# NONLINEAR SECOND ORDER EVOLUTION INCLUSIONS WITH NONCOERCIVE VISCOSITY TERM

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ABSTRACT. In this paper we deal with a second order nonlinear evolution inclusion, with a nonmonotone, noncoercive viscosity term. Using a parabolic regularization (approximation) of the problem and *a priori* bounds that permit passing to the limit, we prove that the problem has a solution.

## 1. INTRODUCTION

Let T = [0, b] and let  $(X, H, X^*)$  be an evolution triple of spaces, with the embedding of X into H being compact (see Section 2 for definitions).

In this paper, we study the following nonlinear evolution inclusion:

(1) 
$$\left\{\begin{array}{l} u''(t) + A(t, u'(t)) + Bu(t) \in F(t, u(t), u'(t)) \text{ for almost all } t \in T, \\ u(0) = u_0, \ u'(0) = u_1. \end{array}\right\}$$

In the past, such multi-valued problems were studied by Gasinski [3], Gasinski and Smolka [6, 7], Migórski *et al.* [11, 12, 13, 14], Ochal [15], Papageorgiou, Rădulescu and Repovš [16, 17], Papageorgiou and Yannakakis [18, 19]. The works of Gasinski [3], Gasinski and Smolka [6, 7] and Ochal [15], all deal with hemivariational inequalities, that is,  $F(t, x, y) = \partial J(x)$  with  $J(\cdot)$  being a locally Lipschitz functional and  $\partial J(\cdot)$  denoting the Clarke subdifferential of  $J(\cdot)$ . In Papageorgiou and Yannakakis [18, 19], the multivalued term F(t, x, y) is general (not necessarily of the subdifferential type) and depends also on the time derivative of the unknown function  $u(\cdot)$ . With the exception of Gasinski and Smolka [7], in all the other works the viscosity term  $A(t, \cdot)$  is assumed to be coercive or zero. In the work of Gasinski and Smolka [7], the viscosity term is autonomous (that is, time independent) and  $A: X \to X^*$  is linear and bounded.

In this work, the viscosity term  $A: T \times X \to X^*$  is time dependent, noncoercive, nonlinear and nonmonotone in  $x \in X$ . In this way, we extend and improve the result of Gasinski and Smolka [7]. Our approach uses a kind of parabolic regularization of the inclusion, analogous to the one used by Lions [10, p. 346] in the context of semilinear hyperbolic equations.

### 2. MATHEMATICAL BACKGROUND AND HYPOTHESES

Let V, Y be Banach spaces and assume that V is embedded continuously and densely into Y (denoted by  $V \hookrightarrow Y$ ). Then we have the following properties:

(i)  $Y^*$  is embedded continuously into  $V^*$ ;

(ii) if V is reflexive, then  $Y^* \hookrightarrow V^*$ .

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The following notion is a useful tool in the theory of evolution equations.

**Definition 1.** By an "evolution triple" (or "Gelfand triple") we understand a triple of spaces  $(X, H, X^*)$  such that

- (a) X is a separable reflexive Banach space and  $X^*$  is its topological dual;
- (b) *H* is a separable Hilbert space identified with its dual  $H^*$ , that is,  $H = H^*$  (pivot space);
- (c)  $X \hookrightarrow H$ .

Then from the initial remarks we have

$$X \hookrightarrow H = H^* \hookrightarrow X^*.$$

In what follows, we denote by  $|| \cdot ||$  the norm of X, by  $| \cdot |$  the norm of H and by  $|| \cdot ||_*$  the norm of  $X^*$ . Evidently we can find  $\hat{c}_1, \hat{c}_2 > 0$  such that

$$|\cdot| \leq \hat{c}_1 ||\cdot||$$
 and  $||\cdot||_* \leq \hat{c}_2 |\cdot|$ 

By  $(\cdot, \cdot)$  we denote the inner product of H and by  $\langle \cdot, \cdot \rangle$  the duality brackets for the pair  $(X^*, X)$ . We have

(2) 
$$\langle \cdot, \cdot \rangle |_{H \times X} = (\cdot, \cdot).$$

Let 1 . The following space is important in the study of problem (1):

$$W_p(0,b) = \left\{ u \in L^p(T,X) : u' \in L^{p'}(T,X^*) \right\} \left( \frac{1}{p} + \frac{1}{p'} = 1 \right).$$

Here u' is understood in the distributional sense (weak derivative). We know that  $L^p(T, X)^* = L^{p'}(T, X^*)$  (see, for example, Gasinski and Papageorgiou [4, p. 129]). Suppose that  $u \in W_p(0, b)$ . If we view  $u(\cdot)$  as an  $X^*$ -valued function, then  $u(\cdot)$  is absolutely continuous, hence differentiable almost everywhere and this derivative coincides with the distributional one. So,  $u' \in L^{p'}(T, X^*)$  and we can say

$$W_p(0,b) \subseteq AC^{1,p'}(T,X^*) = W^{1,p'}((0,b),X^*).$$

The space  $W_p(0, b)$  is equipped with the norm

$$||u||_{W_p} = \left[ ||u||_{L^p(T,X)}^p + ||u'||_{L^{p'}(T,X^*)}^p \right]^{\frac{1}{p}} \text{ for all } u \in W_p(0,b).$$

Evidently, another equivalent norm on  $W_p(0, b)$  is

$$|u|_{W_p} = ||u||_{L^p(T,X)} + ||u'||_{L^p(T,X^*)}$$
 for all  $u \in W_p(0,b)$ .

With any of the above norms,  $W_p(0, b)$  becomes a separable reflexive Banach space. We have that

- (3)  $W_p(0,b) \hookrightarrow C(T,H);$
- (4)  $W_p(0,b) \hookrightarrow L^p(T,H)$  and the embedding is compact.

The elements of  $W_p(0, b)$  satisfy an integration by parts formula which will be useful in our analysis.

**Proposition 2.** If  $u, v \in W_p(0, b)$  and  $\xi(t) = (u(t), v(t))$  for all  $t \in T$ , then  $\xi(\cdot)$  is absolutely continuous and  $\frac{d\xi}{dt}(t) = \langle u'(t), v(t) \rangle + \langle u(t), v'(t) \rangle$  for almost all  $t \in T$ .

Now suppose that  $(\Omega, \Sigma, \mu)$  is a finite measure space,  $\Sigma$  is  $\mu$  – complete and Y is a separable Banach space. A multifunction (set-valued function)  $F : \Omega \to 2^Y \setminus \{\emptyset\}$ is said to be "graph measurable", if

Gr 
$$F = \{(\omega, y) \in \Omega \times Y : y \in F(\omega)\} \in \Sigma \times B(Y),$$

with B(Y) being the Borel  $\sigma$ -field of Y.

If  $F(\cdot)$  has closed values, then graph measurability is equivalent to saying that for every  $y \in Y$  the  $\mathbb{R}_+$ -valued function

$$\omega \mapsto d(y, F(\omega)) = \inf\{||y - v||_Y : v \in F(\omega)\}$$

is  $\Sigma$ -measurable.

Given a graph measurable multifunction  $F : \Omega \to 2^Y \setminus \{\emptyset\}$ , the Yankov-von Neumann-Aumann selection theorem (see Hu and Papageorgiou [8, p. 158]) implies that  $F(\cdot)$  admits a measurable selection, i.e. that there exists  $f : \Omega \to Y$  a  $\Sigma$ measurable function such that  $f(\omega) \in F(\omega)$   $\mu$ -almost everywhere. In fact, we can find an entire sequence  $\{f_n\}_{n\geq 1}$  of measurable selections such that  $F(\omega) \subseteq \overline{\{f_n(\omega)\}}_{n\geq 1}$   $\mu$ -almost everywhere.

For  $1 \leq p \leq \infty$ , we define

$$S_F^p = \{ f \in L^p(\Omega, Y) : f(\omega) \in F(\omega) \ \mu\text{-almost everywhere} \}.$$

It is easy to see that  $S_F^p \neq \emptyset$  if and only if  $\omega \mapsto \inf\{||v||_Y : v \in F(\omega)\}$  belongs to  $L^p(\Omega)$ . This set is "decomposable" in the sense that if  $(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$ , then

$$\chi_A f_1 + \chi_{A^c} f_2 \in S_F^p.$$

Finally, for a sequence  $\{C_n\}_{n \ge 1}$  of nonempty subsets of Y, we define

$$w - \limsup_{n \to \infty} C_n = \{ y \in Y : y = w - \lim_{k \to \infty} y_{n_k}, y_{n_k} \in C_{n_k}, n_1 < n_2 < \dots < n_k < \dots \}.$$

For more details on the notions discussed in this section, we refer to Gasinski and Papageorgiou [4], Roubiček [20], Zeidler [21] (for evolution triples and related notations) and Hu and Papageorgiou [8] (for measurable multifunctions).

Let V be a reflexive Banach space and  $A: V \to V^*$  a map. We say that A is "pseudomonotone", if A is continuous from every finite dimensional subspace of V into  $V_w^*$  (= the dual V<sup>\*</sup> equipped with the weak topology) and if

$$v_n \xrightarrow{w} v$$
 in V,  $\limsup_{n \to \infty} \langle A(v_n), v_n - v \rangle \leq 0$ 

then

$$\langle A(v), v - y \rangle \leq \liminf_{n \to \infty} \langle A(v_n), v_n - y \rangle$$
 for all  $y \in V$ .

An everywhere defined maximal monotone operator is pseudomonotone. If V is finite dimensional, then every continuous map  $A: V \to V^*$  is pseudomonotone. In what follows, for any Banach space Z, we will use the following notations:

$$P_{f(c)}(Z) = \{C \subseteq Z : C \text{ is nonempty, closed (and convex})\},\$$
  
$$P_{(w)k(c)}(Z) = \{C \subseteq Z : C \text{ is nonempty, (weakly-) compact (and convex})}\}.$$

The hypotheses on the data of problem (1) are the following:

 $H(A): A: T \times T \to X^*$  is a map such that

- (i) for all  $y \in X, t \mapsto A(t, y)$  is measurable;
- (ii) for almost all  $t \in T$ , the map  $y \mapsto A(t, y)$  is pseudomonotone;

- (iii)  $||A(t,y)||_* \leq a_1(t) + c_1||y||^{p-1}$  for almost all  $t \in T$  and all  $y \in X$ , with  $a_1 \in L^{p'}(T), c_1 > 0, 2 \leq p < \infty$ ;
- (iv)  $\langle A(t,y), y \rangle \ge 0$  for almost all  $t \in T$  and all  $y \in X$ .

 $H(B): B \in \mathscr{L}(X, X^*), \langle Bx, y \rangle = \langle x, By \rangle$  for all  $x, y \in X$  and  $\langle Bx, x \rangle \ge c_0 ||x||^2$  for all  $x \in X$  and some  $c_0 > 0$ .

- $H(F): F: T \times H \times H \to P_{f_c}(H)$  is a multifunction such that
  - (i) for all  $x, y \in H$ ,  $t \mapsto F(t, x, y)$  is graph measurable;
- (ii) for almost all  $t \in T$ , the graph  $\operatorname{Gr} F(t, \cdot, \cdot)$  is sequentially closed in  $H \times H_w \times H_w$  (here  $H_w$  denotes the Hilbert space H furnished with the weak topology);
- (iii)  $|F(t, x, y)| = \sup\{|h| : h \in F(t, x, y)\} \leq a_2(t)(1 + |x| + |y|)$  for almost all  $t \in T$  and all  $x, y \in H$  with  $a_2 \in L^2(T)_+$ .

**Definition 3.** We say that  $u \in C(T, X)$  is a "solution" of problem (1) with  $u_0 \in X$ ,  $u_1 \in H$ , if

- $u' \in W_p(0,b)$  and
- there exists  $f \in S^2_{F(\cdot,u(\cdot),u'(\cdot))}$  such that

$$\left\{ \begin{array}{l} u''(t) + A(t, u'(t)) + Bu(t) = f(t) \text{ for almost all } t \in T, \\ u(0) = u_0, u'(0) = u_1. \end{array} \right\}$$

In what follows, we denote by  $S(u_0, u_1)$  the set of solutions of problem (1). Recalling that  $W_p(0, b) \hookrightarrow C(T, H)$  (see (3)), we have that

$$S(u_0, u_1) \subseteq C^1(T, H).$$

By Troyanski's renorming theorem (see Gasinski and Papageorgiou [4, p. 911]) we may assume without loss of generality that both X and  $X^*$  are locally uniformly convex. Let  $\mathcal{F}: X \to X^*$  be the duality map of X defined by

$$\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x \rangle = ||x||^2 = ||x^*||_*^2\}.$$

We know that  $\mathcal{F}(\cdot)$  is single-valued and a homeomorphism (see Gasinski and Papageorgiou [4, p. 316] and Zeidler [21, p. 861]).

For every  $r \ge p$ , let  $K_r : X \to X^*$  be the map defined by

$$K_r(y) = ||y||^{r-2} \mathcal{F}(y)$$
 for all  $y \in X$ .

## 3. EXISTENCE THEOREM

Given  $\epsilon > 0$ , we consider the following perturbation (parabolic regularization) of problem (1):

$$\left\{\begin{array}{l}u''(t) + A(t, u'(t)) + \epsilon K_r(u'(t)) + Bu(t) \in F(t, u(t), u'(t)) \text{ for a.a. } t \in T, \\u(0) = u_0, \ u'(0) = u_1.\end{array}\right\}$$

Consider the map  $A_{\epsilon}: T \times X \to X^*$  defined by

$$A_{\epsilon}(t,y) = A(t,y) + \epsilon K_r(y)$$
 for all  $t \in T$ , and all  $y \in X$ .

This map has the following properties:

- (i) for all  $y \in X$ , the map  $t \mapsto A_{\epsilon}(t, y)$  is measurable;
- (ii) for almost all  $t \in T$ , the map  $y \mapsto A_{\epsilon}(t, y)$  is pseudomonotone;

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- (iii)  $||A_{\epsilon}(t,y)||_{*} \leq \hat{a}_{1}(t) + \hat{c}_{1}||y||^{r-1}$  for almost all  $t \in T$ , all  $y \in X$  and with  $\hat{a}_1 \in L^{p'}(T), \ \hat{c}_1 > 0 \ (\text{recall that } r \ge p \text{ and } \frac{1}{r} + \frac{1}{r'} = 1);$ (iv)  $\langle A_{\epsilon}(t, y), y \rangle \ge \epsilon ||y||^r \text{ for all } t \in T, \text{ all } y \in X.$

So, in problem (1) the viscosity term  $A_{\epsilon}(t, \cdot)$  is coercive. Therefore we can apply Theorem 1 of Papageorgiou and Yannakakis [18] and we obtain the following existence result for the approximate (regularized) problem (5).

**Proposition 4.** If hypotheses H(A), H(B), H(F) hold and  $u_0 \in X, u_1 \in H$ , then problem (5) admits a solution  $u_{\epsilon} \in W^{1,r}((0,b),X) \cap C^1(T,H)$  with

$$u'_{\epsilon} \in W_r(0,b).$$

To produce a solution for the original problem (1), we have to pass to the limit as  $\epsilon \to 0^+$ . To do this, we need to have a priori bounds for the solutions  $u_{\epsilon}(\cdot)$ which are independent of  $\epsilon \in (0, 1]$  and  $r \ge p$ .

**Proposition 5.** If hypotheses H(A), H(B), H(F) hold,  $u_0 \in X, u_1 \in H$  and  $u(\cdot)$ is a solution of (5), then there exists  $M_0 > 0$  which is independent of  $\epsilon \in (0, 1]$  and  $r \ge p$  for which we have

$$||u||_{C(T,X)}, \ ||u'||_{C(T,H)}, \ \epsilon^{\frac{1}{r}}||u'||_{L^{r}(T,X)}, \ ||u''||_{L^{2}(T,X^{*})} \leqslant M_{0}.$$

*Proof.* It follows from Proposition 4 that  $u' \in W_r(0,b)$  and that there exists  $f \in W_r(0,b)$  $S_{F(\cdot,u(\cdot),u'(\cdot))}^2$  such that

$$u''(t) + A(t, u'(t)) + \epsilon K_r(u'(t)) + Bu(t) = f(t)$$
for almost all  $t \in T$ .

We act with  $u'(t) \in X$ . Then

(6) 
$$\langle u''(t), u'(t) \rangle + \langle A(t, u'(t)), u'(t) \rangle + \epsilon \langle K_r(u'(t)), u'(t) \rangle = (f(t), u'(t))$$
  
for almost all  $t \in T$  (see (2)).

We examine separately each summand on the left-hand side of (6). Recall that  $u'_r \in W_r(0,b)$ . So from Proposition 2 (the integration by parts formula), we have

(7) 
$$\langle u''(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} |u'(t)|^2 \text{ for almost all } t \in T.$$

Hypothesis H(A)(iv) and the definition of the duality map, imply that

(8) 
$$\langle A(t, u'(t)), u'(t) \rangle + \epsilon \langle K_r(u'(t)), u'(t) \rangle \ge \epsilon ||u'(t)||^r$$
 for almost all  $t \in T$ .

By hypothesis H(B), we have

(9) 
$$\langle Bu(t), u'(t) \rangle = \frac{1}{2} \frac{d}{dt} \langle Bu(t), u(t) \rangle$$
 for almost all  $t \in T$ .

We return to (6) and use (7), (8), (9). We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u'(t)|_2 + \epsilon||u'(t)||^r + \frac{1}{2}\frac{d}{dt}\langle Bu(t), u(t)\rangle \leqslant (f(t), u'(t)) \text{ for a.a. } t \in T, \\ \Rightarrow & \frac{1}{2}|u'(t)|^2 + \epsilon \int_0^t ||u'(s)||^r ds + c_0||u(t)||^2 \\ (10)\leqslant & \int_0^t (f(s), u'(s))ds + \frac{1}{2}|u_1|^2 + \frac{1}{2}||B||_{\mathscr{L}}||u_0||^2 \text{ (see hypothesis } H(B)). \end{aligned}$$

Using hypothesis H(F)(iii), we get

$$\int_{0}^{t} (f(s), u'(s)) ds$$

$$\leq \int_{0}^{t} [a_{2}(s) + a_{2}(s) (|u(s)| + |u'(s)|)] |u'(s)| ds$$

$$(11) \qquad \leq \int_{0}^{t} |u'(s)|^{2} ds + \int_{0}^{t} a_{2}(s)^{2} ds + \int_{0}^{t} a_{2}(s)^{2} [|u(s)|^{2} + |u'(s)|^{2}] ds.$$

Recall that  $u\in W^{1,r}((0,b),X)$  (see Proposition 4). So,  $u\in AC^{1,r}(T,H)$  and we can write

$$u(t) = u_0 + \int_0^t u'(s)ds \text{ for all } t \in T$$

 $(12) \Rightarrow |u(t)|^2 \leq 2|u_0|^2 + 2b \int_0^t |u'(s)|^2 ds \text{ for all } t \in T \text{ (using Jensen's inequality).}$ 

We use (12) in (11) and obtain

$$\int_{0}^{t} (f(s), u'(s)) ds$$

$$\leqslant \quad ||a_{2}||_{2}^{2} + \int_{0}^{t} \left[1 + a_{2}(s)^{2}\right] |u'(s)|^{2} ds + \int_{0}^{t} 2a_{2}(s)^{2} \left[|u_{0}|^{2} + b \int_{0}^{s} |u'(\tau)|^{2} d\tau\right] ds$$

$$(13) \qquad c_{2} + \int_{0}^{t} \eta(s) |u'(s)|^{2} ds + 2b \int_{0}^{t} a_{2}(s)^{2} \int_{0}^{s} |u'(\tau)|^{2} d\tau ds$$
for some  $c_{2} > 0$  and  $\eta \in L^{1}(T)$ .

We use (13) in (10) and have

$$\frac{1}{2}|u'(t)|^2 + \epsilon \int_0^t ||u'(s)||^p ds + c_0||u(t)||^2$$
(14)  $\leq c_3 + \int_0^t \eta(s)|u'(s)|^2 ds + 2b \int_0^t a_2(s)^2 \int_0^s |u'(\tau)|^2 d\tau ds$  for some  $c_3 > 0$ .

Invoking Proposition 1.7.87 of Denkowski, Migórski and Papageorgiou [2, p. 128] we can find M > 0 (independent of  $\epsilon \in (0, 1]$  and  $r \ge p$ ) such that

$$|u'(t)|^2 \leqslant M \text{ for all } t \in T,$$
  
$$\Rightarrow \quad ||u'||_{C(T,H)} \leqslant M_1 = M^{\frac{1}{2}}.$$

Using this bound in (14), we can find  $M_2 > 0$  (independent of  $\epsilon \in (0, 1]$  and  $r \ge p$ ) such that

$$||u||_{C(T,X)} \leq M_2$$
 and  $\epsilon^{\frac{1}{r}} ||u'||_{L^r(T,X)} \leq M_2$ .

Finally, directly from (5), we see that there exists  $M_3 > 0$  (independent of  $\epsilon \in (0, 1]$  and  $r \ge p$ ) such that

$$||u''||_{L^{r'}}(T,X^*) \leqslant M_3$$

We set  $M_0 = \max\{M_1, M_2, M_3\} > 0$  and get the desired bound.

The bounds produced in Proposition 5 permit passing to the limit as  $\epsilon \to 0^+$  to produce a solution for problem (1).

**Theorem 6.** If hypotheses H(A), H(B), H(F) hold and  $u_0 \in X, u_1 \in H$ , then  $S(u_0, u_1) \neq \emptyset$ .

*Proof.* Let  $\epsilon_n \to 0^+$  and let  $u_n = u_{\epsilon_n}$  be solutions of the "regularized" problem (5) (see Proposition 4). Because of the bounds established in Proposition 5 and by passing to a suitable subsequence if necessary, we can say that

(15) 
$$\begin{cases} u_n \xrightarrow{w^*} u \text{ in } L^{\infty}(T, X), \ u_n \xrightarrow{w} u \text{ in } C(T, H), \ u_n \to u \text{ in } L^r(T, H) \\ u'_n \xrightarrow{w^*} y \text{ in } L^{\infty}(T, H), \ u''_n \xrightarrow{w} v \text{ in } L^{r'}(T, X^*) \text{ (see (3) and (4)).} \end{cases}$$

Recall that  $u_n \in AC^{1,r}(T,H)$  for all  $n \in \mathbb{N}$  and so

$$u_n(t) = u_0 + \int_0^t u'_n(s)ds \text{ for all } t \in T,$$
  

$$\Rightarrow \quad u(t) = u_0 + \int_0^t y(s)ds \text{ for all } t \in T \text{ (see (15))},$$
  

$$\Rightarrow \quad u \in AC^{1,r}(T,H) \text{ and } u' = y.$$

Since  $u_n \in W_r(0, b)$  for all  $n \in \mathbb{N}$ , we have

 $v = y' = u'' \in L^{r'}(T, X^*)$  (see Hu and Papageorgiou [9, p. 6]).

Let  $a: L^r(T, X) \to L^{r'}(T, X^*)$  be the nonlinear map defined by

$$u(u)(\cdot) = A(\cdot, u(\cdot))$$
 for all  $u \in L^r(T, X)$ .

Also, let  $\hat{K}_r : L^r(T, X) \to L^{r'}(T, X^*)$  be defined by

$$\hat{K}_r(u)(\cdot) = ||u(\cdot)||^{r-2} \mathscr{F}(u(\cdot))$$
 for all  $u \in L^r(T, X)$ .

Both maps are continuous and monotone, hence maximal monotone (see Gasinski and Papageorgiou [4, Corollary 3.2.32, p. 320]).

Finally, let  $\hat{B} \in \mathscr{L}(L^r(T, X), L^{r'}(T, X^*))$  be defined by

$$\hat{B}(u)(\cdot) = B(u(\cdot))$$
 for all  $u \in L^r(T, X)$ .

We have

(16) 
$$u_n'' + a(u_n') + \epsilon_n \hat{K}_r(u_n') + \hat{B}u_n = f_n \text{ in } L^r(T, X^*)$$
with  $f_n \in S^2_{F(\cdot, u_n(\cdot), u_n'(\cdot))}$  for all  $n \in \mathbb{N}$ .

From (15) we have

(17) 
$$u_n \xrightarrow{w} u \text{ in } L^r(T, X),$$
$$\Rightarrow \quad \hat{B}u_n \xrightarrow{w} \hat{B}u \text{ in } L^{r'}(T, X^*) \text{ as } n \to \infty.$$

Also, we have

$$\begin{aligned} \|\hat{K}_{r}(u'_{n})\|_{L^{r'}(T,X^{*})} &= \|u'_{n}\|_{L^{r}(T,X)}^{r-1}, \\ \Rightarrow \quad \epsilon_{n}\|\hat{K}_{r}(u'_{n})\|_{L^{r'}(T,X^{*})} &= \epsilon_{n}^{\frac{1}{r}} \left(\epsilon_{n}^{\frac{1}{r}}\|u'_{n}\|_{L^{r}(T,X)}\right)^{r-1} \text{ (recall that } \frac{1}{r} + \frac{1}{r'} = 1) \\ &\leq \epsilon_{n}^{\frac{1}{r}}M_{0}^{r-1} \text{ for all } n \in \mathbb{N} \text{ (see Proposition 5)} \\ (18) \Rightarrow \quad \epsilon_{n}\|\hat{K}_{r}(u'_{r})\|_{L^{r'}(T,X^{*})} \to 0 \text{ as } n \to \infty \\ \text{From (15) and since } v = u'', \text{ we have} \\ (19) \qquad \qquad u''_{n} \xrightarrow{w} u'' \text{ in } L^{r'}(T,X^{*}). \end{aligned}$$

Finally, hypothesis H(F)(iii) and Proposition 5 imply that

$${f_n}_{n \ge 1} \subseteq L^2(T, H)$$
 is bounded

By passing to a subsequence if necessary, we may assume that

$$f_n \xrightarrow{w} f$$
 in  $L^2(T, H)$ .

Invoking Proposition 3.9 of Hu and Papageorgiou [8, p. 694], we have

$$f(t) \in \overline{\operatorname{conv}} w - \limsup_{n \to \infty} \{f_n(t)\}$$

$$(20) \qquad \leqslant \overline{\operatorname{conv}} w - \limsup_{n \to \infty} F(t, u_n(t), u'_n(t)) \text{ for almost all } t \in T \text{ (see (16))}.$$

From (15) we see that

$$u'_n \xrightarrow{w} u'$$
 in  $W^{1,r'}((0,b), X^*)$ .

Recall that  $W^{1,r'}((0,b),X^*) \hookrightarrow C(T,X^*)$ . So, it follows that

(21) 
$$u'_n \xrightarrow{w} u' \text{ in } C(T, X^*) \\ \Rightarrow u'_n(t) \xrightarrow{w} u'(t) \text{ in } X^* \text{ for all } t \in T.$$

On the other hand, by Proposition 5 we have

$$|u'_n(t)| \leq M_0$$
 for all  $t \in T$ , all  $n \in \mathbb{N}$ .

So, by passing to a subsequence (a priori the subsequence depends on  $t \in T$ ), we have

$$\begin{aligned} u'_n(t) &\xrightarrow{w} \hat{y}(t) \text{ in } H \\ \Rightarrow & \hat{y}(t) = u'(t) \text{ for all } t \in T \text{ (see (21)).} \end{aligned}$$

Hence for the original sequence we have

(22) 
$$u'_n(t) \xrightarrow{w} u'(t)$$
 in  $H$  for all  $t \in T$ .

We know that  $\{u_n\}_{n \ge 1} \subseteq W_r(0, b)$  is bounded (see Proposition 5) and recall that  $W_r(0, b) \hookrightarrow L^r(T, H)$  compactly (see (4)). From this compact embedding and from (22), we obtain

(23) 
$$u_n(t) \to u(t)$$
 in  $H$  for all  $t \in T$  as  $n \to \infty$ .

From (20), (22), (23) and hypothesis H(F)(iii) we infer that

$$\begin{split} f(t) &\in F(t, u(t), u'(t)) \text{ for almost all } t \in T, \\ \Rightarrow \quad f \in S^2_{F(\cdot, u(\cdot), u'(\cdot))}. \end{split}$$

In what follows, we denote by  $((\cdot, \cdot))$  the duality brackets for the pair

$$(L^r(T, X^*), L^r(T, X)).$$

Acting with  $u'_n - u' \in L^r(T, X)$  on (16), we have

$$((u_n'', u_n' - u')) + ((a(u_n'), u_n' - u')) + ((\epsilon_n \hat{K}_r(u_n'), u_r' - u')) + ((\hat{B}u_n, u_n' - u'))$$
  
(24) =  $\int_0^b (f_n, u_n' - u') dt$  for all  $n \in \mathbb{N}$ .

Note that

$$((u_n'', u_n' - u')) = \int_0^b \langle u_n'', u_n' - u' \rangle dt$$
  

$$= \int_0^b \langle u_n'' - u'', u_n' - u' \rangle dt + ((u'', u_n' - u'))$$
  

$$= \int_0^b \frac{1}{2} \frac{d}{dt} |u_n' - u'|^2 dt + ((u'', u_n' - u')) \text{ (see Proposition 2)}$$
  

$$= \frac{1}{2} |u_n'(b) - u'(b)|^2 + ((u'', u_n' - u'))$$
  
(since  $u_n'(0) = u'(0) = u_1$  for all  $n \in \mathbb{N}$ , see (22))  
(25)  $\Rightarrow \liminf_{n \to \infty} ((u_n'', u_n' - u')) = \frac{1}{2} \liminf_{n \to \infty} |u_n'(b) - u'(b)|^2 \ge 0.$ 

Also we have

$$((\hat{B}(u_n - u), u'_n - u')) = \int_0^b \frac{1}{2} \frac{d}{dt} \langle B(u_n - u), u_n - u \rangle dt$$
$$\frac{1}{2} \langle B(u_n - u)(b), (u_n - u)(b) \rangle \ge 0 \text{ (see hypothesis } H(B))$$
$$(26) \qquad \Rightarrow \quad ((\hat{B}u, u'_n - u')) \le ((\hat{B}u_n, u'_n - u')) \text{ for all } n \in \mathbb{N}.$$

Recall that

$$\epsilon_n^{\frac{1}{2}} ||u_n||_{L^r(T,X)} \leq M_0 \text{ for all } n \in \mathbb{N} \text{ all } r \geq p \text{ (see Proposition 5).}$$

Suppose that  $r_m \to +\infty$ ,  $r_m \ge p$  for all  $m \in \mathbb{N}$ . Then for every  $n \in \mathbb{N}$ ,  $\epsilon_n^{\frac{1}{r_m}} \to 1$  as  $m \to \infty$ . Invoking Problem 1.175 of Gasinski and Papageorgiou [5], we can find  $\{m_n\}_{n\ge 1}$  with  $m_n \to +\infty$  such that

$$\epsilon_n^{\frac{1}{r_{m_n}}} \to 1 \text{ as } n \to \infty.$$

Therefore there exists  $n_0 \in \mathbb{N}$  such that

$$\frac{1}{2} \leqslant \epsilon_n^{\frac{1}{r_{m_n}}} \text{ for all } n \geqslant n_0,$$
  
$$\frac{1}{2} ||u_n'||_{L^{r_{m_n}}(T,X)} \leqslant M_0 \text{ for all } n \geqslant n_0,$$
  
$$\Rightarrow ||u_n'||_{L^p(T,X)} \leqslant 2M_0 \text{ for all } n \geqslant n_0 \text{ (recall that } r_{m_n} \geqslant p).$$

On account of (15) and since y = u', we have

(27) 
$$u'_n \xrightarrow{w} u' \text{ in } L^p(T, X).$$

Then from (26) and (27) it follows that

(28) 
$$0 \leq \liminf_{n \to \infty} ((\hat{B}u_n, u'_n - u')).$$

In addition, we have

(29) 
$$\epsilon_n \hat{K}_p(u'_n) \to 0 \text{ in } L^{p'}(T, X^*) \text{ as } n \to \infty \text{ (see (18))}.$$

By Proposition 5 and (27) it follows that

$$\{u'_n\}_{n \ge 1} \subseteq W_p(0, b) \text{ is bounded},$$
  
 
$$\Rightarrow \ \{u'_n\}_{n \ge 1} \subseteq L^p(T, H) \text{ is relatively compact (see (4))}.$$

Therefore we have

(30) 
$$u'_{n} \to u' \text{ in } L^{p}(T, H) \text{ (see (27))},$$
$$\Rightarrow \int_{0}^{b} (f_{n}, u'_{n} - u') dt \to 0 \text{ as } n \to \infty \text{ (recall that } p \ge 2).$$

If in (24) we pass to the limit as  $n \to \infty$  and use (25), (28), (29), (30), then

$$\limsup_{n \to \infty} ((a(u'_n), u'_n - u')) \leqslant 0.$$

Invoking Theorem 2.35 of Hu and Papageorgiou [9, p. 41], we have

(31) 
$$a(u_n) \xrightarrow{w} a(u') \text{ in } L^{p'}(T, X^*) \text{ as } n \to \infty.$$

In (24) we pass to the limit as  $n \to \infty$  and use (15) (with v = u'') (27), (29), (31). We obtain

$$\begin{split} u'' + a(u') + Bu &= f, \ u(0) = u_0, u'(0) = u_1, f \in S^2_{F(\cdot, u(\cdot), u'(\cdot))}, \\ \Rightarrow \quad u \in S(u_0, u_1) \neq \emptyset. \end{split}$$

The proof is now complete.

3.1. An example. We illustrate the main abstract result of this paper with a hyperbolic boundary value problem. Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain. We consider the following boundary value problem (32)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(a(t,z)|Du_t|^{p-2}Du_t\right) + \beta(z)u_t - \Delta u = f(t,z,u) + \gamma u_t \text{ in } T \times \Omega, \\ u|_{T \times \partial \Omega} = 0, \ u(0,z) = u_0(z), \ u_t(0,z) = u_1(z), \end{cases}$$
with  $u_t = \frac{\partial u}{\partial t}, \ 2 \leqslant p \leqslant \infty, \ \gamma > 0.$ 

The forcing term  $f(t, z, \cdot)$  need not to be continuous. So, following Chang [1], to deal with (32), we replace it by a multivalued problem (partial differential inclusion), by filling in the gaps at the discontinuity points of  $f(t, z, \cdot)$ . So we define

$$f_l(t, z, x) = \liminf_{x' \to x} f(t, z, x') \text{ and } f_u(t, z, x) = \limsup_{x' \to x} f(t, z, x').$$

Then we replace (32) by the following partial differential inclusion (33)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(a(t,z)|Du_t|^{p-2}Du_t\right) + \beta(z)u_t - \Delta u \in [f_l(t,z,u), f_u(t,z,u)] \text{ in } T \times \Omega, \\ u|_{T \times \partial \Omega} = 0, \ u(0,z) = u_0(z), \ u_t(0,z) = u_1(z). \end{cases}$$

Our hypotheses on the data of (33) are the following:

 $H(a): a \in L^{\infty}(T \times \Omega), a(t, z) \ge 0$  for almost all  $(t, z) \in T \times \Omega$ .

 $H(\beta): \beta \in L^{\infty}(\Omega), \ \beta(z) \ge 0$  for almost all  $z \in \Omega$ .

 $H(f): f: T \times \Omega \times \mathbb{R} \to \mathbb{R}$  is a function such that

- (i)  $f_l$ ,  $f_u$  are superpositionally measurable (that is, for all  $u : T \times \Omega \to \mathbb{R}$  measurable, the functions  $(t, z) \mapsto f_l(t, z, u(t, z))$ ,  $f_u(t, z, u(t, z))$  are both measurable);
- (ii) there exists  $a \in L^2(T \times \Omega)$  such that

$$|f(t,z,x)| \leq a_2(t,z)(1+|x|)$$
 for almost all  $(t,z) \in T \times \Omega$ , all  $x \in \mathbb{R}$ .

Let  $X = W_0^{1,p}(\Omega)$ ,  $H = L^2(\Omega)$  and  $X^* = W^{-1,p'}(\Omega)$ . Then  $(X, H, X^*)$  is an evolution triple with  $X \hookrightarrow H$  compactly (by the Sobolev embedding theorem).

Let  $A: T \times X \to X^*$  be defined by

$$\langle A(t,u),h\rangle = \int_{\Omega} a(t,z) |Du|^{p-2} (Du,Dh)_{\mathbb{R}^N} dz + \int_{\Omega} \beta(z) uhdz \text{ for all } u,h \in W_0^{1,p}(\Omega)$$

Then A(t, u) is measurable in  $t \in T$ , continuous and monotone in  $u \in W_0^{1,p}(\Omega)$ (hence, maximal monotone) and  $\langle A(t, u), u \rangle \ge 0$  for almost all  $t \in T$ , all  $u \in W_0^{1,p}(\Omega)$ .

Let  $B \in \mathscr{L}(X, X^*)$  be defined by

$$\langle Bu,h\rangle = \int_{\Omega} (Du,Dh)_{\mathbb{R}^N} dz$$
 for all  $u,h \in W_0^{1,p}(\Omega)$ .

Clearly, B satisfies hypothesis H(B).

Finally, let  $G(t, z, x) = [f_l(t, z, x), f_u(t, z, x)]$  and set

$$F(t, u, v) = S^2_{G(t, \cdot, u(\cdot))} + \gamma v \text{ for all } u, v \in L^2(\Omega).$$

Hypothesis H(f) implies that F satisfies H(F).

Using A(t, u), Bu and F(t, u, v) as defined above, we can rewrite problem (33) as the equivalent second order nonlinear evolution inclusion (1). Assuming that  $u_0 \in W_0^{1,p}(\Omega)$  and that  $u_1 \in L^2(\Omega)$ , we can use Theorem 6 and infer that problem (30) has a solution  $u \in C^1(T, L^2(\Omega)) \cap C(T, W^{1,p}(\Omega))$  with  $\frac{\partial u}{\partial t} \in L^p(\Omega, W_0^{1,p}(\Omega))$ and  $\frac{\partial^2 u}{\partial t} \in L^{p'}(\Omega, W^{-1,p'}(\Omega))$ .

Note that if a = 0, f(t, z, x) = x and  $\gamma = 0$ , then we have the Klein-Gordon equation. If  $f(t, z, x) = f(x) = \eta \sin x$  with  $\eta > 0$ , then we have the sine Gordon equation.

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