# Bernstein type inequalities for self-normalized martingales with applications

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#### Abstract

For self-normalized martingales with conditionally symmetric differences, de la Peña [\[6](#page-15-0)] established the Gaussian type exponential inequalities. Bercu and Touati [\[2\]](#page-15-1) extended de la Peña's inequalities to martingales with differences heavy on left. In this paper, we establish Bernstein type exponential inequalities for self-normalized martingales with differences bounded from below. Moreover, applications to self-normalized sums, t-statistics and autoregressive processes are discussed.

*Keywords:* martingales; self-normalized processes; exponential inequalities; autoregressive processes

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### 1. Introduction

Let  $(\xi_i)_{i\geq 1}$  be a sequence of zero-mean independent random variables satisfying  $\xi_i \leq 1$  for all i. Denote  $S_n = \sum_{i=1}^n \xi_i$  the partial sums of  $(\xi_i)_{i\geq 1}$ . Bennett [\[1\]](#page-15-2) proved the following Bernstein type inequality: for all  $x > 0$ ,

<span id="page-0-0"></span>
$$
\mathbb{P}(S_n \ge xv^2) \le \exp\bigg\{-\frac{x^2v^2}{2(1+x/3)}\bigg\},\tag{1}
$$

where  $v^2 = \text{Var}(S_n)$  is the variance of  $S_n$ . The importance of Bernstein type inequalities comes from the fact that they combine both the Gaussian trends and exponentially decaying rate. To see this, we rewrite the last inequality in the following form: for all  $x > 0$ ,

$$
\mathbb{P}(S_n \ge x) \le \exp\bigg\{-\frac{x^2}{2(v^2 + x/3)}\bigg\}.\tag{2}
$$

It is easy to see that the last bound behaves as  $\exp{-\frac{x^2}{2v^2}}$  $\frac{x^2}{2v^2}$  for moderate  $x = o(v^2)$ , while it is exponentially decaying to 0 as  $x \to \infty$ .

The generalizations of [\(1\)](#page-0-0) to martingales have attracted certain interest. Assume that  $(\xi_i, \mathcal{F}_i)_{i=0,\dots,n}$ is a sequence of martingale differences. If  $\xi_i \leq 1$ , Freedman [\[15](#page-15-3)] showed that [\(1\)](#page-0-0) holds also when  $\mathbb{P}(S_n \geq xv^2)$  is replaced by  $\mathbb{P}(S_n \geq xv^2, \langle S \rangle_n \leq v^2)$ , where  $\langle S \rangle_n$  is the conditional variance of  $S_n$ . De la Peña [\[6\]](#page-15-0), Dzhaparidze and van Zanten [\[10](#page-15-4)] and Fan et al. [\[12](#page-15-5), [14](#page-15-6)] extended Freedman's inequality

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to martingales with non-bounded differences. Recently, Rio [\[20](#page-15-7)] gave a refinements on Freedman's inequality.

Despite the fact that the case of martingale is well studied, there are only a few results on Bernstein type inequalities for self-normalized martingales  $S_n/[S_n]$ , where  $[S_n]$  is the squared variance of  $S_n$ . Among them, let us recall the following exponential inequalities of de la Peña [\[6\]](#page-15-0). Assume that  $(\xi_i, \mathcal{F}_i)_{i=0,\dots,n}$  is a sequence of conditionally symmetric martingale differences. Recall that  $\xi_i$  is called conditionally symmetric if  $\mathcal{L}(\xi_i|\mathcal{F}_{i-1}) = \mathcal{L}(-\xi_i|\mathcal{F}_{i-1})$  for all i, where  $\mathcal{L}(\xi_i|\mathcal{F}_{i-1})$  stands for the regular version of the conditional distribution of  $\xi_i$  given a  $\sigma$ -field  $\mathcal{F}_{i-1}$ . De la Peña [\[6](#page-15-0)] have established the following exponential inequalities for self-normalized martingales: for all  $x > 0$ ,

<span id="page-1-0"></span>
$$
\mathbb{P}\left(\frac{S_n}{[S]_n} \ge x\right) \le \sqrt{\mathbb{E}\left[\exp\left\{-\frac{1}{2}x^2[S]_n\right\}\right]},\tag{3}
$$

and, for all  $x, y > 0$ ,

<span id="page-1-1"></span>
$$
\mathbb{P}\left(\frac{S_n}{[S]_n} \ge x, [S]_n \ge y\right) \le \exp\left\{-\frac{1}{2}x^2y\right\},\tag{4}
$$

where  $[S]_n = \sum_{i=1}^n \xi_i^2$  the squared variance of  $S_n$ . In the i.i.d. case,  $[S]_n/n$  usually converges almost surely to the variance of the random variables. Thus [\(3\)](#page-1-0) and [\(4\)](#page-1-1) can be regarded as Gaussian type inequalities.

The inequalities of de la Peña have been extended to the martingales with differences heavy on left. Recall that an integrable random variable X is called heavy on left if  $\mathbb{E}X = 0$  and, for all  $a > 0$ ,  $\mathbb{E}[T_a(X)] \leq 0$ , where

$$
T_a(X) = \min(|X|, a) \operatorname{sign}(X)
$$

<span id="page-1-2"></span>is the truncated version of X. Clearly, conditionally symmetric martingale differences are heavy on left. Bercu and Touati [\[2](#page-15-1)] have obtained the following extension of de la Peña's inequity [\(3\)](#page-1-0): for all  $x > 0$ ,

$$
\mathbb{P}\left(\frac{S_n}{[S]_n} \ge x\right) \le \inf_{p>1} \left(\mathbb{E}\left[\exp\left\{-\frac{1}{2}(p-1)x^2[S]_n\right\}\right]\right)^{1/p}.\tag{5}
$$

They also showed that [\(4\)](#page-1-1) holds for martingales with differences heavy on left. In the particular case  $p = 2$ , inequality [\(5\)](#page-1-2) reduces to inequality [\(3\)](#page-1-0) under the conditional symmetric assumption. Similar results for self-normalized martingales  $S_n/\sqrt{[S]_n}$  can also be found in Bercu and Touati [\[2](#page-15-1)].

Exponential inequalities for self-normalized martingales have a lot of applications. We refer to de la Peña, Klass and Lai [\[7](#page-15-8)] for autoregressive processes. Bercu and Touati [\[2](#page-15-1)] applied such type inequalities to parameter estimations of linear regressions, autoregressive processes and branching processes. For more applications of such type inequalities, we refer to the monographs of de la Peña, Lai and Shao [\[8](#page-15-9)] and Bercu, Delyon and Rio [\[3](#page-15-10)].

In this paper, we aim to establish Bernstein type inequalities for self-normalized martingales with differences bounded from below. It is obvious that a random variable is bounded from below does not imply that it is heavy on left. Our results for self-normalized martingales are analogues to the inequalities [\(3\)](#page-1-0) - [\(5\)](#page-1-2). Applications to self-normalized sums, t-statistics and autoregressive processes are also discussed.

The paper is organized as follows. We present our main results in Section 2. In Section 3, we discuss the applications, and prove our main results in Section 4.

#### 2. Main results

Let  $(\xi_i, \mathcal{F}_i)_{i=0,\dots,n}$  be a finite sequence of real-valued square integrable martingale differences defined on a probability space  $(\Omega, \mathcal{F}, P)$ , where  $\xi_0 = 0$  and  $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \ldots \subseteq \mathcal{F}_n \subseteq \mathcal{F}$  are increasing  $\sigma$ -fields. So by definition, we have  $\mathbb{E}[\xi_i|\mathcal{F}_{i-1}] = 0, i = 1, \ldots, n$ . Set

$$
S_0 = 0 \quad \text{and} \quad S_k = \sum_{i=1}^k \xi_i \tag{6}
$$

for  $k = 1, \ldots, n$ . Then  $S = (S_k, \mathcal{F}_k)_{k=1,\ldots,n}$  is a martingale. Let  $[S]$  and  $\langle S \rangle$  be, respectively, the squared variance and the conditional variance of the martingale S, that is

$$
[S]_0 = 0, \qquad [S]_k = \sum_{i=1}^k \xi_i^2,
$$

and

<span id="page-2-2"></span>
$$
\langle S \rangle_0 = 0, \qquad \langle S \rangle_k = \sum_{i=1}^k \mathbb{E}[\xi_i^2 | \mathcal{F}_{i-1}], \quad k = 1, ..., n. \tag{7}
$$

Our main result is the following Bernstein type inequalities for self-normalized martingales with differences bounded from below. It is worth to be mentioned that the inequalities are new even for independent random variables.

**Theorem 2.1.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. Then for all*  $x > 0$ *,* 

<span id="page-2-0"></span>
$$
\mathbb{P}\left(\frac{S_n}{[S]_n} \ge x\right) \le \inf_{p>1} \left(\mathbb{E}\left[\exp\left\{-(p-1)\left(x-\log(1+x)\right)[S]_n\right\}\mathbf{1}_{\{S_n \ge x[S]_n\}}\right]\right)^{1/p} \tag{8}
$$

$$
\leq \inf_{p>1} \left( \mathbb{E} \bigg[ \exp \bigg\{ -(p-1) \frac{x^2}{2(1+x)} [S]_n \bigg\} \mathbf{1}_{\{S_n \geq x [S]_n\}} \bigg] \right)^{1/p}, \tag{9}
$$

*and, for all*  $y > 0$ *,* 

<span id="page-2-1"></span>
$$
\mathbb{P}\left(\frac{S_n}{[S]_n} \ge x, [S]_n \ge y\right) \le \exp\left\{-\left(x - \log(1+x)\right)y\right\} \tag{10}
$$

$$
\leq \exp\bigg\{-\frac{x^2y}{2(1+x)}\bigg\}.\tag{11}
$$

Clearly, inequality [\(9\)](#page-2-0) implies that for all  $x > 0$ ,

$$
\mathbb{P}\bigg(\frac{S_n}{[S]_n} \ge x\bigg) \le \inf_{p>1} \bigg(\mathbb{E}\bigg[\exp\bigg\{-(p-1)\frac{x^2}{2(1+x)}[S]_n\bigg\}\bigg]\bigg)^{1/p},
$$

which is an analogues to de la Peña's inequity  $(3)$  and the inequality of Bercu and Touati  $(5)$ .

Denote  $B_n^2 = \sum_{i=1}^n \mathbb{E}\xi_i^2$ . It is easy to see that for all  $x > 0$  and all  $0 < \varepsilon < 1$ ,

$$
\mathbb{P}\bigg(\frac{S_n}{[S]_n} \geq x\bigg) \leq \mathbb{P}\bigg(\frac{S_n}{[S]_n} \geq x, [S]_n \geq B_n^2(1-\varepsilon)\bigg) + \mathbb{P}\bigg([S]_n < B_n^2(1-\varepsilon)\bigg) \n= \mathbb{P}\bigg(\frac{S_n}{[S]_n} \geq x, [S]_n \geq B_n^2(1-\varepsilon)\bigg) + \mathbb{P}\bigg(\sum_{i=1}^n (\xi_i^2 - \mathbb{E}\xi_i^2) < -B_n^2\varepsilon\bigg).
$$

The first term of the last bound can be estimated by [\(11\)](#page-2-1). For the second term of the last bound, notice that  $(\xi_i^2 - \mathbb{E}\xi_i^2)_{i=1,\dots,n}$  are centered random variables bounded from below, and they are independent once  $(\xi_i)_{i=1,\dots,n}$  are independent. Thus we need the following Bernstein type exponential inequalities for centered random variables bounded from below.

<span id="page-3-2"></span>**Theorem 2.2.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. Then for all*  $x > 0$ *,* 

<span id="page-3-0"></span>
$$
\mathbb{P}\left(\frac{S_n}{\langle S\rangle_n} \leq -x\right) \leq \inf_{p>1} \left( \mathbb{E}\left[\exp\left\{-(p-1)\left((1+x)\log(1+x)-x\right)\langle S\rangle_n\right\} \mathbf{1}_{\{S_n \leq -x\langle S\rangle_n\}}\right] \right)^{1/p} (12)
$$
\n
$$
\leq \inf_{p>1} \left( \mathbb{E}\left[\exp\left\{-(p-1)\frac{x^2}{2(1+x/3)}\langle S\rangle_n\right\} \mathbf{1}_{\{S_n \leq -x\langle S\rangle_n\}}\right] \right)^{1/p} . \tag{13}
$$

Inequality [\(13\)](#page-3-0) implies that for all  $x > 0$ ,

<span id="page-3-1"></span>
$$
\mathbb{P}\left(\frac{S_n}{\langle S \rangle_n} \le -x\right) \le \inf_{p>1} \left( \mathbb{E}\left[\exp\left\{-\left(p-1\right)\frac{x^2}{2(1+x/3)}\langle S \rangle_n\right\}\right] \right)^{1/p}.\tag{14}
$$

It seems that the bound [\(14\)](#page-3-1) is usually decreasing in p. For instance, consider the independent case. When  $(\xi_i)_{i=1,\dots,n}$  are independent random variables, we have  $\langle S \rangle_n = \text{Var}(S_n)$ , where  $\text{Var}(S_n)$  stands for the variance of  $S_n$ . Then the bound [\(14\)](#page-3-1) is decreasing in p.

For more exponential inequalities similar to that of Theorem [2.2,](#page-3-2) we refer to Theorem 1.3 of de la Peña [\[6](#page-15-0)]. In particular, de la Peña proved [\(13\)](#page-3-0) with  $p = 2$ . Moreover, de la Peña also proved the following Bernstein type exponential inequalities: for all  $x, y > 0$ ,

<span id="page-3-3"></span>
$$
\mathbb{P}\left(\frac{S_n}{\langle S \rangle_n} \le -x, \ \langle S \rangle_n \ge y\right) \ \le \ \exp\left\{-\left((1+x)\log(1+x)-x\right)y\right\} \tag{15}
$$

<span id="page-3-5"></span><span id="page-3-4"></span>
$$
\leq \exp\bigg\{-\frac{x^2y}{2(1+x/3)}\bigg\}.\tag{16}
$$

It is easy to see that the inequalities [\(15\)](#page-3-3) and [\(16\)](#page-3-3) are respectively the counterparts of [\(10\)](#page-2-1) and [\(11\)](#page-2-1) for  $S_n/\langle S \rangle_n$ .

Notice that in the independent case, the bounds [\(14\)](#page-3-1) and [\(16\)](#page-3-3) are exactly Bernstein's bound [\(1\)](#page-0-0). Thus [\(14\)](#page-3-1) and [\(16\)](#page-3-3) can be regarded as Bernstein type inequalities for martingales.

The following deviation inequality for self-normalized martingales has its independent interest.

**Theorem 2.3.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. Then for all*  $b > 0, M \geq 1$  *and*  $x > 0$ *,* 

$$
\mathbb{P}\left(\frac{S_n}{\sqrt{S_n}} \ge x, \, b \le \sqrt{S_n} \le bM\right) \le \sqrt{e}\left(1 + 2(1+x)\ln M\right) \exp\left\{-\frac{x^2}{2(1+x/b)}\right\}.\tag{17}
$$

Similarly, when  $[S]_n$  in the left hand side of [\(17\)](#page-3-4) is replaced by  $\langle S \rangle_n$ , we have the following inequality for normalized martingales. Such type inequalities are due to Liptser and Spokoiny [\[19](#page-15-11)].

**Theorem 2.4.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. Then for all*  $b > 0, M \geq 1$  *and*  $x > 0$ *,* 

<span id="page-4-0"></span>
$$
\mathbb{P}\left(\frac{S_n}{\sqrt{\langle S \rangle_n}} \le -x, \, b \le \sqrt{\langle S \rangle_n} \le bM\right) \le \sqrt{e} \Big(1 + 2(1+x)\ln M\Big) \exp\bigg\{-\frac{x^2}{2(1+x/(3b))}\bigg\}.\tag{18}
$$

It is interesting to see that in the independent case, inequality [\(18\)](#page-4-0) with  $b = \sqrt{\text{Var}(S_n)}$  and  $M = 1$ reduces to exactly Bennett's inequality, up to an absolute constant  $\sqrt{e}$ . Thus the bound [\(18\)](#page-4-0) is rather tight.

#### 3. Applications

#### *3.1. Self-scaling sums*

As an application of Theorem [2.1,](#page-2-2) we consider the self-normalized sums of i.i.d. random variables.

**Theorem 3.1.** *Assume that*  $(\xi_i)_{i\geq 1}$  *are i.i.d. random variables. Assume that* 

<span id="page-4-2"></span><span id="page-4-1"></span>
$$
\xi_1 \ge -1 \quad \text{and} \quad \mathbb{E}\xi_1^{2p} < \infty,
$$

where  $1 < p \le 2$ . *Denote*  $\sigma^2 = \mathbb{E}\xi_1^2$ . *Then for all*  $x > 0$  *and*  $y \in (0, \sigma^2)$ ,

$$
\mathbb{P}\bigg(\frac{S_n}{[S]_n} \ge x\bigg) \le \exp\bigg\{-\frac{x^2(\sigma^2 - y)}{2(1 + x)}n\bigg\} + \exp\bigg\{-\frac{1}{4}\frac{(p - 1)y^{p/(p-1)}}{(\mathbb{E}\xi_1^{2p})^{1/(p-1)}}n\bigg\}.
$$

*In particular, it implies that for all*  $x \in (0, 1)$ ,

$$
\mathbb{P}\bigg(\frac{S_n}{[S]_n} \ge x\bigg) \le \exp\bigg\{-\frac{\sigma^2 x^2}{2(1+2x^{(p-1)/p})}n\bigg\} + \exp\bigg\{-\frac{(p-1)x}{4(\mathbb{E}\xi_1^{2p}/\sigma^{2p})^{1/(p-1)}}n\bigg\}.
$$

By the last theorem, we have the following moderate deviation result: for any  $x > 0$  and  $\alpha \in (0, \frac{1}{2})$  $(\frac{1}{2}),$ 

$$
\lim_{n \to \infty} \frac{1}{n^{1-2\alpha}} \log \mathbb{P}\left(\frac{S_n}{[S]_n} \ge \frac{x}{n^{\alpha}}\right) \le -\frac{1}{2}\sigma^2 x^2.
$$

For more such type moderate deviation results, we refer to Shao [\[21\]](#page-15-12) and Jing et al. [\[16](#page-15-13)], where the authors established the moderate deviation principles for self-normalized sums  $S_n/\sqrt{[S]_n}$ .

#### *3.2. Student's t-statistics*

Consider Student's t-statistic  $T_n$  defined by

$$
T_n = \sqrt{n} \,\bar{\xi} / \left(\frac{1}{n-1} \sum_{j=1}^n (\xi_j - \bar{\xi})^2\right)^{1/2},
$$

where  $\bar{\xi} = \sum_{i=1}^n \xi_i/n$ . Clearly,  $T_n$  and  $S_n/\sqrt{[S]_n}$  are closely related via the following identity:

<span id="page-5-1"></span><span id="page-5-0"></span>
$$
T_n = \frac{S_n}{\sqrt{[S]_n}} \left( \frac{n-1}{n - (S_n/\sqrt{[S]_n})^2} \right)^{1/2}.
$$
 (19)

Since  $x/(n-x^2)^{1/2}$  is increasing on  $(-\sqrt{n}, \sqrt{n})$ , it follows from [\(19\)](#page-5-0) that

$$
\{T_n \ge x\} = \left\{\frac{S_n}{\sqrt{[S]_n}} \ge x \left(\frac{n}{n+x^2-1}\right)^{1/2}\right\}.
$$
 (20)

The above fact was pointed out by Efron [\[11\]](#page-15-14). With the help of [\(20\)](#page-5-1), the following large deviation type result for t-statistic is an immediate consequence of Theorem [2.3.](#page-3-5)

**Theorem 3.2.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. Then for all*  $b > 0, M \geq 1$  *and*  $x > 0$ *,* 

$$
\mathbb{P}(T_n \ge x, b \le \sqrt{[S]_n} \le bM)
$$
  
\n
$$
\le \sqrt{e} \left( 1 + 2x \left( \frac{n}{n+x^2-1} \right)^{1/2} \ln M \right) \exp \left\{ - \frac{x^2 \left( \frac{n}{n+x^2-1} \right)}{2 \left( 1 + x \left( \frac{n}{n+x^2-1} \right)^{1/2} / b \right)} \right\}.
$$
 (21)

#### *3.3. Autoregressive processes*

The model of autoregressive process can be expressed as follows: for all  $n \geq 0$ , by

<span id="page-5-4"></span>
$$
X_{n+1} = \theta X_n + \varepsilon_{n+1} \tag{22}
$$

where  $X_n$  and  $\varepsilon_n$  are the observations and driven noises, respectively. We assume that  $(\varepsilon_n)$  is a sequence of i.i.d. centered random variables with variation  $\sigma^2 > 0$  and  $X_0 = \varepsilon_0$ . We can estimate the unknown parameter  $\theta$  by the least-squares estimator given by, for all  $n \geq 1$ ,

<span id="page-5-5"></span>
$$
\hat{\theta}_n = \frac{\sum_{k=1}^n X_{k-1} X_k}{\sum_{k=1}^n X_{k-1}^2} \tag{23}
$$

Bercu and Touati [\[2](#page-15-1)] has established the convergence rate of  $\hat{\theta}_n - \theta$  when  $X_0$  and  $(\varepsilon_n)$  are normal random variables. Here, we would like to give a convergence rate of  $\hat{\theta}_n - \theta$  for the case that the driven noises  $(\varepsilon_n)$  are bounded. Applying Theorem [2.2](#page-3-2) and de la Peña's inequality [\(16\)](#page-3-3), we have the following exponential inequalities.

**Theorem 3.3.** Assume that  $|\varepsilon_i| \leq C$  for some positive constant C and all i. If  $|\theta| < 1$ , then for all  $x > 0$ ,

<span id="page-5-3"></span>
$$
\mathbb{P}\left(|\hat{\theta}_n - \theta| \ge x\right) \le 2 \inf_{p>1} \left( \mathbb{E}\left[\exp\left\{- (p-1) \frac{x^2}{2(\sigma^2 + \frac{xC^2}{3(1-|\theta|)})} \sum_{k=1}^n X_{k-1}^2 \right\} \right] \right)^{1/p},\tag{24}
$$

*and, for all*  $x, y > 0$ *,* 

<span id="page-5-2"></span>
$$
\mathbb{P}\left(|\hat{\theta}_n - \theta| \ge x, \sum_{k=1}^n X_{k-1}^2 \ge y\right) \le 2 \exp\left\{-\frac{x^2 y}{2(\sigma^2 + \frac{x C^2}{3(1 - |\theta|)})}\right\}.
$$
 (25)

Inequality  $(25)$  is similar to an exponential inequalities of de la Peña, Klass and Lai [\[7](#page-15-8)], which states that when  $(\varepsilon_n)$  are the standard normal random variables, it holds for all  $x, y > 0$ ,

<span id="page-6-1"></span>
$$
\mathbb{P}\bigg(|\hat{\theta}_n - \theta| \ge x, \sum_{k=1}^n X_{k-1}^2 \ge y\bigg) \le 2 \exp\bigg\{-\frac{1}{2}x^2y\bigg\}.
$$
 (26)

By Theorem [2.4,](#page-4-1) we obtain the following result.

**Theorem 3.4.** Assume that  $|\varepsilon_i| \leq C$  for some positive constant C and all i. If  $|\theta| < 1$ , then for all  $b > 0, M \ge 1$  *and*  $x > 0$ *,* 

$$
\mathbb{P}\left(|\widehat{\theta}_n - \theta|\sqrt{\Sigma_{k=1}^n X_{k-1}^2} \ge x, \ b \le \sqrt{\Sigma_{k=1}^n X_{k-1}^2} \le bM\right)
$$
  

$$
\le 2\sqrt{e}\left(1 + 2(1 + \frac{x}{\sigma})\ln M\right) \exp\left\{-\frac{x^2}{2(\sigma^2 + \frac{xC^2}{3(1-|\theta|)})}\right\}.
$$
 (27)

### 4. Proofs of Theorems

#### *4.1. Preliminary lemmas*

The following technical lemma is from Fan et al. [\[13](#page-15-15)]. For reader's convenience, we shall give a proof following [\[13](#page-15-15)].

**Lemma 4.1.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. For any*  $\lambda \in [0, 1)$ *, denote* 

<span id="page-6-0"></span>
$$
U_n(\lambda) = \exp \left\{ \lambda S_n + (\lambda + \log(1 - \lambda)) [S]_n \right\}.
$$

*Then*  $(U_i(\lambda), \mathcal{F}_i)_{i=0,\dots,n}$  *is a supermartingale, and satisfies that for all*  $\lambda \in [0,1)$ *,* 

$$
\mathbb{E}\big[U_n(\lambda)\big] \le 1. \tag{28}
$$

Proof. Assume  $\xi_i \geq -1$  and  $\lambda \in [0, 1)$ , then  $\lambda \xi_i \geq -\lambda > -1$ . We consider the function

$$
f(x) = \frac{\log(1+x) - x}{x^2/2}, \quad x > -1,
$$

it is increasing in  $x$ , we obtain that

$$
\log(1 + \lambda \xi_i) \geq \lambda \xi_i + \frac{1}{2} (\lambda \xi_i)^2 f(-\lambda)
$$
  
=  $\lambda \xi_i + \xi_i^2 (\lambda + \log(1 - \lambda)).$ 

Therefore, we have

$$
\exp\left\{\lambda \xi_i + \xi_i^2 (\lambda + \log(1 - \lambda))\right\} \le 1 + \lambda \xi_i.
$$

Since  $\mathbb{E}\xi_i = 0$ , it follows that

$$
\mathbb{E}\Big[\exp\big\{\lambda\xi_i+(\lambda+\log(1-\lambda))\xi_i^2\big\}\Big]\leq 1.
$$

For all  $\lambda \in [0,1)$  and  $n \geq 0$ , we have

$$
U_n(\lambda) = U_{n-1}(\lambda) \exp \left\{ \lambda \xi_n + (\lambda + \log(1 - \lambda)) \xi_n^2 \right\}.
$$

Hence, we deduce that for all  $\lambda \in [0,1)$ ,

$$
\mathbb{E}[U_n(\lambda)|\mathcal{F}_{n-1}] = U_{n-1}(\lambda)\mathbb{E}\Big[\exp\Big\{\lambda\xi_n + (\lambda + \log(1-\lambda))\xi_n^2\Big\}\Big|\mathcal{F}_{n-1}\Big]\Big\}\leq U_{n-1}(\lambda),
$$

which means  $(U_i(\lambda), \mathcal{F}_i)_{i=0,\dots,n}$  is a positive supermartingale. Moreover, it holds

$$
\mathbb{E}[U_n(\lambda)] \le \mathbb{E}[U_{n-1}(\lambda)] \le \dots \le \mathbb{E}[U_1(\lambda)] \le 1.
$$

This completes the proof of Lemma [4.1.](#page-6-0)  $\Box$ 

In order to prove Theorem [2.2,](#page-3-2) we need the following lemma of Freedman [\[15\]](#page-15-3).

**Lemma 4.2.** *Assume that*  $\xi_i \geq -1$  *for all*  $i \in [1, n]$ *. Denote* 

$$
W_n(\lambda) = \exp\Big\{-\lambda S_n - (e^{\lambda} - 1 - \lambda)\langle S \rangle_n\Big\}, \quad \lambda \ge 0.
$$

*Then*  $(W_i(\lambda), \mathcal{F}_i)_{i=0,\dots,n}$  *is a supermartingale, and satisfies that* 

<span id="page-7-2"></span>
$$
\mathbb{E}\big[W_n(\lambda)\big] \le 1. \tag{29}
$$

## *4.2. Proof of Theorem [2.1](#page-2-2)*

We follow the method of Bercu and Touati [\[2](#page-15-1)]. Let  $A_n = \{S_n \geq x[S]_n\}, x > 0$ . By Markov's inequality, Hölder's inequality and Lemma [4.1,](#page-6-0) we have for all  $\lambda \in [0,1)$  and  $q > 1$ ,

<span id="page-7-0"></span>
$$
\mathbb{P}(A_n) \leq \mathbb{E}\bigg[\exp\bigg\{\frac{\lambda}{q}\Big(S_n - x[S]_n\Big)\bigg\}1_{A_n}\bigg]
$$
\n
$$
= \mathbb{E}\bigg[\exp\bigg\{\frac{1}{q}\Big(\lambda S_n + (\lambda + \log(1 - \lambda))[S]_n\Big)\bigg\}\exp\bigg\{\frac{1}{q}\Big(-\lambda - \log(1 - \lambda) - \lambda x\Big)[S]_n\bigg\}1_{A_n}\bigg]
$$
\n
$$
\leq \left(\mathbb{E}\bigg[\exp\bigg\{\frac{p}{q}\Big(-\lambda - \log(1 - \lambda) - \lambda x\Big)[S]_n\bigg\}1_{A_n}\bigg]\right)^{1/p}\bigg(\mathbb{E}\big[U_n(\lambda)\big]\bigg)^{1/q}
$$
\n
$$
\leq \left(\mathbb{E}\bigg[\exp\bigg\{-\frac{p}{q}\Big(\lambda + \log(1 - \lambda) + \lambda x\Big)[S]_n\bigg\}1_{A_n}\bigg]\right)^{1/p},\tag{30}
$$

where  $p = 1 + p/q$ . Consequently, as  $p/q = p - 1$ , we can deduce from [\(30\)](#page-7-0) that

$$
\mathbb{P}(A_n) \leq \inf_{p>1} \left( \mathbb{E} \bigg[ \exp \bigg\{ -(p-1)(\lambda + \log(1-\lambda) + \lambda x)[S]_n \bigg\} \mathbf{1}_{A_n} \bigg] \right)^{1/p}.
$$

The right hand side of the last inequality attains its minimum at

$$
\underline{\lambda} = \lambda(x) = \frac{x}{1+x},
$$

<span id="page-7-1"></span>

therefore we obtain

$$
\mathbb{P}(A_n) \leq \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ -(p-1)(x - \log(1+x))[S]_n \right\} \right] \mathbf{1}_{A_n} \right)^{1/p}.
$$

Using the following inequality

<span id="page-8-0"></span>
$$
x - \log(1+x) \ge \frac{x^2}{2(1+x)}, \quad x > 0,\tag{31}
$$

we deduce that

$$
\mathbb{P}(A_n) \leq \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ -(p-1)(x - \log(1+x)) [S]_n \right\} \mathbf{1}_{A_n} \right] \right)^{1/p}
$$
  

$$
\leq \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ -(p-1) \frac{x^2}{2(1+x)} [S]_n \right\} \mathbf{1}_{A_n} \right] \right)^{1/p},
$$

which gives the first two desired inequalities.  $\hfill \Box$ 

Next we prove the last two desired inequalities. Denote  $B_n = \{S_n \geq x[S]_n, [S]_n \geq y\}$ . By an argument similar to the proof of [\(30\)](#page-7-0), we deduce that for all  $q > 1$ ,

$$
\mathbb{P}(B_n) \leq \left( \mathbb{E} \left[ \exp \left\{ \frac{p}{q} \left( -\underline{\lambda} - \log(1 - \underline{\lambda}) - \underline{\lambda} x \right) [S]_n \right\} \mathbf{1}_{B_n} \right] \right)^{1/p} \left( \mathbb{E} [U_n(t)] \right)^{1/q}
$$
  

$$
\leq \left( \mathbb{E} \left[ \exp \left\{ -\frac{p}{q} \left( \underline{\lambda} + \log(1 - \underline{\lambda}) + \underline{\lambda} x \right) y \right\} \right] \right)^{1/p}
$$
  

$$
= \exp \left\{ -\frac{p-1}{p} \left( x - \log(1+x) \right) y \right\}.
$$

Therefore, by [\(31\)](#page-8-0), it holds

$$
\mathbb{P}(B_n) \leq \inf_{p>1} \exp \left\{ -\frac{p-1}{p} \left( x - \log(1+x) \right) y \right\}
$$

$$
= \exp \left\{ -\left( x - \log(1+x) \right) y \right\}
$$

$$
\leq \exp \left\{ -\frac{x^2 y}{2(1+x)} \right\},
$$

which gives the last two desired inequalities.

*4.3. Proof of Theorem [2.2](#page-3-2)*

For all  $x > 0$ , denote by

$$
D_n = \{-S_n \ge x \langle S \rangle_n\}.
$$

By exponential Markov's inequality, we deduce that for all  $\lambda \in [0,3)$  and  $q > 1$ ,

$$
\mathbb{P}(D_n) \leq \mathbb{E}\bigg[\exp\bigg\{\frac{\lambda}{q}\bigg(-S_n - x\langle S \rangle_n\bigg)\bigg\}\mathbf{1}_{D_n}\bigg] \n= \mathbb{E}\bigg[\exp\bigg\{-\frac{\lambda}{q}S_n - \frac{e^{\lambda}-1-\lambda}{q}\langle S \rangle_n\bigg\}\exp\bigg\{\bigg(\frac{e^{\lambda}-1-\lambda}{q}-\frac{\lambda x}{q}\bigg)\langle S \rangle_n\bigg\}\mathbf{1}_{D_n}\bigg].
$$

Using Hölder's inequality and Lemma [4.2,](#page-7-1) we have for all  $\lambda \in [0,3)$  and  $q > 1$ ,

<span id="page-9-0"></span>
$$
\mathbb{P}(D_n) \leq \left( \mathbb{E} \left[ \exp \left\{ \left( \frac{p(e^{\lambda} - 1 - \lambda)}{q} - \frac{p\lambda x}{q} \right) \langle S \rangle_n \right\} \mathbf{1}_{D_n} \right] \right)^{1/p} \left( \mathbb{E} \left[ W_n(t) \right] \right)^{1/q} \leq \left( \mathbb{E} \left[ \exp \left\{ \frac{p}{q} \left( e^{\lambda} - 1 - \lambda - \lambda x \right) \langle S \rangle_n \right\} \mathbf{1}_{D_n} \right] \right)^{1/p}, \tag{32}
$$

where  $p = 1 + p/q$ . Consequently, as  $p/q = p - 1$ , we can deduce from [\(32\)](#page-9-0) that

<span id="page-9-1"></span>
$$
\mathbb{P}(D_n) \le \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ (p-1) \left( e^{\lambda} - 1 - \lambda - \lambda x \right) \langle S \rangle_n \right\} \mathbf{1}_{D_n} \right] \right)^{1/p} . \tag{33}
$$

The right hand side of the last inequality attains its minimum at

$$
\overline{\lambda} = \lambda(x) := \log(1+x).
$$

Substituting  $\overline{\lambda} = \lambda(x)$  in [\(33\)](#page-9-1), we obtain

<span id="page-9-2"></span>
$$
\mathbb{P}(D_n) \le \inf_{p>1} \left( \mathbb{E} \bigg[ \exp \bigg\{ -(p-1) \Big( (1+x) \log(1+x) - x \Big) \langle S \rangle_n \bigg\} \mathbf{1}_{D_n} \bigg] \right)^{1/p} . \tag{34}
$$

Using the inequality

$$
e^{\lambda} - 1 - \lambda \le \frac{\lambda^2}{2(1 - \lambda/3)}, \quad \lambda \in [0, 3), \tag{35}
$$

we get for all  $x\geq 0,$ 

$$
(1+x)\log(1+x) - x = \inf_{\lambda \ge 0} \left( e^{\lambda} - 1 - \lambda - \lambda x \right)
$$
  

$$
\le \inf_{\lambda \ge 0} \left( \frac{\lambda^2}{2(1-\lambda/3)} - \lambda x \right)
$$
  

$$
= \exp \left\{ -\frac{x^2}{1+x/3 + \sqrt{1+2x/3}} \right\}
$$
  

$$
\le \exp \left\{ -\frac{x^2}{2(1+x/3)} \right\},
$$

where the last inequality follows from the fact  $\sqrt{1 + 2x/3} \le 1 + x/3$ . Thus, from [\(34\)](#page-9-2), we obtain for all  $x \geq 0$ ,

$$
\mathbb{P}(D_n) \leq \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ -(p-1) \left( (1+x) \log(1+x) - x \right) \langle S \rangle_n \right\} \mathbf{1}_{D_n} \right] \right)^{1/p}
$$
  
\n
$$
\leq \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ -(p-1) \frac{x^2}{1 + x/3 + \sqrt{1 + 2x/3}} \langle S \rangle_n \right\} \mathbf{1}_{D_n} \right] \right)^{1/p}
$$
  
\n
$$
\leq \inf_{p>1} \left( \mathbb{E} \left[ \exp \left\{ -(p-1) \frac{x^2}{2(1 + x/3)} \langle S \rangle_n \right\} \mathbf{1}_{D_n} \right] \right)^{1/p}.
$$
  
\nThis proves (12) and (14).

## *4.4. Proof of Theorem [2.3](#page-3-5)*

The proof of Theorem [2.3](#page-3-5) is based on a modified method of Liptser and Spokoiny [\[19\]](#page-15-11). Given  $a > 1$ , introduce the geometric series  $b_k = ba^k$  and define random events

$$
C_k = \left\{ \frac{S_n}{\sqrt{[S]_n}} \ge x, \ b_k \le \sqrt{[S]_n} < b_{k+1} \right\}, \quad k = 0, 1, \dots, K,
$$

where K stands for the integer part of  $\log_a M$ . Clearly, it holds

$$
\left\{\frac{S_n}{\sqrt{[S]_n}} \ge x, \ b \le \sqrt{[S]_n} \le bM\right\} \subseteq \bigcup_{k=0}^K C_k,\tag{36}
$$

which leads to

<span id="page-10-0"></span>
$$
\mathbb{P}\left(\frac{S_n}{\sqrt{[S]_n}} \ge x, \ b \le \sqrt{[S]_n} \le bM\right) \le \sum_{k=0}^K \mathbb{P}(C_k). \tag{37}
$$

Notice that

$$
\lambda + \log(1 - \lambda) \ge -\frac{\lambda^2}{2(1 - \lambda)}, \quad \lambda \in [0, 1).
$$

For any  $\lambda \in [0, 1)$ , the last inequality and [\(29\)](#page-7-2) together implies that

$$
\mathbb{E}\Big[\exp\Big\{\lambda S_n - \frac{\lambda^2}{2(1-\lambda)}[S]_n\Big\}\mathbf{1}_{C_k}\Big] \le 1.
$$

Next, taking  $\lambda_k = x/(x + b_k)$ , for any  $x > 0$ , we obtain

$$
1 \geq \mathbb{E}\bigg[\exp\bigg\{\frac{x}{x+b_k}S_n - \frac{x^2}{2b_k(x+b_k)}[S]_n\bigg\}\mathbf{1}_{C_k}\bigg]
$$
  
\n
$$
\geq \mathbb{E}\bigg[\exp\bigg\{\frac{x^2}{x+b_k}\sqrt{[S]_n} - \frac{x^2}{2b_k(x+b_k)}[S]_n\bigg\}\mathbf{1}_{C_k}\bigg]
$$
  
\n
$$
\geq \mathbb{E}\bigg[\exp\bigg\{\inf_{b_k\leq c  
\n
$$
\geq \mathbb{E}\bigg[\exp\bigg\{\frac{x^2b_{k+1}}{x+b_k} - \frac{x^2b_{k+1}^2}{2b_k(x+b_k)}\bigg\}\mathbf{1}_{C_k}\bigg],
$$
$$

which implies that

$$
\mathbb{P}(C_k) \le \exp\left\{-\frac{x^2}{1+x/b_k}\left(a-\frac{a^2}{2}\right)\right\}
$$
  
 
$$
\le \exp\left\{-\frac{x^2}{1+x/b}\left(a-\frac{a^2}{2}\right)\right\}.
$$

Finally, we may pick a to make the right-hand side of the last bound possibly small. This leads to the choice  $a = 1 + \frac{1}{1+x}$ , so that

$$
x^{2}\left(a-\frac{a^{2}}{2}\right) = x^{2}\left\{1+\frac{1}{1+x} - \frac{1}{2}\left(1+\frac{1}{1+x}\right)^{2}\right\} \ge \frac{1}{2}(x^{2}-1).
$$

Since  $\log(1+\frac{1}{1+x}) \geq \frac{1}{2(1+x)}$  $\frac{1}{2(1+x)}$  for  $x \ge 0$ , we obtain  $\log_a M \le 2(1+x) \ln M$  and [\(17\)](#page-3-4) follows by [\(37\)](#page-10-0).  $\Box$ 

# *4.5. Proof of Theorem [2.4](#page-4-1)*

The proof of Theorem [2.4](#page-4-1) is similar to the proof of Theorem [2.3.](#page-3-5) Given  $a > 1$ , introduce the geometric series  $b_k = ba^k$  and define random events

$$
H_k = \left\{ \frac{-S_n}{\sqrt{\langle S \rangle_n}} \ge x, \ b_k \le \sqrt{\langle S \rangle_n} < b_{k+1} \right\}, \quad k = 0, 1, \dots, K,
$$

where K stands for the integer part of  $\log_a M$ . Clearly, it holds

$$
\left\{\frac{-S_n}{\sqrt{\langle S \rangle_n}} \ge x, \ b \le \sqrt{\langle S \rangle_n} \le bM\right\} \subseteq \bigcup_{k=0}^K H_k,\tag{38}
$$

which leads to

<span id="page-11-0"></span>
$$
\mathbb{P}\left(\frac{-S_n}{\sqrt{\langle S \rangle_n}} \ge x, \ b \le \sqrt{\langle S \rangle_n} \le bM\right) \le \sum_{k=0}^K \mathbb{P}(H_k). \tag{39}
$$

Notice that

$$
e^{\lambda}-1-\lambda \ge \frac{\lambda^2}{2(1-\lambda/3)}, \quad \lambda \in [0,3).
$$

For any  $\lambda \in [0, 3)$ , the last inequality and Lemma [4.2](#page-7-1) together implies that

$$
\mathbb{E}\Big[\exp\Big\{\lambda(-S_n)-\frac{\lambda^2}{2(1-\lambda/3)}\langle S\rangle_n\Big\}\mathbf{1}_{H_k}\Big]\leq 1.
$$

Next, taking  $\lambda_k = x/(b_k + x/3)$ , for any  $x > 0$ , we obtain

$$
1 \geq \mathbb{E}\bigg[\exp\bigg\{\frac{x}{b_k + x/3}(-S_n) - \frac{x^2}{b_k(b_k + x/3)}\langle S\rangle_n\bigg\}\mathbf{1}_{H_k}\bigg]
$$
  
\n
$$
\geq \mathbb{E}\bigg[\exp\bigg\{\frac{x^2}{b_k + x/3}\sqrt{\langle S\rangle_n} - \frac{x^2}{b_k(b_k + x/3)}\langle S\rangle_n\bigg\}\mathbf{1}_{H_k}\bigg]
$$
  
\n
$$
\geq \mathbb{E}\bigg[\exp\bigg\{\inf_{b_k \leq c < b_{k+1}}\bigg(\frac{x^2c}{b_k + x/3} - \frac{x^2c^2}{b_k(b_k + x/3)}\bigg)\bigg\}\mathbf{1}_{H_k}\bigg]
$$
  
\n
$$
\geq \mathbb{E}\bigg[\exp\bigg\{\frac{x^2b_{k+1}}{b_k + x/3} - \frac{x^2b_{k+1}^2}{b_k(b_k + x/3)}\bigg\}\mathbf{1}_{H_k}\bigg],
$$

which implies that

$$
\mathbb{P}(H_k) \le \exp\left\{-\frac{x^2}{1+\frac{x}{3b_k}}a + \frac{x^2}{(1+\frac{x}{3b_k})}\frac{a^2}{2}\right\} \n\le \exp\left\{-\frac{x^2}{1+\frac{x}{3b_k}}\left(a-\frac{a^2}{2}\right)\right\} \n\le \exp\left\{-\frac{x^2}{1+\frac{x}{3b}}\left(a-\frac{a^2}{2}\right)\right\}.
$$

Finally, taking  $a = 1 + \frac{1}{1+x}$ , we obtain the desired inequality from [\(39\)](#page-11-0), with an argument similar to the proof of Theorem [2.3.](#page-3-5)<sup> $1^{\mu}$ </sup>

#### *4.6. Proof of Theorem [3.1](#page-4-2)*

In the proof of Theorem [3.1,](#page-4-2) we need the following lemma of Chen et al. [\[5\]](#page-15-16).

**Lemma 4.3.** Let  $(\zeta_i)_{i\geq 1}$  be independent nonnegative random variables with  $\mathbb{E}\zeta_i^p < \infty$ , where  $1 < p \leq$ 2. Then for all  $0 < y < \sum_{i=1}^n \mathbb{E}\zeta_i$ 

<span id="page-12-0"></span>
$$
\mathbb{P}\bigg(\sum_{i=1}^{n} \zeta_i \le \sum_{i=1}^{n} \mathbb{E}\zeta_i - y\bigg) \le \exp\bigg\{-\frac{(p-1)y^{p/(p-1)}}{4(\sum_{i=1}^{n} \mathbb{E}\zeta_i^p)^{1/(p-1)}}\bigg\}.
$$
\n(40)

Now, we are in position to prove Theorem [3.1.](#page-4-2) For all  $x > 0$  and  $y \in (0, \sigma^2)$ , we have

<span id="page-12-1"></span>
$$
\mathbb{P}\left(\frac{S_n}{[S]_n} \ge x\right) = \mathbb{P}\left(\frac{S_n}{[S]_n} \ge x, [S]_n \ge n(\sigma^2 - y)\right) + \mathbb{P}\left(\frac{S_n}{[S]_n} \ge x, [S]_n < n(\sigma^2 - y)\right) \le \exp\left\{-\frac{x^2(\sigma^2 - y)}{2(1 + x)}n\right\} + \mathbb{P}\left([S]_n - n\sigma^2 < -ny\right). \tag{41}
$$

By Lemma [4.3,](#page-12-0) we have for all  $y \in (0, \sigma^2)$ ,

<span id="page-12-2"></span>
$$
\mathbb{P}\left([S]_n - n\sigma^2 < -ny\right) \le \exp\left\{-\frac{(p-1)(ny)^{p/(p-1)}}{4(n\mathbb{E}\xi_1^{2p})^{1/(p-1)}}\right\}
$$
\n
$$
= \exp\left\{-\frac{1}{4}(p-1)\frac{y^{p/(p-1)}}{(\mathbb{E}\xi_1^{2p})^{1/(p-1)}}n\right\}.
$$
\n(42)

Combining [\(41\)](#page-12-1) and [\(42\)](#page-12-2) together, we obtain the first desired inequality. Taking  $y = x^{(p-1)/p} \sigma^2$ , we obtain the second desired inequality.

#### *4.7. Proof of Theorem [3.3](#page-5-3)*

By [\(22\)](#page-5-4), we have  $X_k = \sum_{i=0}^k \theta^{k-i} \varepsilon_i$ . Since  $|\theta| < 1$  and  $|\xi_i| \leq C$ , we deduce that for all k,

$$
|X_k| \le C \sum_{i=0}^k |\theta|^{k-i} \le \frac{C}{1-|\theta|}.
$$

From [\(22\)](#page-5-4) and [\(23\)](#page-5-5), it is easy to see that for all  $n \geq 1$ ,

$$
\hat{\theta}_n - \theta = \frac{\sum_{k=1}^n X_{k-1} \varepsilon_k}{\sum_{k=1}^n X_{k-1}^2}.
$$
\n(43)

For any  $i = 1, \ldots, n$ , set

$$
\xi_i = X_{i-1}\varepsilon_i(1-|\theta|)/C^2
$$
 and  $\mathcal{F}_i = \sigma(\varepsilon_k, 0 \le k \le i)$ .

Then  $(\xi_i, \mathcal{F}_i)_{i=1,\dots,n}$  is a sequence of martingale differences and satisfies

$$
|\xi_i| \leq 1
$$

and

$$
\langle S \rangle_n = \sum_{k=1}^n \mathbb{E} \big[ \xi_i^2 \big| \mathcal{F}_{i-1} \big] = \frac{\sigma^2 (1 - |\theta|)^2}{C^4} \sum_{k=1}^n X_{i-1}^2.
$$

Thus we have

$$
\hat{\theta}_n - \theta = \frac{(1 - |\theta|)\sigma^2}{C^2} \frac{S_n}{\langle S \rangle_n}.
$$

Applying inequality [\(14\)](#page-3-1) to  $(\xi_i, \mathcal{F}_i)_{i=1,\dots,n}$ , we deduce that for all  $x > 0$ ,

<span id="page-13-0"></span>
$$
\mathbb{P}\left(\hat{\theta}_n - \theta \leq -x\right) = \mathbb{P}\left(\frac{S_n}{\langle S \rangle_n} \leq -\frac{xC^2}{(1-|\theta|)\sigma^2}\right)
$$
\n
$$
\leq \inf_{p>1} \left(\mathbb{E}\left[\exp\left\{-\left(p-1\right)\frac{x^2}{2(\sigma^2 + xC^2/(3(1-|\theta|)))}\sum_{k=1}^n X_{k-1}^2\right\}\right]\right)^{1/p}.\tag{44}
$$

Similarly, applying inequality [\(14\)](#page-3-1) to  $(-\xi_i, \mathcal{F}_i)_{i=1,\dots,n}$ , we have for all  $x > 0$ ,

<span id="page-13-1"></span>
$$
\mathbb{P}\left(\hat{\theta}_n - \theta \ge x\right) \le \inf_{p > 1} \left( \mathbb{E}\left[\exp\left\{-(p-1)\frac{x^2}{2(\sigma^2 + xC^2/(3(1-|\theta|)))}\sum_{k=1}^n X_{k-1}^2\right\} \right] \right)^{1/p}.\tag{45}
$$

Combining [\(44\)](#page-13-0) and [\(45\)](#page-13-1) together, we obtain for all  $x > 0$ ,

$$
\mathbb{P}\left(|\hat{\theta}_n - \theta| \geq x\right) \leq 2 \inf_{p>1} \left( \mathbb{E}\left[\exp\left\{-\left(p-1\right) \frac{x^2}{2(\sigma^2 + xC^2/(3(1-|\theta|)))} \sum_{k=1}^n X_{k-1}^2\right\} \right] \right)^{1/p},
$$

which gives the first desired inequality. Applying de la Peña's inequality [\(16\)](#page-3-3) to  $(\xi_i, \mathcal{F}_i)_{i=1,\dots,n}$ , we get for all  $x, y > 0$ ,

<span id="page-13-2"></span>
$$
\mathbb{P}\left(\hat{\theta}_n - \theta \le -x, \sum_{k=1}^n X_{k-1}^2 \ge y\right) = \mathbb{P}\left(\frac{S_n}{\langle S \rangle_n} \le -\frac{xC^2}{(1-|\theta|)\sigma^2}, \langle S \rangle_n \ge y\frac{(1-|\theta|)^2\sigma^2}{C^4}\right)
$$

$$
\le \exp\left\{-\frac{x^2y}{2(\sigma^2 + xC^2/(3(1-|\theta|)))}\right\}.
$$
(46)

Similarly, applying de la Peña's inequality [\(16\)](#page-3-3) to  $(-\xi_i, \mathcal{F}_i)_{i=1,\dots,n}$ , we have for all  $x, y > 0$ ,

<span id="page-14-0"></span>
$$
\mathbb{P}\left(\hat{\theta}_n - \theta \ge x, \ \sum_{k=1}^n X_{k-1}^2 \ge y\right) \le \exp\left\{-\frac{x^2y}{2(\sigma^2 + xC^2/(3(1-|\theta|)))}\right\}.
$$
 (47)

Combining [\(46\)](#page-13-2) and [\(47\)](#page-14-0) together, we obtain for all  $x, y > 0$ ,

$$
\mathbb{P}\bigg(|\hat{\theta}_n-\theta|\geq x,\ \sum_{k=1}^n X_{k-1}^2\geq y\bigg)\leq 2\exp\bigg\{-\frac{x^2y}{2(\sigma^2+xC^2/(3(1-|\theta|)))}\bigg\},\
$$

which gives the second desired inequality.  $\Box$ 

## *4.8. Proof of Theorem [3.4](#page-6-1)*

Recall the notations in the proof of Theorem [3.3.](#page-5-3) It is easy to see that

$$
(\widehat{\theta}_n - \theta) \sqrt{\Sigma_{k=1}^n X_{k-1}^2} = \frac{\sum_{k=1}^n X_{k-1} \varepsilon_k}{\sqrt{\sum_{k=1}^n X_{k-1}^2}} = \sigma \frac{S_n}{\sqrt{\langle S \rangle_n}}.
$$

Therefore, by Theorem [2.4,](#page-4-1) for all  $b > 0, M \ge 1$  and  $x > 0$ ,

$$
\mathbb{P}\left((\widehat{\theta}_{n}-\theta)\sqrt{\Sigma_{k=1}^{n}X_{k-1}^{2}}\leq-x,\ b\leq\sqrt{\Sigma_{k=1}^{n}X_{k-1}^{2}}\leq bM\right)
$$
  

$$
\leq\mathbb{P}\left(\frac{S_{n}}{\sqrt{\langle S\rangle_{n}}}\leq-\frac{x}{\sigma},\ b\frac{(1-|\theta|)\sigma}{C^{2}}\leq\sqrt{\langle S\rangle_{n}}\leq bM\frac{(1-|\theta|)\sigma}{C^{2}}\right)
$$
  

$$
\leq\sqrt{e}\left(1+2(1+\frac{x}{\sigma})\ln M\right)\exp\left\{-\frac{x^{2}}{2(\sigma^{2}+xC^{2}/(3b(1-|\theta|)))}\right\}
$$

Similarly, the same bound holds for the tail probabilities

$$
\mathbb{P}\bigg(\big(\widehat{\theta}_n - \theta\big)\sqrt{\Sigma_{k=1}^n X_{k-1}^2} \ge x, \ b \le \sqrt{\Sigma_{k=1}^n X_{k-1}^2} \le bM\bigg).
$$

Hence, we have for all  $b > 0, M \ge 1$  and  $x > 0$ ,

$$
\mathbb{P}\bigg(\big|\widehat{\theta}_n - \theta\big|\sqrt{\Sigma_{k=1}^n X_{k-1}^2} \ge x, \ b \le \sqrt{\Sigma_{k=1}^n X_{k-1}^2} \le bM\bigg)
$$
  

$$
\le 2\sqrt{e}\bigg(1 + 2(1 + \frac{x}{\sigma})\ln M\bigg)\exp\bigg\{-\frac{x^2}{2(\sigma^2 + xC^2/(3b(1-|\theta|)))}\bigg\},
$$

which gives the desired inequality.

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