

Semisimple pointed isogeny graphs for abelian varieties

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Abstract

In this paper we show that if $\phi_i : A_i \rightarrow A$ is a semisimple pointed K -rational ℓ -isogeny graph of order n for a prime ℓ , then the group of ℓ -torsion points $A[\ell](\overline{K})$ contains a subspace of dimension n generated by K -rational points. We also show that the same result is true for elliptic curves without the semisimplicity condition. Furthermore, we give an explicit counterexample for abelian varieties of higher dimension to show that the semisimplicity condition is indeed necessary.

1 Introduction

Let ℓ be a prime. Let A be an abelian variety over a field K of characteristic $p \neq \ell$ and let $G = \text{Gal}(K(A[\ell])/K)$. Suppose we have n isogenies $\phi_i : A_i \rightarrow A$ over K , whose kernels are generated by K -rational points of order ℓ . We refer to such a collection of isogenies as a K -rational pointed ℓ -isogeny graph. If the corresponding injections of function fields $K(A) \rightarrow K(A_i)$ are linearly disjoint, we say that the pointed isogeny graph is of order n .

Elliptic curves give rise to the simplest examples of nontrivial pointed isogeny graphs. Here we have the following theorem:

Theorem 3.1. *Let*

$$E_1 \xrightarrow{\phi_1} E \xleftarrow{\phi_2} E_2$$

be a K -rational pointed ℓ -isogeny graph of order two. Then $E[\ell](\overline{K}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$ as G_K -modules.

For abelian varieties of dimension $g > 1$, the situation is slightly more complicated. Here we will assume that the representation

$$\rho_\ell : G_K \rightarrow \text{GL}_{2g}(\mathbb{F}_\ell) \tag{1}$$

is *semisimple*, in the sense that G_K -invariant subspaces admit G_K -invariant complements. This condition is satisfied for instance when $\ell \nmid |G|$, where $G = G_K/\ker(\rho_\ell)$. In the semisimple case, we have the following theorem:

Theorem 4.1. *Let $\phi_i : A_i \rightarrow A$ be a semisimple pointed K -rational ℓ -isogeny graph of order n . There then exists an injection of G_K -modules*

$$(\mathbb{Z}/\ell\mathbb{Z})^n \rightarrow A[\ell](\overline{K}). \tag{2}$$

Here $(\mathbb{Z}/\ell\mathbb{Z})^n$ has the trivial G_K -action. The semisimplicity condition is indeed necessary as we will see in Section 4, where we give an explicit counterexample over \mathbb{Q} using products of elliptic curves.

The problem arose from considering descent problems and their applications to finding number fields K with a class group having a high ℓ -part (see [Hel14]). Given a K -rational point on A , one can pull this point back through all n isogenies. This gives n extensions and by class field theory one obtains a part of a generalized class group. Imposing some extra conditions can make most of these extensions unramified everywhere, which then gives subgroups of the ordinary class group.

The paper is structured as follows: we begin by introducing the necessary concepts in Section 2. After this, we will consider the case of elliptic curves in Section 3 and prove Theorem 3.1. In Section 4, we consider abelian varieties of dimension greater than one and prove Theorem 4.1. We will then give a counterexample to show that the semisimplicity condition in Theorem 4.1 is necessary.

2 Preliminaries

Let ℓ be a prime and let K be a field of characteristic $p \neq \ell$. Let K^s be a separable closure of K with Galois group $G_K := \text{Gal}(K^s/K)$. We refer the reader to [Mil08], [vdGM18] and [Mum12] for an introduction to abelian varieties. We will give a quick summary of the facts needed here.

An abelian variety over K is an algebraic group A that is geometrically integral and proper over K . Such a variety A is always commutative and projective, see [Mil08, Corollary 1.4 and Theorem 6.4]. For any $n \in \mathbb{Z}_{>0}$, we let $A[n]$ be the kernel of the multiplication map $[n] : A \rightarrow A$. We then have

Lemma 2.1. *Let A be an abelian variety of dimension g over K . If $p \nmid n$, then $A[n]$ is étale over K and $A[n](K^s) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.*

Proof. See [Liu06, Chapter 7, Theorem 4.38]. Note that the fact that $A[n]$ is étale over K implies that the geometric points $A[n](\overline{K})$ are defined over K^s . \square

We now specialize to a prime $n = \ell$ and we will use the notation $A[\ell](\overline{K})$, which is equal to $A[\ell](K^s)$ by Lemma 2.1. The group $A[\ell](\overline{K})$ is naturally an \mathbb{F}_ℓ -vector space of dimension $2g$ and for any subspace $V \subset A[\ell](\overline{K})$, we denote its dimension by $\dim(V)$.

An isogeny of abelian varieties is a homomorphism $\phi : A \rightarrow A'$ such that ϕ is finite and surjective. We define the *degree* of ϕ by $\deg(\phi) := |(\ker \phi)(\overline{K})|$. There is then a *dual* isogeny $\hat{\phi} : A' \rightarrow A$ such that $\hat{\phi} \circ \phi = [m]_A$. We will only be interested in separable isogenies in this paper. That is, isogenies for which the injection of function fields $K(A') \rightarrow K(A)$ is a separable field extension. For any K -rational point $P \in A[\ell](K)$, we consider the corresponding isogeny $\phi : A \rightarrow A / \langle P \rangle$ over K . Let us write $A' = A / \langle P \rangle$. Then ϕ induces a finite injection of function fields $K(A') \rightarrow K(A)$ of degree ℓ .

Definition 2.1. Let A_i/K be a finite collection of abelian varieties over K with K -rational points $P_i \in A_i[\ell](K)$ and assume that they induce isogenies to a single abelian variety A . That is, we have isogenies $\phi_i : A_i \rightarrow A$. We then say that they form a pointed K -rational ℓ -isogeny graph, or just: pointed isogeny graph.

Definition 2.2. Let $\phi_i : A_i \rightarrow A$ be a set of n isogenies forming a pointed K -rational ℓ -isogeny graph. We say that the pointed isogeny graph is of order n if the field extensions

$$K(A) \rightarrow K(A_i) \tag{3}$$

are linearly disjoint.

Remark 2.1. We have the following equivalent conditions for a pointed isogeny graph to be of order n :

1. The field $K_n := \prod_{i=1}^n K(A_i)$ has dimension ℓ^n as a vector space over $K(A)$.
2. The induced map $\bigotimes_{i=1}^n K(A_i) \rightarrow K_n$ is an injection.

We have a natural action of G_K on $A[\ell](\overline{K})$. This gives a representation

$$\rho_\ell : G_K \rightarrow \text{GL}_{2g}(\mathbb{F}_\ell). \tag{4}$$

We say that ρ_ℓ is *semisimple* if every G_K -invariant subspace $V \subset \mathbb{F}_\ell^{2g} = A[\ell](\overline{K})$ has a G_K -invariant complement. That is, there exists a G_K -invariant subspace V' such that $V \oplus V' = \mathbb{F}_\ell^{2g}$. Here G_K acts through the representation ρ_ℓ .

Definition 2.3. A pointed isogeny graph $\phi_i : A_i \rightarrow A$ is said to be *semisimple* if the representation ρ_ℓ is semisimple.

Remark 2.2. The representation is semisimple for example when $\ell \nmid |G|$, where $G = \text{Gal}(K(A[\ell])/K)$, see [Ser77, Chapter 1, Theorem 1].

3 Pointed isogeny graphs for elliptic curves

Let E/K be an elliptic curve over a field K of characteristic $p \neq \ell$. We will prove the following

Theorem 3.1. *Let*

$$E_1 \xrightarrow{\phi_1} E \xleftarrow{\phi_2} E_2$$

be a K -rational pointed ℓ -isogeny graph of order two. Then $E[\ell](\overline{K}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$ as G_K -modules.

Proof. Let $\{P_1, Q_1\}$ be a basis for $E_1[\ell](\overline{K})$ and $\{P_2, Q_2\}$ a basis for $E_2[\ell](\overline{K})$ such that $\ker(\phi_i) = \langle P_i \rangle$. We claim that $R_1 := \phi_1(Q_1)$ and $R_2 := \phi_2(Q_2)$ are linearly independent and K -rational. Write $H_i = \langle R_i \rangle$.

Lemma 3.1.

$$H_1 \neq H_2. \quad (5)$$

Proof. Let $H = H_1 \cap H_2$. Since $H \subset H_i$, we have induced morphisms

$$E \rightarrow E/H \rightarrow E/H_i \rightarrow E, \quad (6)$$

where the last arrow is given by ϕ_i . This gives rise to injections of function fields

$$K(E) \rightarrow K(E_i) \rightarrow K(E/H) \rightarrow K(E). \quad (7)$$

By assumption, we have that $K(E_1) \cdot K(E_2)$ has dimension ℓ^2 over $K(E)$. In other words, the last injection is an isomorphism and thus $H = (1)$, giving the Lemma. \square

Lemma 3.2. *The R_i are K -rational.*

Proof. Let $\sigma \in G_K$. Consider $R'_1 := \hat{\phi}_2(R_1)$. Then $R'_1 \in \ker(\phi_2)$, meaning that it is K -rational. This gives $\sigma(R'_1) = R'_1$ and thus $\sigma(R_1) - R_1 \in H_2$. But $H_1 = \langle R_1 \rangle$ is G_K -invariant (as it is the kernel of $\hat{\phi}_1$) and thus $\sigma(R_1) - R_1 \in H_1 \cap H_2 = (1)$. Similarly, we have $\sigma(R_2) = R_2$, yielding the Lemma. \square

In other words, we now have two K -rational points $R_i \in E[\ell](K)$ which are linearly independent by Lemma 3.1. This gives the theorem. \square

4 Semisimple pointed isogeny graphs

We now turn to pointed K -rational ℓ -isogeny graphs for abelian varieties of dimension $g > 1$. These isogeny graphs $A_i \rightarrow A$ can exist without an injection of G_K -modules $(\mathbb{Z}/\ell\mathbb{Z})^n \rightarrow A[\ell](\overline{K})$, as we will see after Theorem 4.1. If the representation

$$\rho_\ell : G_K \rightarrow \mathrm{GL}_{2g}(\mathbb{F}_\ell) \quad (8)$$

is *semisimple* however, we have the following theorem:

Theorem 4.1. *Let A be an abelian variety of dimension $g > 1$ and let $A_i \rightarrow A$ be a semisimple K -rational ℓ -isogeny graph of order n . There then exists an injection*

$$(\mathbb{Z}/\ell\mathbb{Z})^n \rightarrow A[\ell](\overline{K}) \quad (9)$$

of G_K -modules.

Proof. Let $\{P_{1,i}, \dots, P_{2g,i}\}$ be a basis of $A_i[\ell](\overline{K})$ such that $\ker(\phi_i) = \langle P_{1,i} \rangle$. Define

$$H_i := \phi_i(\mathrm{Span}\{P_{2,i}, P_{3,i}, \dots, P_{2g,i}\}) = \phi_i(A_i[\ell](\overline{K})). \quad (10)$$

Note that H_i is G_K -invariant, since it is the kernel of the dual isogeny $\hat{\phi}_i$. Let J be any nonempty subset of $I := \{1, 2, \dots, n\}$. We define

$$H_J := \bigcap_{j \in J} H_j. \quad (11)$$

Thus if $J_1 \subseteq J_2$ then

$$H_{J_1} \supseteq H_{J_2}. \quad (12)$$

Note that any H_J is again G_K -invariant, since it is an intersection of G_K -invariant subspaces.

We will start by proving the following

Lemma 4.1. *Let $J \subseteq I = \{1, \dots, n\}$ and consider the subgroup H_J . Then*

$$\dim(H_J) = 2g - \#J. \quad (13)$$

Proof. We will prove this by induction on the order of J . For $J = \{i\}$, we have $\dim(H_i) = 2g - 1$ by definition of the H_i . Suppose that the statement is true for subsets of order k and let J be a subset of order $k + 1$. Consider any $j \in J$ and let $J' = J \setminus \{j\}$. We consider two cases:

1. $H_{J'} \subseteq H_j$,
2. $H_{J'} \not\subseteq H_j$.

Suppose that $H_{J'} \subseteq H_j$. We then have the following induced injections of function fields

$$K(A) \rightarrow K(A_j) \rightarrow K(A/H_{J'}) \rightarrow K(A). \quad (14)$$

Note that $K(A/H_{J'})/K(A)$ has degree $\ell^{|J'|}$ by the induction hypothesis. Note also that the product

$$\prod_{i \in J} K(A_i) \quad (15)$$

is contained in $K(A/H_{J'})$ and has degree $\ell^{|J|}$. But $\ell^{|J|} > \ell^{|J'|}$, yielding a contradiction. We thus conclude that $H_{J'} \not\subseteq H_j$. There is then an $x \in H_{J'}$ such that $x \notin H_j$. But then $\langle x \rangle + H_j = A[\ell](\overline{K})$, since H_j is $2g - 1$ -dimensional. We then also have that $H_J + H_j = A[\ell](\overline{K})$, since $\langle x \rangle \subseteq H_J$. Applying the inclusion/exclusion formula, we obtain

$$\dim(H_J) = 2g - \dim(H_j) + \dim(H_{J'}) = 2g - (2g - 1) + \dim(H_{J'}) = 1 + \dim(H_{J'}). \quad (16)$$

This then gives $\dim(H_J) = 1 + \dim(H_{J'}) = 1 + 2g - |J'| = 2g - |J|$, as desired. By induction, we conclude that the formula holds for all subsets of I , yielding the Lemma. \square

Now consider the inclusions $H_{I \setminus i} \supset H_I$, where $I = \{1, \dots, n\}$. By applying the semisimplicity assumption to every inclusion, we obtain G_K -invariant subspaces W_i such that

$$H_I \oplus W_i = H_{I \setminus i}. \quad (17)$$

By Lemma 4.1, we then find that $\dim(W_i) = 1$ and we can thus write $W_i = \langle Q_i \rangle$ for $Q_i \in A[\ell](\overline{K})$. Note that $W_i \cap H_i = (1)$ for every i .

Lemma 4.2. *The Q_i are K -rational.*

Proof. Consider the points $Q'_i := \hat{\phi}_i(Q_i) \in A_i[\ell](\overline{K})$. They are K -rational, since they are in the kernel of $\hat{\phi}_i$. We thus find that $\sigma(Q_i) - Q_i \in \ker(\hat{\phi}_i) = H_i$. But the W_i are invariant, so $\sigma(Q_i) - Q_i \in W_i \cap H_i = (1)$. This yields $\sigma(Q_i) = Q_i$, as desired. \square

Lemma 4.3. *The Q_i are linearly independent.*

Proof. We will prove this by induction on i . For $i = 1$, we have that Q_1 is not trivial by construction (see Equation 17). We now suppose that the Lemma is true for i and we will show that Q_{i+1} is independent of the other Q_k for $k \leq i$. Suppose for a contradiction that

$$Q_{i+1} = \sum_{j=1}^i c_j Q_j. \quad (18)$$

By construction, we find that $Q_j \in H_{i+1}$ for every $j \leq i$. But then $Q_{i+1} \in H_{i+1}$, a contradiction. By induction, we conclude that the Q_i are linearly independent. \square

We now have a set $\{Q_i\}$ of n linearly independent, K -rational points in $A[\ell](\overline{K})$. These give a natural injection of G_K -modules $(\mathbb{Z}/\ell\mathbb{Z})^n \rightarrow A[\ell](\overline{K})$, which finishes the proof. \square

We note that the condition on the Galois group in Theorem 4.1 is necessary by the following counterexample. Let $\phi_1 : E'_1 \rightarrow E_1$ and $\phi_2 : E'_2 \rightarrow E_2$ be two isogenies of degree ℓ whose kernels are generated by K -rational points. Suppose furthermore that E_1 and E_2 contain no K -rational ℓ -torsion points. We then have the isogeny graph

$$E'_1 \times E_2 \xrightarrow{(\phi_1, id)} E_1 \times E_2 \xleftarrow{(id, \phi_2)} E_1 \times E'_2.$$

Note that by construction, $(E_1 \times E_2)[\ell]$ does not contain any K -rational points. More explicitly, let $\ell = 3$ and $K := \mathbb{Q}$. We then have the following universal families

$$\begin{aligned} E_3 &: y^2 + wxy + vy = x^3, \\ E'_3 &: y^2 + wxy + vy = x^3 - 5wvx - v(w^3 + 7v). \end{aligned}$$

Here $P := (0, 0) \in E_3(\mathbb{Q}(w, v))$ is a point of order three and $E'_3 = E_3 / \langle P \rangle$. Taking values $(v_0, w_0) \in \mathbb{Q}^2$ of (v, w) such that E'_3 has no rational three torsion then yields the desired curve. For instance, $v_0 = 2$ and $w_0 = 1$ work. Taking $E'_1 = E'_2$ and $E_1 = E_2 = E'_1 / \langle P \rangle$ then yields the explicit counterexample over \mathbb{Q} . The formulas for the universal families can be found in [Hel14, Page 15].

References

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