f implies that it is surjective, for étale locally on S it admits sections, and that it is universally open (EGA IV 14.4.1), for S is normal. Hence, by EGA O 17.3.3+IV 6.1.5, the statement "S is regular" is equivalent to the statement "f is flat".

Suppose given S' a locally noetherian algebraic space with regular formal fibers and given a regular surjective morphism $S' \to S$. Then S' is normal and $X \times_S S'$ is regular, for $S' \to S$ and $X \times_S S' \to X$ are regular morphisms. And $f \times_S S'$ is equidimensional with sections étale locally on S'. It amounts evidently the same to verifying "S' is regular and $f \times_S S'$ is flat".

In particular, the question is local on S for the étale topology. Thus, assume S affine and then local and f given a section. Now S being excellent, its subset R consisting of all those points at which S is regular is open. Localizing at a maximal point of $S - R$ if this set is not empty, one can suppose S of dimension ≥ 2 and regular outside its closed point s. One can suppose furthermore S complete. For, the completion S' of S along s is excellent and the projection $S' \to S$ regular surjective and one replaces (S, X, f) by $(S', X \times_S S', f \times_S S')$.

Observe that S is pure along s. Indeed, let E be a locally constant constructible sheaf of sets on $S - \{s\}$ for the étale topology. If j denotes the open immersion of $X - f^{-1}(s)$ in X, $j_* f^* E$ is locally constant constructible on X (SGA 2 X 3.4), for X is regular and for as f is equidimensional $f^{-1}(s)$ is of codimension ≥ 2 in X. Take $g : S \to X$ a section of f which one assumes is given. Then $g^* j_* f^* E$ extends E to S. A similar argument shows that S is also parafactorial along s, namely (EGA IV 21.13.11), factorial. (If S is of dimension 2, purity suffices to imply regularity by Mumford.)

Let the given section of f be $g : S \to X$. Replacing X by an étale neighborhood of $g(s)$, one can suppose X affine and that every irreducible component of $f^{-1}(s)$ contains $g(s)$ and, taking a further finite étale base change $S' \to S$ if necessary, that every irreducible component of $f^{-1}(s)$ is geometrically irreducible.

Choose a maximal point a of $f^{-1}(s)$. Let $\widetilde{X} = \mathrm{Spec}(\mathcal{O}_{X,a})$ and $\overline{X}$ its completion along a; the projection $\overline{X} \to \widetilde{X}$ is regular, for $\widetilde{X}$ is excellent. Notice that

— *The morphism* $f : \widetilde{X} - \{a\} \to S - \{s\}$ *is regular and bijective* :

As S is regular outside s and as X is regular, f is smooth along $g(S - \{s\})$. Consider an arbitrary point v of $S - \{s\}$, choose a spectra V of discrete valuation ring and a morphism $V \to S$ such that the generic point v' (resp. closed point s') of V is mapped to v (resp. s), and let $X_V = X \times_S V$, $f_V = f \times_S V$, $g_V = g \times_S V$. Since f_V is universally open and is smooth at $g_V(v')$, every sufficiently small open neighborhood U of $g_V(s')$ in X_V has smooth geometrically connected fiber over v'; hence, the localization of U at each maximal point of the closed fiber $U_{s'}$ is irreducible of dimension 1. As f is flat outside $f^{-1}(s)$ and as every irreducible component of $f^{-1}(s)$ is geometrically irreducible, the assertion follows.

Let $k = k(s)$. By Artin [1] 4.7 one finds a connected affine k-scheme of finite type T and a point $t \in T(k)$ such that T is k-smooth outside t and such that S is isomorphic to the completion of T along t; by means of this isomorphism k is a chosen coefficient field of $\Gamma(S, \mathcal{O})$.

As S is normal, T is normal at t and thus normal. As moreover T has a k-rational point, it is geometrically connected over k. Let k' be a coefficient field of the formally smooth k-algebra $\Gamma(\overline{X}, \mathcal{O})$. Consider the induced k'-algebra homomorphism

$$\Gamma(T, \mathcal{O}) \otimes_k k' \to \Gamma(\overline{X}, \mathcal{O})$$

and the k'-morphism

$$p : \overline{X} \to \overline{T'}$$

where $\overline{T'}$ is the completion of $T' := T_{k'}$ along t', the unique k'-point of T' above t; the morphism $\overline{T'} \to S$ induced from $T' \to T$ is regular, for it is formally smooth and S excellent.

This morphism p is finite by the choice that a is a maximal point of $f^{-1}(s)$, and p is surjective, for $\overline{T'}$ is normal of the same dimension as that of $\overline{X}$.

If $q : \overline{X} \to \widetilde{X}$ denotes the projection, note that

$$i := (q, p) : \overline{X} \to \widetilde{X} \times_S \overline{T'}$$

is a closed immersion; it is a regular closed immersion since $\overline{X}$ and $\widetilde{X} \times_S \overline{T'}$ are regular. In particular, the conormal module of i, N is free over $\mathcal{O}_{\overline{X}}$. On writing p as the composition

$$\overline{X} \xrightarrow{i} \widetilde{X} \times_S \overline{T'} \xrightarrow{f \times_S \overline{T'}} \overline{T'}$$

one deduces the canonical exact sequence

$$H^{-1}(L_p) \to N \to q^*\Omega^1_{\widetilde{X}/S} \to \Omega^1_p \to 0$$

where L_p is the cotangent complex of p and where $H^{-1}(L_p) \to N$ is 0, the $\mathcal{O}_{\overline{X}}$-module $H^{-1}(L_p)$ (resp. N) being torsion (resp. free). By approximation, the injection $N \to q^*\Omega^1_{\widetilde{X}/S}$ is isomorphic to an injection of the form

$$q^*(\alpha) : q^*\widetilde{N} \to q^*\Omega^1_{\widetilde{X}/S}$$

where $\widetilde{N}$ is a free $\mathcal{O}_{\widetilde{X}}$-module and $\alpha : \widetilde{N} \to \Omega^1_{\widetilde{X}/S}$ an injective $\mathcal{O}_{\widetilde{X}}$-homomorphism. So $\Omega^1_p = q^*\omega$ with $\omega = \mathrm{Coker}(\alpha)$.

To finish, it is sufficient to show that

— *The morphism $p : \overline{X} \to \overline{T'}$ is étale outside the closed point of $\overline{X}$.*

For then p is étale, as the argument proving the purity of S along s applies to $(\overline{T'}, X \times_S \overline{T'}, f \times_S \overline{T'})$, and then, as $\overline{X}$ is regular, $\overline{T'}$ and S are regular.

Write again a (resp. t') for the closed point of $\overline{X}$ (resp. $\overline{T'}$). The finite surjective morphism

$$p : \overline{X} - \{a\} \to \overline{T'} - \{t'\}$$

is flat, since $\overline{T'} - \{t'\}$ and $\overline{X} - \{a\}$ are regular. By SGA 2 X 3.4 it remains to show that p is unramified, that is, $\Omega^1_p = q^*\omega$ vanishes, at every codimension 1 point of $\overline{X}$. In view that

$$f : \widetilde{X} - \{a\} \to S - \{s\}$$

is regular and bijective, it amounts to showing that

$$V(h\mathcal{O}_{\overline{X}}) \cap (\overline{X} - \{a\})$$

is reduced for each prime element h of the factorial ring $\Gamma(S, \mathcal{O})$. Both $\overline{X} \to \widetilde{X}$ and $\widetilde{X} - \{a\} \to S - \{s\}$ being regular morphisms, this is clear, hence the proof.

□

References

1. M. Artin. *The implicit function theorem in algebraic geometry.* Algebraic Geometry, *Bombay*, 1969.

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