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Equivalence of Coniveaus

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Abstract

On a smooth projective variety over \mathbb{C} , there is the coniveau from the coniveau filtration, which is called geometric coniveau. On the same variety, there is another coniveau from the maximal sub-Hodge structure, which is called Hodge coniveau. In this paper we show they are equivalent.

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1 Introduction

Let X be a smooth projective variety over the complex numbers. There is an associated compact complex manifold denoted by the same letter X. Then such X is equipped with the Euclidean topology, which has the well-known \mathbb{Z} module – cohomology group ^I. The question we are trying to answer: What and how does algebro-geometric structure on X determine the structures of the cohomology group? The structures on the cohomology may be expressed as filtrations of subspaces of the linear space, obtained from the cohomology tensored with \mathbb{Q} . In this paper we study two well-known filtrations

1.1 Result

Let X be a complex projective manifold of dimension n. Let p, k be whole numbers. We'll denote the coniveau filtration of coniveau p and degree 2p+kby

$$N^{p}H^{2p+k}(X) \subset H^{2p+k}(X;\mathbb{Q})$$

$$(1.1)$$

and the linear span of sub-Hodge structures of coniveau p and degree 2p + k by

$$M^{p}H^{2p+k}(X) \subset H^{2p+k}(X;\mathbb{Q}).$$
(1.2)

In this paper we prove that

Theorem 1.1.

$$N^{p}H^{2p+k}(X) = M^{p}H^{2p+k}(X)$$
(1.3)

for all X, p, k.

Remark Geometric coniveau is the algebro-geometric index used to describe certain subgroups of cohomology – coniveau filtration, while Hodge coniveau is the index, depending on non-algebraic structures, and used to describe another subgroups of cohomology –maximal sub-Hodge structures. Theorem 1.1 says these two descriptions give the same subgroups, i.e. two indexes are equivalent.

^IOther topological structures do not concern this paper.

1.2 Idea of the proof

Our proof is based on four proved facts which will be introduced with references in Appendix:

- (1) Intersection of currents exists;
- (2) Theorem 1.1 holds for surfaces and 3-folds;
- (3) Generlized Lefschetz standard conjecture is true or equivalently Lefschetz standard conjecture is true;
- (4) The projection from the Cartesian product is supportive.

With the facts (1), (3) and (4), there is a process of manipulations that reduces the equality (1.3) to the same equality on 3-folds through multiple transformations between

$$X \times E$$
 and X ,

where E is an elliptic curve. This paper is the presentation of this process ^{II}. The argument in this paper is so soft that it sometimes obscures the principle: the roots of structures of the cohomology lie beyond the category of cohomology. The four facts above are closer to this principle. But without the content of this paper they lack transparency in the connection to theorem 1.1. Thus it should be considered as the last step in the proof of theorem 1.1. Coming back to the technical transformations in this paper, our proof is an inductive reasoning on the dimension of the manifold. So starting from the fact (2), we assume theorem 1.1 holds for all X of dim(X) < n. Let's prove it for X of dim(X) = n. First we deal with cohomology classes of degree $\neq n$. Applying the fact (3), we can easily reduce theorem 1.1 to the middle dimension. On the middle dimensional cohomology our strategy is to focus on different cycles of different degrees, different coniveaus and study them in a different space

$$X \times E. \tag{1.4}$$

The following is the sketch of the process. Let $a, a' \in H^1(E; \mathbb{Q})$ be a standard basis such that

$$a \cup a = 0 = a' \cup a', a \cup a' = 1.$$

^{II}There is a different type of interplays between $X \times C$ and X for a curve C observed by Grothendieck ([2]) and carried out by Voision ([3]), and it is limited to sub-Hodge structures of levels ≤ 1 . Our interplay deals with higher levels.

We denote the intersection number between two cycles in the space S by

 $(\cdot, \cdot)_S$.

To have the induction going forward, we must first assume

$$H^1(X; \mathbb{Q}) \neq 0.$$

Let $\alpha \in M^p H^{2p+k}(X)$ be non-zero such that 2p + k = n is the middle dimension. Then it is well-known that it suffices to show that there is a cycle $\beta \in N^p H^{2p+k}(X)$ such that the intersection number

$$(\alpha,\beta)_X \neq 0. \tag{1.5}$$

We call any cycle β satisfying (1.5) a dual of α . (This is because

$$N^p H^{2p+k}(X) \subset M^p H^{2p+k}(X)$$

and the non-zero intersection (1.5) implies

$$\dim(M^p H^{2p+k}(X)) \le \dim(N^p H^{2p+k}(X)).$$

). To find such a β we begin with the different cycle

$$\alpha \otimes a' \in M^p H^{2p+k+1}(X \times E). \tag{1.6}$$

By the Poincaré duality, a generic vector

$$\theta \in M^p H^{2p+k+1}(X \times E)$$

is a dual of $\alpha \otimes a'$, i.e. it satisfies the intersection formula,

$$(\alpha \otimes a', \theta)_{X \times E} \neq 0. \tag{1.7}$$

Next we focus on this generic θ . Using transformations between $X \times E$ and X, supported by the four facts and the assumption

$$H^1(X;\mathbb{Q}) \neq 0,$$

we turn θ form Hodge leveled to geometrically leveled, i.e. we prove that

$$\theta \in N^p H^{2p+k+1}(X \times E).$$

At last, we use the notion of intersection currents, the fact (1) to extract/construct a cycle class

$$\beta \in N^p H^{2p+k}(X), \tag{1.8}$$

from the Künneth's decomposition of θ such that the intersection number

$$(\alpha,\beta)_X \neq 0. \tag{1.9}$$

Once theorem 1.1 holds for X with non-zero $H^1(X; \mathbb{Q})$, it will hold for all X through the projection $X \times E \to X$.

2 Proof

First we use induction on n, the dimension of X satisfying

$$H^1(X;\mathbb{Q}) \neq 0. \tag{2.1}$$

Recall the indices p, q, k satisfying

$$p + q + k = n.$$

The cases for surfaces and threefolds are proved in the Appendix B. So we assume that the theorem 1.1 holds for all X satisfying

$$0 \le \dim(X) \le n - 1$$

where $n \ge 4$. Next we consider the case dim(X) = n. By [1] and [2],

$$N^p H^{2p+k}(X) \subset M^p H^{2p+k}(X)$$

for all p. Applying the fact (3), we obtain

$$N^{p}H^{2p+k}(X) = (N^{q}H^{2q+k}(X))^{\vee}.$$

for all p, q. Then it suffices to prove that the intersection pairing gives the injectivity of

$$M^p H^{2p+k}(X) \rightarrow (N^q H^{2q+k}(X))^{\vee}$$
 (2.2)

In the following subsections we prove it in all cases.

Notation:

(1) In the rest of the paper including appendix, we let $u \in H^2(X; \mathbb{Z})$ be a hyperplane section class represented by a generic hyperplane section V of a polarization of X. Furthermore $V^h, h > 0$ denotes the generic complete intersection in the projective space by the plane sections.

(2) We say classes and representatives of classes are \mathcal{N}_k leveled or have geometric level k if the classes are in $N^p H^{2p+k}(X)$. The index p is the geometric coniveau in the abstract. Similarly they are \mathcal{M}_k leveled or have Hodge level k if the classes are in $M^p H^{2p+k}(X)$ and p is the Hodge coniveau in the abstract.

2.1 Non middle dimension

This section does not use the assumption in the formula (2.1).

Proposition 2.1. The map (2.2) is injective for $p + q = n - k, p \neq q$.

Proof. Suppose q > p. Let $\alpha \in M^p H^{2p+k}(X)$ be a non-zero cycle. Let

$$h = q - p > 0.$$

Then by the hard Lefschetz theorem $\alpha u^h \neq 0$ in $H^{2q+k}(X; \mathbb{Q})$. Let

$$Z = X \cap V^h$$

be a smooth plane section of X and

$$i: Z \hookrightarrow X$$
 (2.3)

be the inclusion map. Note Z is irreducible. Then applying lemma 6.2, [4], we obtain that

$$\alpha u^h = i_! \circ i^*(\alpha).$$

Hence $i^*(\alpha) \neq 0$ in $H^{2p+k}(Z; \mathbb{Q})$. By the proposition 5.2, [4]

 $i^*(\alpha)$

is also \mathcal{M}_k leveled. Since h > 0, we can apply the inductive assumption to the variety Z to obtain an \mathcal{N}_k leveled cycle β such that

$$(i^*(\alpha),\beta)_Z \neq 0. \tag{2.4}$$

Then applying lemma 6.2, [4], we have

$$(\alpha, i_!(\beta))_X = (i^*(\alpha), \beta)_Z \neq 0 \tag{2.5}$$

Notice by the proposition 5.2, [4], $i_!(\beta)$ is \mathcal{N}_k leveled. Thus the proposition in this case is proved.

Next we consider the case q < p. Let h = p - q > 0. We start with

$$\alpha \in M^p H^{2p+k}(X).$$

Using hard Lefschetz theorem there is a $\alpha_h \in H^{2q+k}(X; \mathbb{Q})$ such that

$$\alpha = \alpha_h u^h. \tag{2.6}$$

By the same argument above we obtain a \mathcal{N}_k leveled cycle β in $H^{2p+k}(X;\mathbb{Q})$ such that

$$(\alpha_h, \beta)_X \neq 0. \tag{2.7}$$

Now applying the fact (3), there is an \mathcal{N}_k leveled cycle $\beta_h \in H^{2q+k}(X;\mathbb{Q})$ such that

$$\beta_h u^h = \beta. \tag{2.8}$$

Then the formula (2.7) becomes

$$(\alpha_h, \beta_h u^h)_X = (\alpha_h u^h, \beta_h)_X = (\alpha, \beta_h)_X \neq 0.$$
(2.9)

where β_h is \mathcal{N}_k leveled. Thus we complete the proof for the case $p \neq q$.

2.2 Middle dimension

This section uses assumption in the formula (2.1).

Proposition 2.2. The map (2.2) is injective for p = q.

Proof. It suffices to prove that for an \mathcal{M}_k leveled cycle $\alpha \in M^p H^{2p+k}(X; \mathbb{Q})$, there is an \mathcal{N}_k leveled cycle

$$\beta \in N^p H^{2p+k}(X; \mathbb{Q})$$

such that

$$(\alpha,\beta)_X \neq 0.$$

Since p = 0 is a trivial case, we consider two cases: 1). $p \ge 2$; 2). p = 1. Case 1: $p \ge 2$. Let E be an elliptic curve and

$$Y = X \times E.$$

Also let

 $P:Y\to X$

be the projection.

Let $\alpha \in M^p H^n(X)$. So k = n - 2p. Let $a, a' \in H^1(E; Q)$ be a standard basis, i.e.

$$a \cup a' = 1, a \cup a = a' \cup a' = 0.$$

Let

$$\Lambda \subset H^n(X; \mathbb{Q}) \tag{2.10}$$

be the sub-Hodge structure of X, containing α . Then

$$\Lambda \otimes H^1(E;\mathbb{Q}) \tag{2.11}$$

is the sub-Hodge structure of Y of level k + 1 containing $\alpha \otimes a'$. Thus

$$\alpha \otimes a' \in M^p H^{2p+k+1}(Y). \tag{2.12}$$

By Poincaré duality, there is a $\theta \in M^p H^{2p+k+1}(Y)$ such that

$$(\alpha \otimes a', \theta)_Y \neq 0. \tag{2.13}$$

Let θ be generic in $M^p H^{2p+k+1}(Y)$. Next we prove that θ is \mathcal{N}_{k+1} leveled III

^{III} In general, turning from \mathcal{M} leveled to \mathcal{N} leveled is the theorem 1.1. But now we'll only prove it in a special setting.

Now we consider the Gysin homomorphism

$$P_!: H^{\bullet}(Y; \mathbb{C}) \to H^{\bullet-2}(X; \mathbb{C}).$$
(2.14)

Notice that if n is odd,

$$M^{p-1}H^{2p+k-1}(X)$$

is non-zero because it contains a non-zero cycle $u^{\frac{n-1}{2}}$. If n is even, it contains subspace $H^1(X; \mathbb{Q})u^{\frac{n}{2}-1}$ which is also non-zero by the assumption. Hence the

$$im(P_{!}) = M^{p-1}H^{2p+k-1}(X) \neq 0.$$

Since θ is generic in the linear space $M^p H^{2p+k+1}(Y)$, $P_!(\theta) \neq 0$.

Now we obtained a non-zero cycle

$$P_{!}(\theta) \in M^{p-1} H^{2p+k-1}(X).$$
(2.15)

Notice 2p + k - 1 = n - 1 which is less than middle dimension of X. Applying the hard Lefschetz theorem on X, $P_1(\theta)u$ is non-zero in

$$M^p H^{2p+k+1}(X).$$

Let

$$i: X_{n-1} \hookrightarrow X$$
 (2.16)

be the inclusion map of a smooth hyperplane section $X_{n-1} = V \cap X$. Then by lemma 6.2, [4]

$$i_! \circ i^*(P_!(\theta)) = P_!(\theta)u. \tag{2.17}$$

Because $P_1(\theta)u$ is non-zero, neither is

$$i^*(P_!(\theta)).$$

Notice

$$i^*(P_!(\theta)) \in M^{p-1}H^{2p+k-1}(X_{n-1}).$$
 (2.18)

and

$$\dim(X_{n-1}) = n - 1.$$

By the induction

$$i^*(P_!(\theta)) \in N^{p-1}H^{2p+k-1}(X_{n-1}).$$
 (2.19)

Hence by the formula (2.17)

$$P_{!}(\theta)u \in N^{p}H^{2p+k+1}(X_{n-1}).$$
(2.20)

Applying the fact (3), we obtain that

$$P_{!}(\theta) \in N^{p-1} H^{2p+k-1}(X).$$
(2.21)

Next we show that

Claim 2.3.

$$\theta \in N^p H^{2p+k+1}(Y). \tag{2.22}$$

Proof of claim 2.3: The argument of claim 2.3 below follows the principle: it is in the category beyond the cohomology. We consider cellular cycles. Let T'_{θ} be a cellular cycle on Y representing θ . By the fact (4) (proved in Appendix C), there is another singular cycle T''_{θ} on Y finite to X such that

$$T''_{\theta} = T'_{\theta} + dK \tag{2.23}$$

Then the push-forward $P_{\#}(T''_{\theta})$ is again a cellular cycle of dimension n + 1 in X. Considering the cohomology, by the formula (2.21), we know that the cohomology class of $P_{\#}(T''_{\theta})$ is

$$P_{!}(\theta)) \in N^{p-1} H^{2p+k-1}(X).$$
(2.24)

Hence we have formula

$$P_{\#}(T_{\theta}'') = T_a + dL, \qquad (2.25)$$

where T_a is a non-zero cellular cycle supported on an algebraic set Z' of dimension at most p + k + 1, and L is a singular chain (This shows that the non vanishing $P_1(\theta)$ leads to the existence of Z'). Now we let

$$T_{\theta} = T_{\theta}'' - dL \times \{e\} \tag{2.26}$$

where $e \in E$ is a point. Because $P: T_{\theta} \to X$ is again 1-to-1 on each Euclidean open set, the singular cycle T_{θ} must lie in the algebraic set $Z' \times E = Z$ of

codimension p-1. The following graph summarizes what we obtained in the category of singular cycles.

Spaces	Cohomology	Singular Cycles		$Algebraic\ subsets$
Y	heta	$T_{ heta}$	\subset	Z
$\downarrow P$	$\downarrow P_{!}$	$\downarrow P_*$		$\downarrow P$
X	$P_{!}(\theta)$	T_a	\subset	Z'
	× /			(2.27)

Next argument returns to the category of cohomology. Let \tilde{Z} be the smooth resolution of Z. We have the following composition map j:

$$j: \tilde{Z} \to Z \to Y.$$
 (2.28)

By corollary 8.2.8, [1], there is an exact sequence

$$H^{k+3}(\tilde{Z};\mathbb{Q}) \xrightarrow{j_!} H^{2p+k+1}(Y;\mathbb{Q}) \rightarrow H^{2p+k+1}(Y-Z;\mathbb{Q}).$$
 (2.29)

Since non-zero T_{θ} is supported on Z, θ is in the kernel of

$$H^{2p+k+1}(Y;\mathbb{Q}) \rightarrow H^{2p+k+1}(Y-Z;\mathbb{Q}).$$
 (2.30)

Hence there is a class

$$\theta_{\tilde{Z}} \in M^1 H^{k+3}(\tilde{Z}) \tag{2.31}$$

such that

$$j_!(\theta_{\tilde{Z}}) = \theta. \tag{2.32}$$

In the following we discuss a couple of cases for the class $\theta_{\tilde{Z}}$ on \tilde{Z} , whose dimension is

p + k + 2 = 2 - p + n.

(a) If the coniveau p > 2, then k + 4 < dim(Z) < n. By the induction

$$\theta_{\tilde{Z}} \in N^1 H^{3+k}(\tilde{Z}). \tag{2.33}$$

Then by [4], the geometric level of a cycle under the Gysin homomorphism $j_{!}$ must be preserved. Thus we obtain that,

$$j_!(\theta_{\tilde{Z}}) = \theta \in N^p H^{2p+k+1}(Y).$$

$$(2.34)$$

This proves the claim 2.3 in case (a).

(b) If p = 2, then Z has dimension n = k + 4. Thus k + 3 is not a middle dimension for \tilde{Z} . Then we consider the Lefschetz isomorphism

$$u: M^1 H^{k+3}(\tilde{Z}) \to M^2 H^{k+5}(\tilde{Z})$$

$$(2.35)$$

where u is a hyperplane section class represented by the hyperplane V. Let

$$l: V \cap \tilde{Z} \hookrightarrow \tilde{Z}$$

be the inclusion map. Then

$$l^*(\theta_{\tilde{Z}}) \tag{2.36}$$

is a class on $V \cap \tilde{Z}$ which must be \mathcal{M}_{n-2} leveled. Since $V \cap \tilde{Z}$ has dimension

$$k+3 = n-1,^{IV}$$

and $V \cap \tilde{Z}$ satisfies assumption 2.1, we apply the induction to obtain that

$$l^*(\theta_{\tilde{Z}}) \tag{2.37}$$

is \mathcal{N}_{n-3} leveled in $V \cap \tilde{Z}$. Notice

$$l_! \circ l^*(\theta_{\tilde{Z}}) = \theta_{\tilde{Z}} \cdot u \tag{2.38}$$

Hence $\theta_{\tilde{Z}} \cdot u$ is \mathcal{N}_{n-3} leveled in \tilde{Z} . Now we use the fact (3) to obtain

 $\theta_{ ilde{Z}}$

is \mathcal{N}_{n-2} leveled in \tilde{Z} . In the coniveau, it says

$$\theta_{\tilde{Z}} \in N^1 H^{3+k}(Z)$$

Then by the formula (2.34) we complete the proof for the claim 2.3.

Applying the claim 2.3, we obtain a non-empty algebraic set W of dimension at most p + k + 1 such that θ is Poincaré dual to a cellular cycle

$$T_{\theta} \subset W. \tag{2.39}$$

Next argument is called "descending construction". It extracts a lower algebraically leveled cycle from θ . This argument occurs in the category of

^{IV}This shows that the lowest n for our method is 4. Our method does not apply to the case n = 2 or 3.

currents. Cellular chains above represent currents of integrations over the chains. We denote the associated currents by the same letters. Applying the Künneth decomposition, T_{θ} must be in the form of

$$T_{\theta} = B \otimes b + B' \otimes b' + \varsigma + dK \in \mathcal{D}'(X \times E)$$
(2.40)

where B, B' represent singular cycles in X, whose cohomology class have Hodge levels k, b, b' represent a, a', dK is exact and ς is the sum of currents in the form $\zeta \otimes c$ with deg(c) = 0, 2. Let

$$\beta = \langle B \rangle, \beta' = \langle B' \rangle,$$

where $\langle \cdot \rangle$ denotes the cohomology class. Let b'' be a closed 1-current in E such that the intersection satisfy

$$[b'' \wedge b'] = 0, [b'' \wedge b] = \{e\}$$

where $e \in E$. The the intersection of currents from the fact (1) yields

$$[(X \otimes b'') \wedge T_{\theta}] = [(X \otimes b'') \wedge dK] + B \otimes \{e\}$$
(2.41)

is a current supported on W. Let \tilde{W} be a smooth resolution of the scheme W. We obtain the diagram

where the top sequence is the Gysin exact sequence, and ν_1 , which is a Gysin map, is the composition of Gysin maps q_1, P_1 . By (2.41), cohomology of $B \otimes \{e\}$, denoted by

$$\beta \otimes \langle \{e\} \rangle$$

is in the kernel of R. Hence it has a preimage

$$\phi \in H^{k+2}(\tilde{W}; \mathbb{Q}).$$

Because $q_!$ is an algebraic correspondence, ϕ can be chosen to have Hodge level k. (this is the strictness of the morphism of Hodge structures). Since the $dim(\tilde{W}) = n + 1 - p < n$, the inductive assumption says the Hodge level

is the geometric level. The Gysin image $\nu_{!}(\phi)$ then also has geometric level k. Looking back at the formula (2.41), the class

$$P_! \langle B \otimes \{e\} \rangle$$

is β . Hence β is the class $\nu_{!}(\phi)$ which is \mathcal{N}_{k} leveled. On the other hand the intersection number

$$(\alpha \otimes a', \theta)_Y = (\alpha, \beta)_X \neq 0.$$
(2.43)

This completes the proof of proposition 2.2 for the case $p \neq 1$.

Case 2: Coniveau p = 1.

Now we deal with the minor case when p = 1. In this case we already theorem 1.1 for $p \neq 1$. We consider $\alpha \in M^1H^n(X)$ where n = dim(X)is any whole number. Then as before E is an elliptic curve, $Y = X \times E$ and $a, a' \in H^1(E; \mathbb{Q})$ form a standard basis in the cohomology ring. In the following we'll use the projection $P: Y \to X$, but on a different type of cycles. First

$$\alpha \otimes 1 \in M^1 H^n(Y). \tag{2.44}$$

Let $\theta \in M^2 H^{n+2}(Y)$ be its generic dual. Since we proved

$$M^2 H^{n+2}(Y) = N^2 H^{n+2}(Y)$$

(geometric coniveau is 2) we obtain that

$$\theta \in N^2 H^{n+2}(Y).$$

Now we apply the Künneth decomposition,

$$\theta = \beta \otimes \omega + \beta_i \otimes a + \beta' \otimes a' + \gamma \otimes 1 \tag{2.45}$$

where ω is the fundamental class of E. Because $P_1(\theta)$ and θ will have the same geometric level, $P_1(\theta)$ lies in

$$N^1H^n(X).$$

Looking back to the formula (2.45), $P_{!}(\theta) = \beta$. This shows

$$\beta \in N^1 H^n(X).$$

On the other hand, we see that

$$(\alpha \otimes 1, \theta)_Y = (\alpha, \beta)_X \neq 0. \tag{2.46}$$

This completes injectivity of the map (2.2) in the case of $H^1(X; \mathbb{Q}) \neq 0$.

Proof. of theorem 1.1: Proposition 2.1, 2.2 show theorem 1.1 is correct for all X with non-zero $H^1(X; \mathbb{Q})$. Assume X is arbitrary and may not satisfy $H^1(X; \mathbb{Q}) \neq 0$. Notice that $X \times E$ has non-zero first cohomology. Thus theorem 1.1 holds on $X \times E$. Let $\alpha \in M^p H^{2p+k}(X)$. Then

$$\alpha \otimes \omega \in M^{p+1} H^{2p+k+2}(X \times E).$$
(2.47)

where ω is the fundamental class of E. By the proved theorem 1.1 for $X \times E$,

$$\alpha \otimes \omega \in N^{p+1} H^{2p+k+2}(X \times E).$$
(2.48)

Then for the Gysin image, we have

$$P_!(\alpha \otimes \omega) \in N^p H^{2p+k}(X), \tag{2.49}$$

where $P: X \times E \to X$ is the projection. Since $P_!(\alpha \otimes \omega) = \alpha$, we complete the proof of theorem 1.1.

A Intersection of currents

Denote the real vector space of real currents of degree i by \mathcal{D}'^i . Let

$$\mathcal{R}(X) \subset \mathcal{D}^{\prime i}(X) \times \mathcal{D}^{\prime j}(X) \tag{A.1}$$

be the subset of currents satisfying some expected condition (de Rham condition in [6]). Then we show that there is a well-defined homomorphism \wedge

$$\mathcal{R}(X) \to \mathcal{D}^{\prime(i+j)}(X)$$
 (A.2)

such that \wedge is reduced to the cap product and the algebraic intersection. The new notion of intersection \wedge is a variant depending on the variant de Rham data \mathcal{U} of holomorphic coordinates charts.

Nevertheless carrying the \mathcal{U} , the intersection satisfies basic properties:

- (a) graded commutativity,
- (b) associativity,
- (c) continuity,

- (d) topologicity, (i.e. coincides with the cap product)
- (e) algebraicity (i.e. coincides with algebraic intersection),
- (f) Supportivity (i.e. the support of the intersection is the intersection of the supports).

For a full exploration of this notion, we refer the readers to [6].

B Surfaces and threefolds

Proposition B.1. Theorem 1.1 holds for all X of $dim(X) \leq 3$.

Proof. When X is a curve, the proposition is trivial. so we consider Case 1: dim(X) = 2.

We have

$$\mathcal{N}_0(X) = \sum_{i=0}^2 N^i H^{2i}(X; \mathbb{Q}).$$

By the Lefschetz theorem on (1,1) classes,

$$\sum_{i=0}^{2} N^{i} H^{2i}(X) = \sum_{i=0}^{2} M^{i} H^{2i}(X) = \mathcal{M}_{0}(X).$$

Now we consider the level 1.

$$\mathcal{N}_1(X) = \mathcal{N}_0 \oplus N^0 H^1(X) \oplus N^1 H^3(X).$$

Thus

$$\mathcal{N}_1(X) = \mathcal{M}_0 \oplus H^1(X; \mathbb{Q}) \oplus N^1 H^3(X).$$

By the fact (3),

$$N^{1}H^{3}(X) \simeq N^{0}H^{1}(X) = H^{1}(X; \mathbb{Q}) = M^{0}H^{1}(X).$$

By the Poincaré duality,

$$M^0 H^1(X) \simeq M^1 H^3(X).$$

Thus because $M^0H^1(X) = H^1(X; \mathbb{Q}),$

$$N^1 H^3(X) \simeq M^1 H^3(X).$$

The maximal level k = 2 is a trivial case. Now we conclude theorem 1.1 for dim(X) = 2.

Case 2: dim(X) = 3. In this case, the only non trivial assertion is

$$M^{1}H^{3}(X) = N^{1}H^{3}(X).$$
(B.1)

This is a non-trivial case of the generalized Hodge conjecture of level 1 on threefolds for which a well-known example was constructed by Grothendieck in [2]. Let's start with Voisin's construction. Suppose $L \subset H^3(X; \mathbb{Q})$ is a sub-Hodge structure of coniveau 1. In [3], Voisin showed that there is a smooth curve C, and a Hodge cycle

$$\tilde{\Psi} \in Hdg^4(C \times X) \tag{B.2}$$

such that

$$\tilde{\Psi}_*(H^1(C;\mathbb{Q})) = L. \tag{B.3}$$

where $\tilde{\Psi}_*$ is defined as the Gysin image

$$P_!\bigg(\tilde{\Psi}\cup(\bullet)\otimes 1)\bigg),$$

with the projection $P: X \times C \to X$. Notice $P_!(\tilde{\Psi})$ is a Hodge cycle in X. By the assumption it is algebraic on X, i.e there is a closed current $T_{\tilde{\Psi}}$ on $X \times C$ representing the class $\tilde{\Psi}$ such that

$$P_*(T_{\tilde{\Psi}}) = S_a + bK \tag{B.4}$$

where S_a is a current of integration over the algebraic cycle S, and bK is an exact current of dimension 4 in X. (adjust $\tilde{\Psi}$ so S_a is non-zero). Consider another current in $C \times X$

$$T := T_{\tilde{\Psi}} - [e] \otimes bK \tag{B.5}$$

denoted by T, where [e] is a current of evaluation at a point $e \in C$. By the fact (4), we can adjust the exact current on the right hand side of (B. 5) to have

$$P(supp(T)) = supp(P_*(T)).$$
(B.6)

Let Θ be a collection of closed currents on C representing the classes in $H^1(C; \mathbb{Q})$. Then by the correspondence of currents in [6],

$$T_*(\Theta), \tag{B.7}$$

is a family of currents supported on the support of the current

$$P_*(T) = S_a. \tag{B.8}$$

which is the integration over an algebraic cycle, i.e. the the family of currents are all supported on the algebraic set |S|. This is known as a criterion for coniveau filtration^V, i.e. for $\beta \in T_*(\Theta)$, the cohomology $\langle \beta \rangle$ of β satisfies

$$\langle \beta \rangle \in ker \left(H^3(X; \mathbb{Q}) \to H^3(X - |S|; \mathbb{Q}) \right)$$
 (B.9)

Using the fact (1), cohomology of the currents in $T_*(\Theta)$ consists of all classes in L. This shows $L \subset N^1 H^3(X)$. We complete the proof.

C Generalized Lefschetz standard conjecture

Theorem C.1. The map

$$\begin{array}{rccc} u_a^{q-p} : N^p H^{2p+k}(X) & \to & N^q H^{2q+k}(X) \\ \alpha & \to & \alpha \cdot u^{q-p}. \end{array} \tag{C.1}$$

is an isomorphism on coniveau filtration for

$$p+q = n-k, p \le q, k \ge 0.$$

The speculation of the truth of the theorem will be referred to as the generalized Lefschetz standard conjecture. It turns out to be equivalent to the Grothendeick's Lefschetz standard conjecture over \mathbb{C} .

The theorem, therefore the standard conjectures over \mathbb{C} , is proved by using the fact (1). We refer readers to [5].

^VFor instance, see [4].

D Supportive projection

Let X be a compact manifold of dimension n. A p-cell S consists of three elements: a p dimensional polyhedron Δ^p in \mathbb{R}^v (an open set), an orientation of \mathbb{R}^v , and a C^{∞} map f of \mathbb{R}^v to X restricted to a one-to-one map on Δ^p . A chain C is a linear combination of cells. The support |C| of C is the image of all Δ^p in X. A point in C is a point in |C|.

Let Y be another compact manifold of dimension m. Let

$$P: Y \times X \to X \tag{D.1}$$

be the projection.

Definition D.1. Let C be a C^{∞} p-chain of $Y \times X$. Let $a \in P(|C|)$. If

 $P^{-1}(a) \cap |C|$

is a finite set, we say C is finite at a. If C is finite at all points, we say C is finite to X.

Proposition D.2. For any C^{∞} p-cell S in the coordinates chart of $Y \times X$ with $p < \dim(X)$, S is homotopic to a chain that has the same boundary and is finite to X.

Proof. Let

$$\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n$$

be the coordinate's charts for $Y, X, Y \times X$ respectively. Since we are dealing with a single cell, we may replace the polyhedron by the unit ball B. Let B_{ϵ} be a ball with radius ϵ that is sufficiently small. Then we can use multiple barycentric subdivisions to divide S to a chain $\sum_{i=0}^{N} c_i$ where c_0 is represented by B_{ϵ} , and rest of cells c_i are supported on the image of $B - B_{\epsilon}$. We use the same notation c_i to denote the polyhedron representing the cell c_i . Composing each cell map

$$c_i \rightarrow Y \times X$$
 (D.2)

with the projection P, we obtain a map denoted by f_i from subsets $c_i \subset B$ to \mathbb{R}^n . Let $\bigcup_{i \neq 0} c_i = c$, f_c be the united f_i for all i. Let $r, \gamma_1, \cdots, \gamma_{p-1}$ be the polar coordinates of $\mathbb{R}^p - B_{\epsilon}$, i.e. $(r, r\gamma_1, \cdots, r\gamma_{p-1})$ are the rectangular coordinates of \mathbb{R}^p . We denote $(\gamma_1, \cdots, \gamma_{p-1})$ by γ . Next we consider the homotopy of the maps

$$\mathbb{R}^p - B_\epsilon \to \mathbb{R}^n$$

in polar coordinates

$$tf_c + (1-t)h, \quad t \in [0,1]$$
 (D.3)

where

$$h(r,\gamma) = f_c(r,\gamma) + (1-r)g(r,\gamma).$$

By this homotopy, $f_c(r, \gamma)$ is homotopic to $h(r, \gamma)$ whose boundary $h(1, \gamma)$ is $f_c(1, \gamma)$. The Jacobian of h at $r \neq 1$ varies with the Jacobian of $(1-r)g(r, \gamma)$. Hence the Jacobian of h is non zero for all the bounded r, γ of B except for r = 1. By the inverse function theorem, each $c_i, i \neq 0$ is on-to-one to its image in X. We may assume the center 0 of B maps to an arbitrary point of S. Then the above proof also showed there is a homotopy making the center one-to-one to X. Overall we obtain a homotopy that fixes the boundary of S and deform the interior of cells to these that are finite to X. This completes the proof.

Proposition D.3. For any cellular cycle S in $Y \times X$, of dimension

S is homopotic to a cycle finite to X.

Proof. For each cell S_i of S, by proposition D.2, there is a homotopical chain c^i that is finite to X and agree with S_i on the all faces of S_i . Thus we can glue all c^i along their faces to obtain a cellular cycle S' that is homotopic to S. Since there are only finitely many such chains c_i^j , the projection $S' \to X$ must also be finite on the interior of each cell.

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