Spectral Picture For Rationally Multicyclic Subnormal Operators

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Abstract

For a pure bounded rationally cyclic subnormal operator S on a separable complex Hilbert space H, [Conway and Elias \(1993](#page-15-0)) shows that $clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S)))$. This paper examines the property for rationally multicyclic (N-cyclic) subnormal operators. We show: (1) There exists a 2-cyclic irreducible subnormal operator S with $\text{cos}(\sigma(S) \setminus \sigma_e(S)) \neq$ $\text{clos}(Int(\sigma(S)))$. (2) For a pure rationally N-cyclic subnormal operator S on H with the minimal normal extension M on $K \supset H$, let $\mathcal{K}_m = clos(span\{(M^*)^k x : x \in \mathcal{H}, 0 \leq k \leq m\})$. Suppose $M|_{K_{N-1}}$ is pure, then $clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S))).$

1 Introduction

Let H be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the space of bounded linear operators on H. An operator $S \in \mathcal{L}(\mathcal{H})$ is subnormal if there exist a separable complex Hilbert space K containing H and a normal operator $M_z \in \mathcal{L}(\mathcal{K})$ such that $M_z \mathcal{H} \subset \mathcal{H}$ and $S = M_z |_{\mathcal{H}}$. By the spectral theorem of normal operators, we assume that

$$
\mathcal{K} = \bigoplus_{i=1}^{m} L^2(\mu_i) \tag{1-1}
$$

where $\mu_1 >> \mu_2 >> ... >> \mu_m$ (m may be ∞) are compactly supported finite positive measures on the complex plane C, and M_z is multiplication by z on K. For $H = (h_1, ..., h_m) \in \mathcal{K}$ and $G = (g_1, ..., g_m) \in \mathcal{K}$, we define

$$
\langle H(z), G(z) \rangle = \sum_{i=1}^{m} h_i(z) \overline{g_i(z)} \frac{d\mu_i}{d\mu_1}, \ |H(z)|^2 = \langle H(z), H(z) \rangle. \tag{1-2}
$$

The inner product of H and G in $\mathcal K$ is defined by

$$
(H, G) = \int \langle H(z), G(z) \rangle \, d\mu_1(z). \tag{1-3}
$$

 M_z is the minimal normal extension if

$$
\mathcal{K} = clos\left(span(M_z^{*k} x : x \in \mathcal{H}, k \ge 0) \right). \tag{1-4}
$$

We will always assume that M_z is the minimal normal extension of S and K satisfies (1-1) to (1-4). For details about the functional model above and basic knowledge of subnormal operators, the reader shall consult Chapter II of the book [Conway \(1991](#page-15-1)).

For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, $\sigma_e(T)$ the essential spectrum of T, T^* its adjoint, $ker(T)$ its kernel, and $Ran(T)$ its range. For a subset $A \subset \mathbb{C}$, we set $Int(A)$ for its

interior, $\text{cos}(A)$ for its closure, A^c for its complement, and $\overline{A} = \{\overline{z} : z \in A\}$. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0, 1)$. Let P denote the set of polynomials in the complex variable z. For a compact subset $K \subset \mathbb{C}$, let $Rat(K)$ be the set of all rational functions with poles off K and let $R(K)$ be the uniform closure of $Rat(K)$.

A subnormal operator S on H is pure if for every non-zero invariant subspace I of $S(SI \subset I)$, the operator $S|_I$ is not normal. For $F_1, F_2, ..., F_N \in \mathcal{H}$, let

$$
R^2(S|F_1, F_2, ..., F_N) = clos\{r_1(S)F_1 + r_2(S)F_2 + ... + r_N(S)F_N\}
$$

in H, where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$ and let

$$
P^{2}(S|F_{1}, F_{2},..., F_{N}) = clos\{p_{1}(S)F_{1} + p_{2}(S)F_{2} + ... + p_{N}(S)F_{N}\}
$$

in H, where $p_1, p_2, ..., p_N \in \mathcal{P}$. A subnormal operator S on H is rationally multicyclic (N−cyclic) if there are N vectors $F_1, F_2, ..., F_N \in \mathcal{H}$ such that

$$
\mathcal{H} = R^2(S|F_1, F_2, \ldots, F_N)
$$

and for any $G_1, ..., G_{N-1} \in \mathcal{H}$,

$$
\mathcal{H} \neq R^2(S|G_1, G_2, ..., G_{N-1}).
$$

We call N is the rationally cyclic multiple of S. S is multicyclic (N –cyclic) if

$$
\mathcal{H} = P^2(S|F_1, F_2, \ldots, F_N)
$$

and for any $G_1, ..., G_{N-1} \in \mathcal{H}$,

$$
\mathcal{H} \neq P^2(S|G_1, G_2, ..., G_{N-1}).
$$

We call N is the cyclic multiple of S. In this case, $m \leq N$ where m is as in (1-1).

Let μ be a compactly supported finite positive measure on the complex plane $\mathbb C$ and let spt(μ) denote the support of μ . For a compact subset K with spt(μ) ⊂ K, let $R^2(K, \mu)$ be the closure of $Rat(K)$ in $L^2(\mu)$. Let $P^2(\mu)$ denote the closure of P in $L^2(\mu)$.

If S is rationally cyclic, then S is unitarily equivalent to multiplication by z on $R^2(\sigma(S), \mu_1)$, where $m = 1$ and $F_1 = 1$. We may write $R^2(S|F_1) = R^2(\sigma(S), \mu_1)$. If S is cyclic, then S is unitarily equivalent to multiplication by z on $P^2(\mu_1)$. We may write $P^2(S|F_1) = P^2(\mu_1)$.

For a rationally N-cyclic subnormal operator S with cyclic vectors $F_1, F_2, ..., F_N$ and $\lambda \in$ $\sigma(S)$, we denote the map

$$
E(\lambda) : \sum_{i=1}^{N} r_i(S) F_i \to \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \dots \\ r_N(\lambda) \end{bmatrix},
$$
\n(1-5)

where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$. If $E(\lambda)$ is bounded from K to (\mathbb{C}^N) where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$. If $E(\lambda)$ is bounded from K to $(\mathbb{C}^N, \|\cdot\|_{1,N})$, where $\|x\|_{1,N} = \sum_{i=1}^N |x_i|$ for $x \in \mathbb{C}^N$, then every component in the right hand side extends to a bounded linear functional on H and we will call λ a bounded point evaluation for S. We use $bpe(S)$ to denote the set of bounded point evaluations for S . The set $bpe(S)$ does not depend on the choices of cyclic vectors $F_1, F_2, ..., F_N$ (see Corollary 1.1 in [Mbekhta et al. \(2016](#page-15-2))). A point $\lambda_0 \in int(bpe(S))$ is called an analytic bounded point evaluation for S if there is a neighborhood $B(\lambda_0, \delta) \subset bpe(S)$ of λ_0 such that $E(\lambda)$ is analytic as a function of λ on $B(\lambda_0, \delta)$ (equivalently (1-5) is uniformly bounded for $\lambda \in B(\lambda_0, \delta)$. We use $abpe(S)$ to denote the set of analytic bounded point evaluations for S. The set $abpe(S)$ does not depend on the choices of cyclic vectors $F_1, F_2, ..., F_N$ (also see Remark 3.1 in [Mbekhta et al. \(2016](#page-15-2))). Similarly, for an N-cyclic subnormal operator S, we can define $bpe(S)$ and $abpe(S)$ if we replace $r_1, r_2, ..., r_N \in Rat(\sigma(S))$ in (1-5) by $p_1, p_2, ..., p_N \in \mathcal{P}$.

For $N = 1$, [Thomson \(1991](#page-15-3)) proves a remarkable structural theorem for $P^2(\mu)$.

Thomson's Theorem. *There is a Borel partition* $\{\Delta_i\}_{i=0}^{\infty}$ *of sptµ such that the space* $P^2(\mu|\Delta_i)$ *contains no nontrivial characteristic functions and*

$$
P^2(\mu) = L^2(\mu|_{\Delta_0}) \oplus \left\{\oplus_{i=1}^{\infty} P^2(\mu|_{\Delta_i})\right\}.
$$

Furthermore, if U_i *is the open set of analytic bounded point evaluations for* $P^2(\mu|_{\Delta_i})$ *for* $i \geq 1$ *, then* U_i *is a simply connected region and the closure of* U_i *contains* Δ_i *.*

[Conway and Elias \(1993](#page-15-0)) extends some results of Thomson's Theorem to the space $R^2(K, \mu)$, while [Brennan \(2008](#page-15-4)) expresses $R^2(K,\mu)$ as a direct sum that includes both Thomson's theorem and results of [Conway and Elias \(1993](#page-15-0)). For a compactly supported complex Borel measure ν of C, by estimating analytic capacity of the set $\{\lambda : |\mathcal{C}\nu(\lambda)| \geq c\}$, where $\mathcal{C}\nu$ is the Cauchy transform of ν (see Section 3 for definition), [Brennan \(2006. English\)](#page-15-5), [Aleman et al. \(2009](#page-15-6)), and [Aleman et al. \(2010\)](#page-15-7) provide interesting alternative proofs of Thomson's theorem. Both their proofs rely on X. Tolsa's deep results on analytic capacity. There are other related research papers for $N = 1$ in the history. For example, [Brennan \(1979\)](#page-15-8), [Hruscev \(1979. Russian\)](#page-15-9), [Brennan and Militzer \(2011\)](#page-15-10), and [Yang \(2016](#page-16-0)), etc.

Thomson's Theorem shows in Theorem 4.11 of [Thomson \(1991](#page-15-3)) that $abpe(S) = bpe(S)$ for a cyclic subnormal operator S (See also Chap VIII Theorem 4.4 in [Conway \(1991](#page-15-1))). Corollary 5.2 in [Conway and Elias \(1993](#page-15-0)) proves that the result holds for rationally cyclic subnormal operators. For $N > 1$, [Yang \(2018\)](#page-16-1) extends the result to rationally N-cyclic subnormal operators.

It is shown in Theorem 2.1 of Conway and Elias (1993) that if S is a pure rationally cyclic subnormal operator, then

$$
clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S))). \qquad (1-6)
$$

This leads us to examine if (1-6) holds for a rationally N−cyclic subnormal operator.

A Gleason part of $R(K)$ is a maximal set Ω in $\mathbb C$ such that for $x, y \in \Omega$, if e_x and e_y denote the functionals evaluation at x and y respectively, then $||e_x - e_y||_{R(K)^*} < 2$. [Olin and Thomson](#page-15-11) (1980) shows that a compact set K can be the spectrum of an irreducible subnormal operator if and only if $R(K)$ has only one non-trivial Gleason part Ω and $K = clos(\Omega)$. [McGuire \(1988](#page-15-12)) and [Feldman and McGuire \(2003](#page-15-13)) construct irreducible subnormal operators with a prescribed spectrum, approximate point spectrum, essential spectrum, and the (semi) Fredholm index. Our first result is to construct a (rationally) 2-cyclic irreducible subnormal operator for a prescribed spectrum and essential spectrum. Consequently we show that (1-6) may not hold for a (rationally) N −cyclic irreducible subnormal operator with cyclic multiple $N > 1$.

Theorem 1. Let K and K_e be two compact subsets of $\mathbb C$ such that $R(K)$ has only one nontrival *Gleason part* Ω , $K = clos(\Omega)$, and $\partial K \subset K_e \subset K$. *Then there exists a rationally 2-cyclic irreducible subnormal operator* S *such that* $\sigma(S) = K$, $\sigma_e(S) = K_e$, and $ind(S - \lambda) = -1$ for $\lambda \in K \setminus K_e$. If, in particular, $\mathbb{C} \setminus K$ has only one component, then S can be constructed as a *2-cyclic irreducible subnormal operator.*

Let $K = clos(\mathbb{D})$ and $K_e = \partial \mathbb{D} \cup clos(\frac{1}{2} \mathbb{D})$. We see that

$$
clos(K \setminus K_e) = \{ z : \frac{1}{2} \le |z| \le 1 \} \neq clos(Int(K)) = clos(1)
$$

From Theorem [1,](#page-2-0) we get the following result.

Corollary 1. *There exists a 2-cyclic irreducible subnormal operator* S *such that (1-6) does not hold.*

In the second part of this paper, we will investigate certain classes of rationally $N-$ cyclic subnormal operators that have the property $(1-6)$. Let S be a rationally N-cyclic subnormal operator on $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$. Let ψ be a smooth function with compact support. Define

$$
\mathcal{K}_n^{\psi} = clos\left\{\psi^m x : x \in \mathcal{H}, 0 \leq m \leq n\right\},\
$$

then

$$
\mathcal{H} \subset \mathcal{K}_1^{\psi} \subset \ldots \subset \mathcal{K}_n^{\psi} \subset \ldots \subset \mathcal{K}
$$

and $M_z|_{\mathcal{K}_n^{\psi}}$ is a subnormal operator.

Definition 1. A subnormal operator satisfies the property (N, ψ) if the following conditions are *met:*

(1) S is a pure (rationally) N-*cyclic subnormal operator on* $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$.

(2) ψ *a smooth function with compact support and Area(* $\sigma(S) \cap {\bar{\partial}} \psi = 0$ *) = 0. Let M_z <i>on* K *be the minimal normal extension of* S *satisfying* $(1-1)$ to $(1-4)$, then $M_z|_{K_{N-1}^{\psi}}$ is also a pure *subnormal operator.*

Theorem 2. Let $N > 1$ and let S be a pure subnormal operator on H satisfying the property (N, ψ) , *then there exist bounded open subsets* U_i *for* $1 \leq i \leq N$ *such that*

$$
\sigma_e(S) = \bigcup_{i=1}^N \partial U_i, \ \sigma(S) = \bigcup_{i=1}^N clos(U_i),
$$

and

$$
ind(S - \lambda) = -i
$$

for $\lambda \in U_i$ *and* $i = 1, 2, ...N$ *. Consequently,*

$$
\sigma(S) = clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S))).
$$

An important special case is that $\psi = \bar{z}$. In section 3, we will provide several examples of subnormal operators that satisfy the property (N, ψ) . We prove Theorem [1](#page-2-0) in section 2 and Theorem [2](#page-3-0) in section 3.

2 Spectral Pictures for Irreducible Rationally 2-Cyclic Subnormal Operators

In this section, we assume that K is a compact subset of \mathbb{C} , $Int(K) \neq \emptyset$, and $R(K)$ has only one nontrival Gleason part Ω with $K = clos(\Omega)$. Theorem 5 and Corollary 6 in [McGuire](#page-15-12) [\(1988\)](#page-15-12) constructs a representing measure ν of $R(K)$ at $z_0 \in Int(K)$ with support on ∂K such that S_{ν} on $R^2(K, \nu)$ is irreducible, $\sigma(S_{\nu}) = K$, $\sigma_e(S_{\nu}) = \partial K$, and $ind(S_{\nu} - \lambda) = -1$ for $\lambda \in Int(K) = \sigma(S_{\nu}) \setminus \sigma_e(S_{\nu})$. From Theorem 6.2 in [Gamelin \(1969\)](#page-15-14), we get

$$
L^{2}(\nu) = R^{2}(K, \nu) \oplus N^{2} \oplus \overline{R_{0}^{2}(K, \nu)}
$$
\n(2-1)

where $\overline{R_0^2(K,\nu)} = \{\bar{r} : r(z_0) = 0 \text{ and } r \in R^2(K,\nu)\}.$ The operator M_z , multiplication by z on $L^2(\nu)$, can be written as the following matrix with respect to (2-1):

$$
M_z = \begin{bmatrix} S_{\nu}, & A, & B \\ 0, & C, & D \\ 0, & 0, & T_{\nu}^* \end{bmatrix}
$$

where T_{ν} , multiplication by \bar{z} on $R_0^2(K, \nu)$, is an irreducible rationally cyclic subnormal operator with $\sigma(T_{\nu}) = \bar{K}$, $\sigma_e(T_{\nu}) = \partial \bar{K}$, and $ind(T_{\nu} - \lambda) = -1$ for $\lambda \in Int(\bar{K})$. Let

$$
S=\begin{bmatrix} S_\nu, & A \\ 0, & C \end{bmatrix},
$$

then S is the dual of T_{ν} . From the properties of dual subnormal operators (see, for example, Theorem 2.4 in Feldman and McGuire (2003) , we see that S is an irreducible subnormal operator with $\sigma(S) = K$, $\sigma_e(S) = \partial K$, and $ind(S - \lambda) = -1$ for $\lambda \in Int(K)$.

The following lemma, due to [Cowen and Douglas \(1978\)](#page-15-15) on page 194, allows us to choose eigenvectors for S^* in a co-analytic manner whenever the Fredholm index function for S is -1 . **Lemma 1.** *If* $X \in L(\mathcal{H})$ *and* $ind(X - \lambda) = -1$ *for all* $\lambda \in G := \sigma(X) \setminus \sigma_e(X)$ *, then there exists a co-analytic function* $h: G \to H$ *that is not identically zero on any component of* G *such that* $h(\lambda) \in \text{ker}(X - \lambda)^*$. In particular, for every $x \in \mathcal{H}$, the function $\lambda \to (x, h(\lambda))$ is analytic on G.

Using Lemma [1,](#page-3-1) we conclude that there exists a co-analytic function $k_{\lambda} \in \mathcal{H} := R^2(K, \nu) \oplus N^2$ such that $(S - \lambda)^* k_\lambda = 0$ on $Int(K)$. Let δ_λ be the point mass measure at λ . Let $K_e \subset K$ be a compact subset of $\mathbb C$ such that $\partial K \subset K_e$. Let $\{\lambda_n\} \subset K_e \cap Int(K)$ with $K_e \cap Int(K) \subset$ $\text{clos}(\{\lambda_n\})$. Define

$$
\mu = \nu + \sum_{n=1}^{\infty} c_n \delta_{\lambda_n},\tag{2-2}
$$

where $c_n > 0$ and $\sum_{n=1}^{\infty} c_n ||k_{\lambda_n}||^2 = 1$. Let M_z^1 be the multiplication by z operator on $L^2(\mu)$. **Lemma 2.** Define an operator T from H to $L^2(\mu)$ by

$$
Tf(z) = \begin{cases} f(z), & z \in \partial K \\ (f, k_{\lambda_n}), & z = \lambda_n. \end{cases}
$$
 (2-3)

Then T *is a bounded linear one to one operator with closed range. Set* $\mathcal{H}_1 = Ran(T)$, *then* T *is invertible from* \mathcal{H} *to* \mathcal{H}_1 , $M_z^1 \mathcal{H}_1 \subset \mathcal{H}_1$, $S_1 = M_z^1|_{\mathcal{H}_1}$ *is an irreducible subnormal operator such invertible from H* to *H*₁, M_z *H*₁ \subset *H*₁, *S*₁ = M_z _{*i*} \uparrow *H*₁ *is an irreduct that* $S_1 = TST^{-1}$ *, and* M_z^1 *is the minimal normal extension of* S_1 *.*

Proof: By definition, we get

$$
||f||_{L^{2}(\nu)}^{2} \leq ||Tf||_{L^{2}(\mu)}^{2} = ||f||_{L^{2}(\nu)}^{2} + \sum_{n=1}^{\infty} c_{n} |(f, k_{\lambda_{n}})|^{2} \leq 2||f||_{L^{2}(\nu)}^{2}.
$$

Therefore, T is a bounded linear operator and invertible from H to \mathcal{H}_1 . Since $(zf, k_{\lambda n}) =$ $\lambda_n(f, k_{\lambda_n}),$ we see that $M_z^1 \mathcal{H}_1 \subset \mathcal{H}_1$ and $S_1 = TST^{-1}$. Since $(Tk_{\lambda_n})(\lambda_n) = ||k_{\lambda_n}||^2 > 0$, clearly, we have

$$
L^{2}(\mu) = clos (span{\bar{z}}^{m}x : x \in \mathcal{H}_{1}, m \ge 0).
$$

Therefore, M_z^1 is the minimal normal extension of S_1 .

It remains to prove that S_1 is irreducible. Let N_1 and N_2 be two reducing subspaces of S_1 such that $\mathcal{H}_1 = N_1 \oplus N_2$. Then for $f_1 \in N_1$ and $f_2 \in N_2$, we have

$$
(zn f1, zm f2) = \int zn \bar{z}m f1 \bar{f}2 d\mu = 0
$$

for $n, m = 0, 1, 2, \dots$ This implies $f_1(z)\overline{f_2}(z) = 0$ a.e. μ . By the definition of T, we see that $(T^{-1}f_1)(z)\overline{(T^{-1}f_2)}(z) = 0$ a.e. ν . Hence, $\mathcal{H} = T^{-1}N_1 \oplus T^{-1}N_2$. $T^{-1}N_1$ and $T^{-1}N_2$ are reducing subspaces of S. By the construction, T_{ν} is irreducible (Corollary 6 in [McGuire \(1988](#page-15-12))), so S, as the dual T_{ν} , is irreducible (see, for example, Theorem 2.4 in [Feldman and](#page-15-13) McGuire [\(2003](#page-15-13))). This means that $N_1 = 0$ or $N_2 = 0$. The lemma is proved.

We write the operator M_z^1 as the following:

$$
M_z^1 = \begin{bmatrix} S_1, & A_1 \\ 0, & T_1^* \end{bmatrix} \tag{2-4}
$$

Then T_1 , as a dual of S_1 , is irreducible.

Lemma 3. Let μ be as in(2-2) and let \mathcal{H}_1 be as in Lemma [2.](#page-4-0) Define

$$
F(z) = \begin{cases} \bar{z} - \bar{z}_0, & z \in \partial K, \\ 0, & z \in Int(K). \end{cases}
$$
 (2-5)

and

$$
G_n(z) = \begin{cases} k_{\lambda_n}(z), & z \in \partial K, \\ -1/c_n, & z = \lambda_n, \\ 0, & z = \lambda_m, \ m \neq n. \end{cases}
$$
 (2-6)

Then

$$
\mathcal{H}_1^{\perp} = clos\left(\operatorname{span}\{r(\bar{z})F, G_j, 1 \leq j < \infty, r \in Rat(K)\}\right).
$$

Proof: It is straightforward to check, from $(2-1)$, $(2-2)$, and $(2-3)$, that $F, G_j \in \mathcal{H}_1^{\perp}$. Now let $H(z) \perp clos (span{r(\bar{z})F, G_j, 1 \leq j < \infty, r \in Rat(K)})$, then

$$
\int H(z)r(z)\bar{F}(z)d\mu = \int H(z)r(z)(z-z_0)d\nu = 0
$$

for $r \in Rat(K)$. From (2-1), we see that the function $H|_{\partial K} \in \mathcal{H}$. It follows from $\int H(z)\overline{G}_j(z)d\mu =$ 0 that $H(\lambda_j) = (H|_{\partial K}, k_{\lambda_j})$. Thus, $H(z) \in \mathcal{H}_1$. The lemma is proved.

Lemma 4. Let μ , T_1 , F , and G_n be as in (2-2), (2-4), (2-5) and (2-6), respectively. Then there *exists a sequence of positive numbers* {an} *satisfying*

$$
\sum_{n=1}^{\infty} a_n ||G_n|| < \infty, \ G = \sum_{n=1}^{\infty} a_n G_n,
$$

and

$$
\mathcal{H}_1^{\perp} = clos\left(span\{r(\bar{z})F(z) + p(\bar{z})G(z): r \in Rat(K), p \in \mathcal{P}\}\right).
$$

Therefore, T_1 *is a rationally 2-cyclic irreducible subnormal operator with*

$$
\sigma(T_1) = \bar{K}, \ \sigma_e(T_1) = \bar{K}_e, \ \text{and} \ \text{ind}(T_1 - \lambda) = -1, \ \lambda \in \bar{K} \setminus \bar{K}_e. \tag{2-7}
$$

Proof: Notice that

$$
\int f(z)(z-\lambda_n)\bar{k}_{\lambda_n}(z)d\nu=0
$$

for $f \in \mathcal{H}$. We conclude, from (2-1), that $(\bar{z} - \bar{\lambda}_n) k_{\lambda_n}(z) \in \overline{R_0^2(K, \nu)}$. Hence, there are $\{r_n\} \subset$ $R^2(K,\nu)$ such that

$$
k_{\lambda_n}(z) = \frac{r_n(\bar{z})}{\bar{z} - \bar{\lambda}_n}(\bar{z} - \bar{z}_0).
$$

We will recursively choose $\{a_n\}$. First choose $a_1 = 1$. Then we assume that $a_1, a_2, ..., a_n$ have been chosen. Now we will choose a_{n+1} . Let

$$
p_k(z) = \frac{\Pi_{j \neq k, 1 \leq j \leq n} (z - \bar{\lambda}_j)}{a_k \Pi_{j \neq k, 1 \leq j \leq n} (\bar{\lambda}_k - \bar{\lambda}_j)},
$$

for $k = 1, 2, ..., n$. Denote

$$
q_{1k}(z) = p_k(z) \sum_{j \neq k, 1 \leq j \leq n} \frac{a_j}{z - \overline{\lambda}_j} r_j(z)
$$

and

$$
q_{2k}(z) = \frac{a_k(p_k(z) - p_k(\bar{\lambda}_k))}{z - \bar{\lambda}_k} r_k(z).
$$

So $p_k \in \mathcal{P}$ and $q_{1k}, q_{2k} \in R^2(K, \nu)$ for $k = 1, 2, ..., n$. Clearly,

$$
p_k(\bar{z})\sum_{j=1}^n a_jG_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))(\bar{z}-\bar{z}_0) = \frac{r_k(\bar{z})(\bar{z}-\bar{z}_0)}{\bar{z}-\bar{\lambda}_k}, \ z \in \partial K.
$$

Hence,

$$
p_k(\bar{z})\sum_{j=1}^n a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F(z) = G_k(z), \ a.e. \ \mu.
$$

We have the following calculation:

$$
\int \left| p_k(\bar{z}) \sum_{j=1}^{n+1} a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z})) F(z) - G_k(z) \right|^2 d\mu
$$

=
$$
\int |p_k(\bar{z}) a_{n+1} G_{n+1}(z)|^2 d\mu
$$

$$
\leq \left(\frac{a_{n+1}}{a_k} \right)^2 \frac{(4D^2)^{n-1}}{\prod_{j \neq k, 1 \leq j \leq n} |\lambda_k - \lambda_j|^2} ||G_{n+1}||^2
$$

where $D = \max\{|z| : z \in K\}$. Now set

 $\ddot{}$

$$
a_{n+1} = \min\left(\frac{1}{2^{n+1}}, \min_{1 \le k \le n} \frac{a_k \Pi_{j \ne k, 1 \le j \le n} \min(1, |\lambda_k - \lambda_j|)}{4^n \max(1, D)^{n-1}}\right) / \max(1, \|G_{n+1}\|).
$$
 (2-8)

So we have chosen all $\{a_n\}$. From (2-8), we have the following calculation.

$$
\left\| p_k \sum_{i=n+2}^{\infty} a_j G_j \right\|
$$

\n
$$
\leq \frac{(2D)^{n-1}}{a_k \Pi_{j \neq k, 1 \leq j \leq n} |\lambda_k - \lambda_j|} \sum_{i=n+2}^{\infty} \frac{a_k \Pi_{j \neq k, 1 \leq j \leq i-1} \min(1, |\lambda_k - \lambda_j|)}{4^{i-1} \max(1, D)^{i-2}}
$$

\n
$$
\leq \frac{1}{2^{n+2}}.
$$

Therefore,

$$
||p_k(\bar{z})G - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F - G_k(z)||
$$

\n
$$
\leq ||p_k(\bar{z})\sum_{j=1}^{n+1} a_j G_j - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F - G_k(z)|| + ||p_k(\bar{z})\sum_{j=n+2}^{\infty} a_j G_j||
$$

\n
$$
\leq \frac{1}{2^n}.
$$

Hence,

$$
G_k \in clos(span{r(\bar{z})F(z) + p(\bar{z})G(z) : r \in Rat(K), p \in \mathcal{P}}), k = 1, 2,
$$

Since T_1 is the dual of S_1 , we see that $\sigma(M_z^1) \subset \sigma_e(S_1) \cup \overline{\sigma_e(T_1)}$ (see, for example, Theorem 2.4 in [Feldman and McGuire \(2003](#page-15-13))), $\sigma_e(S_1) = \partial K$, and $\sigma_e(T_1) \supset \partial \bar{K}$. So (2-7) follows. This completes the proof.

Proof of Theorem [1:](#page-2-0) It follows from Lemma [4.](#page-5-0)

3 Spectral Picture of a Class of Rationally Multicyclic Subnormal Operators

In this section, we will prove Theorem [2.](#page-3-0) First we provide some examples of subnormal operators that have the property (N, ψ) in Definition [1.](#page-3-2)

Example 1. *Every pure subnormal operator* S *on* H *with finite rank self-commutator has the property* (N, ψ) . *Notice that the structure of such subnormal operators has been established based on Xia's model (see [Xia \(1996\)](#page-16-2) and [Yakubovich \(1998\)](#page-16-3)).*

Proof: Assume that M_z on K is the minimal normal extension satisfying (1-1) to (1-4). Define the self-commutator as the following

$$
D = [S^*, S] = S^*S - SS^*.
$$

The element $x \in \text{ker}(D)$ if and only if $\overline{z}x \in \mathcal{H}$. This implies $S\text{ker}(D) \subset \text{ker}(D)$. Therefore,

$$
S^*Ran(D) \subset Ran(D). \tag{3-1}
$$

Let

$$
\mathcal{H}_0 = clos\left(span(S^n f : f \in Ran(D), \ n \ge 0)\right),\
$$

then $S|_{\mathcal{H}_0}$ is N-cyclic subnormal where $N = dim(Ran(D)).$

On the other hand,

$$
S^*S^nD = SS^*S^{n-1}D + DS^{n-1}D,
$$

hence, we can recursively show that $S^*S^nRan(D) \subset \mathcal{H}_0$ since (3-1). So $S^*\mathcal{H}_0 \subset \mathcal{H}_0$. This implies that

$$
S(\mathcal{H}\ominus\mathcal{H}_0)\subset\mathcal{H}\ominus\mathcal{H}_0
$$

and $S|_{\mathcal{H}\ominus\mathcal{H}_0}$ is normal. Since S is pure, we conclude that $\mathcal{H} = \mathcal{H}_0$ and S is N-cyclic. From $(3-1)$, we see that there is a polynomial p such that

$$
\bar{p}(S^*|_{\text{Ran}(D)})=0.
$$

Therefore,

$$
p(S): \mathcal{H} \to \ker(D).
$$

Hence,

$$
||M_z^* p(S)f|| = ||M_z p(S)f|| = ||Sp(S)f|| = ||S^* p(S)f||
$$

for $f \in \mathcal{H}$. This implies $\bar{z}p\mathcal{H} \subset \mathcal{H}$. Let $\psi = \bar{z}p$, then $Area\{\bar{\partial}\psi = 0\} = Area\{z : p(z) = 0\} = 0$, $\mathcal{K}_{N-1}^{\psi} = \mathcal{H}$, and S satisfies the property (N, ψ) in Definition [1.](#page-3-2)

Example 2. In Lemma [4,](#page-5-0) if $K = clos(\mathbb{D})$ and $K_e = (\partial \mathbb{D}) \cup (\frac{1}{2} \partial \mathbb{D})$, then the operator T_1 is a 2-cyclic irreducible subnormal operator satisfying the property $(2, \psi)$ where $\psi = |z|^4 - \frac{5}{4}|z|^2$.

Proof: For $f \in \mathcal{H}_1$, we get

$$
\psi f = (|z|^2 - 1)(|z|^2 - \frac{1}{4})f + \frac{1}{4}f = \frac{1}{4}f
$$

since $spt(\mu) \subset K_e$. Hence, $\mathcal{K}_1^{\psi} = \mathcal{H}_1$. On the other hand,

$$
Area\{\bar{\partial}\psi=0\}\leq Area\left(\{0\}\cup\{|z|=\frac{5}{8}\}\right)=0.
$$

Therefore, the operator T_1 satisfies the property $(2, \psi)$.

In the remaining section, we assume that $N > 1$ and S is a pure rationally N-cyclic subnormal operator on $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$ and M_z on K, which satisfies (1-1) to (1-4), is the minimal normal extension of S. Moreover, S satisfies the property (N, ψ) in Definition [1.](#page-3-2) Let U_k be the set of $\lambda \in Int(\sigma(S))$ such that $Ran(S - \lambda)$ is closed and $dim (ker(S - \lambda)^*) = k$, where $k = 1, 2, ..., N$.

Lemma 5. *If* $1 \leq k \leq N$, $\delta > 0$, $B(\lambda_0, 2\delta) \subset Int(\sigma(S))$, *I is an index subset of* $\{1, 2, ..., N\}$ *with size* $N - k$, $F = \sum_{i=1}^{N} r_i F_i$ *where* $r_i \in Rat(\sigma(S))$, and $\{a_{ls}(\lambda)\}_{1 \leq l \leq N-k, 1 \leq s \leq k}$ *are analytic on* $B(\lambda_0, 2\delta)$ *such that*

$$
\sup_{1 \leq s \leq k, \lambda \in B(\lambda_0, \delta)} |r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda)| \leq M \|F\| \tag{3-2}
$$

and

$$
F_{i_l}(z) = \sum_{s=1}^{k} a_{ls}(z) F_{j_s}(z), \ a.e \ \mu_1|_{B(\lambda_0, \delta)}, \tag{3-3}
$$

where $i_l \in I$ *and* $j_s \notin I$. *Then* $\lambda_0 \in \bigcup_{i=k}^N U_k$.

Proof: From $(3-3)$, we get

$$
\int_{B(\lambda_0,\delta)} |F|^2 d\mu_1 = \int_{B(\lambda_0,\delta)} \left| \sum_{s=1}^k \left(r_{j_s}(z) + \sum_{l=1}^{N-k} a_{ls}(z) r_{i_l}(z) \right) F_{j_s}(z) \right|^2 d\mu_1.
$$

Using (3-2) and the maximal modulus principle,

$$
\sup_{1 \leq s \leq k, \lambda \in B(\lambda_0, \delta)} \left| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) \right| \leq \frac{M}{\delta} ||(S - \lambda_0)F||.
$$

Hence,

$$
\int |F|^2 d\mu_1 \leq \int_{B(\lambda_0,\delta)^c} |F|^2 d\mu_1 + \left(\sum_{j \notin I} ||F_j||\right)^2 \sup_{1 \leq s \leq k, \lambda \in B(\lambda_0,\delta)} \left| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) \right|^2.
$$

Therefore,

$$
||F|| \leq M_1 ||(S - \lambda_0)F||,
$$

where

$$
M_1^2 = \left(1 + \left(\sum_{j \notin I} ||F_j||\right)^2\right) \left(\frac{M}{\delta}\right)^2.
$$

So $Ran(S - \lambda_0)$ is closed. On the other hand, there are k linearly independent $k_{\lambda}^j \in \mathcal{H}$ such that

$$
r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) = \int \left\langle F(z), k_\lambda^j(z) \right\rangle d\mu_1(z)
$$

where $j_s \notin I$ and $\lambda \in B(\lambda_0, \delta)$. This implies

$$
dim(Ker(S - \lambda_0)^*) \geq k.
$$

Therefore, $\lambda_0 \in \bigcup_{i=k}^N U_i$.

Let ν be a compactly supported finite measure on \mathbb{C} . The transform

$$
\mathcal{C}_{\psi}^{i} \nu(z) = \int \frac{(\psi(w) - \psi(z))^{i}}{w - z} d\nu(w)
$$

is continuous at each point z with $|\nu|(\{z\})=0$ and $i>0$. For $i=0$, the transformation

$$
\mathcal{C}_{\psi}^{0}(\nu) = \mathcal{C}(\nu) = \int \frac{1}{w - z} d\nu(w)
$$

is the Cauchy transform of ν . Let $M^G(z)$ be the following N by N matrix,

$$
M^G(z) = \left[\mathcal{C}_{\psi}^{i-1}(\langle F_j, G \rangle \mu_1) \right]_{N \times N}
$$

where we assume that $G \perp \mathcal{K}_{N-1}^{\psi}$ or equivalently G satisfies the following conditions

$$
\bar{\psi}^n G \perp \mathcal{H}, \ n = 0, 1, 2, ..., N - 1. \tag{3-4}
$$

The set $W^G\subset \mathbb{C}$ is defined by:

$$
W^G = \{\lambda: \int \frac{1}{|z-\lambda|} |\langle F_i(z), G(z) \rangle| d\mu_1(z) < \infty, 1 \le i \le N\}.
$$

Let

$$
\Omega^G = Int(\sigma(S)) \cap W^G \cap \{\lambda : |det(M^G(\lambda))| > 0\}.
$$
\n(3-5)

Then for $\lambda \in \Omega^G$, the matrix

$$
\left[\mathcal{C}(\langle F_j \psi^{i-1}, G \rangle \mu_1) \right]_{N \times N} \tag{3-6}
$$

is invertible. By Construction, we see that

$$
det(M^G(z)) = 0 \ a.e. \ Area|_{(clos(\Omega^G))^c}.
$$

Lemma 6. *Using above notations, we conclude that*

$$
\Omega^G \subset abpe(S).
$$

Hence, by Lemma [5,](#page-7-0) we get $\Omega^G \subset U_N$.

Proof: Using $(3-4)$, $(3-5)$, and $(3-6)$, we see that the lemma is a direct application of Theorem 2 in [Yang \(2018\)](#page-16-1).

Let $A = {\lambda_n : \mu_1({\lambda_n}) > 0}$ be the set of atoms for μ_1 . Now let us define the matrix $M_j^G(z)$ to be a submatrix of $M^G(z)$ by eliminating the first row and j column. Let $B_j^G(z)$ be the j column of the matrix $M^{G}(z)$ by eliminating the first row. Define

$$
\Omega_j^G = \left(Int(\sigma(S)) \cap A^c \cap \{ z : \ |det(M_j^G(z))| > 0 \} \right) \setminus clos(\Omega^G). \tag{3-7}
$$

Notice that $M_j^G(\lambda)$ is continuous at each $\lambda \in \Omega_j^G$. On Ω_j^G , we can define the following function vector

$$
a_j(z) = [a_{ij}(z)]_{(N-1)\times 1} = (M_j^G(z))^{-1} B_j^G(z).
$$
\n(3-8)

Lemma 7. Let $G, \Omega^G, \Omega_j^G,$ and $a_j(z)$ be as in $(3-4)$, $(3-5)$, $(3-7)$, and $(3-8)$, respectively. *Then for* $\lambda_0 \in \Omega_j^G$, *there exists* $\delta > 0$ *such that* $a_j(z)$ *equals an analytic function on* $B(\lambda_0, \delta) \subset$ $Int(\sigma(S)$ almost everywhere with respect to the area measure. Moreover,

$$
\mathcal{C}(\langle F_j, G \rangle \mu)(z) = \sum_{k=1}^{j-1} a_{kj}(z) \mathcal{C}(\langle F_k, G \rangle \mu)(z) + \sum_{k=j+1}^{N} a_{k-1,j}(z) \mathcal{C}(\langle F_k, G \rangle \mu)(z), \ a.e. \ Area|_{B(\lambda_0, \delta)},
$$
\n(3-9)

and

$$
\langle F_j, G \rangle = \sum_{k=1}^{j-1} a_{kj}(z) \langle F_k, G \rangle + \sum_{k=j+1}^{N} a_{k-1,j}(z) \langle F_k, G \rangle, \ a.e. \mu|_{B(\lambda_0, \delta)}.
$$
 (3-10)

Proof: Without loss of generality, we assume that $j = N$. For $z \in Int(\sigma(S) \cap W^G \cap \Omega_N^G)$ write

$$
M^G(z) = \left[\begin{array}{cc} A_N^G(z) & c_N^G(z) \\ M_N^G(z) & B_N^g(z) \end{array} \right]
$$

where

$$
A_N^G(z) = [\mathcal{C}(\langle F_1, G \rangle \mu_1)(z), \mathcal{C}(\langle F_2, G \rangle \mu_1)(z), ..., \mathcal{C}(\langle F_{N-1}, G \rangle \mu_1)(z)]
$$

and

$$
c_N^G(z) = C(\langle F_N, G \rangle \mu_1)(z).
$$

By construction of Ω_N^G , we conclude that

$$
det(M^{G}(z)) = (A_{N}^{G}(z)(M_{N}^{G}(z))^{-1}B_{N}^{G}(z) - c_{N}^{G}(z))det(M_{N}^{G}(z)) = 0
$$
 a.e. $Area|_{\Omega_{N}^{G}}.$

Therefore,

$$
c_N^G(z) = A_N^G(z) (M_N^G(z))^{-1} B_N^G(z) \ a.e. \ Area|_{\Omega_N^G}.
$$
 (3-11)

Let $\nu_i = \langle F_i, G \rangle \mu_1$ and $H_{i,m}(z) = \frac{m^2}{\pi} \nu_i(B(z, \frac{1}{m}))$, then the functions $H_{i,m}(z)$ are bounded with compact supports. We have

$$
\mathcal{C}(H_{i,m}dA)(w) = \int_{|\lambda - w| \geq \frac{1}{m}} \frac{1}{\lambda - w} d\nu_i(\lambda) + \int_{|\lambda - w| < \frac{1}{m}} \frac{m^2 |\lambda - w|^2}{\lambda - w} d\nu_i(\lambda).
$$

Hence,

$$
|\mathcal{C}(H_{i,m}dA)(w) - \mathcal{C}\nu_i(w)| \leq 2\int_{|w-z| < 1/m} \frac{1}{|w-z|} d|\nu_i|(z) \ a.e. \ Area
$$

and

$$
\lim_{m \to \infty} C(H_{i.m} dA)(w) = C\nu_i(w), \ a.e. \ Area.
$$

Let $C_0 > 0$ be a constant such that $|\psi(z) - \psi(w)| \leq C_0 |z - w|$. We estimate $C^1_{\psi}(\nu_i)$ as the following,

$$
\begin{split} &\left| \mathcal{C}^1_\psi(H_{i,m} dA)(w) - \mathcal{C}^1_\psi \nu_i(w) \right| \\ &= \left| \frac{m^2}{\pi} \int\int_{|z-\lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z-w} dA(z) d\nu_i(\lambda) - \mathcal{C}^1_\psi \nu_i(w) \right| \\ &\leq \left| \frac{m^2}{\pi} \int_{|\lambda-w| \geq \frac{1}{\sqrt{m}}} \int_{|z-\lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z-w} dA(z) d\nu_i(\lambda) - \int_{|\lambda-w| \geq \frac{1}{\sqrt{m}}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w} d\nu_i(\lambda) \right| \\ &+ \left| \frac{m^2}{\pi} \int_{|\lambda-w| < \frac{1}{\sqrt{m}}} \int_{|z-\lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z-w} dA(z) d\nu_i(\lambda) \right| + \left| \int_{|\lambda-w| < \frac{1}{\sqrt{m}}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w} d\nu_i(\lambda) \right| \end{split}
$$

Notice that

$$
\frac{m^2}{\pi}\int_{|\lambda-w|\geq \frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}}\frac{1}{z-w}dA(z)d\nu_i(\lambda)=\int_{|\lambda-w|\geq \frac{1}{\sqrt{m}}}\frac{1}{\lambda-w}d\nu_i(\lambda).
$$

We get

$$
\begin{split} &| \mathcal{C}^1_\psi(H_{i,m} dA)(w) - \mathcal{C}^1_\psi \nu_i(w) | \\ & \leq \left| \frac{m^2}{\pi} \int_{|\lambda - w| \geq \frac{1}{\sqrt{m}}} \int_{|z - \lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(\lambda)}{z - w} dA(z) d\nu_i(\lambda) \right| + 2C_0 |\nu_i| (B(w, \frac{1}{\sqrt{m}})) \\ & \leq \frac{m^2}{\pi} \int_{|\lambda - w| \geq \frac{1}{\sqrt{m}}} \int_{|z - \lambda| < \frac{1}{m}} \frac{C_0 |z - \lambda|}{|w - \lambda| - |z - \lambda|} dA(z) d\nu_i(\lambda) + 2C_0 |\nu_i| (B(w, \frac{1}{\sqrt{m}})) \\ & \leq C_0 \frac{\frac{1}{m}}{\frac{1}{\sqrt{m}} - \frac{1}{m}} |\nu_i| (B(w, \frac{1}{\sqrt{m}})^c) + 2C_0 |\nu_i| (B(w, \frac{1}{\sqrt{m}})) \\ & \leq \frac{C_0}{\sqrt{m} - 1} ||\nu_i|| + 2C_0 |\nu_i| (B(w, \frac{1}{\sqrt{m}})). \end{split}
$$

Therefore,

$$
\lim_{m \to \infty} \mathcal{C}_{\psi}^1(H_{i,m} dA)(w) = \mathcal{C}_{\psi}^1 \nu_i(w)
$$

for $w \notin A$. For $\lambda_0 \in \Omega_N^G$ and $\epsilon > 0$, we can choose a $\delta > 0$ and m_0 such that

$$
|\mathcal{C}_{\psi}^{1}(H_{i,m}dA)(w) - \mathcal{C}_{\psi}^{1}\nu_{i}(w)|
$$

\n
$$
\leq 2C_{0}|\nu_{i}|(B(w, \frac{1}{\sqrt{m}})) + \frac{C_{0}}{\sqrt{m}-1}||\nu_{i}||
$$

\n
$$
\leq 2C_{0}|\nu_{i}|(B(\lambda_{0}, \delta + \frac{1}{\sqrt{m}})) + \frac{C_{0}}{\sqrt{m}-1}||\nu_{i}||
$$

\n
$$
<\epsilon
$$

where $w \in B(\lambda_0, \delta) \setminus A$ and $m \geq m_0$. Since $C^1_{\psi} \nu_i(w)$ is continuous at λ_0 , δ can be chosen to ensure

$$
|\mathcal{C}_{\psi}^1 \nu_i(w) - \mathcal{C}_{\psi}^1 \nu_i(\lambda_0)| < \epsilon
$$

where $w \in B(\lambda_0, \delta) \setminus A$. It is easy to verify that $C^1_\psi(H_{i,m} dA)$ is a smooth function. For $k > 1$, clearly $\mathcal{C}_{\psi}^{k} \nu_{i}(w)$ is a smooth function. Define

$$
M_N^{Gm}(z) = \begin{bmatrix} C^1_\psi(H_{1,m} dA), & C^1_\psi(H_{2,m} dA), & \dots, & C^1_\psi(H_{N-1,m} dA) \\ C^2_\psi(\nu_1), & C^2_\psi(\nu_2), & \dots, & C^2_\psi(\nu_{N-1}) \\ \dots, & \dots, & \dots, & \dots \\ C^{N-1}_\psi(\nu_1), & C^{N-1}_\psi(\nu_2), & \dots, & C^{N-1}_\psi(\nu_{N-1}) \end{bmatrix}
$$

.

We can choose ϵ small enough so that

 $M_N^{Gm}(w)$, $M_N^G(w)$

are invertible for $w \in B(\lambda_0, \delta) \setminus A$ and $m > m_0$. Define

$$
B_N^{Gm}(z) = \begin{bmatrix} C^1_{\psi}(H_{N,m}dA) \\ C^2_{\psi}(\nu_N) \\ \dots \\ C^{N-1}_{\psi}(\nu_N) \end{bmatrix},
$$

$$
A_N^{Gm}(z) = [C(H_{1,m}dA), C(H_{2,m}dA), ..., C(H_{N-1,m}dA)]
$$

and

$$
c_N^{Gm}(z) = \mathcal{C}(H_{N,m}dA)(z).
$$

For a smooth function ϕ with compact support in $B(\lambda_0, \delta)$, using the definition (3-8) and Lebesgue's Dominated Convergence Theorem, we get the following calculation,

$$
\int \bar{\partial}\phi(z)a_N(z)dA(z)
$$
\n
$$
= \lim_{m \to \infty} \int \bar{\partial}\phi(z) \left((M_N^{Gm}(z))^{-1} B_N^{Gm}(z) \right) dA(z)
$$
\n
$$
= - \lim_{m \to \infty} \int \phi(z)\bar{\partial} \left((M_N^{Gm}(z))^{-1} B_N^{Gm}(z) \right) dA(z)
$$
\n
$$
= \lim_{m \to \infty} \int \phi(z)(M_N^{Gm}(z))^{-1} \left((\bar{\partial}M_N^{Gm}(z)) (M_N^{Gm}(z))^{-1} B_N^{Gm}(z) - \bar{\partial}B_N^{Gm}(z) \right) dA(z).
$$
\n(3-12)

On the other hand,

$$
\bar{\partial}M_N^{Gm}(z) = \bar{\partial}\psi(z) \begin{bmatrix}\n-C(H_{1,m}dA), & -C(H_{2,m}dA), & \dots, & -C(H_{N-1,m}dA) \\
-2C^1_{\psi}(\nu_1), & -2C^1_{\psi}(\nu_2), & \dots, & -2C^1_{\psi}(\nu_{N-1}) \\
\vdots & \vdots & \ddots & \vdots \\
-(N-1)C^{N-2}(\nu_1), & -(N-1)C^{N-2}(\nu_2), & \dots, & -(N-1)C^{N-2}(\nu_{N-1})\n\end{bmatrix}.
$$

Therefore,

$$
(\bar{\partial}M_N^{Gm}(z))(M_N^{Gm}(z))^{-1} = -\bar{\partial}\psi(z)\left[\begin{array}{cccc}A_N^{Gm}(z) & (M_N^{Gm}(z))^{-1}, & & \\ 2, & 0, & ..., & 0, & 0 \\ ..., & ..., & ..., & ..., & ..., & ... \\ 0, & 0, & ..., & N-1, & 0\end{array}\right].
$$

Hence,

$$
(\bar{\partial}M_N^{Gm}(z))(M_N^{Gm}(z))^{-1}B_N^{Gm}(z) - \bar{\partial}B_N^{Gm}(z) = -\bar{\partial}\psi(z)\begin{bmatrix} A_N^{Gm}(z)(M_N^{Gm}(z))^{-1}B_N^{Gm} - c_N^{Gm} \\ 0 & \dots & 0 \\ 0 & 0 & \end{bmatrix}.
$$

Using (3-11), we see that

$$
\lim_{m \to \infty} \left(A_N^{Gm}(z) (M_N^{Gm}(z))^{-1} B_N^{Gm} - c_N^{Gm} \right) = 0 \text{ a.e. } Area|_{B(\lambda_0, \delta)}.
$$

Since each component of the above vector function is less than

$$
M\int \frac{1}{|w-z|}d|\nu_i|(z) \ a.e. \ Area|_{B(\lambda_0,\delta)},
$$

applying Lebesgue's Dominated Convergence Theorem to the last step of (3-12), we conclude

$$
\int \bar{\partial}\phi(z)a_N(z)dA(z)=0.
$$

By Weyl's lemma, we see that $a_N(z)$ is analytic on $B(\lambda_0, \delta)$. From equation (3-8), we get

$$
\mathcal{C}_{\psi}^1(F_N, G)\mu_1)(z) = \sum_{k=1}^{N-1} a_{kj}(z) \mathcal{C}_{\psi}^1(F_k, G)\mu_1)(z), \ a.e. \ Area|_{B(\lambda_0, \delta)}.
$$

The above equation implies (3-9) since

$$
\bar{\partial} \mathcal{C}_{\psi}^{1}(\nu_{i})(z) = -\mathcal{C}(\nu_{i})(z) \ a.e. \ Area.
$$

For equation (3-10), let ϕ be a smooth function with compact support in $B(\lambda_0, \delta)$ and let ν be a compactly supported finite measure, we get

$$
\int \bar{\partial}\phi(z)\mathcal{C}\nu(z)dA(z) = \pi \int \phi(z)d\nu(z).
$$

Apply the above equation to the both sides of the equation (3-9) for $j = N$ and using

$$
\bar{\partial}\phi(z)a_{kj}(z)=\bar{\partial}(\phi(z)a_{kj}(z)),\ z\in B(\lambda_0,\delta),
$$

we conclude

$$
\int \phi \langle F_N, G \rangle d\mu_1 = \int \phi \sum_{k=1}^{N-1} a_{kj} \langle F_k, G \rangle d\mu_1.
$$

Hence the equation (3-10) follows. This completes the proof of the lemma.

Corollary 2. Let G, Ω^G , and Ω_i^G be as in Lemma [7.](#page-9-0) Suppose $G \perp \mathcal{K}_{N-1}^{\psi}$ (satisfies (3-4)). *Then* $\Omega_i^G \subset U_{N-1} \cup U_N$.

Proof: Without loss of generality, we assume that $j = N$. From Lemma [7,](#page-9-0) for $\lambda_0 \in \Omega_N^G$, there exists $\delta > 0$ such that $B(\lambda_0, \delta) \subset Int(\sigma(S))$ and the equations (3-9) and (3-10) hold, which imply (3-3). For $r_1, r_2, ..., r_N \in Rat(\sigma(S))$, let

$$
F = \sum_{i=1}^{N} r_i F_i.
$$

Notice that

$$
r_i(\lambda)\mathcal{C}_{\psi}^k\langle F_i, G \rangle \mu_1) = \mathcal{C}_{\psi}^k\langle r_i F_i, G \rangle \mu_1)
$$

since $G \perp \mathcal{K}_{N-1}^{\psi}$. Then

$$
\sum_{i=1}^N r_i(\lambda) \mathcal{C}_{\psi}^k(\langle F_i, G \rangle \mu_1)(\lambda) = \mathcal{C}_{\psi}^k(\langle F, G \rangle \mu_1)(\lambda),
$$

for $k = 1, 2, ..., N - 1$. Now using the equation (3-9) for $\lambda \in B(\lambda_0, \delta) \setminus A$, we get

$$
\sum_{i=1}^{N-1} (r_i(\lambda) + a_{Ni}(\lambda) r_N(\lambda)) C^k_{\psi}(\langle F_i, G \rangle \mu_1)(\lambda) = C^k_{\psi}(\langle F, G \rangle \mu_1)(\lambda),
$$

equivalently,

$$
M_N^G(\lambda) \begin{bmatrix} r_1(\lambda) + a_{N1}(\lambda)r_N(\lambda) \\ r_2(\lambda) + a_{N2}(\lambda)r_N(\lambda) \\ \dots \\ r_{N-1}(\lambda) + a_{N,N-1}(\lambda)r_N(\lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{\psi}^1(\langle F, G \rangle \mu_1)(\lambda) \\ \mathcal{C}_{\psi}^2(\langle F, G \rangle \mu_1)(\lambda) \\ \dots \\ \mathcal{C}_{\psi}^{N-1}(\langle F, G \rangle \mu_1)(\lambda) \end{bmatrix}.
$$

where the inverse of $M_N^G(\lambda)$ is bounded on $B(\lambda_0, \delta) \setminus A$ and a_{Ni} are analytic on $B(\lambda_0, \delta)$. Therefore, there exists a positive constant M such that

$$
\sup_{1 \le k \le N-1, \lambda \in B(\lambda_0, \frac{\delta}{2})} |r_k(\lambda) + a_{Nk}(\lambda)r_N(\lambda)| \le M ||F||,
$$

which implies (3-2). Hence, Lemma 3.1 implies $\Omega_N^G \subset U_{N-1} \cup U_N$.

Now let us recursively construct other sets such as Ω_{ij}^G for a given $G \perp \mathcal{K}_{N-1}^{\psi}$. We will only describe the algorithm for $k = N - 2$ and the other cases will follow recursively. Let $E_N^G = \Omega^G$

and $E_{N-1}^G = \bigcup_{i=1}^N \Omega_i^G$. Let M_{ij}^G be an $N-2$ by $N-2$ submatrix of M^G by eliminating the first two rows and the i and j columns. Define

$$
\Omega_{ij}^G = (Int(\sigma(S)) \cap A^c \cap \{z : |det(M_{ij}^G(z))| > 0\}) \setminus clos(E_N^G \cup E_{N-1}^G).
$$

Without loss of generality, let us assume that $i = N - 1$ and $j = N$. Similar to Lemma [7,](#page-9-0) one can prove that for $\lambda_0 \in \Omega_{N-1,N}^G$, there exist $\delta > 0$, analytic functions $a_i(z)$ and $b_i(z)$ on $B(\lambda_0, \delta) \subset Int(\sigma(S))$ such that

$$
F_{N-1} = \sum_{i=1}^{N-2} a_i(z) F_i(z), \ F_N = \sum_{i=1}^{N-2} b_i(z) F_i(z), \ a.e. \mu_1|_{B(\lambda_0, \delta)}, \tag{3-13}
$$

and there exists a constant $M > 0$ such that

$$
\sup_{1 \le k \le N-2, \lambda \in B(\lambda_0, \frac{\delta}{2})} |r_k(\lambda) + a_k(\lambda)r_{N-1}(\lambda) + b_k(\lambda)r_N(\lambda)| \le M ||F||, \tag{3-14}
$$

where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$ and $F = \sum_{i=1}^{N} r_i F_i$. (3-13) and (3-14) are the same as (3-2) and (3-3) for the case $k = N - 2$. Let

$$
E_{N-2}^{G} = \cup_{i\n(3-15)
$$

Corollary 3. Let E_{N-2}^G be as in (3-15). Suppose $G \perp \mathcal{K}_{N-1}^{\psi}$ (satisfies (3-4)). Then

$$
E_{N-2}^G\subset U_{N-2}\cup U_{N-1}\cup U_N.
$$

The proof is the same as Corollary [2.](#page-12-0) Therefore we can recursively construct E_k^G for $k =$ $1, 2, ..., N$ such that

$$
E_k^G \subset \bigcup_{i=k}^N U_i \tag{3-16}
$$

where the proof for $k = N$ is from Lemma [6,](#page-8-0) $k = N - 1$ is from Corollary [2,](#page-12-0) and $k = N - 2$ is from Corollary [3.](#page-13-0)

The following theorem proves, under the conditions S satisfies the property (N, ψ) , the set $\cup_{k=1}^N E_k^G$ is big.

Theorem 3. Let E_i^G be constructed for $i = 1, 2, ..., N$ as above. Suppose $\{G_j\} \subset (\mathcal{K}_{N-1}^{\psi})^{\perp}$ is a *dense subset, then*

$$
spt\mu_1 \subset clos\left(\bigcup_{i=1}^N\bigcup_{j=1}^\infty E_i^{G_j}\right).
$$

Proof: First we prove

$$
\mu_1\left(Int(\sigma(S))\setminus clos\left(\bigcup_{i=1}^N\bigcup_{j=1}^\infty E_i^{G_j}\right)\right)=0.
$$

Suppose that $B(\lambda_0, \delta) \subset Int(\sigma(S))$ and $B(\lambda_0, \delta) \cap clos\left(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j}\right) = \emptyset$, then by construction of $E_i^{G_j}$, we conclude that

$$
\mathcal{C}_{\psi}^{N-1}(\langle F_i, G_j \rangle \mu_1)(z) = 0
$$

on $B(\lambda_0, \delta)$, where $i = 1, 2, ..., N$. By taking $\overline{\partial}$ in the sense of distribution, we see that

$$
\mathcal{C}(\langle F_i,G_j\rangle\mu_1)(z)=0
$$

a.e. Area on $B(\lambda_0, \delta)$ since $Area({\bar \partial \psi = 0} \cap \sigma(S)) = 0$, where $i = 1, 2, ..., N$. For a smooth function ϕ with compact support in $B(\lambda_0, \delta)$,

$$
\int \phi(z)\langle F_i, G_j \rangle d\mu_1 = \frac{1}{\pi} \int \bar{\partial} \phi(z) \mathcal{C}(\langle F_i, G_j \rangle \mu_1)(z) dA(z) = 0.
$$

Therefore,

$$
\langle F_i(z), G_j(z) \rangle = 0. \ a.e. \ \mu_1|_{B(\lambda_0, \delta)} \tag{3-17}
$$

where $i = 1, 2, ..., N$. From (1-4), we see that for $P \in \bigoplus_{k=1}^{m} L^2(\mu_k|_{B(\lambda_0, \delta)})$, (3-17) implies $(P, G_j) = 0$. Therefore,

$$
\oplus_{k=1}^m L^2(\mu_k|_{B(\lambda_0,\delta)}) \subset \mathcal{K}_{N-1}^{\psi}.
$$

Hence, $\mu_1|_{B(\lambda_0,\delta)} = 0$ since $M_z|_{\mathcal{K}_{N-1}^{\psi}}$ is pure.

Now assume $B(\lambda_0, \delta) \cap clos(Int(\sigma(S))) = \emptyset$. For $N > 1$, the function $\mathcal{C}_{\psi}^{N-1}(\langle F_i, G_j \rangle \mu_1)(z)$ is continuous on $\mathbb{C} \setminus A$ and is zero on $\mathbb{C} \setminus \sigma(S)$. Hence,

$$
\mathcal{C}_{\psi}^{N-1}(\langle F_i, G_j \rangle \mu_1)(z) = 0
$$

on $B(\lambda_0, \delta) \setminus A$, where $i = 1, 2, ..., N$. Using the same proof as above, we see that $\mu_1|_{B(\lambda_0, \delta)} = 0$. This implies $spt\mu_1 \subset clos(Int(\sigma(S))).$ The theorem is proved.

Proof of Theorem [2:](#page-3-0) From (3-16) and Theorem [3,](#page-13-1) we get

$$
\bigcup_{i=1}^N \partial U_i \subset \sigma_e(S) \subset spt(\mu_1) \subset clos\left(\bigcup_{i=1}^N U_i\right).
$$

This implies

$$
\sigma_e(S) = \bigcup_{i=1}^N \partial U_i
$$

since $\sigma_e(S) \cap U_i = \emptyset$. This completes the proof.

For a positive finite measure μ with compact support on \mathbb{C} , definite

$$
P^{2}(\mu|1,\bar{z},...,\bar{z}^{N-1}) = clos\{p_{1}(z) + p_{2}(z)\bar{z} + ... + p_{N}(z)\bar{z}^{N-1} : p_{1},p_{2},...,p_{N} \in \mathcal{P}\}
$$

and $S_{N,\mu}$ as the multiplication by z on $P^2(\mu|1,\bar{z},...,\bar{z}^{N-1})$. Then $S_{N,\mu}$ is a multicyclic subnormal operator with the minimal normal extension M_{μ} , the multiplication by z, on $L^2(\mu)$.

Corollary 4. Suppose that $S_{2,\mu}$ on $P^2(\mu|1,\bar{z},\bar{z}^2)$ is pure, then the operator $S_{1,\mu}$ on $P^2(\mu|1,\bar{z})$ *satisfies*

$$
\sigma(S_{1,\mu}) = clos(\sigma(S_{1,\mu}) \setminus \sigma_e(S_{1,\mu})).
$$

Proof: Since

$$
\mathcal{K}_1^{\bar{z}} = clos(span(\bar{z}^k P^2(\mu | 1, \bar{z}) : 0 \le k \le 1)) = P^2(\mu | 1, \bar{z}, \bar{z}^2)
$$

and $S_{2,\mu}$ on $P^2(\mu|1,\bar{z},\bar{z}^2)$ is pure. Therefore, the result follows from Theorem [2.](#page-3-0)

It seems strong to assume that $S_{2,\mu}$ on $P^2(\mu|1,\bar{z},\bar{z}^2)$ is pure in the corollary. We believe that the condition can be reduced to assume that $S_{1,\mu}$ on $P^2(\mu|1,\bar{z})$ is pure. However, we are not able to prove the result under the weaker conditions. We will leave it as an open problem for further research.

Problem 1. Does Corollary [4](#page-14-0) hold under the weaker assumption that $S_{1,\mu}$ on $P^2(\mu|1,\bar{z})$ is *pure?*

Corollary 5. Let S on H be a pure rationally N -cyclic subnormal operator with $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$ and let M_z be its minimal normal extension on K satisfying $(1-1)$ to $(1-4)$. Suppose that there *exists a smooth function* ψ *on* $\mathbb C$ *such that* $Area({\lbrace \bar{\partial}\psi = 0 \rbrace} \cap \sigma(S)) = 0$ *and* $\psi(M_z) \mathcal{H} \subset \mathcal{H}$. *Then there exist bounded open subsets* U_i *for* $1 \leq i \leq N$ *such that*

$$
\sigma_e(S) = \bigcup_{i=1}^N \partial U_i, \ \sigma(S) \setminus \sigma_e(S) = \bigcup_{i=1}^N U_i,
$$

and

$$
dimker(S - \lambda)^* = i.
$$

for $\lambda \in U_i$.

Notice that Example [1](#page-6-0) and [2](#page-7-1) are special cases of Corollary [5.](#page-14-1) It seems that further results could be obtained for the special cases where S satisfies the conditions of Corollary [5.](#page-14-1) Moreover, we might be able to combine the methodology in [McCarthy and Yang \(1997](#page-15-16)) to obtain the structural models for the class of subnormal operators, which might extend Xia's model for subnormal operators with finite rank self-commutators.

Problem 2. *Can the structure of subnormal operators in Corollary [5](#page-14-1) be characterized?*

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