

APPLICATION BMO TYPE SPACE TO PARABOLIC EQUATIONS OF NAVIER-STOKES TYPE WITH THE NEUMANN BOUNDARY CONDITION

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ABSTRACT. Let \mathcal{L} be a Neumann operator of the form $\mathcal{L} = -\Delta_N$ acting on $L^2(\mathbb{R}^n)$. Let $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ denote the BMO space on \mathbb{R}^n associated to the Neumann operator \mathcal{L} . In this article we will show that a function $f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ is the trace of the solution of $\mathbb{L}u = u_t + \mathcal{L}u = 0$, $u(x, 0) = f(x)$, where u satisfies a Carleson-type condition

$$\sup_{x_B, r_B} r_B^{-n} \int_0^{r_B^2} \int_{B(x_B, r_B)} |\nabla u(x, t)|^2 dx dt \leq C < \infty,$$

for some constant $C > 0$. Conversely, this Carleson condition characterizes all the \mathbb{L} -carolic functions whose traces belong to the space $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$. This result extends the analogous characterization founded by E. Fabes and U. Neri in (*Duke Math. J.* **42** (1975), 725-734) for the classical BMO space of John and Nirenberg. Furthermore, based on the characterization of $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ space mentioned above, we prove global well-posedness for parabolic equations of Navier-Stokes type with the Neumann boundary condition under smallness condition on initial data $u_0 \in \text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$, which is motivated by the work of P. Auscher and D. Frey (*J. Inst. Math. Jussieu* **16**(5) (2017), 947-985).

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In Harmonic Analysis, to study a (suitable) function $f(x)$ on \mathbb{R}^n is to consider a harmonic function on \mathbb{R}_+^{n+1} which has the boundary value as $f(x)$. A standard choice for such a harmonic function is the Poisson integral $e^{-t\sqrt{-\Delta}}f(x)$ and one recovers $f(x)$ when letting $t \rightarrow 0^+$, where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is the Laplace operator. In other words, one obtains $u(x, t) = e^{-t\sqrt{-\Delta}}f(x)$ as the solution of the equation

$$\begin{cases} \partial_t u + \Delta u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

This approach is intimately related to the study of singular integrals. In [15], the authors studied the classical case $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

It is well known that the BMO space, i.e. the space of functions of bounded mean oscillation, is natural substitution to study singular integral at the end-point space $L^\infty(\mathbb{R}^n)$. A celebrated theorem of Fefferman and Stein [10] states that a BMO function is the trace of the solution of $\partial_t u + \Delta u = 0$, $u(x, 0) = f(x)$, whenever u satisfies

$$(1.1) \quad \sup_{x_B, r_B} r_B^{-n} \int_0^{r_B^2} \int_{B(x_B, r_B)} |t \nabla u(x, t)|^2 \frac{dx dt}{t} < \infty,$$

2010 *Mathematics Subject Classification.* 42B35, 42B37, 35J10, 47F05.

Key words and phrases. Neumann operator, BMO space, Carleson measure, Navier-Stokes equation.

M. H. Yang was supported by the Postdoctoral Science Foundation of Jiangxi Province (grant no. 2017KY23) and Educational Commission Science Programm of Jiangxi Province (grant no. GJJ170345). C. Zhang was supported by the Natural Science Foundation of Zhejiang Province (Grant No. LY18A010006) and the National Natural Science Foundation of China (Grant No. 11401525).

where $\nabla = (\nabla_x, \partial_t) = (\partial_1, \dots, \partial_n, \partial_t)$. Conversely, Fabes, Johnson and Neri [7] showed that condition above characterizes all the harmonic functions whose traces are in $BMO(\mathbb{R}^n)$ in 1976. The study of this topic has been widely extended to more general operators such as elliptic operators and Schrödinger operators (instead of the Laplacian), for more general initial data spaces such as Morrey spaces and for domains other than \mathbb{R}^n such as Lipschitz domains. For these generalizations, see [2, 6, 8, 9, 13].

In [8], Fabes and Neri further generalized the above characterization to caloric functions (temperature), that is the authors proved that a BMO function f is the trace of the solution of

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

whenever u satisfies

$$(1.2) \quad \sup_{x_B, r_B} r_B^{-n} \int_0^{r_B^2} \int_{B(x_B, r_B)} |\nabla u(x, t)|^2 dx dt < \infty,$$

where $\nabla = (\nabla_x, \partial_t)$; and, conversely, the condition (1.2) characterizes all the carolic functions whose traces are in $BMO(\mathbb{R}^n)$. The authors in [11] made a complete conclusion, related to harmonic functions and carolic functions, about this subject.

We denote by Δ_n the Laplacian on \mathbb{R}^n . Next we recall the Neumann Laplacian on \mathbb{R}_+^n and \mathbb{R}_-^n . Consider the Neumann problem on the half line $(0, \infty)$:

$$(1.3) \quad \begin{cases} u_t - \Delta u = 0, & 0 < x < \infty, 0 < t < \infty, \\ u(x, 0) = f(x), & 0 < x < \infty, \\ u_x(0, t) = 0, & 0 < t < \infty. \end{cases}$$

Denote this corresponding Laplacian by Δ_{1, N_+} and we see that

$$u(x, t) = e^{t\Delta_{1, N_+}}(f)(x).$$

For $n > 1$, we write $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times \mathbb{R}_+$. And we definition the Neumann Laplacian on \mathbb{R}_+^n by $\Delta_{n, N_+} = \Delta_{n-1} + \Delta_{1, N_+}$; where Δ_{n-1} is the Laplacian on \mathbb{R}^{n-1} and Δ_{1, N_+} is the Laplacian corresponding to (1.3), for more results related to this topic, we refer the readers to see [5, 12]. Similarly we can definition Neumann Laplacian Δ_{n, N_-} on \mathbb{R}^n . Now let Δ_N be the uniquely determined unbounded operator acting on $L^2(\mathbb{R}^n)$ such that $(\Delta_N f)_+ = \Delta_{N_+} f_+$ and $(\Delta_N f)_- = \Delta_{N_-} f_-$ for all $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f_+ \in W^{1,2}(\mathbb{R}_+^n)$ and $f_- \in W^{1,2}(\mathbb{R}_-^n)$. Then Δ_N is a positive self-adjoint operator and

$$(\exp(t\Delta_N)f)_+ = \exp(t\Delta_{N_+})f_+; \quad (\exp(t\Delta_N)f)_- = \exp(t\Delta_{N_-})f_-.$$

The operator Δ_N is a self-adjoint operator on $L^2(\mathbb{R}^n)$. Hence Δ_N generates the Δ_N -heat semigroup

$$\mathcal{T}_t f(x) = e^{t\Delta_N} f(x) = \int_{\mathbb{R}^n} p_{t, \Delta_N}(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^n), \quad t > 0.$$

The heat kernel of $\exp(t\Delta_N)$, denoted by $p_{t, \Delta_N}(x, y)$, is then given as:

$$p_{t, \Delta_N}(x, y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x'-y'|^2}{4t}} (e^{-\frac{|x_n-y_n|^2}{4t}} - e^{-\frac{|x_n+y_n|^2}{4t}}) H(x_n y_n),$$

where $H : \mathbb{R} \rightarrow \{0, 1\}$ is the Heaviside given by

$$H(t) = 0, \quad t < 0; \quad H(t) = 1, \quad t \geq 0,$$

$x' = (x_1, x_1, \dots, x_{n-1})$ and $y' = (y_1, y_1, \dots, y_{n-1})$. It is well-known that the semigroup kernels $p_{t,\Delta_N}(x, y)$ of the operators $e^{t\Delta_N}$ satisfies Gaussian bounds:

$$0 \leq p_{t,\Delta_N}(x, y) \leq h_t(x - y)$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$, where

$$h_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

is the kernel of the classical heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$ on \mathbb{R}^n . For the classical heat semigroup associated with Laplacian, see [14]. For $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, it is well known that $u(x, t) = e^{t\Delta_N} f(x)$, $t > 0$, $x \in \mathbb{R}^n$, is a solution to the equation

$$(1.4) \quad \mathbb{L}u = \partial_t u - \Delta_N u = 0 \quad \text{in } \mathbb{R}_+^{n+1}$$

with the boundary data $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. The equation $\mathbb{L}u = 0$ is interpreted in the weak sense via a sesquilinear form, that is, $u \in W_{\text{loc}}^{1,2}(\mathbb{R}_\mathfrak{N}^n \times \mathbb{R}^+)$ is a weak solution of $\mathbb{L}u = 0$ if it satisfies

$$\int_{\mathbb{R}_\mathfrak{N}^n \times \mathbb{R}^+} \nabla_x u(x, t) \cdot \nabla_x \psi(x, t) dx dt - \int_{\mathbb{R}_\mathfrak{N}^n \times \mathbb{R}^+} u(x, t) \partial_t \psi(x, t) dx dt = 0,$$

for any $\psi(x, t) \in C_0^1(\mathbb{R}_\mathfrak{N}^n \times \mathbb{R}^+)$ and $\mathfrak{N} \in \{+, -\}$. In the sequel, we call such a function u an \mathbb{L} -carolic function associated to the operator \mathbb{L} .

The first main aim of this article is to study a similar characterization to (1.2) for the Neumann operator Δ_N . In a word, we are interested in deriving the characterization of the solution to the equation $\mathbb{L}u = 0$ in \mathbb{R}_+^{n+1} having boundary values with BMO type data ($\text{BMO}_{\Delta_N}(\mathbb{R}^n)$). We now recall the definition and some fundamental properties of $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ from [5, 12].

Define

$$M = \left\{ f \in L_{\text{loc}}^1(\mathbb{R}^n) : \exists d > 0, \text{ s.t. } \int_{\mathbb{R}^n} \frac{|f(x)|^2}{1 + |x|^{n+d}} dx < \infty \right\}.$$

Definition 1.1 ([5], Definition 2.2). We say that $f \in M$ is of bounded mean oscillation associated with Δ_N , abbreviated as $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$, if

$$(1.5) \quad \|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} = \sup_{B(y,r)} \frac{1}{|B(y,r)|} \int_{B(y,r)} |f(x) - \exp(-r^2 \Delta_N) f(x)| dx < \infty,$$

where the supremum is taken over all balls $B(y, r)$ in \mathbb{R}^n . The smallest bound for which (1.5) is satisfied is then taken to be the norm of f in this space, and is denoted by $\|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)}$.

Let us introduce a new function class on half plane \mathbb{R}_+^{n+1} or \mathbb{R}_-^{n+1} .

Definition 1.2 (Temperature Mean Oscillation for Δ_N). A C^1 -functions $u(x, t)$ defined on $\mathbb{R}_+^n \times (0, \infty)$ belongs to the class $\text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))$, if $u(x, t)$ is the solution of

$$(1.6) \quad \begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}_+^n, 0 < t < \infty, \\ \partial_{x_n} u(x', 0, t) = 0, & x' \in \mathbb{R}^{n-1}, 0 < x_n, t < \infty, \end{cases}$$

such that

$$(1.7) \quad \|u\|_{\text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))}^2 = \sup_{B(x_B, r_B) \subset \mathbb{R}_+^n \times (0, \infty)} r_B^{-n} \int_0^{r_B^2} \int_{B(x_B, r_B)} |\nabla u(x, t)|^2 dx dt < \infty,$$

where $\nabla = (\nabla_x, \partial_t)$. Similarly, we can define $\text{TMO}_{\Delta_{N_-}}(\mathbb{R}_-^n \times (0, \infty))$, the temperature mean oscillation for Δ_{N_-} .

We say that $u(x, t) \in \text{TMO}_{\Delta_N}(\mathbb{R}^n \times (0, \infty))$, if $u_+(x, t) = u(x, t)\chi_{\mathbb{R}_+^n \times (0, \infty)}(x, t) \in \text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))$ and $u_-(x, t) = u(x, t)\chi_{\mathbb{R}_-^n \times (0, \infty)}(x, t) \in \text{TMO}_{\Delta_{N_-}}(\mathbb{R}_-^n \times (0, \infty))$ with

$$\|u\|_{\text{TMO}_{\Delta_N}(\mathbb{R}^{n+1})} = \max \left\{ \|u_+\|_{\text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))}, \|u_-\|_{\text{TMO}_{\Delta_{N_-}}(\mathbb{R}_-^n \times (0, \infty))} \right\} < \infty.$$

Theorem 1.3. (1) *If $f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$, then the function $u(x, t) = e^{t\Delta_N} f(x) \in \text{TMO}_{\Delta_N}(\mathbb{R}^{n+1})$ with*

$$\|u\|_{\text{TMO}_{\Delta_N}(\mathbb{R}^{n+1})} \approx \|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)}.$$

(2) *If $u \in \text{TMO}_{\Delta_N}(\mathbb{R}^{n+1})$, then there exists some $f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ such that $u(x, t) = e^{t\Delta_N} f(x)$, and*

$$\|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \leq C \|u\|_{\text{TMO}_{\Delta_N}(\mathbb{R}^{n+1})}$$

with some constant $C > 0$ independent of u and f .

(3) *We say that a tempered distribution $f \in \text{BMO}_{\Delta_N}^{-1}$ if*

$$\sup_{B(x_B, r_B) \subset \mathbb{R}^n \times (0, \infty)} r_B^{-n} \int_0^{r_B^2} \int_{B(x_B, r_B)} |e^{t\Delta_N} f(x)|^2 dx dt < \infty.$$

According the definition above, it follows that a tempered distribution $f \in \mathbb{R}^n$ belongs to $\nabla \cdot (\text{BMO}_{\Delta_N}(\mathbb{R}^n))^n$ if and only if there are $f_j \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{j=n} \partial_j f_j$. That is,

$$\nabla \cdot (\text{BMO}_{\Delta_N}(\mathbb{R}^n))^n = \text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n).$$

As an application of Theorem 1.3, we shall consider the well-posedness of parabolic equations of Navier-Stokes type with the Neumann boundary condition, which borrows from [1] completely. That is, for a.e. $x \in \mathbb{R}^n$, consider the equation

$$(1.8) \quad \begin{cases} \partial_t u - \Delta u = \text{div}_x f(u^2(t, x)), & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \frac{\partial u(x, t)}{\partial x_n} = 0, & x_n = 0, \end{cases}$$

where $\Omega = \mathbb{R}_+^n \cup \mathbb{R}_-^n$, we assume that $f : \mathbb{R} \rightarrow \mathbb{R}^n$ is globally Lipschitz continuous, and satisfies

$$|f(x)| \leq C|x|, \quad x \in \mathbb{R}.$$

Without loss of generality, we assume that $f(x) = x$ in this paper. The mild solutions of the system (1.8) is

$$(1.9) \quad u(., t) = e^{t\Delta_N} u_0 - \int_0^t e^{(t-s)\Delta_N} \text{div}_x(u^2(s, .)) ds.$$

Clearly, from Theorem 1.3, we know that the divergence of a vector field with components in $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ is in $\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$. To make sense to the free evolution term $e^{t\Delta_N} u_0$ from (1.9), recall that in the case of the system (1.9) (with the Neumann Laplacian in the background), the adapted value space consists of divergence free elements u_0 in $\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$ and is characterized by $e^{t\Delta_N} u_0$ in the path space. Thus, we let

$$\varepsilon := \{u(x, t) \text{ measurables in } \mathbb{R}^n \times (0, \infty) : \|u\|_\varepsilon < \infty\},$$

with

$$(1.10) \quad \|u\|_\varepsilon = \left\| t^{1/2} u \right\|_{L^\infty(\mathbb{R}^n \times (0, \infty))} + \|u\|_{T^{\infty, 2}(\mathbb{R}_+^{n+1})},$$

where

$$\|u\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} = \sup_{x \in \mathbb{R}^n, t \in (0, \infty)} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |u(y, s)|^2 dy ds \right)^{1/2}.$$

Our second result in this paper reads as follows.

Theorem 1.4. *Let $u_0 \in \text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$. There exists $\epsilon > 0$, such that, if $\|u_0\|_{\text{BMO}_{\Delta_N}^{-1}} < \epsilon$, then the system (1.8) has a global mild solution $u \in \mathcal{E}$, which is unique one in the closed all $\{u \in \mathcal{E} : \|u\|_{\mathcal{E}} \leq 2\epsilon\}$.*

This article is organized as follows. In Section 2, we give the proof of our characterization theorem, Theorem 1.3. In Section 4, we give the proof of Theorem 1.4 by using Theorem 1.3 as an application.

Throughout the article, the letters “ c ” and “ C ” will denote (possibly different) constants which are independent of the essential variables.

2. PROOF OF CHARACTERIZATION THEOREM

In this section, we will give the proof of Theorem 1.3.

Proof of Theorem 1.3. (1) Since $f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$, $f_{+,e} \in \text{BMO}(\mathbb{R}^n)$ and $f_{-,e} \in \text{BMO}(\mathbb{R}^n)$. And we have

$$e^{t\Delta_N} f = (e^{t\Delta_N} f)_+ + (e^{t\Delta_N} f)_- = e^{t\Delta_{N_+}} f_+ + e^{t\Delta_{N_-}} f_-.$$

Hence, letting $B = B(x_B, r_B)$,

$$\begin{aligned} \int_0^{r_B^2} \int_B |\nabla_x e^{t\Delta_N} f(x)|^2 dx dt &\leq 2 \int_0^{r_B^2} \int_B |\nabla_x e^{t\Delta_{N_+}} f_+(x)|^2 dx dt + 2 \int_0^{r_B^2} \int_B |\nabla_x e^{t\Delta_{N_-}} f_-(x)|^2 dx dt \\ &= 2 \int_0^{r_B^2} \int_B |\nabla_x e^{t\Delta} f_{+,e}(x)|^2 dx dt + 2 \int_0^{r_B^2} \int_B |\nabla_x e^{t\Delta} f_{-,e}(x)|^2 dx dt \\ &\leq C|B| \|f_{+,e}\|_{\text{BMO}(\mathbb{R}^n)}^2 + C|B| \|f_{-,e}\|_{\text{BMO}(\mathbb{R}^n)}^2 \\ &\leq C|B| \|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)}^2, \end{aligned}$$

where we have used the result in [8].

(2) If $u \in \text{TMO}_{\Delta_N}(\mathbb{R}_+^{n+1})$, then $u_+ \in \text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))$ and $u_- \in \text{TMO}_{\Delta_{N_-}}(\mathbb{R}_-^n \times (0, \infty))$. Since $u_+ \in \text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))$, letting $u_{+,e}$ be the even extension of u_+ on \mathbb{R}_+^{n+1} . Then, by the result in [8], there exists an even function $f^1 \in \text{BMO}(\mathbb{R}^n)$ such that $u_{+,e}(x, t) = e^{t\Delta} f^1(x)$ for $x \in \mathbb{R}^n$, and $\|f^1\|_{\text{BMO}(\mathbb{R}^n)} \leq C\|u_{+,e}\|_{\text{TMO}_{\Delta}(\mathbb{R}_+^{n+1})}$. Then $f_+^1(x) \in \text{BMO}_r(\mathbb{R}_+^n)$ and, for any $x \in \mathbb{R}_+^n$,

$$u(x, t) = u_+(x, t) = u_{+,e}(x, t) = e^{t\Delta} f^1(x) = e^{t\Delta} f_{+,e}^1(x) = e^{t\Delta_{N_+}} f_+^1(x)$$

with

$$\|f_+^1\|_{\text{BMO}_r(\mathbb{R}_+^n)} \leq \|f^1\|_{\text{BMO}(\mathbb{R}^n)} \leq C\|u_{+,e}\|_{\text{TMO}_{\Delta}(\mathbb{R}_+^{n+1})} \leq C\|u\|_{\text{TMO}_{\Delta_{N_+}}(\mathbb{R}_+^n \times (0, \infty))}$$

and

$$\partial_{x_n} e^{t\Delta_{N_+}} f_+^1(x', 0) = 0.$$

Also, since $u_- \in \text{TMO}_{\Delta_{N_-}}(\mathbb{R}_-^n \times (0, \infty))$, letting $u_{-,e}$ be the even extension of u_- on \mathbb{R}_-^{n+1} . Then, by the result in [8], there exists an even function $\bar{f}^1 \in \text{BMO}(\mathbb{R}^n)$ such that $u_{-,e}(x, t) = e^{t\Delta} \bar{f}^1(x)$ for $x \in \mathbb{R}^n$, and $\|\bar{f}^1\|_{\text{BMO}(\mathbb{R}^n)} \leq C\|u_{-,e}\|_{\text{TMO}_{\Delta}(\mathbb{R}_+^{n+1})}$. Then $\bar{f}_-^1(x) \in \text{BMO}_r(\mathbb{R}_-^n)$ and, for any $x \in \mathbb{R}_-^n$,

$$u(x, t) = u_-(x, t) = u_{-,e}(x, t) = e^{t\Delta} \bar{f}^1(x) = e^{t\Delta} \bar{f}_{-,e}^1(x) = e^{t\Delta_{N_-}} \bar{f}_-^1(x)$$

with

$$\|\bar{f}_-\|_{\text{BMO}_t(\mathbb{R}_+^n)} \leq \|\bar{f}_-\|_{\text{BMO}(\mathbb{R}^n)} \leq C\|u_{-,e}\|_{\text{TMO}_{\Delta}(\mathbb{R}_+^{n+1})} \leq C\|u\|_{\text{TMO}_{\Delta_N}(\mathbb{R}^n \times (0,\infty))}$$

and

$$\partial_{x_n} e^{t\Delta_N} \bar{f}_-(x', 0) = 0.$$

Let $f(x) = f_+^1(x) + \bar{f}_-(x)$, for any $x \in \mathbb{R}^n$. Then $f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ and

$$e^{t\Delta_N} f(x) = (e^{t\Delta_N} f)_+(x) + (e^{t\Delta_N} f)_-(x) = e^{t\Delta_N} f_+^1(x) + e^{t\Delta_N} \bar{f}_-(x) = u(x, t)$$

for any $x \in \mathbb{R}^n$. And we have

$$\|f\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)} \leq \max \left\{ \|f_+^1\|_{\text{BMO}_t(\mathbb{R}_+^n)}, \|\bar{f}_-\|_{\text{BMO}_t(\mathbb{R}_+^n)} \right\} \leq C\|u\|_{\text{TMO}_{\Delta_N}(\mathbb{R}_+^{n+1})}$$

and

$$\partial_{x_n} e^{t\Delta_N} f(x', 0) = \partial_{x_n} e^{t\Delta_N} f_+^1(x', 0) + \partial_{x_n} e^{t\Delta_N} \bar{f}_-(x', 0) = 0.$$

(3) For any $f \in \nabla \cdot \text{BMO}_{\Delta_N}(\mathbb{R}^n)$, there exists $f_1, f_2, \dots, f_n \in \text{BMO}_{\Delta_N}(\mathbb{R}^d)$ such that $f = \sum_{j=1}^{j=n} \partial_j f_j$, we have

$$\|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \leq \sum_{j=1}^{j=n} \|\partial_j f_j\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \leq \sum_{j=1}^{j=n} \|f_j\|_{\text{BMO}_{\Delta_N}(\mathbb{R}^n)}.$$

On the other hand, if $f \in \text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$ and $f_{j,k} = \partial_j \partial_k (-\Delta)^{-1} f$, It is suffice to prove we $f_{j,k} \in \text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$. In fact, by the term (3.2) below, we have that

$$\begin{aligned} \|\partial_j \partial_k (-\Delta_N)^{-1} f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^d)} &\leq \frac{1}{2} \|\partial_j \partial_k (-\Delta_N)^{-1} f_{+,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)}^2 + \frac{1}{2} \|\partial_j \partial_k (-\Delta_N)^{-1} f_{-,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)}^2 \\ &\leq \frac{1}{2} \|f_{+,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)}^2 + \frac{1}{2} \|f_{-,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)}^2 \lesssim \|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)}^2 \end{aligned}$$

Thus we have $f_k = -\partial_k (-\Delta)^{-1} f \in \text{BMO}_{\Delta_N}(\mathbb{R}^n)$ and $f = \sum_{k=1}^{k=n} \partial_k f_k$. And we end the proof. \square

3. PROOF OF WELL-POSEDNESS OF PARABOLIC EQUATIONS WITH NEUMANN BOUNDARY CONDITION

In this section, we shall prove the well-posedness of the system (1.8) with initial data in $\text{BMO}_{\Delta_N}^{-1}$, which can be obtained by combining the characterization of the $\text{BMO}_{\Delta_N}(\mathbb{R}^n)$ derived in Theorem 1.3 and the Banach contraction mapping principle. Therefore it is necessary to examine the linear and nonlinear terms of (1.8). To do so, we need the following functions estimates.

Proposition 3.1. *According to all the notations as in (1.10), one has*

$$(3.1) \quad \|f\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \approx \|f_{+,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \|f_{-,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})},$$

here $f_{\pm,e}$ is the even extension of the restriction of f from \mathbb{R}_{\pm}^n . Namely, $f \in T^{\infty,2}(\mathbb{R}_+^{n+1})$ if and only if $f_{+,e} \in T^{\infty,2}(\mathbb{R}_+^{n+1})$ and $f_{-,e} \in T^{\infty,2}(\mathbb{R}_+^{n+1})$. Similarly, we also have

$$(3.2) \quad \|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \approx \|f_{+,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)} + \|f_{-,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)},$$

$$(3.3) \quad \|f\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} \approx \|f_{+,e}\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + \|f_{-,e}\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})},$$

$$(3.4) \quad \|t^{1/2} f\|_{L^{\infty}((0,\infty) \times \mathbb{R}^n)} \approx \|t^{1/2} f_{+,e}\|_{L^{\infty}((0,T) \times \mathbb{R}^n)} + \|t^{1/2} f_{-,e}\|_{L^{\infty}((0,\infty) \times \mathbb{R}^n)},$$

and there exist a positive constant C such that

$$(3.5) \quad \|t^{1/2} \exp(t\Delta_N) f\|_{L^{\infty}((0,\infty) \times \mathbb{R}^n)} \leq C \|f_{+,e}\|_{\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^n)} + C \|f_{-,e}\|_{\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^n)}.$$

Proof. Let us begin with the proof of the inequality (3.1), for any $t > 0$,

$$\begin{aligned}
(3.6) \quad & t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |f(y, s)|^2 dy ds \\
& \leq t^{-n/2} \int_0^t \int_{B(x, \sqrt{t}), y \in \mathbb{R}_+^n} |f_{+,e}(y, s)|^2 dy ds + t^{-n/2} \int_0^t \int_{B(x, \sqrt{t}), y \in \mathbb{R}_-^n} |f_{-,e}(y, s)|^2 dy ds \\
& \leq t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |f_{+,e}(y, s)|^2 dy ds + t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |f_{-,e}(y, s)|^2 dy ds \\
& \leq \|f_{+,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^2 + \|f_{-,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^2.
\end{aligned}$$

Conversely,

$$\begin{aligned}
& t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |f_{+,e}(y, s)|^2 dy ds \\
& \leq t^{-n/2} \int_0^t \int_{B(x, \sqrt{t}), y \in \mathbb{R}_+^n} |f_+(y, s)|^2 dy ds + t^{-n/2} \int_0^t \int_{B(\tilde{x}, \sqrt{t}), y \in \mathbb{R}_+^n} |f_+(y, s)|^2 dy ds \\
& \leq 2 \|f\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^2,
\end{aligned}$$

and, similarly,

$$t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |f_{-,e}(y, s)|^2 dy ds \leq 2 \|f\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^2.$$

These together with the estimates (3.6) imply

$$\|f\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \leq \|f_{+,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \|f_{-,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \leq 2\sqrt{2} \|f\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})},$$

and which completes the proof of (3.1).

For the term (3.2), obviously,

$$\begin{aligned}
& t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |\exp(t\Delta_N)f(y, s)|^2 dy ds \\
& \leq \frac{1}{2} t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |\exp(t\Delta)f_{+,e}(y, s)|^2 dy ds \\
& \quad + \frac{1}{2} t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |\exp(t\Delta)f_{-,e}(y, s)|^2 dy ds \\
& \leq \frac{1}{2} \|f_{+,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)}^2 + \frac{1}{2} \|f_{-,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)}^2.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |\exp(t\Delta)f_{+,e}(y, s)|^2 dy ds \\
& \leq 2t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |\exp(t\Delta_N)f(y, s)|^2 dy ds \leq 2 \|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)}^2
\end{aligned}$$

and

$$t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |\exp(t\Delta)f_{-,e}(y, s)|^2 dy ds \leq 2 \|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)}^2.$$

Finally, we have equivalent norms

$$\frac{\sqrt{2}}{4} \|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \leq \|f_{+,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)} + \|f_{-,e}\|_{\text{BMO}^{-1}(\mathbb{R}^n)} \leq \frac{\sqrt{2}}{2} \|f\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)}.$$

Collecting terms above, we have the desired estimate (3.2) and thus complete the proof of (3.2). The terms (3.3)-(3.5) can be proved similarly, here we omit the details. \square

Proposition 3.2. *Suppose $u_0 \in \text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)$, then we have that*

$$\|\exp(t\Delta_N)u_0\|_{\varepsilon} \lesssim \|u_0\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)}.$$

Proof. By Lemma 3.1 and $\text{BMO}^{-1}(\mathbb{R}^n) \hookrightarrow \dot{\text{B}}_{\infty,\infty}^{-1}(\mathbb{R}^n)$, where $\dot{\text{B}}_{\infty,\infty}^{-1}(\mathbb{R}^n)$ is homogeneous Besov space (the definition of homogeneous Besov spaces, see [16]), it thus follows that

$$\begin{aligned} \|\exp(t\Delta_N)u_0\|_{\varepsilon} &= \|t^{1/2} \exp(t\Delta_N)u_0\|_{L^\infty((0,\infty)\times\mathbb{R}^n)} \\ &\quad + \sup_{x \in \mathbb{R}^n, t \in (0,\infty)} \left(t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |\exp(s\Delta_N)u_0(y,s)|^2 dy ds \right)^{1/2} \\ &\leq \|u_{0,+e}\|_{\dot{\text{B}}_{\infty,\infty}^{-1}(\mathbb{R}^n)} + \|u_{0,-e}\|_{\dot{\text{B}}_{\infty,\infty}^{-1}(\mathbb{R}^n)} + \|u_0\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \lesssim \|u_0\|_{\text{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)}. \end{aligned}$$

Collecting terms, we have the desired estimate and thus complete the proof of Proposition 3.2. \square

Let $\alpha := u \times v$ and $u, v \in \varepsilon$ and

$$\mathcal{A}(\alpha)(t) := \int_0^t e^{(t-s)\Delta_N} \text{div}_x \alpha(s, \cdot) ds.$$

Proposition 3.3. *Let $u, v \in \varepsilon$. A positive constant C exists such that*

$$\left\| t^{\frac{1}{2}} \mathcal{A}(\alpha) \right\|_{L^\infty((0,\infty)\times\mathbb{R}^n)} \leq C \|\alpha\|_{T^{\infty,1}(\mathbb{R}^n)} + C \|\alpha(s)\|_{L^\infty((0,\infty)\times\mathbb{R}^n)}.$$

Proof. We first split the integral as

$$\mathcal{A}(\alpha)(t) := \int_0^{t/2} e^{(t-s)\Delta_N} \text{div}_x \alpha(s, \cdot) ds + \int_{t/2}^t e^{(t-s)\Delta_N} \text{div}_x \alpha(s, \cdot) ds$$

From the definition of $p_{t-s,\Delta_N}(x, y)$, using the decay of the heat kernel at ∞ , the divergence term can be estimated as following

$$\begin{aligned} &|\nabla \exp((t-s)\Delta_N)\alpha(s, x)| \\ &\lesssim \int_{\mathbb{R}_+^n} |\nabla p_{t-s,\Delta_N}(x, y)| |\alpha_+(s, y)| dy \\ &\lesssim \int_{\mathbb{R}_+^n} \left(\sum_{i=1}^{n-1} \left| \frac{\partial}{\partial x_i} p_{t-s,\Delta_N}(x, y) \right| + \left| \frac{\partial}{\partial x_n} p_{t-s,\Delta_N}(x, y) \right| \right) |\alpha_+(s, y)| dy \\ &\lesssim \int_{\mathbb{R}_+^n} \sum_{i=1}^n \left| \frac{x_i - y_i}{2t} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \right| |\alpha_+(s, y)| dy \\ &\quad + \int_{\mathbb{R}_+^n} e^{-\frac{|x_n+y_n|^2}{4t}} \left(\sum_{i=1}^{n-1} \left| \frac{x_i - y_i}{2t} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x'-y'|^2}{4t}} \right| + \left| \frac{x_n + y_n}{2t} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x'-y'|^2}{4t}} \right| \right) |\alpha_+(s, y)| dy \\ &\lesssim \int_{\mathbb{R}^n} |\nabla p_{t-s,\Delta}(x, y)| |\alpha_{+,e}(s, y)| dy \end{aligned}$$

$$\lesssim \int_{\mathbb{R}^n} (t-s)^{-n/2} \left(1 + (t-s)^{-1/2} |x-y|\right)^{-n-1} |\alpha_{+,e}(s,y)| dy.$$

We can also obtain similar estimates for any $x \in \mathbb{R}_-^n$,

$$\begin{aligned} |\nabla \exp(t\Delta_N)\alpha(s,x)| &\lesssim \int_{\mathbb{R}^n} |\nabla p_{t-s,\Delta}(x,y)| |\alpha_{-,e}(s,y)| dy \\ &\lesssim \int_{\mathbb{R}^n} (t-s)^{-n/2} \left(1 + (t-s)^{-1/2} |x-y|\right)^{-n-1} |\alpha_{-,e}(s,y)| dy. \end{aligned}$$

Since $0 < s < t/2$ implies $t-s \sim t$, using the fact that every cube of side length \sqrt{t} is contained in a ball of radius \sqrt{t} , we find that

$$\begin{aligned} &\sup_{t>0} \left| t^{\frac{1}{2}} \int_0^{t/2} e^{(t-s)\Delta_N} \operatorname{div}_x \alpha(s,x) ds \right| \\ &\leq \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla p_{t-s,\Delta}(x,y)| |\alpha_{+,e}(s,y)| dy ds \right| \\ &\quad + \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla p_{t-s,\Delta}(x,y)| |\alpha_{-,e}(s,y)| dy ds \right| \\ &\leq \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_0^{t/2} \sum_{k \in \mathbb{Z}^n} t^{-n/2} (1+|k|)^{-n} |\alpha_{+,e}(s,y)| dy ds \right| \\ &\quad + \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_0^{t/2} \sum_{k \in \mathbb{Z}^n} t^{-n/2} (1+|k|)^{-n} |\alpha_{-,e}(s,y)| dy ds \right| \\ &\leq \sum_{k \in \mathbb{Z}^n} (1+|k|)^{-n} \left(\|\alpha_{+,e}\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + \|\alpha_{-,e}\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} \right). \end{aligned}$$

Now we look at $t/2 < s < t$, the estimates of the kernel of the heat semigroup imply that

$$\begin{aligned} &\sup_{t>0} \left| t^{\frac{1}{2}} \int_{t/2}^t e^{(t-s)\Delta_N} \operatorname{div}_x \alpha(s,x) ds \right| \\ &\leq \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla p_{t-s,\Delta}(x,y)| |\alpha_{+,e}(s,y)| dy ds \right| \\ &\quad + \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_0^{t/2} \int_{\mathbb{R}^n} |\nabla p_{t-s,\Delta}(x,y)| |\alpha_{-,e}(s,y)| dy ds \right| \\ &\leq \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_{t/2}^t \frac{1}{(4\pi(t-s))^{n/2}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{|x-y|}{2s(t-s)} dy \right) \|s\alpha_{+,e}(s,y)\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} ds \right| \\ &\quad + \sup_{t>0, x \in \mathbb{R}^n} \left| t^{\frac{1}{2}} \int_{t/2}^t \frac{1}{(4\pi(t-s))^{n/2}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{|x-y|}{2s(t-s)} dy \right) \|s\alpha_{-,e}(s,y)\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} ds \right| \\ &\leq \|s\alpha_{-,e}(s,y)\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} + \|s\alpha_{+,e}(s,y)\|_{L^\infty((0,\infty) \times \mathbb{R}^n)}. \end{aligned}$$

Resuming the above estimates, the proof of Proposition 3.3 is completed. \square

To derive the Carleson measure estimate, we write

$$\mathcal{A}(\alpha)(t) := \int_0^t e^{-(t-s)\mathcal{L}} \operatorname{div}_x \alpha(s, \cdot) ds$$

$$\begin{aligned}
&= \int_0^t e^{-(t-s)\mathcal{L}} L\mathcal{T} \left(s^{1/2} \alpha(s, \cdot) \right) ds + \int_0^\infty e^{-(t+s)\mathcal{L}} (\operatorname{div}_x \alpha(s, \cdot)) ds \\
&\quad - \int_t^\infty e^{-(t-s)\mathcal{L}} s^{-1/2} (\operatorname{div}_x s^{1/2} \alpha(s, \cdot)) ds \\
&=: \mathcal{A}_1(\alpha)(t) + \mathcal{A}_2(\alpha)(t) + \mathcal{A}_3(\alpha)(t),
\end{aligned}$$

where $\mathcal{L} = -\Delta_N$ and

$$\mathcal{T}F(s, x) =: (s\mathcal{L})^{-1}(I - e^{-2s\mathcal{L}})\operatorname{div}F(s, x).$$

Note that, the operator Δ_N satisfies Gaussian bound and has a bounded H_∞ -calculus in $L^2(\mathbb{R}^n)$.

For the estimate on \mathcal{A}_1 , we apply the following two lemmas. The first one is an extension of [3, Theorem 3.2] using the structure of the maximal regularity operator.

Lemma 3.4. *The operator*

$$\begin{aligned}
\mathcal{M}^+ &: T^{\infty,2}(\mathbb{R}_+^{n+1}) \rightarrow T^{\infty,2}(\mathbb{R}_+^{n+1}), \\
(\mathcal{M}^+F)(t, \cdot) &:= \int_0^t L e^{-(t-s)\mathcal{L}} F(s, \cdot) ds,
\end{aligned}$$

is bounded.

Proof. According to [4, Lemma 1,19], $(t\mathcal{L}e^{t\mathcal{L}})_{t>0}$ satisfies Gaussian estimates, therefore in particular the weaker $\mathcal{L}^2(\mathbb{R}^n)$ off-diagonal estimates of [3, Theorem 3.2]. Hence, we can apply [3, Theorem 3.2] to obtain that $\mathcal{M}^+ : T^{\infty,2}(\mathbb{R}_+^{n+1}) \rightarrow T^{\infty,2}(\mathbb{R}_+^{n+1})$. The proof of Lemma 3.4 is thus completed. \square

Lemma 3.5. *The operator*

$$\mathcal{T}F(s, x) := T_s F(s, \cdot)(x), \quad \forall (s, x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

with

$$T_s =: (s\mathcal{L})^{-1}(I - e^{-2s\mathcal{L}})\operatorname{div}F(s, x)$$

is bounded from $T^{\infty,2}(\mathbb{R}_+^{n+1})$ to $T^{\infty,2}(\mathbb{R}_+^{n+1})$.

Proof. Since

$$T_s =: -s^{-1/2} \int_0^{2s} e^{-\mu\mathcal{L}} \operatorname{div}d\mu,$$

the estimate

$$\begin{aligned}
&t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| s^{-1/2} \int_0^\infty e^{-\mu\mathcal{L}} \operatorname{div}d\mu f(y, s) \right|^2 dy ds \\
&\leq \frac{1}{2} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| s^{-1/2} \int_0^{2s} e^{-\mu\mathcal{L}} \operatorname{div}d\mu f_{+,e}(y, s) \right|^2 dy ds \right) \\
&\quad + \frac{1}{2} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| s^{-1/2} \int_0^{2s} e^{-\mu\mathcal{L}} \operatorname{div}d\mu f_{-,e}(y, s) \right|^2 dy ds \right) \\
&\leq \frac{1}{2} \left\| s^{-1/2} \int_0^{2s} e^{-\mu\mathcal{L}} \operatorname{div}f_{+,e} d\mu \right\|_{T^{\infty,2}(\mathbb{R}^n)} + \frac{1}{2} \left\| s^{-1/2} \int_0^{2s} e^{-\mu\mathcal{L}} \operatorname{div}f_{-,e} d\mu \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \\
&\leq \frac{1}{2} \|f_{+,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \frac{1}{2} \|f_{-,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}
\end{aligned}$$

holds true. Therefore, the desired estimate now follows due to the Proposition 3.1. We complete the proof of Lemma 3.5. \square

An application of Fubini's theorem yields

$$\begin{aligned} \langle \mathcal{A}_2 F, G \rangle &= \int_{\mathbb{R}^n} \int_0^\infty e^{(t+s)\Delta_N} \operatorname{div} F(s, y) ds \cdot \overline{G(t, y)} dt dy \\ &= \int_{\mathbb{R}^n} \int_0^\infty F(s, y) \overline{\nabla e^{(t+s)\Delta_N} G(t, y)} dt ds dy \\ &= \left\langle F, \nabla e^{s\Delta_N} \int_0^\infty e^{t\Delta_N} G(t, \cdot) dt \right\rangle = \langle F, \mathcal{A}_2^* G \rangle. \end{aligned}$$

We thus have the following lemma.

Lemma 3.6. *There exists a positive constant $c > 0$ such that*

$$\|(\mathcal{A}_2^* G)(s, \cdot)\|_{T^{1,\infty}(\mathbb{R}_+^{n+1})} = \left\| \nabla e^{s\Delta_N} \int_0^\infty e^{t\Delta_N} G(t, \cdot) dt \right\|_{T^{1,\infty}(\mathbb{R}_+^{n+1})} \leq \|G\|_{T^{1,2}(\mathbb{R}_+^{n+1})}$$

Proof. To derive the estimate of $\mathcal{A}_2^* G$ in the $T^{1,\infty}(\mathbb{R}_+^{n+1})$ norm, we employ the theory of Hardy spaces associated with operators Δ_N . Recall that a Hardy-type space associated to Δ_N was introduced in [5], defined by

$$(3.7) \quad H_{\Delta_N}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : S_{\Delta_N}(f)(x) \in L^1(\mathbb{R}^n)\}$$

in the norm of

$$\|f\|_{H_{\Delta_N}^1(\mathbb{R}^n)} = \|S_{\Delta_N}(f)\|_{L^1(\mathbb{R}^n)},$$

where

$$S_{\Delta_N}(f)(x) = \left(\int_0^\infty \int_{|y-x|<t} |t^2 \Delta_N \exp(t^2 \Delta_N) f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

By the characterization of $H_{\Delta_N}^1(\mathbb{R}^n)$, we find that

$$\begin{aligned} & t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| \nabla e^{s\Delta_N} \int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right| dy ds \\ & \leq \frac{1}{2} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| \nabla e^{s\Delta} \left[\int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right]_{+,e} \right| dy ds \right) \\ & \quad + \frac{1}{2} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| \nabla e^{s\Delta} \left[\int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right]_{-,e} \right| dy ds \right) \\ & \leq \frac{1}{2} \left\| \left[\nabla \int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right]_{+,e} \right\|_{H^1(\mathbb{R}^n)} + \frac{1}{2} \left\| \left[\nabla \int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right]_{-,e} \right\|_{H^1(\mathbb{R}^n)} \\ & \leq \left\| \nabla \int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right\|_{H_{\Delta_N}^1(\mathbb{R}^n)}. \end{aligned}$$

From the definition of $H_{\Delta_N}^1(\mathbb{R}^n)$, we have

$$\begin{aligned} & \left\| \nabla \int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right\|_{H_{\Delta_N}^1(\mathbb{R}^n)} \\ & = \int_{\mathbb{R}^n} \left| \left(\int_0^\infty \int_{|y-x|<t} |t^2 \Delta_N \exp(t^2 \Delta_N) \left(\nabla \int_0^\infty e^{\theta\Delta_N} G(\theta, y) d\theta \right)|^2 \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}(x) \right| dx \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| \nabla e^{s\Delta} \left[\int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right]_{+,e} \right| dy ds \right) \\
&\quad + \frac{1}{2} \left(t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} \left| \nabla e^{-s\Delta} \left[\int_0^\infty e^{\theta\Delta_N} G(\theta, \cdot) d\theta \right]_{-,e} \right| dy ds \right) \\
&\lesssim \left\| \left[\nabla \int_0^\infty e^{\theta\Delta} G_{+,e}(\theta, \cdot) d\theta \right] \right\|_{H^1(\mathbb{R}^n)} + \left\| \left[\nabla \int_0^\infty e^{\theta\Delta} G_{+,e}(\theta, \cdot) d\theta \right] \right\|_{H^1(\mathbb{R}^n)} \\
&\lesssim \|G_{+,e}\|_{T^{1,2}(\mathbb{R}_+^{n+1})} + \|G_{-,e}\|_{T^{1,2}(\mathbb{R}_+^{n+1})}.
\end{aligned}$$

We complete the proof of Lemma 3.6. \square

Lemma 3.7. *Let*

$$RF(\cdot, s) = \int_s^\infty e^{(s+\tau)\Delta_N} \tau^{-\frac{1}{2}} \operatorname{div} F(\cdot, \tau) d\tau.$$

Then we have

$$\|RF(\cdot, t)\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \leq \|F(\cdot, t)\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}.$$

Proof. We can see that for any $t > 0$,

$$\begin{aligned}
&t^{-n/2} \int_0^t \int_{B(x, \sqrt{t})} |RF(\cdot, s)|^2 dy ds \\
&\leq \left(t^{-n/2} \int_0^t \int_{|y-x| < \sqrt{t}, y \in \mathbb{R}_+^n} \left| \int_s^\infty e^{(s+\tau)\Delta} \tau^{-\frac{1}{2}} \operatorname{div} F(\cdot, \tau) d\tau \right|^2 dy ds \right) \\
&\quad + \left(t^{-n/2} \int_0^t \int_{|y-x| < \sqrt{t}, y \in \mathbb{R}_+^n} \left| \int_s^\infty e^{(s+\tau)\Delta} \tau^{-\frac{1}{2}} \operatorname{div} F(\cdot, \tau) d\tau \right|^2 dy ds \right) \\
&\leq \frac{1}{2} \left\| s^{-1/2} \int_0^{2s} e^{\mu\Delta} \operatorname{div} f_{+,e} d\mu \right\|_{T^{\infty,2}(\mathbb{R}^n)} + \frac{1}{2} \left\| s^{-1/2} \int_0^{2s} e^{\mu\Delta} \operatorname{div} f_{-,e} d\mu \right\|_{T^{\infty,2}(\mathbb{R}^n)} \\
&\leq \frac{1}{2} \|f_{+,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \frac{1}{2} \|f_{-,e}\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})},
\end{aligned}$$

which completes the proof of Lemma 3.7. \square

Proposition 3.8. *Let $u, v \in \mathcal{E}$. Then we have that*

$$\|\mathcal{A}(\alpha)\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \leq C \|\alpha\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + C \left\| s^{\frac{1}{2}} \alpha(s) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}.$$

Proof. For $\mathcal{A}(\alpha)$, it follows from Lemmas 3.4-3.7 that

$$\begin{aligned}
\mathcal{A}(\alpha)(t) &\leq \mathcal{A}_1(\alpha)(t) + \mathcal{A}_2(\alpha)(t) + \mathcal{A}_3(\alpha)(t) \\
&\leq \left\| T_s \left(s^{1/2} \alpha(s, \cdot) \right) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \left\| s^{1/2} \alpha(s, \cdot) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \sup_{\|G\|_{T^{1,2}(\mathbb{R}_+^{n+1})} \leq 1} \langle \alpha(s, \cdot), \mathcal{A}_2^* G \rangle \\
&\leq \left\| T_s \left(s^{1/2} \alpha(s, \cdot) \right) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \left\| s^{1/2} \alpha(s, \cdot) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \\
&\quad + \sup_{\|G\|_{T^{1,2}(\mathbb{R}_+^{n+1})} \leq 1} \left\| (\mathcal{A}_2^* G)(s, \cdot) \right\|_{T^{1,\infty}(\mathbb{R}_+^{n+1})} \|\alpha\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} \\
&\leq C \|\alpha\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + C \left\| s^{\frac{1}{2}} \alpha(s) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}.
\end{aligned}$$

This completes the proof of Proposition 3.8. \square

Proof of the Theorem 1.4. Now we define the operator

$$\Theta(u) := e^{t\Delta_N} u_0 - \int_0^t e^{(t-s)\Delta_N} \operatorname{div}_x(u^2(s, \cdot)) ds.$$

For the existence of small global solution, we adopt the space

$$D_{2\epsilon} = \{u \in \mathcal{E}; \|u\|_{\mathcal{E}} < 2\epsilon\}.$$

It suffices to show that $\Theta(u)$ is a contraction operator mapping \mathcal{E} into itself. Indeed, it is readily seen that

$$\|\exp(t\Delta_N)u_0\|_{\mathcal{E}} \lesssim \|u_0\|_{\operatorname{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \leq \epsilon$$

by Proposition 3.2 and provided that $\|u_0\|_{\operatorname{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} \leq \epsilon$. Let $u, v \in \mathcal{E}$. It follows from Propositions 3.1, 3.3 and 3.8 that

$$\begin{aligned} \|\Theta(u)\|_{\mathcal{E}} &\leq \|u_0\|_{\operatorname{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} + C \sum_{i=1}^{i=3} \|\mathcal{A}_i(u \times v)(t)\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \|u \times v\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + C \|s(u \times v)(s)\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} \\ &\leq \|u_0\|_{\operatorname{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} + C \|(u \times v)_{+,e}\|_{T^{\infty,1}(\mathbb{R}^n)} + C \left\| s^{\frac{1}{2}}(u \times v)_{+,e}(s) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \left\| s(u \times v)_{+,e}(s) \right\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} \\ &\quad + C \|(u \times v)_{-,e}\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + C \left\| s^{\frac{1}{2}}(u \times v)_{-,e}(s) \right\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \left\| s(u \times v)_{-,e}(s) \right\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} \\ &\leq \|u_0\|_{\operatorname{BMO}_{\Delta_N}^{-1}(\mathbb{R}^n)} + C \|u\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^2 + C \left\| s^{\frac{1}{2}}u \right\|_{L^\infty((0,\infty) \times \mathbb{R}^n)} \|u\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \left\| s^{\frac{1}{2}}u \right\|_{L^\infty((0,\infty) \times \mathbb{R}^n)}^2 \\ &\leq \epsilon + C \|u\|_{\mathcal{E}_\infty}^2 < 2\epsilon \end{aligned}$$

and

$$\begin{aligned} &\|\Theta(u) - \Theta(v)\|_{\mathcal{E}} \\ &\leq \|\Phi(u - v, u)\|_{\mathcal{E}} + \|\Phi(v, u - v)\|_{\mathcal{E}} \\ &\leq \|(u - v)u\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + \|(u - v)v\|_{T^{\infty,1}(\mathbb{R}_+^{n+1})} + \|s(u - v)u\|_{L^\infty} + \|s(u - v)v\|_{L^\infty} \\ &\quad + \|s^{1/2}(u - v)v\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \|s^{1/2}(u - v)v\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} \\ &\leq \|u - v\|_{T^{\infty,2}} \|(u, v)\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})} + \|s^{\frac{1}{2}}(u - v)\|_{L^\infty} (\|s^{\frac{1}{2}}v\|_{L^\infty} + \|s^{\frac{1}{2}}u\|_{L^\infty}) \\ &\quad + \|s^{1/2}(u - v)\|_{L^\infty} \|s^{1/2}v\|_{L^\infty}^{\frac{1}{2}} \|u - v\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^{\frac{1}{2}} \|v\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^{\frac{1}{2}} \\ &\quad + \|s^{1/2}(u - v)\|_{L^\infty} \|s^{1/2}u\|_{L^\infty}^{\frac{1}{2}} \|u - v\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^{\frac{1}{2}} \|u\|_{T^{\infty,2}(\mathbb{R}_+^{n+1})}^{\frac{1}{2}} \\ &\leq C \|u - v\|_{\mathcal{E}} \|(u, v)\|_{\mathcal{E}} \leq C\epsilon \|u - v\|_{\mathcal{E}}, \end{aligned}$$

provided that $u \in D_{2\epsilon}$ and $C\epsilon < 1$, the map $\Theta : D_{2\epsilon} \rightarrow D_{2\epsilon}$ is a contraction and has a fixed point in $D_{2\epsilon}$, which is the unique solution u for the integral equation satisfying $\|u\|_{\mathcal{E}} \leq 2\epsilon$. Hence there exists a unique solution $u \in \mathcal{E}$ satisfying $u = \Theta(u)$ due to the Banach contraction principle. The proof of Theorem 1.4 is complete. \square

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