NONLINEAR SECOND ORDER EVOLUTION INCLUSIONS WITH NONCOERCIVE VISCOSITY TERM

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Abstract. In this paper we deal with a second order nonlinear evolution inclusion, with a nonmonotone, noncoercive viscosity term. Using a parabolic regularization (approximation) of the problem and a priori bounds that permit passing to the limit, we prove that the problem has a solution.

1. INTRODUCTION

Let $T = [0, b]$ and let (X, H, X^*) be an evolution triple of spaces, with the embedding of X into H being compact (see Section [2](#page-0-0) for definitions).

In this paper, we study the following nonlinear evolution inclusion:

(1)
$$
\begin{cases} u''(t) + A(t, u'(t)) + Bu(t) \in F(t, u(t), u'(t)) \text{ for almost all } t \in T, \\ u(0) = u_0, u'(0) = u_1. \end{cases}
$$

In the past, such multi-valued problems were studied by Gasinski [\[3\]](#page-10-0), Gasinski and Smolka $[6, 7]$ $[6, 7]$, Migórski et al. $[11, 12, 13, 14]$ $[11, 12, 13, 14]$ $[11, 12, 13, 14]$ $[11, 12, 13, 14]$, Ochal $[15]$, Papageorgiou, Rădulescu and Repovš [\[16,](#page-11-5) [17\]](#page-11-6), Papageorgiou and Yannakakis [\[18,](#page-11-7) [19\]](#page-11-8). The works of Gasinski [\[3\]](#page-10-0), Gasinski and Smolka [\[6,](#page-10-1) [7\]](#page-10-2) and Ochal [\[15\]](#page-11-4), all deal with hemivariational inequalities, that is, $F(t, x, y) = \partial J(x)$ with $J(\cdot)$ being a locally Lipschitz functional and $\partial J(\cdot)$ denoting the Clarke subdifferential of $J(\cdot)$. In Papageorgiou and Yannakakis [\[18,](#page-11-7) [19\]](#page-11-8), the multivalued term $F(t, x, y)$ is general (not necessarily of the subdifferential type) and depends also on the time derivative of the unknown function $u(\cdot)$. With the exception of Gasinski and Smolka [\[7\]](#page-10-2), in all the other works the viscosity term $A(t, \cdot)$ is assumed to be coercive or zero. In the work of Gasinski and Smolka [\[7\]](#page-10-2), the viscosity term is autonomous (that is, time independent) and $A: X \to X^*$ is linear and bounded.

In this work, the viscosity term $A: T \times X \to X^*$ is time dependent, noncoercive, nonlinear and nonmonotone in $x \in X$. In this way, we extend and improve the result of Gasinski and Smolka [\[7\]](#page-10-2). Our approach uses a kind of parabolic regularization of the inclusion, analogous to the one used by Lions [\[10,](#page-11-9) p. 346] in the context of semilinear hyperbolic equations.

2. Mathematical Background and Hypotheses

Let V, Y be Banach spaces and assume that V is embedded continuously and densely into Y (denoted by $V \hookrightarrow Y$). Then we have the following properties:

(i) Y^* is embedded continuously into V^* ;

(ii) if V is reflexive, then $Y^* \hookrightarrow V^*$.

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The following notion is a useful tool in the theory of evolution equations.

Definition 1. By an "evolution triple" (or "Gelfand triple") we understand a triple of spaces (X, H, X^*) such that

- (a) X is a separable reflexive Banach space and X^* is its topological dual;
- (b) H is a separable Hilbert space identified with its dual H^* , that is, $H = H^*$ (pivot space);
- (c) $X \hookrightarrow H$.

Then from the initial remarks we have

$$
X \hookrightarrow H = H^* \hookrightarrow X^*.
$$

In what follows, we denote by $|| \cdot ||$ the norm of X, by $|| \cdot ||$ the norm of H and by $|| \cdot ||_*$ the norm of X^* . Evidently we can find $\hat{c}_1, \hat{c}_2 > 0$ such that

$$
|\cdot| \leq \hat{c}_1 ||\cdot|| \text{ and } ||\cdot||_* \leq \hat{c}_2 |\cdot|.
$$

By (\cdot, \cdot) we denote the inner product of H and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X) . We have

(2)
$$
\langle \cdot, \cdot \rangle |_{H \times X} = (\cdot, \cdot).
$$

Let $1 < p < \infty$. The following space is important in the study of problem [\(1\)](#page-0-1):

$$
W_p(0,b) = \left\{ u \in L^p(T,X) : u' \in L^{p'}(T,X^*) \right\} \left(\frac{1}{p} + \frac{1}{p'} = 1 \right).
$$

Here u' is understood in the distributional sense (weak derivative). We know that $L^p(T, X)^* = L^{p'}(T, X^*)$ (see, for example, Gasinski and Papageorgiou [\[4,](#page-10-3) p. 129]). Suppose that $u \in W_p(0, b)$. If we view $u(\cdot)$ as an X^{*}-valued function, then $u(\cdot)$ is absolutely continuous, hence differentiable almost everywhere and this derivative coincides with the distributional one. So, $u' \in L^{p'}(T, X^*)$ and we can say

$$
W_p(0,b) \subseteq AC^{1,p'}(T,X^*) = W^{1,p'}((0,b),X^*).
$$

The space $W_p(0, b)$ is equipped with the norm

$$
||u||_{W_p} = \left[||u||^p_{L^p(T,X)} + ||u'||^p_{L^{p'}(T,X^*)}\right]^{\frac{1}{p}} \text{ for all } u \in W_p(0,b).
$$

Evidently, another equivalent norm on $W_p(0, b)$ is

$$
|u|_{W_p}=||u||_{L^p(T,X)}+||u'||_{L^p(T,X^*)}\text{ for all }u\in W_p(0,b).
$$

With any of the above norms, $W_p(0, b)$ becomes a separable reflexive Banach space. We have that

- (3) $W_p(0, b) \hookrightarrow C(T, H);$
- (4) $W_p(0, b) \hookrightarrow L^p(T, H)$ and the embedding is compact.

The elements of $W_p(0, b)$ satisfy an integration by parts formula which will be useful in our analysis.

Proposition 2. If $u, v \in W_p(0, b)$ and $\xi(t) = (u(t), v(t))$ for all $t \in T$, then $\xi(\cdot)$ is absolutely continuous and $\frac{d\xi}{dt}(t) = \langle u'(t), v(t) \rangle + \langle u(t), v'(t) \rangle$ for almost all $t \in T$.

Now suppose that (Ω, Σ, μ) is a finite measure space, Σ is μ – *complete* and Y is a separable Banach space. A multifunction (set-valued function) $F: \Omega \to 2^Y \setminus \{\emptyset\}$ is said to be "graph measurable", if

$$
\operatorname{Gr} F = \{ (\omega, y) \in \Omega \times Y : y \in F(\omega) \} \in \Sigma \times B(Y),
$$

with $B(Y)$ being the Borel σ -field of Y.

If $F(\cdot)$ has closed values, then graph measurability is equivalent to saying that for every $y \in Y$ the \mathbb{R}_+ -valued function

$$
\omega \mapsto d(y, F(\omega)) = \inf \{ ||y - v||_Y : v \in F(\omega) \}
$$

is Σ-measurable.

Given a graph measurable multifunction $F : \Omega \to 2^Y \setminus \{\emptyset\}$, the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [\[8,](#page-10-4) p. 158]) implies that $F(\cdot)$ admits a measurable selection, i.e. that there exists $f : \Omega \to Y$ a Σ measurable function such that $f(\omega) \in F(\omega)$ *µ*-almost everywhere. In fact, we can find an entire sequence $\{f_n\}_{n\geq 1}$ of measurable selections such that $F(\omega) \subseteq$ ${f_n(\omega)}_{n\geqslant 1}$ μ -almost everywhere.

For $1 \leqslant p \leqslant \infty$, we define

$$
S_F^p = \{ f \in L^p(\Omega, Y) : f(\omega) \in F(\omega) \text{ μ-almost everywhere} \}.
$$

It is easy to see that $S_F^p \neq \emptyset$ if and only if $\omega \mapsto \inf\{||v||_Y : v \in F(\omega)\}\)$ belongs to $L^p(\Omega)$. This set is "decomposable" in the sense that if $(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$, then

$$
\chi_A f_1 + \chi_{A^c} f_2 \in S_F^p.
$$

Finally, for a sequence $\{C_n\}_{n\geq 1}$ of nonempty subsets of Y, we define

$$
w-\limsup_{n\to\infty}C_n = \{y\in Y: y = w-\lim_{k\to\infty}y_{n_k}, y_{n_k}\in C_{n_k}, n_1 < n_2 < \cdots < n_k < \cdots\}.
$$

For more details on the notions discussed in this section, we refer to Gasinski and Papageorgiou $[4]$, Roubiček $[20]$, Zeidler $[21]$ (for evolution triples and related notations) and Hu and Papageorgiou [\[8\]](#page-10-4) (for measurable multifunctions).

Let V be a reflexive Banach space and $A: V \to V^*$ a map. We say that A is "pseudomonotone", if A is continuous from every finite dimensional subspace of V into V_w^* (= the dual V^* equipped with the weak topology) and if

$$
v_n \xrightarrow{w} v \text{ in } V, \limsup_{n \to \infty} \langle A(v_n), v_n - v \rangle \leq 0
$$

then

$$
\langle A(v), v - y \rangle \leq \liminf_{n \to \infty} \langle A(v_n), v_n - y \rangle \text{ for all } y \in V.
$$

An everywhere defined maximal monotone operator is pseudomonotone. If V is finite dimensional, then every continuous map $A: V \to V^*$ is pseudomonotone. In what follows, for any Banach space Z , we will use the following notations:

$$
P_{f(c)}(Z) = \{C \subseteq Z : C \text{ is nonempty, closed (and convex)}\},
$$

$$
P_{(w)k(c)}(Z) = \{C \subseteq Z : C \text{ is nonempty, (weakly-) compact (and convex)}\}.
$$

The hypotheses on the data of problem [\(1\)](#page-1-0) are the following:

 $H(A): A: T \times T \to X^*$ is a map such that

- (i) for all $y \in X$, $t \mapsto A(t, y)$ is measurable;
- (ii) for almost all $t \in T$, the map $y \mapsto A(t, y)$ is pseudomonotone;
- (iii) $||A(t,y)||_* \leq a_1(t) + c_1||y||^{p-1}$ for almost all $t \in T$ and all $y \in X$, with $a_1 \in L^{p'}(T), c_1 > 0, 2 \leq p < \infty;$
- (iv) $\langle A(t, y), y \rangle \geq 0$ for almost all $t \in T$ and all $y \in X$.

 $H(B): B \in \mathscr{L}(X, X^*)$, $\langle Bx, y \rangle = \langle x, By \rangle$ for all $x, y \in X$ and $\langle Bx, x \rangle \geqslant c_0 ||x||^2$ for all $x \in X$ and some $c_0 > 0$.

- $H(F): F: T \times H \times H \to P_{f_c}(H)$ is a multifunction such that
- (i) for all $x, y \in H$, $t \mapsto F(t, x, y)$ is graph measurable;
- (ii) for almost all $t \in T$, the graph $\operatorname{Gr} F(t, \cdot, \cdot)$ is sequentially closed in $H \times$ $H_w \times H_w$ (here H_w denotes the Hilbert space H furnished with the weak topology);
- (iii) $|F(t, x, y)| = \sup\{|h| : h \in F(t, x, y)\} \leq a_2(t)(1 + |x| + |y|)$ for almost all $t \in T$ and all $x, y \in H$ with $a_2 \in L^2(T)_+$.

Definition 3. We say that $u \in C(T, X)$ is a "solution" of problem [\(1\)](#page-0-1) with $u_0 \in C(T, X)$ $X, u_1 \in H, if$

- $u' \in W_p(0, b)$ and
- there exists $f \in S^2_{F(\cdot, u(\cdot), u'(\cdot))}$ such that

$$
\begin{cases}\n u''(t) + A(t, u'(t)) + Bu(t) = f(t) \text{ for almost all } t \in T, \\
 u(0) = u_0, u'(0) = u_1.\n\end{cases}
$$

In what follows, we denote by $S(u_0, u_1)$ the set of solutions of problem (1). Recalling that $W_p(0, b) \hookrightarrow C(T, H)$ (see [\(3\)](#page-1-1)), we have that

$$
S(u_0, u_1) \subseteq C^1(T, H).
$$

By Troyanski's renorming theorem (see Gasinski and Papageorgiou [\[4,](#page-10-3) p. 911]) we may assume without loss of generality that both X and X^* are locally uniformly convex. Let $\mathcal{F}: X \to X^*$ be the duality map of X defined by

$$
\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x \rangle = ||x||^2 = ||x^*||_*^2\}.
$$

We know that $\mathcal{F}(\cdot)$ is single-valued and a homeomorphism (see Gasinski and Papageorgiou [\[4,](#page-10-3) p. 316] and Zeidler [\[21,](#page-11-11) p. 861]).

For every $r \geq p$, let $K_r : X \to X^*$ be the map defined by

$$
K_r(y) = ||y||^{r-2} \mathcal{F}(y)
$$
 for all $y \in X$.

3. Existence Theorem

Given $\epsilon > 0$, we consider the following perturbation (parabolic regularization) of problem [\(1\)](#page-0-1):

$$
(5) \quad
$$

$$
\begin{cases}\n u''(t) + A(t, u'(t)) + \epsilon K_r(u'(t)) + Bu(t) \in F(t, u(t), u'(t)) \text{ for a.a. } t \in T, \\
 u(0) = u_0, u'(0) = u_1.\n\end{cases}
$$

Consider the map $A_{\epsilon}: T \times X \to X^*$ defined by

$$
A_{\epsilon}(t, y) = A(t, y) + \epsilon K_r(y) \text{ for all } t \in T, \text{ and all } y \in X.
$$

This map has the following properties:

- (i) for all $y \in X$, the map $t \mapsto A_{\epsilon}(t, y)$ is measurable;
- (ii) for almost all $t \in T$, the map $y \mapsto A_{\epsilon}(t, y)$ is pseudomonotone;
- (iii) $||A_{\epsilon}(t, y)||_* \leq \hat{a}_1(t) + \hat{c}_1||y||^{r-1}$ for almost all $t \in T$, all $y \in X$ and with $\hat{a}_1 \in L^{p'}(T), \, \hat{c}_1 > 0$ (recall that $r \geqslant p$ and $\frac{1}{r} + \frac{1}{r'}$ $\frac{1}{r'}=1);$
- (iv) $\langle A_{\epsilon}(t, y), y \rangle \geqslant \epsilon ||y||^{r}$ for all $t \in T$, all $y \in X$.

So, in problem [\(1\)](#page-0-1) the viscosity term $A_{\epsilon}(t, \cdot)$ is coercive. Therefore we can apply Theorem 1 of Papageorgiou and Yannakakis [\[18\]](#page-11-7) and we obtain the following existence result for the approximate (regularized) problem [\(5\)](#page-3-0).

Proposition 4. If hypotheses $H(A)$, $H(B)$, $H(F)$ hold and $u_0 \in X, u_1 \in H$, then problem [\(5\)](#page-3-0) admits a solution $u_{\epsilon} \in W^{1,r}((0,b),X) \cap C^1(T,H)$ with

$$
u'_{\epsilon}\in W_r(0,b).
$$

To produce a solution for the original problem [\(1\)](#page-0-1), we have to pass to the limit as $\epsilon \to 0^+$. To do this, we need to have a priori bounds for the solutions $u_{\epsilon}(\cdot)$ which are independent of $\epsilon \in (0, 1]$ and $r \geq p$.

Proposition 5. If hypotheses $H(A), H(B), H(F)$ hold, $u_0 \in X, u_1 \in H$ and $u(\cdot)$ is a solution of [\(5\)](#page-3-0), then there exists $M_0 > 0$ which is independent of $\epsilon \in (0,1]$ and $r \geqslant p$ for which we have

$$
||u||_{C(T,X)}, \ ||u'||_{C(T,H)}, \ \epsilon^{\frac{1}{r}}||u'||_{L^r(T,X)}, \ ||u''||_{L^2(T,X^*)} \leq M_0.
$$

Proof. It follows from Proposition [4](#page-4-0) that $u' \in W_r(0, b)$ and that there exists $f \in$ $S_{F(\cdot,u(\cdot),u'(\cdot))}^2$ such that

$$
u''(t) + A(t, u'(t)) + \epsilon K_r(u'(t)) + Bu(t) = f(t)
$$
 for almost all $t \in T$.

We act with $u'(t) \in X$. Then

(6)
$$
\langle u''(t), u'(t) \rangle + \langle A(t, u'(t)), u'(t) \rangle + \epsilon \langle K_r(u'(t)), u'(t) \rangle = (f(t), u'(t))
$$

for almost all $t \in T$ (see (2)).

We examine separately each summand on the left-hand side of [\(6\)](#page-4-1). Recall that $u'_r \in W_r(0, b)$. So from Proposition [2](#page-1-3) (the integration by parts formula), we have

(7)
$$
\langle u''(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} |u'(t)|^2 \text{ for almost all } t \in T.
$$

Hypothesis $H(A)(iv)$ and the definition of the duality map, imply that

(8)
$$
\langle A(t, u'(t)), u'(t) \rangle + \epsilon \langle K_r(u'(t)), u'(t) \rangle \geq \epsilon ||u'(t)||^r
$$
 for almost all $t \in T$.

By hypothesis $H(B)$, we have

(9)
$$
\langle Bu(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle Bu(t), u(t) \rangle \text{ for almost all } t \in T.
$$

We return to (6) and use (7) , (8) , (9) . We obtain

$$
\frac{1}{2}\frac{d}{dt}|u'(t)|_2 + \epsilon||u'(t)||^r + \frac{1}{2}\frac{d}{dt}\langle Bu(t), u(t)\rangle \le (f(t), u'(t)) \text{ for a.a. } t \in T,
$$

\n
$$
\Rightarrow \frac{1}{2}|u'(t)|^2 + \epsilon \int_0^t ||u'(s)||^r ds + c_0||u(t)||^2
$$

\n
$$
(10) \le \int_0^t (f(s), u'(s))ds + \frac{1}{2}|u_1|^2 + \frac{1}{2}||B||_{\mathcal{L}}||u_0||^2 \text{ (see hypothesis } H(B)).
$$

Using hypothesis $H(F)(iii)$, we get

$$
\int_0^t (f(s), u'(s))ds
$$
\n
$$
\leq \int_0^t [a_2(s) + a_2(s) (|u(s)| + |u'(s)|)] |u'(s)| ds
$$
\n(11)
$$
\leq \int_0^t |u'(s)|^2 ds + \int_0^t a_2(s)^2 ds + \int_0^t a_2(s)^2 [|u(s)|^2 + |u'(s)|^2] ds.
$$

Recall that $u \in W^{1,r}((0, b), X)$ (see Proposition [4\)](#page-4-0). So, $u \in AC^{1,r}(T, H)$ and we can write

$$
u(t) = u_0 + \int_0^t u'(s)ds \text{ for all } t \in T
$$

 $\Rightarrow |u(t)|^2 \leq 2|u_0|^2 + 2b \int_0^t$ 0 (12) $|u(t)|^2 \leq 2|u_0|^2 + 2b \int |u'(s)|^2 ds$ for all $t \in T$ (using Jensen's inequality).

We use (12) in (11) and obtain

$$
\int_0^t (f(s), u'(s))ds
$$
\n
$$
\leqslant ||a_2||_2^2 + \int_0^t \left[1 + a_2(s)^2\right] |u'(s)|^2 ds + \int_0^t 2a_2(s)^2 \left[|u_0|^2 + b \int_0^s |u'(\tau)|^2 d\tau\right] ds
$$
\n
$$
(13) \qquad c_2 + \int_0^t \eta(s) |u'(s)|^2 ds + 2b \int_0^t a_2(s)^2 \int_0^s |u'(\tau)|^2 d\tau ds
$$
\nfor some $c_2 > 0$ and $\eta \in L^1(T)$.

We use (13) in (10) and have

$$
\frac{1}{2}|u'(t)|^2 + \epsilon \int_0^t ||u'(s)||^p ds + c_0||u(t)||^2
$$
\n
$$
(14) \leq c_3 + \int_0^t \eta(s)|u'(s)|^2 ds + 2b \int_0^t a_2(s)^2 \int_0^s |u'(\tau)|^2 d\tau ds \text{ for some } c_3 > 0.
$$

Invoking Proposition 1.7.87 of Denkowski, Migórski and Papageorgiou [\[2,](#page-10-5) p. 128] we can find $M > 0$ (independent of $\epsilon \in (0,1]$ and $r \geq p$) such that

$$
|u'(t)|^2 \leq M \text{ for all } t \in T,
$$

$$
\Rightarrow ||u'||_{C(T,H)} \leq M_1 = M^{\frac{1}{2}}.
$$

Using this bound in [\(14\)](#page-5-3), we can find $M_2 > 0$ (independent of $\epsilon \in (0, 1]$ and $r \geq p$) such that

$$
||u||_{C(T,X)} \le M_2
$$
 and $\epsilon^{\frac{1}{r}}||u'||_{L^r(T,X)} \le M_2$.

Finally, directly from [\(5\)](#page-3-0), we see that there exists $M_3 > 0$ (independent of $\epsilon \in (0,1]$ and $r \geqslant p$) such that

$$
||u''||_{L^{r'}}(T, X^*) \le M_3.
$$

We set $M_0 = \max\{M_1, M_2, M_3\} > 0$ and get the desired bound.

The bounds produced in Proposition [5](#page-4-6) permit passing to the limit as $\epsilon \to 0^+$ to produce a solution for problem [\(1\)](#page-0-1).

Theorem 6. If hypotheses $H(A), H(B), H(F)$ hold and $u_0 \in X, u_1 \in H$, then $S(u_0, u_1) \neq \emptyset$.

Proof. Let $\epsilon_n \to 0^+$ and let $u_n = u_{\epsilon_n}$ be solutions of the "regularized" problem [\(5\)](#page-3-0) (see Proposition [4\)](#page-4-0). Because of the bounds established in Proposition [5](#page-4-6) and by passing to a suitable subsequence if necessary, we can say that

(15)
$$
\left\{\n\begin{array}{ll}\n u_n \xrightarrow{w^*} u \text{ in } L^\infty(T, X), \ u_n \xrightarrow{w} u \text{ in } C(T, H), \ u_n \to u \text{ in } L^r(T, H) \\
 u'_n \xrightarrow{w^*} y \text{ in } L^\infty(T, H), \ u''_n \xrightarrow{w} v \text{ in } L^{r'}(T, X^*) \text{ (see (3) and (4))}.\n\end{array}\n\right\}
$$

Recall that $u_n\in AC^{1,r}(T,H)$ for all $n\in\mathbb{N}$ and so

$$
u_n(t) = u_0 + \int_0^t u'_n(s)ds \text{ for all } t \in T,
$$

\n
$$
\Rightarrow u(t) = u_0 + \int_0^t y(s)ds \text{ for all } t \in T \text{ (see (15))},
$$

\n
$$
\Rightarrow u \in AC^{1,r}(T, H) \text{ and } u' = y.
$$

Since $u_n \in W_r(0, b)$ for all $n \in \mathbb{N}$, we have

 $v = y' = u'' \in L^{r'}(T, X^*)$ (see Hu and Papageorgiou [\[9,](#page-11-12) p. 6]).

Let $a: L^r(T,X) \to L^{r'}(T,X^*)$ be the nonlinear map defined by

$$
a(u)(\cdot) = A(\cdot, u(\cdot))
$$
 for all $u \in L^r(T, X)$.

Also, let $\hat{K}_r: L^r(T, X) \to L^{r'}(T, X^*)$ be defined by

$$
\hat{K}_r(u)(\cdot) = ||u(\cdot)||^{r-2} \mathscr{F}(u(\cdot)) \text{ for all } u \in L^r(T, X).
$$

Both maps are continuous and monotone, hence maximal monotone (see Gasinski and Papageorgiou [\[4,](#page-10-3) Corollary 3.2.32, p. 320]).

Finally, let $\hat{B} \in \mathscr{L}(L^r(T,X),L^{r'}(T,X^*))$ be defined by

$$
\hat{B}(u)(\cdot) = B(u(\cdot))
$$
 for all $u \in L^r(T, X)$.

We have

(16)
$$
u''_n + a(u'_n) + \epsilon_n \hat{K}_r(u'_n) + \hat{B}u_n = f_n \text{ in } L^r(T, X^*)
$$

with $f_n \in S^2_{F(\cdot, u_n(\cdot), u'_n(\cdot))}$ for all $n \in \mathbb{N}$.

From [\(15\)](#page-6-0) we have

(17)
$$
u_n \xrightarrow{w} u \text{ in } L^r(T, X),
$$

$$
\Rightarrow \hat{B}u_n \xrightarrow{w} \hat{B}u \text{ in } L^{r'}(T, X^*) \text{ as } n \to \infty.
$$

Also, we have

$$
||\hat{K}_r(u'_n)||_{L^{r'}(T,X^*)} = ||u'_n||_{L^r(T,X)}^{r-1},
$$

\n
$$
\Rightarrow \epsilon_n ||\hat{K}_r(u'_n)||_{L^{r'}(T,X^*)} = \epsilon_n^{\frac{1}{r}} \left(\epsilon_n^{\frac{1}{r}} ||u'_n||_{L^r(T,X)}\right)^{r-1} \text{ (recall that } \frac{1}{r} + \frac{1}{r'} = 1)
$$

\n
$$
\leq \epsilon_n^{\frac{1}{r}} M_0^{r-1} \text{ for all } n \in \mathbb{N} \text{ (see Proposition 5)}
$$

\n(18) $\Rightarrow \epsilon_n ||\hat{K}_r(u'_r)||_{L^{r'}(T,X^*)} \to 0 \text{ as } n \to \infty$
\nFrom (15) and since $v = u''$, we have
\n
$$
u''_n \xrightarrow{w} u'' \text{ in } L^{r'}(T,X^*).
$$

Finally, hypothesis $H(F)(iii)$ and Proposition [5](#page-4-6) imply that

$$
\{f_n\}_{n\geqslant 1}\subseteq L^2(T,H)
$$
 is bounded.

By passing to a subsequence if necessary, we may assume that

$$
f_n \xrightarrow{w} f
$$
 in $L^2(T, H)$.

Invoking Proposition 3.9 of Hu and Papageorgiou [\[8,](#page-10-4) p. 694], we have

$$
f(t) \in \overline{\text{conv}}\,w - \limsup_{n \to \infty} \{f_n(t)\}
$$

(20)
$$
\leq \overline{\text{conv}}\,w - \limsup_{n \to \infty} F(t, u_n(t), u'_n(t)) \text{ for almost all } t \in T \text{ (see (16))}.
$$

From [\(15\)](#page-6-0) we see that

$$
u'_n \xrightarrow{w} u' \text{ in } W^{1,r'}((0,b),X^*).
$$

Recall that $W^{1,r'}((0,b),X^*) \hookrightarrow C(T,X^*)$. So, it follows that

(21)
$$
u'_n \xrightarrow{w} u' \text{ in } C(T, X^*)
$$

$$
\Rightarrow u'_n(t) \xrightarrow{w} u'(t) \text{ in } X^* \text{ for all } t \in T.
$$

On the other hand, by Proposition [5](#page-4-6) we have

$$
|u_n'(t)|\leqslant M_0 \text{ for all } t\in T, \text{ all } n\in\mathbb{N}.
$$

So, by passing to a subsequence (a priori the subsequence depends on $t \in T$), we have

$$
u'_n(t) \xrightarrow{w} \hat{y}(t) \text{ in } H
$$

\n
$$
\Rightarrow \hat{y}(t) = u'(t) \text{ for all } t \in T \text{ (see (21))}.
$$

Hence for the original sequence we have

(22)
$$
u'_n(t) \xrightarrow{w} u'(t) \text{ in } H \text{ for all } t \in T.
$$

We know that $\{u_n\}_{n\geqslant 1}\subseteq W_r(0,b)$ is bounded (see Proposition [5\)](#page-4-6) and recall that $W_r(0, b) \hookrightarrow L^r(T, H)$ compactly (see [\(4\)](#page-1-1)). From this compact embedding and from [\(22\)](#page-7-1), we obtain

(23)
$$
u_n(t) \to u(t)
$$
 in H for all $t \in T$ as $n \to \infty$.

From [\(20\)](#page-7-2), [\(22\)](#page-7-1), [\(23\)](#page-7-3) and hypothesis $H(F)(iii)$ we infer that

$$
f(t) \in F(t, u(t), u'(t)) \text{ for almost all } t \in T,
$$

$$
\Rightarrow f \in S_{F(\cdot, u(\cdot), u'(\cdot))}^2.
$$

In what follows, we denote by $((\cdot, \cdot))$ the duality brackets for the pair

$$
(L^r(T, X^*), L^r(T, X)).
$$

Acting with $u'_n - u' \in L^r(T, X)$ on [\(16\)](#page-6-1), we have

$$
((u''_n, u'_n - u')) + ((a(u'_n), u'_n - u')) + ((\epsilon_n \hat{K}_r(u'_n), u'_r - u')) + ((\hat{B}u_n, u'_n - u'))
$$

(24) = $\int_0^b (f_n, u'_n - u')dt$ for all $n \in \mathbb{N}$.

Note that

$$
((u''_n, u'_n - u')) = \int_0^b \langle u''_n, u'_n - u' \rangle dt
$$

\n
$$
= \int_0^b \langle u''_n - u'', u'_n - u' \rangle dt + ((u'', u'_n - u'))
$$

\n
$$
= \int_0^b \frac{1}{2} \frac{d}{dt} |u'_n - u'|^2 dt + ((u'', u'_n - u')) \text{ (see Proposition 2)}
$$

\n
$$
= \frac{1}{2} |u'_n(b) - u'(b)|^2 + ((u'', u'_n - u'))
$$

\n(since $u'_n(0) = u'(0) = u_1 \text{ for all } n \in \mathbb{N}, \text{ see (22)}$)
\n
$$
\Rightarrow \liminf_{n \to \infty} ((u''_n, u'_n - u')) = \frac{1}{2} \liminf_{n \to \infty} |u'_n(b) - u'(b)|^2 \ge 0.
$$

Also we have

$$
((\hat{B}(u_n - u), u'_n - u')) = \int_0^b \frac{1}{2} \frac{d}{dt} \langle B(u_n - u), u_n - u \rangle dt
$$

$$
\frac{1}{2} \langle B(u_n - u)(b), (u_n - u)(b) \rangle \ge 0 \text{ (see hypothesis } H(B))
$$

$$
\Rightarrow ((\hat{B}u, u'_n - u')) \le ((\hat{B}u_n, u'_n - u')) \text{ for all } n \in \mathbb{N}.
$$

Recall that

$$
\epsilon_n^{\frac{1}{2}}||u_n||_{L^r(T,X)} \leq M_0
$$
 for all $n \in \mathbb{N}$ all $r \geq p$ (see Proposition 5).

Suppose that $r_m \to +\infty$, $r_m \geqslant p$ for all $m \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, $\epsilon_n^{\frac{1}{r_m}} \to 1$ as $m \to \infty$. Invoking Problem 1.175 of Gasinski and Papageorgiou [\[5\]](#page-10-6), we can find $\{m_n\}_{n\geqslant 1}$ with $m_n\to +\infty$ such that

$$
\epsilon_n^{\frac{1}{r_{m_n}}} \to 1 \text{ as } n \to \infty.
$$

Therefore there exists $n_0 \in \mathbb{N}$ such that

$$
\frac{1}{2} \leqslant e_n^{\frac{1}{r_{m_n}}} \text{ for all } n \geqslant n_0,
$$
\n
$$
\frac{1}{2} ||u'_n||_{L^{r_{m_n}}(T,X)} \leqslant M_0 \text{ for all } n \geqslant n_0,
$$
\n
$$
\Rightarrow ||u'_n||_{L^p(T,X)} \leqslant 2M_0 \text{ for all } n \geqslant n_0 \text{ (recall that } r_{m_n} \geqslant p).
$$

On account of [\(15\)](#page-6-0) and since $y = u'$, we have

(27)
$$
u'_n \xrightarrow{w} u' \text{ in } L^p(T, X).
$$

Then from [\(26\)](#page-8-0) and [\(27\)](#page-8-1) it follows that

(28)
$$
0 \leqslant \liminf_{n \to \infty} ((\hat{B}u_n, u'_n - u')).
$$

In addition, we have

(29)
$$
\epsilon_n \hat{K}_p(u'_n) \to 0 \text{ in } L^{p'}(T, X^*) \text{ as } n \to \infty \text{ (see (18))}.
$$

By Proposition [5](#page-4-6) and [\(27\)](#page-8-1) it follows that

$$
\{u'_n\}_{n\geqslant 1} \subseteq W_p(0, b)
$$
 is bounded,

$$
\Rightarrow \{u'_n\}_{n\geqslant 1} \subseteq L^p(T, H)
$$
 is relatively compact (see (4)).

Therefore we have

(30)
$$
u'_n \to u' \text{ in } L^p(T, H) \text{ (see (27))},
$$

$$
\Rightarrow \int_0^b (f_n, u'_n - u')dt \to 0 \text{ as } n \to \infty \text{ (recall that } p \geqslant 2).
$$

If in [\(24\)](#page-7-4) we pass to the limit as $n \to \infty$ and use [\(25\)](#page-8-2), [\(28\)](#page-8-3), [\(29\)](#page-8-4), [\(30\)](#page-9-0), then

$$
\limsup_{n \to \infty} ((a(u'_n), u'_n - u')) \leq 0.
$$

Invoking Theorem 2.35 of Hu and Papageorgiou [\[9,](#page-11-12) p. 41], we have

(31)
$$
a(u_n) \xrightarrow{w} a(u') \text{ in } L^{p'}(T, X^*) \text{ as } n \to \infty.
$$

In [\(24\)](#page-7-4) we pass to the limit as $n \to \infty$ and use [\(15\)](#page-6-0) (with $v = u''$) [\(27\)](#page-8-1), [\(29\)](#page-8-4), [\(31\)](#page-9-1). We obtain

$$
u'' + a(u') + \hat{B}u = f, \ u(0) = u_0, u'(0) = u_1, f \in S_{F(\cdot, u(\cdot), u'(\cdot))}^2,
$$

\n
$$
\Rightarrow u \in S(u_0, u_1) \neq \emptyset.
$$

The proof is now complete. \Box

3.1. An example. We illustrate the main abstract result of this paper with a hyperbolic boundary value problem. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. We consider the following boundary value problem (39)

(32)
\n
$$
\begin{cases}\n\frac{\partial^2 u}{\partial t^2} - \text{div}\left(a(t, z)|Du_t|^{p-2}Du_t\right) + \beta(z)u_t - \Delta u = f(t, z, u) + \gamma u_t \text{ in } T \times \Omega, \\
u|_{T \times \partial \Omega} = 0, \ u(0, z) = u_0(z), \ u_t(0, z) = u_1(z),\n\end{cases}
$$
\nwith $u_t = \frac{\partial u}{\partial x} \quad 2 \leq n \leq \infty, \ x > 0$

with $u_t = \frac{\partial u}{\partial t}$, $2 \leqslant p \leqslant \infty$, $\gamma > 0$.

The forcing term $f(t, z, \cdot)$ need not to be continuous. So, following Chang [\[1\]](#page-10-7), to deal with [\(32\)](#page-9-2), we replace it by a multivalued problem (partial differential inclusion), by filling in the gaps at the discontinuity points of $f(t, z, \cdot)$. So we define

$$
f_l(t, z, x) = \liminf_{x' \to x} f(t, z, x')
$$
 and $f_u(t, z, x) = \limsup_{x' \to x} f(t, z, x').$

Then we replace [\(32\)](#page-9-2) by the following partial differential inclusion (33)

$$
\begin{cases} \frac{\partial^2 u}{\partial t^2} - \text{div}\left(a(t,z)|Du_t|^{p-2}Du_t\right) + \beta(z)u_t - \Delta u \in [f_l(t,z,u), f_u(t,z,u)] \text{ in } T \times \Omega, \\ u|_{T \times \partial \Omega} = 0, \ u(0,z) = u_0(z), \ u_t(0,z) = u_1(z). \end{cases}
$$

Our hypotheses on the data of [\(33\)](#page-9-3) are the following:

 $H(a): a \in L^{\infty}(T \times \Omega), a(t, z) \geq 0$ for almost all $(t, z) \in T \times \Omega$.

 $H(\beta): \beta \in L^{\infty}(\Omega), \ \beta(z) \geqslant 0$ for almost all $z \in \Omega$.

 $H(f) : f : T \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that

- (i) f_l , f_u are superpositionally measurable (that is, for all $u : T \times \Omega \to \mathbb{R}$ measurable, the functions $(t, z) \mapsto f_l(t, z, u(t, z))$, $f_u(t, z, u(t, z))$ are both measurable);
- (ii) there exists $a \in L^2(T \times \Omega)$ such that

$$
|f(t, z, x)| \leq a_2(t, z)(1 + |x|)
$$
 for almost all $(t, z) \in T \times \Omega$, all $x \in \mathbb{R}$.

Let $X = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $X^* = W^{-1,p'}(\Omega)$. Then (X, H, X^*) is an evolution triple with $X \hookrightarrow H$ compactly (by the Sobolev embedding theorem).

Let $A: T \times X \to X^*$ be defined by

$$
\langle A(t, u), h \rangle = \int_{\Omega} a(t, z) |Du|^{p-2} (Du, Dh)_{\mathbb{R}^N} dz + \int_{\Omega} \beta(z) u h dz \text{ for all } u, h \in W_0^{1, p}(\Omega).
$$

Then $A(t, u)$ is measurable in $t \in T$, continuous and monotone in $u \in W_0^{1,p}(\Omega)$ (hence, maximal monotone) and $\langle A(t, u), u \rangle \geq 0$ for almost all $t \in T$, all $u \in$ $W_0^{1,p}(\Omega)$.

Let $B \in \mathscr{L}(X, X^*)$ be defined by

$$
\langle Bu, h \rangle = \int_{\Omega} (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,p}(\Omega).
$$

Clearly, B satisfies hypothesis $H(B)$.

Finally, let $G(t, z, x) = [f_l(t, z, x), f_u(t, z, x)]$ and set

$$
F(t, u, v) = S_{G(t, \cdot, u(\cdot))}^2 + \gamma v \text{ for all } u, v \in L^2(\Omega).
$$

Hypothesis $H(f)$ implies that F satisfies $H(F)$.

Using $A(t, u)$, Bu and $F(t, u, v)$ as defined above, we can rewrite problem [\(33\)](#page-9-3) as the equivalent second order nonlinear evolution inclusion (1). Assuming that $u_0 \in W_0^{1,p}(\Omega)$ and that $u_1 \in L^2(\Omega)$, we can use Theorem [6](#page-6-3) and infer that problem [\(30\)](#page-9-0) has a solution $u \in C^1(T, L^2(\Omega)) \cap C(T, W^{1,p}(\Omega))$ with $\frac{\partial u}{\partial t} \in L^p(\Omega, W_0^{1,p}(\Omega))$ and $\frac{\partial^2 u}{\partial t} \in L^{p'}(\Omega, W^{-1,p'}(\Omega)).$

Note that if $a = 0, f(t, z, x) = x$ and $\gamma = 0$, then we have the Klein-Gordon equation. If $f(t, z, x) = f(x) = \eta \sin x$ with $\eta > 0$, then we have the sine Gordon equation.

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REFERENCES

- [1] K.C. Chang, The obstacle problem and partial differential equations with discontinuous nonlinear terms, Comm. Pure Appl. Math. 33 (1980), 117-146.
- [2] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic Publishers, Boston, 2003.
- [3] L. Gasinski, Existence of solutions for hyperbolic hemivariational inequalities, J. Math. Anal. Appl. 276 (2002), 723-746.
- [4] L. Gasinski, N.S. Papageorgiou, Nonlinear Analysis, Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [5] L. Gasinski, N.S. Papageorgiou, Exercises in Analysis. Part 1: Nonlinear Analysis, Springer, Heidelberg, 2014.
- [6] L. Gasinski, M. Smolka, An existence theorem for wave-type hyperbolic hemivariational inequalities, Math. Nachr. 242 (2002), 79-90.
- [7] L. Gasinski, M. Smolka, Existence of solutions for wave-type hemivariational inequalities with noncoercive viscosity damping, J. Math. Anal. Appl. 270 (2002), 150-164.
- [8] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis. Volume I: Theory, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.
- [9] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis, Volume II: Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
- [10] J-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.
- [11] Z. Liu, S. Migórski, Noncoercive damping in dynamic hemivariational inequality with application to problem of piezoelectricity, Discrete Contin. Dyn. Syst. Ser. B 9 (2008), no. 1, 129-143.
- [12] S. Migórski, Boundary hemivariational inequalities of hyperbolic type and applications, J. Global Optim. 31 (2005), 505-533.
- [13] S. Migórski, Boundary hemivariational inequalities for a class of dynamic viscoelastic frictional contact problems, Computers Math. Appl. 52 (2006), 677-698.
- [14] S. Migórski, A. Ochal, Vanishing viscosity for hemivariational inequalities modeling dynamic problems in elasticity, Nonlinear Anal. 66 (2007), no. 8, 1840-1852.
- [15] A. Ochal, Existence results for evolution hemivariational inequalities of second order, Nonlinear Anal. 60 (2005), 1369-1391.
- [16] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonhomogeneous hemivariational inequalities with indefinite potential and Robin boundary condition, J. Optim. Theory Appl. 175 (2017), no. 2, 293-323.
- [17] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Sensitivity analysis for optimal control problems governed by nonlinear evolution inclusions, Adv. Nonlinear Anal. 6 (2017), no. 2, 199-235.
- [18] N.S. Papageorgiou, N. Yannakakis, Second order nonlinear evolutions inclusions I: existence and relaxations results, Acta Math. Sinica (English Series) 21 (2005), 977-996.
- [19] N.S. Papageorgiou, N. Yannakakis, Second order nonlinear evolutions inclusions II: structure of the solution set, Acta Math. Sinica (English Series) 22 (2006), 195-206.
- [20] T. Roubiček, Nonlinear Partial Differential Equations with Applications, Birkhäuser, Basel, 2013.
- [21] E. Zeidler, Nonlinear Functional Analysis and its Applications II/A and II/B, Springer, New York, 1990.

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