Spectral Picture For Rationally Multicyclic Subnormal Operators

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Abstract

For a pure bounded rationally cyclic subnormal operator S on a separable complex Hilbert space \mathcal{H} , Conway and Elias (1993) shows that $clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S)))$. This paper examines the property for rationally multicyclic (N-cyclic) subnormal operators. We show: (1) There exists a 2-cyclic irreducible subnormal operator S with $clos(\sigma(S) \setminus \sigma_e(S)) \neq$ $clos(Int(\sigma(S)))$. (2) For a pure rationally N-cyclic subnormal operator S on \mathcal{H} with the minimal normal extension M on $\mathcal{K} \supset \mathcal{H}$, let $\mathcal{K}_m = clos(span\{(M^*)^k x : x \in \mathcal{H}, 0 \leq k \leq m\}$. Suppose $M|_{\mathcal{K}_{N-1}}$ is pure, then $clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S)))$.

1 Introduction

Let \mathcal{H} be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the space of bounded linear operators on \mathcal{H} . An operator $S \in \mathcal{L}(\mathcal{H})$ is subnormal if there exist a separable complex Hilbert space \mathcal{K} containing \mathcal{H} and a normal operator $M_z \in \mathcal{L}(\mathcal{K})$ such that $M_z \mathcal{H} \subset \mathcal{H}$ and $S = M_z|_{\mathcal{H}}$. By the spectral theorem of normal operators, we assume that

$$\mathcal{K} = \bigoplus_{i=1}^{m} L^2(\mu_i) \tag{1-1}$$

where $\mu_1 >> \mu_2 >> \dots >> \mu_m$ (*m* may be ∞) are compactly supported finite positive measures on the complex plane \mathbb{C} , and M_z is multiplication by z on \mathcal{K} . For $H = (h_1, \dots, h_m) \in \mathcal{K}$ and $G = (g_1, \dots, g_m) \in \mathcal{K}$, we define

$$\langle H(z), G(z) \rangle = \sum_{i=1}^{m} h_i(z) \overline{g_i(z)} \frac{d\mu_i}{d\mu_1}, \ |H(z)|^2 = \langle H(z), H(z) \rangle.$$
(1-2)

The inner product of H and G in \mathcal{K} is defined by

$$(H,G) = \int \langle H(z), G(z) \rangle \, d\mu_1(z). \tag{1-3}$$

 M_z is the minimal normal extension if

$$\mathcal{K} = clos\left(span(M_z^{*k}x: x \in \mathcal{H}, k \ge 0)\right).$$
(1-4)

We will always assume that M_z is the minimal normal extension of S and \mathcal{K} satisfies (1-1) to (1-4). For details about the functional model above and basic knowledge of subnormal operators, the reader shall consult Chapter II of the book Conway (1991).

For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, $\sigma_e(T)$ the essential spectrum of T, T^* its adjoint, ker(T) its kernel, and Ran(T) its range. For a subset $A \subset \mathbb{C}$, we set Int(A) for its

interior, clos(A) for its closure, A^c for its complement, and $\overline{A} = \{\overline{z} : z \in A\}$. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0, 1)$. Let \mathcal{P} denote the set of polynomials in the complex variable z. For a compact subset $K \subset \mathbb{C}$, let Rat(K) be the set of all rational functions with poles off K and let R(K) be the uniform closure of Rat(K).

A subnormal operator S on \mathcal{H} is pure if for every non-zero invariant subspace I of S ($SI \subset I$), the operator $S|_I$ is not normal. For $F_1, F_2, ..., F_N \in \mathcal{H}$, let

$$R^{2}(S|F_{1}, F_{2}, ..., F_{N}) = clos\{r_{1}(S)F_{1} + r_{2}(S)F_{2} + ... + r_{N}(S)F_{N}\}$$

in \mathcal{H} , where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$ and let

$$P^{2}(S|F_{1}, F_{2}, ..., F_{N}) = clos\{p_{1}(S)F_{1} + p_{2}(S)F_{2} + ... + p_{N}(S)F_{N}\}$$

in \mathcal{H} , where $p_1, p_2, ..., p_N \in \mathcal{P}$. A subnormal operator S on \mathcal{H} is rationally multicyclic (N-cyclic) if there are N vectors $F_1, F_2, ..., F_N \in \mathcal{H}$ such that

$$\mathcal{H} = R^2(S|F_1, F_2, \dots, F_N)$$

and for any $G_1, ..., G_{N-1} \in \mathcal{H}$,

$$\mathcal{H} \neq R^2(S|G_1, G_2, ..., G_{N-1}).$$

We call N is the rationally cyclic multiple of S. S is multicyclic (N-cyclic) if

$$\mathcal{H} = P^2(S|F_1, F_2, \dots, F_N)$$

and for any $G_1, ..., G_{N-1} \in \mathcal{H}$,

$$\mathcal{H} \neq P^2(S|G_1, G_2, ..., G_{N-1}).$$

We call N is the cyclic multiple of S. In this case, $m \leq N$ where m is as in (1-1).

Let μ be a compactly supported finite positive measure on the complex plane \mathbb{C} and let $spt(\mu)$ denote the support of μ . For a compact subset K with $spt(\mu) \subset K$, let $R^2(K,\mu)$ be the closure of Rat(K) in $L^2(\mu)$. Let $P^2(\mu)$ denote the closure of \mathcal{P} in $L^2(\mu)$.

If S is rationally cyclic, then S is unitarily equivalent to multiplication by z on $R^2(\sigma(S), \mu_1)$, where m = 1 and $F_1 = 1$. We may write $R^2(S|F_1) = R^2(\sigma(S), \mu_1)$. If S is cyclic, then S is unitarily equivalent to multiplication by z on $P^2(\mu_1)$. We may write $P^2(S|F_1) = P^2(\mu_1)$.

For a rationally N-cyclic subnormal operator S with cyclic vectors $F_1, F_2, ..., F_N$ and $\lambda \in \sigma(S)$, we denote the map

$$E(\lambda): \sum_{i=1}^{N} r_i(S)F_i \to \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \\ \dots \\ r_N(\lambda) \end{bmatrix}, \qquad (1-5)$$

where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$. If $E(\lambda)$ is bounded from \mathcal{K} to $(\mathbb{C}^N, \|.\|_{1,N})$, where $\|x\|_{1,N} = \sum_{i=1}^N |x_i|$ for $x \in \mathbb{C}^N$, then every component in the right hand side extends to a bounded linear functional on \mathcal{H} and we will call λ a bounded point evaluation for S. We use bpe(S)to denote the set of bounded point evaluations for S. The set bpe(S) does not depend on the choices of cyclic vectors $F_1, F_2, ..., F_N$ (see Corollary 1.1 in Mbekhta et al. (2016)). A point $\lambda_0 \in int(bpe(S))$ is called an analytic bounded point evaluation for S if there is a neighborhood $B(\lambda_0, \delta) \subset bpe(S)$ of λ_0 such that $E(\lambda)$ is analytic as a function of λ on $B(\lambda_0, \delta)$ (equivalently (1-5) is uniformly bounded for $\lambda \in B(\lambda_0, \delta)$). We use abpe(S) to denote the set of analytic bounded point evaluations for S. The set abpe(S) does not depend on the choices of cyclic vectors $F_1, F_2, ..., F_N$ (also see Remark 3.1 in Mbekhta et al. (2016)). Similarly, for an N-cyclic subnormal operator S, we can define bpe(S) and abpe(S) if we replace $r_1, r_2, ..., r_N \in Rat(\sigma(S))$ in (1-5) by $p_1, p_2, ..., p_N \in \mathcal{P}$.

For N = 1, Thomson (1991) proves a remarkable structural theorem for $P^2(\mu)$.

Thomson's Theorem. There is a Borel partition $\{\Delta_i\}_{i=0}^{\infty}$ of $spt\mu$ such that the space $P^2(\mu|_{\Delta_i})$ contains no nontrivial characteristic functions and

$$P^{2}(\mu) = L^{2}(\mu|_{\Delta_{0}}) \oplus \left\{ \bigoplus_{i=1}^{\infty} P^{2}(\mu|_{\Delta_{i}}) \right\}.$$

Furthermore, if U_i is the open set of analytic bounded point evaluations for $P^2(\mu|_{\Delta_i})$ for $i \ge 1$, then U_i is a simply connected region and the closure of U_i contains Δ_i .

Conway and Elias (1993) extends some results of Thomson's Theorem to the space $R^2(K, \mu)$, while Brennan (2008) expresses $R^2(K, \mu)$ as a direct sum that includes both Thomson's theorem and results of Conway and Elias (1993). For a compactly supported complex Borel measure ν of \mathbb{C} , by estimating analytic capacity of the set $\{\lambda : |\mathcal{C}\nu(\lambda)| \ge c\}$, where $\mathcal{C}\nu$ is the Cauchy transform of ν (see Section 3 for definition), Brennan (2006. English), Aleman et al. (2009), and Aleman et al. (2010) provide interesting alternative proofs of Thomson's theorem. Both their proofs rely on X. Tolsa's deep results on analytic capacity. There are other related research papers for N = 1 in the history. For example, Brennan (1979), Hruscev (1979. Russian), Brennan and Militzer (2011), and Yang (2016), etc.

Thomson's Theorem shows in Theorem 4.11 of Thomson (1991) that abpe(S) = bpe(S) for a cyclic subnormal operator S (See also Chap VIII Theorem 4.4 in Conway (1991)). Corollary 5.2 in Conway and Elias (1993) proves that the result holds for rationally cyclic subnormal operators. For N > 1, Yang (2018) extends the result to rationally N-cyclic subnormal operators.

It is shown in Theorem 2.1 of Conway and Elias (1993) that if S is a pure rationally cyclic subnormal operator, then

$$clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S))).$$
(1-6)

This leads us to examine if (1-6) holds for a rationally N-cyclic subnormal operator.

A Gleason part of R(K) is a maximal set Ω in \mathbb{C} such that for $x, y \in \Omega$, if e_x and e_y denote the functionals evaluation at x and y respectively, then $||e_x - e_y||_{R(K)^*} < 2$. Olin and Thomson (1980) shows that a compact set K can be the spectrum of an irreducible subnormal operator if and only if R(K) has only one non-trivial Gleason part Ω and $K = clos(\Omega)$. McGuire (1988) and Feldman and McGuire (2003) construct irreducible subnormal operators with a prescribed spectrum, approximate point spectrum, essential spectrum, and the (semi) Fredholm index. Our first result is to construct a (rationally) 2-cyclic irreducible subnormal operator for a prescribed spectrum and essential spectrum. Consequently we show that (1-6) may not hold for a (rationally) N-cyclic irreducible subnormal operator with cyclic multiple N > 1.

Theorem 1. Let K and K_e be two compact subsets of \mathbb{C} such that R(K) has only one nontrivial Gleason part Ω , $K = clos(\Omega)$, and $\partial K \subset K_e \subset K$. Then there exists a rationally 2-cyclic irreducible subnormal operator S such that $\sigma(S) = K$, $\sigma_e(S) = K_e$, and $ind(S - \lambda) = -1$ for $\lambda \in K \setminus K_e$. If, in particular, $\mathbb{C} \setminus K$ has only one component, then S can be constructed as a 2-cyclic irreducible subnormal operator.

Let $K = clos(\mathbb{D})$ and $K_e = \partial \mathbb{D} \cup clos(\frac{1}{2}\mathbb{D})$. We see that

$$clos(K \setminus K_e) = \{z : \frac{1}{2} \le |z| \le 1\} \ne clos(Int(K)) = clos(\mathbb{D}).$$

From Theorem 1, we get the following result.

Corollary 1. There exists a 2-cyclic irreducible subnormal operator S such that (1-6) does not hold.

In the second part of this paper, we will investigate certain classes of rationally N-cyclic subnormal operators that have the property (1-6). Let S be a rationally N-cyclic subnormal operator on $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$. Let ψ be a smooth function with compact support. Define

$$\mathcal{K}_n^{\psi} = clos \left\{ \psi^m x : x \in \mathcal{H}, \ 0 \le m \le n \right\},\$$

then

$$\mathcal{H} \subset \mathcal{K}_1^{\psi} \subset \ldots \subset \mathcal{K}_n^{\psi} \subset \ldots \subset \mathcal{K}$$

and $M_z|_{\kappa^{\psi}}$ is a subnormal operator.

Definition 1. A subnormal operator satisfies the property (N, ψ) if the following conditions are met:

(1) S is a pure (rationally) N-cyclic subnormal operator on $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$.

(2) ψ a smooth function with compact support and $Area(\sigma(S) \cap \{\bar{\partial}\psi = 0\}) = 0$. Let M_z on \mathcal{K} be the minimal normal extension of S satisfying (1-1) to (1-4), then $M_z|_{\mathcal{K}_{N-1}^{\psi}}$ is also a pure subnormal operator.

Theorem 2. Let N > 1 and let S be a pure subnormal operator on \mathcal{H} satisfying the property (N, ψ) , then there exist bounded open subsets U_i for $1 \le i \le N$ such that

$$\sigma_e(S) = \bigcup_{i=1}^N \partial U_i, \ \sigma(S) = \bigcup_{i=1}^N clos(U_i),$$

and

$$ind(S-\lambda) = -i$$

for $\lambda \in U_i$ and i = 1, 2, ...N. Consequently,

$$\sigma(S) = clos(\sigma(S) \setminus \sigma_e(S)) = clos(Int(\sigma(S)))$$

An important special case is that $\psi = \bar{z}$. In section 3, we will provide several examples of subnormal operators that satisfy the property (N, ψ) . We prove Theorem 1 in section 2 and Theorem 2 in section 3.

2 Spectral Pictures for Irreducible Rationally 2-Cyclic Subnormal Operators

In this section, we assume that K is a compact subset of \mathbb{C} , $Int(K) \neq \emptyset$, and R(K) has only one nontrival Gleason part Ω with $K = clos(\Omega)$. Theorem 5 and Corollary 6 in McGuire (1988) constructs a representing measure ν of R(K) at $z_0 \in Int(K)$ with support on ∂K such that S_{ν} on $R^2(K,\nu)$ is irreducible, $\sigma(S_{\nu}) = K$, $\sigma_e(S_{\nu}) = \partial K$, and $ind(S_{\nu} - \lambda) = -1$ for $\lambda \in Int(K) = \sigma(S_{\nu}) \setminus \sigma_e(S_{\nu})$. From Theorem 6.2 in Gamelin (1969), we get

$$L^{2}(\nu) = R^{2}(K,\nu) \oplus N^{2} \oplus \overline{R^{2}_{0}(K,\nu)}$$

$$(2-1)$$

where $\overline{R_0^2(K,\nu)} = \{\overline{r} : r(z_0) = 0 \text{ and } r \in R^2(K,\nu)\}$. The operator M_z , multiplication by z on $L^2(\nu)$, can be written as the following matrix with respect to (2-1):

$$M_z = \begin{bmatrix} S_\nu, & A, & B\\ 0, & C, & D\\ 0, & 0, & T_\nu^* \end{bmatrix}$$

where T_{ν} , multiplication by \bar{z} on $\overline{R_0^2(K,\nu)}$, is an irreducible rationally cyclic subnormal operator with $\sigma(T_{\nu}) = \bar{K}$, $\sigma_e(T_{\nu}) = \partial \bar{K}$, and $ind(T_{\nu} - \lambda) = -1$ for $\lambda \in Int(\bar{K})$. Let

$$S = \begin{bmatrix} S_\nu, & A \\ 0, & C \end{bmatrix},$$

then S is the dual of T_{ν} . From the properties of dual subnormal operators (see, for example, Theorem 2.4 in Feldman and McGuire (2003)), we see that S is an irreducible subnormal operator with $\sigma(S) = K$, $\sigma_e(S) = \partial K$, and $ind(S - \lambda) = -1$ for $\lambda \in Int(K)$.

The following lemma, due to Cowen and Douglas (1978) on page 194, allows us to choose eigenvectors for S^* in a co-analytic manner whenever the Fredholm index function for S is -1. Lemma 1. If $X \in L(\mathcal{H})$ and $ind(X - \lambda) = -1$ for all $\lambda \in G := \sigma(X) \setminus \sigma_e(X)$, then there exists a co-analytic function $h: G \to H$ that is not identically zero on any component of G such that $h(\lambda) \in ker(X - \lambda)^*$. In particular, for every $x \in \mathcal{H}$, the function $\lambda \to (x, h(\lambda))$ is analytic on G. Using Lemma 1, we conclude that there exists a co-analytic function $k_{\lambda} \in \mathcal{H} := R^2(K,\nu) \oplus N^2$ such that $(S - \lambda)^* k_{\lambda} = 0$ on Int(K). Let δ_{λ} be the point mass measure at λ . Let $K_e \subset K$ be a compact subset of \mathbb{C} such that $\partial K \subset K_e$. Let $\{\lambda_n\} \subset K_e \cap Int(K)$ with $K_e \cap Int(K) \subset clos(\{\lambda_n\})$. Define

$$\mu = \nu + \sum_{n=1}^{\infty} c_n \delta_{\lambda_n}, \qquad (2-2)$$

where $c_n > 0$ and $\sum_{n=1}^{\infty} c_n ||k_{\lambda_n}||^2 = 1$. Let M_z^1 be the multiplication by z operator on $L^2(\mu)$. Lemma 2. Define an operator T from \mathcal{H} to $L^2(\mu)$ by

$$Tf(z) = \begin{cases} f(z), & z \in \partial K\\ (f, k_{\lambda_n}), & z = \lambda_n. \end{cases}$$
(2-3)

Then T is a bounded linear one to one operator with closed range. Set $\mathcal{H}_1 = Ran(T)$, then T is invertible from \mathcal{H} to \mathcal{H}_1 , $M_z^1 \mathcal{H}_1 \subset \mathcal{H}_1$, $S_1 = M_z^1|_{\mathcal{H}_1}$ is an irreducible subnormal operator such that $S_1 = TST^{-1}$, and M_z^1 is the minimal normal extension of S_1 .

Proof: By definition, we get

$$|f||_{L^{2}(\nu)}^{2} \leq ||Tf||_{L^{2}(\mu)}^{2} = ||f||_{L^{2}(\nu)}^{2} + \sum_{n=1}^{\infty} c_{n} |(f, k_{\lambda_{n}})|^{2} \leq 2||f||_{L^{2}(\nu)}^{2}.$$

Therefore, T is a bounded linear operator and invertible from \mathcal{H} to \mathcal{H}_1 . Since $(zf, k_{\lambda_n}) = \lambda_n(f, k_{\lambda_n})$, we see that $M_z^1 \mathcal{H}_1 \subset \mathcal{H}_1$ and $S_1 = TST^{-1}$. Since $(Tk_{\lambda_n})(\lambda_n) = ||k_{\lambda_n}||^2 > 0$, clearly, we have

$$L^{2}(\mu) = clos\left(span\{\bar{z}^{m}x: x \in \mathcal{H}_{1}, m \geq 0\}\right).$$

Therefore, M_z^1 is the minimal normal extension of S_1 .

It remains to prove that S_1 is irreducible. Let N_1 and N_2 be two reducing subspaces of S_1 such that $\mathcal{H}_1 = N_1 \oplus N_2$. Then for $f_1 \in N_1$ and $f_2 \in N_2$, we have

$$(z^n f_1, z^m f_2) = \int z^n \bar{z}^m f_1 \bar{f}_2 d\mu = 0$$

for n, m = 0, 1, 2, ... This implies $f_1(z)\bar{f}_2(z) = 0$ a.e. μ . By the definition of T, we see that $(T^{-1}f_1)(z)(T^{-1}f_2)(z) = 0$ a.e. ν . Hence, $\mathcal{H} = T^{-1}N_1 \oplus T^{-1}N_2$. $T^{-1}N_1$ and $T^{-1}N_2$ are reducing subspaces of S. By the construction, T_{ν} is irreducible (Corollary 6 in McGuire (1988)), so S, as the dual T_{ν} , is irreducible (see, for example, Theorem 2.4 in Feldman and McGuire (2003)). This means that $N_1 = 0$ or $N_2 = 0$. The lemma is proved.

We write the operator M_z^1 as the following:

$$M_z^1 = \begin{bmatrix} S_1, & A_1 \\ 0, & T_1^* \end{bmatrix}$$
(2-4)

Then T_1 , as a dual of S_1 , is irreducible.

Lemma 3. Let μ be as in(2-2) and let \mathcal{H}_1 be as in Lemma 2. Define

$$F(z) = \begin{cases} \overline{z} - \overline{z}_0, & z \in \partial K, \\ 0, & z \in Int(K). \end{cases}$$
(2-5)

and

$$G_n(z) = \begin{cases} k_{\lambda_n}(z), & z \in \partial K, \\ -1/c_n, & z = \lambda_n, \\ 0, & z = \lambda_m, \quad m \neq n. \end{cases}$$
(2-6)

Then

$$\mathcal{H}_1^{\perp} = clos\left(span\{r(\bar{z})F, G_j, \ 1 \le j < \infty, \ r \in Rat(K)\}\right).$$

Proof: It is straightforward to check, from (2-1), (2-2), and (2-3), that $F, G_j \in \mathcal{H}_1^{\perp}$. Now let $H(z) \perp clos(span\{r(\bar{z})F, G_j, 1 \leq j < \infty, r \in Rat(K)\})$, then

$$\int H(z)r(z)\bar{F}(z)d\mu = \int H(z)r(z)(z-z_0)d\nu = 0$$

for $r \in Rat(K)$. From (2-1), we see that the function $H|_{\partial K} \in \mathcal{H}$. It follows from $\int H(z)\overline{G}_j(z)d\mu = 0$ that $H(\lambda_j) = (H|_{\partial K}, k_{\lambda_j})$. Thus, $H(z) \in \mathcal{H}_1$. The lemma is proved.

Lemma 4. Let μ , T_1 , F, and G_n be as in (2-2), (2-4), (2-5) and (2-6), respectively. Then there exists a sequence of positive numbers $\{a_n\}$ satisfying

$$\sum_{n=1}^{\infty} a_n \|G_n\| < \infty, \ G = \sum_{n=1}^{\infty} a_n G_n,$$

and

$$\mathcal{H}_1^{\perp} = clos\left(span\{r(\bar{z})F(z) + p(\bar{z})G(z): r \in Rat(K), p \in \mathcal{P}\}\right).$$

Therefore, T_1 is a rationally 2-cyclic irreducible subnormal operator with

$$\sigma(T_1) = \bar{K}, \ \sigma_e(T_1) = \bar{K}_e, \ and \ ind(T_1 - \lambda) = -1, \ \lambda \in \bar{K} \setminus \bar{K}_e.$$
(2-7)

Proof: Notice that

$$\int f(z)(z-\lambda_n)\bar{k}_{\lambda_n}(z)d\nu = 0$$

for $f \in \mathcal{H}$. We conclude, from (2-1), that $(\bar{z} - \bar{\lambda}_n)k_{\lambda_n}(z) \in \overline{R_0^2(K,\nu)}$. Hence, there are $\{r_n\} \subset R^2(K,\nu)$ such that

$$k_{\lambda_n}(z) = \frac{r_n(\bar{z})}{\bar{z} - \bar{\lambda}_n} (\bar{z} - \bar{z}_0).$$

We will recursively choose $\{a_n\}$. First choose $a_1 = 1$. Then we assume that $a_1, a_2, ..., a_n$ have been chosen. Now we will choose a_{n+1} . Let

$$p_k(z) = \frac{\prod_{j \neq k, 1 \le j \le n} (z - \lambda_j)}{a_k \prod_{j \neq k, 1 \le j \le n} (\bar{\lambda}_k - \bar{\lambda}_j)},$$

for k = 1, 2, ..., n. Denote

$$q_{1k}(z) = p_k(z) \sum_{j \neq k, 1 \le j \le n} \frac{a_j}{z - \bar{\lambda}_j} r_j(z)$$

and

$$q_{2k}(z) = \frac{a_k(p_k(z) - p_k(\bar{\lambda}_k))}{z - \bar{\lambda}_k} r_k(z).$$

So $p_k \in \mathcal{P}$ and $q_{1k}, q_{2k} \in R^2(K, \nu)$ for k = 1, 2, ..., n. Clearly,

$$p_k(\bar{z})\sum_{j=1}^n a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))(\bar{z} - \bar{z}_0) = \frac{r_k(\bar{z})(\bar{z} - \bar{z}_0)}{\bar{z} - \bar{\lambda}_k}, \ z \in \partial K.$$

Hence,

$$p_k(\bar{z})\sum_{j=1}^n a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F(z) = G_k(z), \ a.e. \ \mu.$$

We have the following calculation:

$$\int \left| p_k(\bar{z}) \sum_{j=1}^{n+1} a_j G_j(z) - (q_{1k}(\bar{z}) + q_{2k}(\bar{z})) F(z) - G_k(z) \right|^2 d\mu$$

=
$$\int |p_k(\bar{z}) a_{n+1} G_{n+1}(z)|^2 d\mu$$

$$\leq \left(\frac{a_{n+1}}{a_k}\right)^2 \frac{(4D^2)^{n-1}}{\prod_{j \neq k, 1 \leq j \leq n} |\lambda_k - \lambda_j|^2} \|G_{n+1}\|^2$$

where $D = \max\{|z|: z \in K\}$. Now set

$$a_{n+1} = \min\left(\frac{1}{2^{n+1}}, \min_{1 \le k \le n} \frac{a_k \prod_{j \ne k, 1 \le j \le n} \min(1, |\lambda_k - \lambda_j|)}{4^n \max(1, D)^{n-1}}\right) / \max(1, \|G_{n+1}\|).$$
(2-8)

So we have chosen all $\{a_n\}$. From (2-8), we have the following calculation.

$$\begin{aligned} \left\| p_k \sum_{i=n+2}^{\infty} a_j G_j \right\| \\ &\leq \frac{(2D)^{n-1}}{a_k \prod_{j \neq k, 1 \leq j \leq n} |\lambda_k - \lambda_j|} \sum_{i=n+2}^{\infty} \frac{a_k \prod_{j \neq k, 1 \leq j \leq i-1} \min(1, |\lambda_k - \lambda_j|)}{4^{i-1} \max(1, D)^{i-2}} \\ &\leq \frac{1}{2^{n+2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|p_{k}(\bar{z})G - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F - G_{k}(z)\| \\ &\leq \left\| p_{k}(\bar{z})\sum_{j=1}^{n+1} a_{j}G_{j} - (q_{1k}(\bar{z}) + q_{2k}(\bar{z}))F - G_{k}(z) \right\| + \left\| p_{k}(\bar{z})\sum_{j=n+2}^{\infty} a_{j}G_{j} \right\| \\ &\leq \frac{1}{2^{n}}. \end{aligned}$$

Hence,

$$G_k \in clos(span\{r(\bar{z})F(z) + p(\bar{z})G(z): r \in Rat(K), p \in \mathcal{P}\}), k = 1, 2, ...$$

Since T_1 is the dual of S_1 , we see that $\sigma(M_z^1) \subset \sigma_e(S_1) \cup \overline{\sigma_e(T_1)}$ (see, for example, Theorem 2.4 in Feldman and McGuire (2003)), $\sigma_e(S_1) = \partial K$, and $\sigma_e(T_1) \supset \partial \overline{K}$. So (2-7) follows. This completes the proof.

Proof of Theorem 1: It follows from Lemma 4.

3 Spectral Picture of a Class of Rationally Multicyclic Subnormal Operators

In this section, we will prove Theorem 2. First we provide some examples of subnormal operators that have the property (N, ψ) in Definition 1.

Example 1. Every pure subnormal operator S on \mathcal{H} with finite rank self-commutator has the property (N, ψ) . Notice that the structure of such subnormal operators has been established based on Xia's model (see Xia (1996) and Yakubovich (1998)).

Proof: Assume that M_z on \mathcal{K} is the minimal normal extension satisfying (1-1) to (1-4). Define the self-commutator as the following

$$D = [S^*, S] = S^*S - SS^*.$$

The element $x \in ker(D)$ if and only if $\overline{z}x \in \mathcal{H}$. This implies $Sker(D) \subset ker(D)$. Therefore,

$$S^*Ran(D) \subset Ran(D). \tag{3-1}$$

Let

$$\mathcal{H}_0 = clos\left(span(S^n f : f \in Ran(D), n \ge 0)\right),$$

then $S|_{\mathcal{H}_0}$ is N-cyclic subnormal where N = dim(Ran(D)).

On the other hand,

$$S^*S^n D = SS^*S^{n-1}D + DS^{n-1}D,$$

hence, we can recursively show that $S^*S^nRan(D) \subset \mathcal{H}_0$ since (3-1). So $S^*\mathcal{H}_0 \subset \mathcal{H}_0$. This implies that

$$S(\mathcal{H} \ominus \mathcal{H}_0) \subset \mathcal{H} \ominus \mathcal{H}_0$$

and $S|_{\mathcal{H} \ominus \mathcal{H}_0}$ is normal. Since S is pure, we conclude that $\mathcal{H} = \mathcal{H}_0$ and S is N-cyclic. From (3-1), we see that there is a polynomial p such that

$$\bar{p}(S^*|_{Ran(D)}) = 0.$$

Therefore,

$$p(S): \mathcal{H} \to ker(D).$$

Hence,

$$||M_z^* p(S)f|| = ||M_z p(S)f|| = ||Sp(S)f|| = ||S^* p(S)f||$$

for $f \in \mathcal{H}$. This implies $\bar{z}p\mathcal{H} \subset \mathcal{H}$. Let $\psi = \bar{z}p$, then $Area\{\bar{\partial}\psi = 0\} = Area\{z : p(z) = 0\} = 0$, $\mathcal{K}_{N-1}^{\psi} = \mathcal{H}$, and S satisfies the property (N, ψ) in Definition 1.

Example 2. In Lemma 4, if $K = clos(\mathbb{D})$ and $K_e = (\partial \mathbb{D}) \cup (\frac{1}{2}\partial \mathbb{D})$, then the operator T_1 is a 2-cyclic irreducible subnormal operator satisfying the property $(2, \psi)$ where $\psi = |z|^4 - \frac{5}{4}|z|^2$.

Proof: For $f \in \mathcal{H}_1$, we get

$$\psi f = (|z|^2 - 1)(|z|^2 - \frac{1}{4})f + \frac{1}{4}f = \frac{1}{4}f$$

since $spt(\mu) \subset K_e$. Hence, $\mathcal{K}_1^{\psi} = \mathcal{H}_1$. On the other hand,

$$Area\{\bar{\partial}\psi=0\} \le Area\left(\{0\} \cup \{|z|=\frac{5}{8}\}\right) = 0.$$

Therefore, the operator T_1 satisfies the property $(2, \psi)$.

In the remaining section, we assume that N > 1 and S is a pure rationally N-cyclic subnormal operator on $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$ and M_z on \mathcal{K} , which satisfies (1-1) to (1-4), is the minimal normal extension of S. Moreover, S satisfies the property (N, ψ) in Definition 1. Let U_k be the set of $\lambda \in Int(\sigma(S))$ such that $Ran(S - \lambda)$ is closed and $dim(ker(S - \lambda)^*) = k$, where k = 1, 2, ..., N.

Lemma 5. If $1 \leq k \leq N$, $\delta > 0$, $B(\lambda_0, 2\delta) \subset Int(\sigma(S))$, I is an index subset of $\{1, 2, ..., N\}$ with size N - k, $F = \sum_{i=1}^{N} r_i F_i$ where $r_i \in Rat(\sigma(S))$, and $\{a_{ls}(\lambda)\}_{1 \leq l \leq N-k, 1 \leq s \leq k}$ are analytic on $B(\lambda_0, 2\delta)$ such that

$$\sup_{1 \le s \le k, \lambda \in B(\lambda_0, \delta)} |r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda)| \le M \|F\|$$
(3-2)

and

$$F_{i_l}(z) = \sum_{s=1}^k a_{ls}(z) F_{j_s}(z), \ a.e \ \mu_1|_{B(\lambda_0,\delta)},$$
(3-3)

where $i_l \in I$ and $j_s \notin I$. Then $\lambda_0 \in \bigcup_{i=k}^N U_k$.

Proof: From (3-3), we get

$$\int_{B(\lambda_0,\delta)} |F|^2 d\mu_1 = \int_{B(\lambda_0,\delta)} \left| \sum_{s=1}^k \left(r_{j_s}(z) + \sum_{l=1}^{N-k} a_{ls}(z) r_{i_l}(z) \right) F_{j_s}(z) \right|^2 d\mu_1.$$

Using (3-2) and the maximal modulus principle,

$$\sup_{1 \le s \le k, \lambda \in B(\lambda_0, \delta)} \left| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) \right| \le \frac{M}{\delta} \| (S - \lambda_0) F \|.$$

Hence,

$$\int |F|^2 d\mu_1 \le \int_{B(\lambda_0,\delta)^c} |F|^2 d\mu_1 + (\sum_{j \notin I} ||F_j||)^2 \sup_{1 \le s \le k, \lambda \in B(\lambda_0,\delta)} \left| r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) \right|^2.$$

Therefore,

$$||F|| \le M_1 ||(S - \lambda_0)F||,$$

where

$$M_1^2 = \left(1 + \left(\sum_{j \notin I} \|F_j\|\right)^2\right) \left(\frac{M}{\delta}\right)^2.$$

So $Ran(S - \lambda_0)$ is closed. On the other hand, there are k linearly independent $k_{\lambda}^j \in \mathcal{H}$ such that

$$r_{j_s}(\lambda) + \sum_{l=1}^{N-k} a_{ls}(\lambda) r_{i_l}(\lambda) = \int \left\langle F(z), k_{\lambda}^j(z) \right\rangle d\mu_1(z)$$

where $j_s \notin I$ and $\lambda \in B(\lambda_0, \delta)$. This implies

$$\dim(Ker(S-\lambda_0)^*) \ge k.$$

Therefore, $\lambda_0 \in \bigcup_{i=k}^N U_i$.

Let ν be a compactly supported finite measure on \mathbb{C} . The transform

$$\mathcal{C}^{i}_{\psi}\nu(z) = \int \frac{(\psi(w) - \psi(z))^{i}}{w - z} d\nu(w)$$

is continuous at each point z with $|\nu|(\{z\}) = 0$ and i > 0. For i = 0, the transformation

$$\mathcal{C}^0_{\psi}(\nu) = \mathcal{C}(\nu) = \int \frac{1}{w - z} d\nu(w)$$

is the Cauchy transform of ν . Let $M^G(z)$ be the following N by N matrix,

$$M^{G}(z) = \left[\mathcal{C}_{\psi}^{i-1}(\langle F_{j}, G \rangle \,\mu_{1})\right]_{N \times N}$$

where we assume that $G \perp \mathcal{K}^{\psi}_{N-1}$ or equivalently G satisfies the following conditions

$$\bar{\psi}^n G \perp \mathcal{H}, \ n = 0, 1, 2, ..., N - 1.$$
 (3-4)

The set $W^G \subset \mathbb{C}$ is defined by:

$$W^G = \{\lambda : \int \frac{1}{|z-\lambda|} |\langle F_i(z), G(z) \rangle| d\mu_1(z) < \infty, \ 1 \le i \le N \}.$$

Let

$$\Omega^G = Int(\sigma(S)) \cap W^G \cap \{\lambda : |det(M^G(\lambda)| > 0\}.$$
(3-5)

Then for $\lambda \in \Omega^G$, the matrix

$$\left[\mathcal{C}(\langle F_j\psi^{i-1},G\rangle\mu_1)\right]_{N\times N} \tag{3-6}$$

is invertible. By Construction, we see that

 $det(M^G(z)) = 0$ a.e. $Area|_{(clos(\Omega^G))^c}$.

Lemma 6. Using above notations, we conclude that

$$\Omega^G \subset abpe(S).$$

Hence, by Lemma 5, we get $\Omega^G \subset U_N$.

Proof: Using (3-4), (3-5), and (3-6), we see that the lemma is a direct application of Theorem 2 in Yang (2018).

Let $A = \{\lambda_n : \mu_1(\{\lambda_n\}) > 0\}$ be the set of atoms for μ_1 . Now let us define the matrix $M_j^G(z)$ to be a submatrix of $M^G(z)$ by eliminating the first row and j column. Let $B_j^G(z)$ be the j column of the matrix $M^G(z)$ by eliminating the first row. Define

$$\Omega_j^G = \left(Int(\sigma(S)) \cap A^c \cap \{z : |det(M_j^G(z))| > 0\} \right) \setminus clos(\Omega^G).$$
(3-7)

Notice that $M_j^G(\lambda)$ is continuous at each $\lambda \in \Omega_j^G$. On Ω_j^G , we can define the following function vector

$$a_j(z) = [a_{ij}(z)]_{(N-1)\times 1} = (M_j^G(z))^{-1} B_j^G(z).$$
(3-8)

Lemma 7. Let G, Ω^G , Ω_j^G , and $a_j(z)$ be as in (3-4), (3-5), (3-7), and (3-8), respectively. Then for $\lambda_0 \in \Omega_j^G$, there exists $\delta > 0$ such that $a_j(z)$ equals an analytic function on $B(\lambda_0, \delta) \subset Int(\sigma(S))$ almost everywhere with respect to the area measure. Moreover,

$$\mathcal{C}(\langle F_j, G \rangle \mu)(z) = \sum_{k=1}^{j-1} a_{kj}(z) \mathcal{C}(\langle F_k, G \rangle \mu)(z) + \sum_{k=j+1}^N a_{k-1,j}(z) \mathcal{C}(\langle F_k, G \rangle \mu)(z), \ a.e. \ Area|_{B(\lambda_0,\delta)},$$
(3-9)

and

$$\langle F_j, G \rangle = \sum_{k=1}^{j-1} a_{kj}(z) \langle F_k, G \rangle + \sum_{k=j+1}^N a_{k-1,j}(z) \langle F_k, G \rangle, \ a.e.\mu|_{B(\lambda_0,\delta)}.$$
 (3-10)

Proof: Without loss of generality, we assume that j = N. For $z \in Int(\sigma(S) \cap W^G \cap \Omega_N^G)$, write

$$M^{G}(z) = \begin{bmatrix} A_{N}^{G}(z) & c_{N}^{G}(z) \\ M_{N}^{G}(z) & B_{N}^{g}(z) \end{bmatrix}$$

where

$$A_N^G(z) = [\mathcal{C}(\langle F_1, G \rangle \mu_1)(z), \mathcal{C}(\langle F_2, G \rangle \mu_1)(z), ..., \mathcal{C}(\langle F_{N-1}, G \rangle \mu_1)(z)]$$

and

$$c_N^G(z) = \mathcal{C}(\langle F_N, G \rangle \mu_1)(z).$$

By construction of Ω_N^G , we conclude that

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$$det(M^{G}(z)) = (A_{N}^{G}(z)(M_{N}^{G}(z))^{-1}B_{N}^{G}(z) - c_{N}^{G}(z))det(M_{N}^{G}(z)) = 0 \ a.e. \ Area|_{\Omega_{N}^{G}}.$$

Therefore,

$$c_N^G(z) = A_N^G(z)(M_N^G(z))^{-1} B_N^G(z) \ a.e. \ Area|_{\Omega_N^G}.$$
(3-11)

Let $\nu_i = \langle F_i, G \rangle \mu_1$ and $H_{i,m}(z) = \frac{m^2}{\pi} \nu_i(B(z, \frac{1}{m}))$, then the functions $H_{i,m}(z)$ are bounded with compact supports. We have

$$\mathcal{C}(H_{i,m}dA)(w) = \int_{|\lambda-w| \ge \frac{1}{m}} \frac{1}{\lambda-w} d\nu_i(\lambda) + \int_{|\lambda-w| < \frac{1}{m}} \frac{m^2 |\lambda-w|^2}{\lambda-w} d\nu_i(\lambda).$$

Hence,

$$|\mathcal{C}(H_{i,m}dA)(w) - \mathcal{C}\nu_i(w)| \le 2 \int_{|w-z|<1/m} \frac{1}{|w-z|} d|\nu_i|(z) \ a.e. \ Area$$

and

$$\lim_{n \to \infty} \mathcal{C}(H_{i.m} dA)(w) = \mathcal{C}\nu_i(w), \ a.e. \ Area.$$

Let $C_0 > 0$ be a constant such that $|\psi(z) - \psi(w)| \leq C_0 |z - w|$. We estimate $C^1_{\psi}(\nu_i)$ as the following,

$$\begin{aligned} &|\mathcal{C}^{1}_{\psi}(H_{i,m}dA)(w) - \mathcal{C}^{1}_{\psi}\nu_{i}(w)| \\ &= \left|\frac{m^{2}}{\pi} \int \int_{|z-\lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w} dA(z) d\nu_{i}(\lambda) - \mathcal{C}^{1}_{\psi}\nu_{i}(w)\right| \\ &\leq \left|\frac{m^{2}}{\pi} \int_{|\lambda-w| \geq \frac{1}{\sqrt{m}}} \int_{|z-\lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w} dA(z) d\nu_{i}(\lambda) - \int_{|\lambda-w| \geq \frac{1}{\sqrt{m}}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w} d\nu_{i}(\lambda)\right| \\ &+ \left|\frac{m^{2}}{\pi} \int_{|\lambda-w| < \frac{1}{\sqrt{m}}} \int_{|z-\lambda| < \frac{1}{m}} \frac{\psi(z) - \psi(w)}{z - w} dA(z) d\nu_{i}(\lambda)\right| + \left|\int_{|\lambda-w| < \frac{1}{\sqrt{m}}} \frac{\psi(\lambda) - \psi(w)}{\lambda - w} d\nu_{i}(\lambda)\right| \end{aligned}$$

Notice that

$$\frac{m^2}{\pi} \int_{|\lambda-w| \ge \frac{1}{\sqrt{m}}} \int_{|z-\lambda| < \frac{1}{m}} \frac{1}{z-w} dA(z) d\nu_i(\lambda) = \int_{|\lambda-w| \ge \frac{1}{\sqrt{m}}} \frac{1}{\lambda-w} d\nu_i(\lambda).$$

We get

$$\begin{aligned} &|\mathcal{C}^{1}_{\psi}(H_{i,m}dA)(w) - \mathcal{C}^{1}_{\psi}\nu_{i}(w)| \\ &\leq \left|\frac{m^{2}}{\pi}\int_{|\lambda-w|\geq\frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}}\frac{\psi(z) - \psi(\lambda)}{z - w}dA(z)d\nu_{i}(\lambda)\right| + 2C_{0}|\nu_{i}|(B(w,\frac{1}{\sqrt{m}})) \\ &\leq \frac{m^{2}}{\pi}\int_{|\lambda-w|\geq\frac{1}{\sqrt{m}}}\int_{|z-\lambda|<\frac{1}{m}}\frac{C_{0}|z-\lambda|}{|w-\lambda| - |z-\lambda|}dA(z)d\nu_{i}(\lambda) + 2C_{0}|\nu_{i}|(B(w,\frac{1}{\sqrt{m}})) \\ &\leq C_{0}\frac{\frac{1}{m}}{\frac{1}{\sqrt{m}} - \frac{1}{m}}|\nu_{i}|(B(w,\frac{1}{\sqrt{m}})^{c}) + 2C_{0}|\nu_{i}|(B(w,\frac{1}{\sqrt{m}})) \\ &\leq \frac{C_{0}}{\sqrt{m} - 1}\|\nu_{i}\| + 2C_{0}|\nu_{i}|(B(w,\frac{1}{\sqrt{m}})). \end{aligned}$$

Therefore,

$$\lim_{m \to \infty} \mathcal{C}^1_{\psi}(H_{i,m} dA)(w) = \mathcal{C}^1_{\psi} \nu_i(w)$$

for $w \notin A$. For $\lambda_0 \in \Omega_N^G$ and $\epsilon > 0$, we can choose a $\delta > 0$ and m_0 such that

$$\begin{aligned} &|\mathcal{C}^{1}_{\psi}(H_{i,m}dA)(w) - \mathcal{C}^{1}_{\psi}\nu_{i}(w)| \\ &\leq 2C_{0}|\nu_{i}|(B(w,\frac{1}{\sqrt{m}})) + \frac{C_{0}}{\sqrt{m}-1}\|\nu_{i}\| \\ &\leq 2C_{0}|\nu_{i}|(B(\lambda_{0},\delta+\frac{1}{\sqrt{m}})) + \frac{C_{0}}{\sqrt{m}-1}\|\nu_{i}\| \\ &<\epsilon \end{aligned}$$

where $w \in B(\lambda_0, \delta) \setminus A$ and $m \ge m_0$. Since $C^1_{\psi}\nu_i(w)$ is continuous at λ_0, δ can be chosen to ensure

$$|\mathcal{C}^1_{\psi}\nu_i(w) - \mathcal{C}^1_{\psi}\nu_i(\lambda_0)| < \epsilon$$

where $w \in B(\lambda_0, \delta) \setminus A$. It is easy to verify that $\mathcal{C}^1_{\psi}(H_{i,m}dA)$ is a smooth function. For k > 1, clearly $\mathcal{C}^k_{\psi}\nu_i(w)$ is a smooth function. Define

$$M_{N}^{Gm}(z) = \begin{bmatrix} \mathcal{C}_{\psi}^{1}(H_{1,m}dA), & \mathcal{C}_{\psi}^{1}(H_{2,m}dA), & \dots, & \mathcal{C}_{\psi}^{1}(H_{N-1,m}dA) \\ \mathcal{C}_{\psi}^{2}(\nu_{1}), & \mathcal{C}_{\psi}^{2}(\nu_{2}), & \dots, & \mathcal{C}_{\psi}^{2}(\nu_{N-1}) \\ \dots, & \dots, & \dots, & \dots \\ \mathcal{C}_{\psi}^{N-1}(\nu_{1}), & \mathcal{C}_{\psi}^{N-1}(\nu_{2}), & \dots, & \mathcal{C}_{\psi}^{N-1}(\nu_{N-1}) \end{bmatrix}.$$

We can choose ϵ small enough so that

$$M_N^{Gm}(w), \ M_N^G(w)$$

are invertible for $w \in B(\lambda_0, \delta) \setminus A$ and $m > m_0$. Define

$$B_N^{Gm}(z) = \begin{bmatrix} \mathcal{C}^1_{\psi}(H_{N,m}dA) \\ \mathcal{C}^2_{\psi}(\nu_N) \\ \dots \\ \mathcal{C}^{N-1}_{\psi}(\nu_N) \end{bmatrix},$$
$$A_N^{Gm}(z) = [\mathcal{C}(H_{1,m}dA), \mathcal{C}(H_{2,m}dA), \dots, \mathcal{C}(H_{N-1,m}dA)]$$

and

$$c_N^{Gm}(z) = \mathcal{C}(H_{N,m}dA)(z).$$

For a smooth function ϕ with compact support in $B(\lambda_0, \delta)$, using the definition (3-8) and Lebesgue's Dominated Convergence Theorem, we get the following calculation,

$$\int \bar{\partial}\phi(z)a_N(z)dA(z)$$

$$= \lim_{m \to \infty} \int \bar{\partial}\phi(z) \left((M_N^{Gm}(z))^{-1} B_N^{Gm}(z) \right) dA(z)$$

$$= -\lim_{m \to \infty} \int \phi(z) \bar{\partial} \left((M_N^{Gm}(z))^{-1} B_N^{Gm}(z) \right) dA(z)$$

$$= \lim_{m \to \infty} \int \phi(z) (M_N^{Gm}(z))^{-1} \left((\bar{\partial} M_N^{Gm}(z)) (M_N^{Gm}(z))^{-1} B_N^{Gm}(z) - \bar{\partial} B_N^{Gm}(z) \right) dA(z).$$
(3-12)

On the other hand,

$$\bar{\partial}M_{N}^{Gm}(z) = \bar{\partial}\psi(z) \begin{bmatrix} -\mathcal{C}(H_{1,m}dA), & -\mathcal{C}(H_{2,m}dA), & \dots, & -\mathcal{C}(H_{N-1,m}dA) \\ -2\mathcal{C}_{\psi}^{1}(\nu_{1}), & -2\mathcal{C}_{\psi}^{1}(\nu_{2}), & \dots, & -2\mathcal{C}_{\psi}^{1}(\nu_{N-1}) \\ \dots, & \dots, & \dots, & \dots \\ -(N-1)\mathcal{C}^{N-2}(\nu_{1}), & -(N-1)\mathcal{C}^{N-2}(\nu_{2}), & \dots, & -(N-1)\mathcal{C}^{N-2}(\nu_{N-1}) \end{bmatrix}.$$

Therefore,

$$(\bar{\partial}M_N^{Gm}(z))(M_N^{Gm}(z))^{-1} = -\bar{\partial}\psi(z) \begin{bmatrix} A_N^{Gm}(z) & (M_N^{Gm}(z))^{-1}, \\ 2, & 0, & \dots, & 0, & 0 \\ \dots, & \dots, & \dots, & \dots, & \dots \\ 0, & 0, & \dots, & N-1, & 0 \end{bmatrix}.$$

Hence,

$$(\bar{\partial}M_N^{Gm}(z))(M_N^{Gm}(z))^{-1}B_N^{Gm}(z) - \bar{\partial}B_N^{Gm}(z) = -\bar{\partial}\psi(z) \begin{bmatrix} A_N^{Gm}(z)(M_N^{Gm}(z))^{-1}B_N^{Gm} - c_N^{Gm} \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Using (3-11), we see that

$$\lim_{m \to \infty} \left(A_N^{Gm}(z) (M_N^{Gm}(z))^{-1} B_N^{Gm} - c_N^{Gm} \right) = 0 \ a.e. \ Area|_{B(\lambda_0, \delta)}.$$

Since each component of the above vector function is less than

$$M\int \frac{1}{|w-z|} d|\nu_i|(z) \text{ a.e. } Area|_{B(\lambda_0,\delta)},$$

applying Lebesgue's Dominated Convergence Theorem to the last step of (3-12), we conclude

$$\int \bar{\partial}\phi(z)a_N(z)dA(z) = 0.$$

By Weyl's lemma, we see that $a_N(z)$ is analytic on $B(\lambda_0, \delta)$. From equation (3-8), we get

$$\mathcal{C}^1_{\psi}\langle F_N, G \rangle \mu_1)(z) = \sum_{k=1}^{N-1} a_{kj}(z) \mathcal{C}^1_{\psi}\langle F_k, G \rangle \mu_1)(z), \ a.e. \ Area|_{B(\lambda_0,\delta)}.$$

The above equation implies (3-9) since

$$\bar{\partial} \mathcal{C}^1_{\psi}(\nu_i)(z) = -\mathcal{C}(\nu_i)(z) \ a.e. \ Area.$$

For equation (3-10), let ϕ be a smooth function with compact support in $B(\lambda_0, \delta)$ and let ν be a compactly supported finite measure, we get

Apply the above equation to the both sides of the equation (3-9) for j = N and using

$$\bar{\partial}\phi(z)a_{kj}(z) = \bar{\partial}(\phi(z)a_{kj}(z)), \ z \in B(\lambda_0, \delta)$$

we conclude

$$\int \phi \langle F_N, G \rangle d\mu_1 = \int \phi \sum_{k=1}^{N-1} a_{kj} \langle F_k, G \rangle d\mu_1.$$

Hence the equation (3-10) follows. This completes the proof of the lemma.

Corollary 2. Let G, Ω^G , and Ω_i^G be as in Lemma 7. Suppose $G \perp \mathcal{K}_{N-1}^{\psi}$ (satisfies (3-4)). Then $\Omega_i^G \subset U_{N-1} \cup U_N$.

Proof: Without loss of generality, we assume that j = N. From Lemma 7, for $\lambda_0 \in \Omega_N^G$, there exists $\delta > 0$ such that $B(\lambda_0, \delta) \subset Int(\sigma(S))$ and the equations (3-9) and (3-10) hold, which imply (3-3). For $r_1, r_2, ..., r_N \in Rat(\sigma(S))$, let

$$F = \sum_{i=1}^{N} r_i F_i.$$

Notice that

$$r_i(\lambda)\mathcal{C}^k_{\psi}\langle F_i, G\rangle\mu_1) = \mathcal{C}^k_{\psi}\langle r_iF_i, G\rangle\mu_1)$$

since $G \perp \mathcal{K}_{N-1}^{\psi}$. Then

$$\sum_{i=1}^{N} r_i(\lambda) \mathcal{C}^k_{\psi}(\langle F_i, G \rangle \mu_1)(\lambda) = \mathcal{C}^k_{\psi}(\langle F, G \rangle \mu_1)(\lambda),$$

for k = 1, 2, ..., N - 1. Now using the equation (3-9) for $\lambda \in B(\lambda_0, \delta) \setminus A$, we get

$$\sum_{i=1}^{N-1} (r_i(\lambda) + a_{Ni}(\lambda)r_N(\lambda))\mathcal{C}^k_{\psi}(\langle F_i, G \rangle \mu_1)(\lambda) = \mathcal{C}^k_{\psi}(\langle F, G \rangle \mu_1)(\lambda),$$

equivalently,

$$M_{N}^{G}(\lambda) \begin{bmatrix} r_{1}(\lambda) + a_{N1}(\lambda)r_{N}(\lambda) \\ r_{2}(\lambda) + a_{N2}(\lambda)r_{N}(\lambda) \\ \dots, \\ r_{N-1}(\lambda) + a_{N,N-1}(\lambda)r_{N}(\lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{\psi}^{1}(\langle F, G \rangle \mu_{1})(\lambda) \\ \mathcal{C}_{\psi}^{2}(\langle F, G \rangle \mu_{1})(\lambda) \\ \dots, \\ \mathcal{C}_{\psi}^{N-1}(\langle F, G \rangle \mu_{1})(\lambda) \end{bmatrix}.$$

where the inverse of $M_N^G(\lambda)$ is bounded on $B(\lambda_0, \delta) \setminus A$ and a_{Ni} are analytic on $B(\lambda_0, \delta)$. Therefore, there exists a positive constant M such that

$$\sup_{1 \le k \le N-1, \lambda \in B(\lambda_0, \frac{\delta}{2})} |r_k(\lambda) + a_{Nk}(\lambda)r_N(\lambda)| \le M \|F\|,$$

which implies (3-2). Hence, Lemma 3.1 implies $\Omega_N^G \subset U_{N-1} \cup U_N$.

Now let us recursively construct other sets such as Ω_{ij}^G for a given $G \perp \mathcal{K}_{N-1}^{\psi}$. We will only describe the algorithm for k = N - 2 and the other cases will follow recursively. Let $E_N^G = \Omega^G$

and $E_{N-1}^G = \bigcup_{i=1}^N \Omega_i^G$. Let M_{ij}^G be an N-2 by N-2 submatrix of M^G by eliminating the first two rows and the *i* and *j* columns. Define

$$\Omega_{ij}^G = (Int(\sigma(S)) \cap A^c \cap \{z : |det(M_{ij}^G(z))| > 0\}) \setminus clos(E_N^G \cup E_{N-1}^G).$$

Without loss of generality, let us assume that i = N - 1 and j = N. Similar to Lemma 7, one can prove that for $\lambda_0 \in \Omega_{N-1,N}^G$, there exist $\delta > 0$, analytic functions $a_i(z)$ and $b_i(z)$ on $B(\lambda_0, \delta) \subset Int(\sigma(S))$ such that

$$F_{N-1} = \sum_{i=1}^{N-2} a_i(z) F_i(z), \ F_N = \sum_{i=1}^{N-2} b_i(z) F_i(z), \ a.e.\mu_1|_{B(\lambda_0,\delta)},$$
(3-13)

and there exists a constant M > 0 such that

$$\sup_{1 \le k \le N-2, \lambda \in B(\lambda_0, \frac{\delta}{2})} |r_k(\lambda) + a_k(\lambda)r_{N-1}(\lambda) + b_k(\lambda)r_N(\lambda)| \le M ||F||,$$
(3-14)

where $r_1, r_2, ..., r_N \in Rat(\sigma(S))$ and $F = \sum_{i=1}^N r_i F_i$. (3-13) and (3-14) are the same as (3-2) and (3-3) for the case k = N - 2. Let

$$E_{N-2}^G = \bigcup_{i < j}^N \Omega_{ij}^G. \tag{3-15}$$

Corollary 3. Let E_{N-2}^G be as in (3-15). Suppose $G \perp \mathcal{K}_{N-1}^{\psi}$ (satisfies (3-4)). Then

$$E_{N-2}^G \subset U_{N-2} \cup U_{N-1} \cup U_N$$

The proof is the same as Corollary 2. Therefore we can recursively construct E_k^G for k=1,2,...,N such that

$$E_k^G \subset \bigcup_{i=k}^N U_i \tag{3-16}$$

where the proof for k = N is from Lemma 6, k = N - 1 is from Corollary 2, and k = N - 2 is from Corollary 3.

The following theorem proves, under the conditions S satisfies the property (N, ψ) , the set $\cup_{k=1}^{N} E_k^G$ is big.

Theorem 3. Let E_i^G be constructed for i = 1, 2, ..., N as above. Suppose $\{G_j\} \subset (\mathcal{K}_{N-1}^{\psi})^{\perp}$ is a dense subset, then

$$spt\mu_1 \subset clos\left(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j}\right)$$

Proof: First we prove

$$\mu_1\left(Int(\sigma(S)) \setminus clos\left(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j}\right)\right) = 0.$$

Suppose that $B(\lambda_0, \delta) \subset Int(\sigma(S))$ and $B(\lambda_0, \delta) \cap clos\left(\bigcup_{i=1}^N \bigcup_{j=1}^\infty E_i^{G_j}\right) = \emptyset$, then by construction of $E_i^{G_j}$, we conclude that

$$\mathcal{C}_{\psi}^{N-1}(\langle F_i, G_j \rangle \mu_1)(z) = 0$$

on $B(\lambda_0, \delta)$, where i = 1, 2, ..., N. By taking $\bar{\partial}$ in the sense of distribution, we see that

$$\mathcal{C}(\langle F_i, G_j \rangle \mu_1)(z) = 0$$

a.e. Area on $B(\lambda_0, \delta)$ since $Area(\{\bar{\partial}\psi = 0\} \cap \sigma(S)) = 0$, where i = 1, 2, ..., N. For a smooth function ϕ with compact support in $B(\lambda_0, \delta)$,

$$\int \phi(z) \langle F_i, G_j \rangle d\mu_1 = \frac{1}{\pi} \int \bar{\partial} \phi(z) \mathcal{C}(\langle F_i, G_j \rangle \mu_1)(z) dA(z) = 0$$

Therefore,

$$\langle F_i(z), G_j(z) \rangle = 0. \ a.e. \ \mu_1|_{B(\lambda_0, \delta)}$$

$$(3-17)$$

where i = 1, 2, ..., N. From (1-4), we see that for $P \in \bigoplus_{k=1}^{m} L^{2}(\mu_{k}|_{B(\lambda_{0}, \delta)})$, (3-17) implies $(P, G_{j}) = 0$. Therefore,

$$\oplus_{k=1}^m L^2(\mu_k|_{B(\lambda_0,\delta)}) \subset \mathcal{K}_{N-1}^{\psi}.$$

Hence, $\mu_1|_{B(\lambda_0,\delta)} = 0$ since $M_z|_{\mathcal{K}_{N-1}^{\psi}}$ is pure.

Now assume $B(\lambda_0, \delta) \cap clos(Int(\sigma(S))) = \emptyset$. For N > 1, the function $\mathcal{C}_{\psi}^{N-1}(\langle F_i, G_j \rangle \mu_1)(z)$ is continuous on $\mathbb{C} \setminus A$ and is zero on $\mathbb{C} \setminus \sigma(S)$. Hence,

$$\mathcal{C}_{\psi}^{N-1}(\langle F_i, G_j \rangle \mu_1)(z) = 0$$

on $B(\lambda_0, \delta) \setminus A$, where i = 1, 2, ..., N. Using the same proof as above, we see that $\mu_1|_{B(\lambda_0, \delta)} = 0$. This implies $spt\mu_1 \subset clos(Int(\sigma(S)))$. The theorem is proved.

Proof of Theorem 2: From (3-16) and Theorem 3, we get

$$\bigcup_{i=1}^{N} \partial U_i \subset \sigma_e(S) \subset spt(\mu_1) \subset clos\left(\bigcup_{i=1}^{N} U_i\right).$$

This implies

$$\sigma_e(S) = \bigcup_{i=1}^N \partial U_i$$

since $\sigma_e(S) \cap U_i = \emptyset$. This completes the proof.

For a positive finite measure μ with compact support on \mathbb{C} , definite

$$P^{2}(\mu|1,\bar{z},...,\bar{z}^{N-1}) = clos\{p_{1}(z) + p_{2}(z)\bar{z} + ... + p_{N}(z)\bar{z}^{N-1}: p_{1}, p_{2}, ..., p_{N} \in \mathcal{P}\}$$

and $S_{N,\mu}$ as the multiplication by z on $P^2(\mu|1, \bar{z}, ..., \bar{z}^{N-1})$. Then $S_{N,\mu}$ is a multicyclic subnormal operator with the minimal normal extension M_{μ} , the multiplication by z, on $L^2(\mu)$.

Corollary 4. Suppose that $S_{2,\mu}$ on $P^2(\mu|1, \bar{z}, \bar{z}^2)$ is pure, then the operator $S_{1,\mu}$ on $P^2(\mu|1, \bar{z})$ satisfies

$$\sigma(S_{1,\mu}) = clos(\sigma(S_{1,\mu}) \setminus \sigma_e(S_{1,\mu})).$$

Proof: Since

$$\mathcal{K}_{1}^{\bar{z}} = clos(span(\bar{z}^{k}P^{2}(\mu|1,\bar{z}):0 \le k \le 1)) = P^{2}(\mu|1,\bar{z},\bar{z}^{2})$$

and $S_{2,\mu}$ on $P^2(\mu|1, \bar{z}, \bar{z}^2)$ is pure. Therefore, the result follows from Theorem 2.

It seems strong to assume that $S_{2,\mu}$ on $P^2(\mu|1, \bar{z}, \bar{z}^2)$ is pure in the corollary. We believe that the condition can be reduced to assume that $S_{1,\mu}$ on $P^2(\mu|1, \bar{z})$ is pure. However, we are not able to prove the result under the weaker conditions. We will leave it as an open problem for further research.

Problem 1. Does Corollary 4 hold under the weaker assumption that $S_{1,\mu}$ on $P^2(\mu|1,\bar{z})$ is pure?

Corollary 5. Let S on \mathcal{H} be a pure rationally N-cyclic subnormal operator with $\mathcal{H} = R^2(S|F_1, F_2, ..., F_N)$ and let M_z be its minimal normal extension on \mathcal{K} satisfying (1-1) to (1-4). Suppose that there exists a smooth function ψ on \mathbb{C} such that $Area(\{\bar{\partial}\psi = 0\} \cap \sigma(S)) = 0$ and $\psi(M_z)\mathcal{H} \subset \mathcal{H}$. Then there exist bounded open subsets U_i for $1 \leq i \leq N$ such that

$$\sigma_e(S) = \bigcup_{i=1}^N \partial U_i, \ \sigma(S) \setminus \sigma_e(S) = \bigcup_{i=1}^N U_i,$$

and

$$dimker(S-\lambda)^* = i.$$

for $\lambda \in U_i$.

Notice that Example 1 and 2 are special cases of Corollary 5. It seems that further results could be obtained for the special cases where S satisfies the conditions of Corollary 5. Moreover, we might be able to combine the methodology in McCarthy and Yang (1997) to obtain the structural models for the class of subnormal operators, which might extend Xia's model for subnormal operators with finite rank self-commutators.

Problem 2. Can the structure of subnormal operators in Corollary 5 be characterized?

References

- A. Aleman, S. Richter, and C. Sundberg. Nontangential limits in $P^t(\mu)$ -spaces and the index of invariant subspaces. Ann. of Math., 169(2):449–490, 2009.
- A. Aleman, S. Richter, and C. Sundberg. A quantitative estimate for bounded point evaluations in $P^t(\mu)$ -spaces. Topics in operator theory. Operators, matrices and analytic functions, Oper. Theory Adv. Appl., 202, Birkhuser Verlag, Basel, 1:1–10, 2010.
- J. E. Brennan. Point evaluations, invariant subspaces and approximation in the mean of polynomials. J. Funct. Anal., 34:407–420, 1979.
- J. E. Brennan. Thomson's theorem on mean-square polynomial approximation, algebra i analiz 17 no.2 (2005), 1-32. Russian. St. Petersburg Math. J., 17(2):217–238, 2006. English.
- J. E. Brennan. The structure of certain spaces of analytic functions. *Comput. Methods Funct.* theory, 8(2):625–640, 2008.
- J. E. Brennan and E. R. Militzer. L^p -bounded point evaluations for polynomials and uniform rational approximation. St. Petersburg Math. J., 22(1):41–53, 2011.
- J. B. Conway. The theory of subnormal operators. Mathematical Survey and Monographs 36, 1991.
- J. B. Conway and N. Elias. Analytic bounded point evaluations for spaces of rational functions. J. Functional Analysis, 117:1–24, 1993.
- M.J. Cowen and R.G. Douglas. Complex geometry and operator theory. *Acta. Math.*, 141: 187–261, 1978.
- N. Feldman and P. McGuire. On the spectral picture of an irreducible subnormal operator II. Proc. Amer. Math. Soc., 131(6):1793–1801, 2003.
- T. W. Gamelin. Uniform algebras. American Mathematical Society, Rhode Island, 1969.
- S. Hruscev. The Brennan alternative for measures with finite entropy. *Izv. Akad. Nauk Armjan.* SSR Ser. Math., 14(3):184–191, 1979. Russian.
- M. Mbekhta, N. Ourchane, and E. H. Zerouali. The interior of bounded point evaluations for rationally cyclic operators. *Mediterr. J. Math.*, 13:1981–1996, 2016.
- J. McCarthy and L. Yang. Subnormal operators and quadrature domains. Adv. Math., 127: 52–72, 1997.
- P. McGuire. On the spectral picture of an irreducible subnormal operator. Proc. Amer. Math. Soc., 104(3):801–808, 1988.
- R. F. Olin and J. E. Thomson. Irreducible operators whose spectra are spectral sets. Pacific J. Math., 91:431–434, 1980.
- J. E. Thomson. Approximation in the mean by polynomials. Ann. of Math., 133(3):477–507, 1991.

- D. Xia. On pure subnormal operators with finite rank self-commutators and related operator tuples. *Integral Eqs. Operator Theory*, 24:106–125, 1996.
- D. Yakubovich. Subnormal operators of finite type ii. structure theorems. *Revista Matematica Iberoamericana*, 14(3):623–681, 1998.
- L. Yang. A note o
o $L^p\mbox{-bounded}$ point evaluations for polynomials. Proc. Amer. Math. Soc., 144
(11):4943–4948, 2016.
- L. Yang. Bounded point evaluations for rationally multicyclic subnormal operators. Journal of Mathematical Analysis and Applications, 458:1059–1072, 2018.