On the piecewise pseudo almost periodic solution of nondensely impulsive integro-differential systems with infinite delay

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Abstract In the theory of neutral differential equations with pulse influence (neutral impulsive differential equations), there are many unsolved problems related to certain results in the theory of integral and integro-differential equations. In this article, we present an result for the existence of the piecewise pseudo almost periodic solution of a class of neutral impulsive nonlinear integro-differential systems with infinite delay. The method used involves result on the theory of integrated semigroup as well as the Sadovskii's fixed point theorem.

Keywords Impulsive integro-differential equations, Pseudo almost periodic solution, Sadovskii's fixed point theorem, Infinitesimal generator of an analytic semigroup.

1 Introduction

Because they can describe various of real processes and phenomena, the delayed impulsive differential equations is an important theory in mathematical modeling of phenomena, we can cited for example in: artificial neural networks, biologies, computer sciences and physics ([1], [6], [7], [8], [25], [30]). We refer the reader to the monographs of Bainov and Simeonov [9], Lakshmikantham

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et al. [23], and Samoilenko and Perestyuk [29]. In [24], Li Shan Liu et al. obtain a unique solution of the first order impulsive integrodifferential equation of mixed type in a Banach space, an explicit iterative approximation of the solution and estimate the approximation sequence. Using the Krasnoselski-Schaefer type fixed point theorem in [12], the authors prove the existence of solutions for impulsive partial neutral functional differential equations with infinite delay in a Banach space. In [32], Stamov et al. employ the contraction mapping principle to proved some sufficient conditions for the existence of almost periodic solutions of impulsive integro-differential neural networks.

On the other hand, many researchers are interested to periodic concept, in particular studied the existence and stability of periodic solutions ([20], [26], [38]) and the references cited therein. However, upon considering longterm dynamical behaviors it is possible for the various components of the model to be periodic with rationally independent periods, and therefore it is more reasonable to consider the various parameters of models to be changing almostperiodically rather than periodically. Thus, it is more reasonable to consider the almost periodic behavior of solutions. The investigation of almost periodic solutions is established to be more accordant with reality. Although it has widespread applications in real life, the generalization to the notion of almost periodicity is not as developed as that of periodic solutions. To the best of authors knowledge, there are a few recent published papers considering the notion of almost periodicity of differential equations with impulses ([31], [32]).

Motivated by the works mentioned above and noting that there is no papers in the literature for pseudo almost periodic solution of nondensely impulsive neutral functional integrodifferential equations with infinite delay, the main purpose of this paper is to establish the existence and of the piecewise pseudo almost periodic (PPAP) solution of an impulsive integro-differential equations in a general Banach space modeled in the form:

$$\frac{d}{dt}\left[x(t) - g\left(t, x_t, \int_0^t h(t, s, x_s)ds\right)\right] = A\left[x(t) - g\left(t, x_t, \int_0^t h(t, s, x_s)ds\right)\right]$$

$$+ f\left(t, x_t, \int_0^{\cdot} k(t, s, x_s) ds\right), \text{ if } t \in J = [0, b] \setminus t_k, \ k = 1, ..., m,$$
(1)

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \text{ if } t = t_k,$$

$$x_0 = \phi \in BM_h,$$
(2)
(3)

$$=\phi \in \mathrm{BM}_h,$$

where

(i) $A: D(A) \subset X \longrightarrow X$ is a non-densely defined $(\overline{D(A)} \neq X)$ closed linear operator and satisfies the Hille-Yosida condition (see Definition 6), (where Xis a Banach space with the norm $\|\cdot\|_X$).

(ii) The nonlinear functions $f, g: J \times BM_h \times X \longrightarrow X, h, k: \Delta \times BM_h \longrightarrow$ X, BM_h is the abstract phase space which will be defined later,

$$\Delta = \{(t,s): 0 \le s \le t \le b\},\$$

 $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of the function x at t_k , respectively.

(iii) $I_k \in C(X, X)$ (k = 1, ..., m, where $m \in \mathbb{Z}_+$) are bounded functions. (iv) For any $x : (-\infty, b] \longrightarrow X$ and $t \in J$, the histories $x_t : (-\infty, b] \longrightarrow X$ defined by $x_t(s) = x(t+s), s \leq 0$ which belong to BM_h.

It is well known that the solution $x(\cdot)$ of the problem (1)-(3) is a piecewise continuous function with points of discontinuity at the moments $t = t_k, k \in \mathbb{Z}$, at which it is continuous from the left, i.e. the following relations are valid:

$$x(t_k^-) = x(t_k)$$

and

$$x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \ \forall k \in \mathbb{Z}.$$

For examples of the linear operators with nondense domain satisfying the Hille-Yosida condition, we can look in the one-dimensional, $E = C([0, 1], \mathbb{R})$ and defined the operator $A : D(A) \longrightarrow E$ by Ay' = y, where $D(A) = \{y \in C^1([0, 1], \mathbb{R}) :$

y(0) = 0. Then $\overline{D(A)} = \{y \in E : y(0) = 0\} \neq E$. In the n-dimensional, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary Γ and define $E = C(\overline{\Omega}, \mathbb{R})$ and the operator $A : D(A) \longrightarrow E$ defined by $Ay = \Delta y$, where $D(A) = \{y \in E : y = 0 \text{ on } \Gamma; \Delta y \in E\}$. Here Δ is the Laplacian in the sense of distributions on Ω . In this case we have $\overline{D(A)} = \{y \in E : y = 0 \text{ on } \Gamma\} \neq E$. See the paper found by Da Prato and Sinestrari [15], Abada et al. [1] for more examples and details concerning the non-densely defined operators.

In the case where $g(\cdot)$ is a null function, $f: J \times BM_h \longrightarrow X$, and A is a nondensely closed defined linear operator generating a C_0 -semigroup of bounded linear operators, the problem (1)-(3) has been investigated on compact intervals in [1]. To the best of our knowledge, no paper in the literature has investigated the existence, uniqueness and globally exponential stability of PPAP solution for system (1)-(3). Hence, our goal in this paper is to study the dynamics of model (1)-(3). By applying Sadovskii's theorem, we give some sufficient conditions ensuring the existence of PPAP solution of system (1)-(3), which are new and complement the previously known results.

The remainder of this paper is organized as follows: In Section 2, we will introduce an abstract phase space BM_h and the concept of integrated semigroups. In Section 3, the concept of pseudo almost periodicity is presented, we derived also, some preliminaries and hypotheses which will be used in the paper. In section 4, we present a new idea of research to prove some criteria for ensuring the existence of the PPAP solution. At last, an illustrative example is given. It should be mentioned that the main results include Theorem 3. The results of the present paper extend to a nondensely defined operator some ones considered in the previous literature. Our results are based on the properties of the analytic semigroup and ideas and techniques in Kellermann [21], Pazy [27], Yosida [36].

Let \mathbb{R} and \mathbb{Z} be the sets of real and integer number. Define the space \mathbf{T} by $\mathbf{T} = \{\{t_k\} : t_k \in (-\infty, +\infty), t_k < t_{k+1}, k = \pm 1, \pm 2, ..., \lim_{k \to \pm \infty} t_k = \pm \infty\}$, we denote the set of unbounded and strictly increasing sequences. Let $A : D(A) \longrightarrow X$ be the infinitesimal generator of a compact analytic semigroup of bounded linear operators $\{S(t)\}_{t\geq 0}$ on a Banach space X with the norm $\|\cdot\|$, and let $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^{-\alpha}$, for $0 \leq \alpha < 1$, as closed linear invertible operator with domain $D((-A)^{-\alpha})$ dense in X. The closedness of $D((-A)^{-\alpha})$ implies that $D((-A)^{-\alpha})$ endowed with the graph norm $\|x\|_X = \|x\| + \|(-A)^{-\alpha}x\|$ is a Banach space. Since $(-A)^{-\alpha}$ is invertible, its graph norm $\|x\|_X$ is equivalent to the norm $|x| = \|(-A)^{-\alpha}x\|$. Thus $D((-A)^{-\alpha})$ equipped with the norm $|\cdot|$ is a Banach space which we denote by X_{α} .

To obtain the existence of the pseudo almost periodic solutions of (1)-(3), we make the following assumptions:

 (H_1) For every $t \in (-\infty, 0]$ the function $t \longrightarrow \phi(t)$ is pseudo almost periodic, and for $t \in J$, the functions $t \longrightarrow f(t, x(\cdot), y(\cdot)), t \longrightarrow g(t, x(\cdot), y(\cdot))$ are pseudo almost periodic for $x(\cdot), y(\cdot)$ pseudo almost periodic and there exist constants $G_i > 0$, (i = 1, 2) and H > 0 such that

(i) $\|g(t,\psi,x) - g(t,\chi,y)\|_X \le G_1 \|\psi - \chi\|_{BM_h} + G_2 \|x - y\|_X, t \in J, \psi, \chi \in BM_h, x, y \in X;$

(ii)
$$\|h(t, s, \psi) - h(t, s, \chi)\|_X \le H \|\psi - \chi\|_{BM_h}, t, s \in J, \ \psi, \chi \in BM_h.$$

(iii) $\|(-A)^{\beta} f(t, x, \psi) - (-A)^{\beta} f(t, y, \chi)\|_{X} \leq F_{1} \|x - y\|_{BM_{h}}, t, s \in J, \ \psi, \chi \in BM_{h}$, where $\beta \in (0, 1)$ and $F_{1} > 0$.

 (H_2) (i) Let A be an infinitesimal operator of an analytic semigroup S(t) and that $0 \in \rho(T)$, such that

$$\left\| (-A)^{1-\alpha} S(t-s) \right\| \le \frac{C_{1-\alpha}}{(t-s)^{1-\alpha}} \exp(-\lambda(t-s)), \ t \in J, \ \alpha \in (0,1), \ \lambda > 0.$$

Denote by $BC(\mathbb{R}, X_{\alpha})$ the space of X_{α} -valued bounded continuous functions.

(ii) The semigroup $\{S'(t)\}_{t\geq 0}$ is pseudo almost periodic, compact on $(D(A), \|\cdot\|)$, whenever t > 0, and there exists $M \geq 1$ such that $\|S'(t)\| \leq M$, for all $t \in J$.

 (H_3) (i) For every $t \in \mathbb{R}$, the functions $t \longrightarrow h(t, s, x_s), t \longrightarrow k(t, s, x_s)$ are pseudo almost periodic for $(s, x_s) \in \mathbb{R} \times X$ and there exist continuous functions $p, q: J \longrightarrow [0, +\infty)$ such that

$$\left\| \int_{0}^{t} h(t,s,\psi) ds \right\|_{X} \le p(t) \|\psi\|_{C}, \quad \left\| \int_{0}^{t} k(t,s,\psi) ds \right\|_{X} \le q(t) \|\psi\|_{C}$$

for every $(t,s) \in \Delta$ and $\psi \in C$.

(ii) $\sup_{t \in J} p(t) < \bar{p}$, and $\sup_{t \in J} q(t) < \bar{q}$, where \bar{p} and \bar{q} are positive constants.

 (H_4) For every positive integer k there exits $\alpha_k \in L^1(J, [0, +\infty))$ such that

$$\sup_{\|\psi\|_C, \|x\| \le k} \|f(t, \psi, x)\|_X \le \alpha_k(t), \quad \forall t \in J,$$

and

$$\liminf_{k \to +\infty} \frac{1}{k} \int_0^b \alpha_k(s) ds = \mu < \infty.$$

 (H_5) (i) For each $k,\,(k=1,...,m),\,I_k\in C(X,\overline{D(A)})$ is pseudo almost periodic sequence,

$$||I_k(x)||_{\overline{D(A)}} \le L_k(||x||_X),$$

and

$$\sum_{0 < t_k < t} L_k \left(\| (x(t_k) \|_X) < \infty, \ \lim_{\sigma \longrightarrow +\infty} \frac{L_k(\sigma)}{\sigma} = \lambda_k < \infty, \right)$$

where $L_k: [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous nondecreasing function.

(ii) $\exists L_I > 0$, $\|I_k(x) - I_k(y)\|_{\overline{D(A)}} \le L_I \|x - y\|_X$.

 (H_6) (i) For each $(t, s) \in \Delta$, the function $k(t, s, \cdot) : BM_h \longrightarrow X$ is continuous and for each $\psi \in BM_h$, the function $k(\cdot, \cdot, \psi) : \Delta \longrightarrow X$ is strongly measurable.

(ii) For each $(\psi, x) \in C \times X$, the function $f(\cdot, \psi, x) : J \longrightarrow X$ is strongly measurable.

(iii) The set of sequences $\{\tau_{k+j} - \tau_k\}, k, j \in \mathbb{Z}, \{\tau_k\} \in \mathbf{T}$ is uniformly pseudo almost periodic and $\inf_k \{\tau_{k+1} - \tau_k\} = \varrho > 0.$

It is well known that $BC(\mathbb{R}, X_{\alpha})$ is a Banach space when it is equipped with the sup norm defined by

$$||x||_{\alpha,\infty} := \sup_{t \in \mathbb{R}} ||x(t)||_{\alpha}, \text{ for } x \in BC(\mathbb{R}, X_{\alpha}),$$

where the α -norm defined by

$$||u||_{\alpha} := ||(-A)^{-\alpha}u||_X$$
, for $u \in \mathcal{D}((-A)^{-\alpha})$.

2 Abstract phase space BM_h and integrated semigroups

2.1 Phase space BM_h

Assume $h: (-\infty, 0] \longrightarrow [0, +\infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(s) ds < \infty$, besides for any a > 0, define

 $BM_a = \{\psi : [-a, 0] \longrightarrow X \text{ such that } \psi(t) \text{ is bounded and measurable} \},\$

and equip the space BM_a by the norm

$$\|\psi\|_{[-a,0]} = \sup_{[-a,0]} \|\psi(s)\|_X, \ \forall \psi \in BM_a.$$

Let us define

$$BM_h = \left\{ \psi : (-\infty, 0] \longrightarrow X \text{ such that for any } c > 0 \ \psi|_{[-c,0]} \in BM_a \\ \text{and} \int_{-\infty}^0 h(s) \|\psi\|_{[s,0]} ds < +\infty \right\}.$$

If BM_h is endowed with the norm,

$$\|\psi\|_{\mathrm{BM}_h} = \int_{-\infty}^0 h(s) \|\psi\|_{[s,0]} ds, \ \forall \psi \in \mathrm{BM}_h$$

then it is clear that $(BM_h, \|\cdot\|_{BM_h})$ is a Banach space.

Now, for all $\sigma > 0$ we consider the space

$$\begin{split} \mathrm{BM}'_h &= \{x: (-\sigma, b] \longrightarrow X, \text{ such that } x_k \in C(J_k, X) \text{ is a piecewise continuous function} \\ \text{with points of discontinuity } \tilde{t} \in [-\sigma, 0] \text{ at which } x(\tilde{t}^-) \text{ and } x(\tilde{t}^+) \text{ exist} \\ \text{and } x(\tilde{t}^-) &= x(\tilde{t}), \ k = 0, 1, ..., m, \ x_0 = \phi \in \mathrm{BM}_h \}, \end{split}$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}], k = 0, 1, ..., m$. Set $\|\cdot\|_b$ be a seminorm in BM'_h defined by

$$\|x\|_{b} = \|x_{0}\|_{\mathrm{BM}_{h}} + \sup\{\|x(s)\|_{X}, s \in [0, b]\}, x \in \mathrm{BM}_{h}'$$

Lemma 1 ([12]) Assume $x \in BM'_h$, then for $t \in J$, $x_t \in BM_h$. Moreover,

$$l\|x(t)\|_{X} \le \|x_{t}\|_{\mathrm{BM}_{h}} \le \|x_{0}\|_{\mathrm{BM}_{h}} + l \sup_{s \in [0,t]} \|x(s)\|, \text{ where } l = \int_{-\infty}^{0} h(t)dt < +\infty.$$

Before proceeding to the our main result, we shall set forth some definitions and hypotheses from ([27], [21], [36]).

2.2 Integrated Semigroups

Definition 1 (semigroup [17]) A semigroup is a set S coupled with a binary operation * ($*: S \times S \longrightarrow S$) which is associative. Associativity can also be realized as F(F(x, y), z) = F(x, F(y, z)) where F(x, y) serves as the mapping from $S \times S$ to S.

A semigroup, unlike a group, need not have an identity element e such that $x * e = x, \forall x \in S$.

Definition 2 (C_0 semigroup [2]) A C_0 semigroup (or strongly continuous semigroups) is a family, $\{S(t)\}_{t\geq 0} = \{S(t) \mid t \in \mathbb{R}_+\}$, of bounded linear operators from X to X satisfying:

(i) S(0) = 0;

(ii) $t \longrightarrow S(t)$ is strongly continuous $(\lim_{t \longrightarrow 0^+} S(t)f = f$ for each $f \in X$ with respect to the norm on X);

(iii) $S(s+t) = S(s)S(t) = \int_0^s (S(t+\sigma) - S(\sigma))d\sigma$, for all $t, s \ge 0$.

Definition 3 (Generator [10]) Let S be a semigroup. The (infinitesimal) generator of S, denoted by A, is given by the equation:

$$Af = \lim_{t \to 0^+} A_t f = \lim_{t \to 0^+} \frac{S(t)f - f}{t},$$

where the limit is evaluated in terms of the norm on X and f is in the domain of A if this limit exists.

Definition 4 Let $\rho(A)$ is the resolvent set of an operator A and I is the identity operator in X. Then A is called a generator of an integrated semigroup, if there exists $\delta \in \mathbb{R}$ such that $(\delta, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $\{S(t)\}_{t\geq 0}$ of linear bounded operators such that S(0) = 0 and $R(\lambda, A) := (\lambda I - A)^{-1} = \lambda \int_0^{+\infty} \exp(-\lambda t)S(t)dt$ exists for all $\lambda > \delta$.

Definition 5 An integrated semigroup $\{S(t)\}_{t\geq 0}$ is called locally Lipschitz continuous if, for all $\delta > 0$ there exists a constant γ such that

$$||S(t) - S(s)||_X \le \gamma |t - s|, \ t, s \in [0, \delta].$$

Moreover it is called non-degenerate if S(t)x = 0, for all $t \ge 0$, implies x = 0.

Definition 6 We say that a linear operator A satisfies the Hille-Yosida condition if there exists $\overline{M} \ge 1$ and $\delta \in \mathbb{R}$ such that $(\delta, +\infty) \subset \rho(A)$ and

$$\sup\left\{(\lambda-\delta)^n \| R(\lambda,A)^n \|_X : n \in \mathbb{N}, \lambda > \delta\right\} \le \overline{M}.$$

Theorem 1 ([21], p.166) The following assertions are equivalent:

(i) A is the generator of a non-degenerate, locally Lipschitz continuous integrated semi-group;

(ii) A satisfies the Hille-Yosida condition.

Throughout this paper we assume that: The operator A satisfies the Hille-Yosida condition: (H_0) .

Lemma 2 The Operator A satisfies the Hille-Yosida condition on X (with M = 1 and $\delta = 0$). Therefore it follows that the operator A generates an integrated semigroup $S(t), t \leq b$. Such that $||S'(t)|| \leq e^{-\kappa t}$, for $t \geq 0$ and some constant $\kappa > 0$.

From [21], since the operator A hold the condition in Definition 6, this operator is the generator of a locally Lipshitz continuous integrated semigroup $\{S(t)\}_{t\geq 0}$ generates C_0 semigroup (see [27]) on $\overline{D(A)}$ such that

$$||S(t)x||_X \le \overline{M}e^{\delta t} ||x||_X,$$

for all $t \ge 0$ and $x \in \overline{D(A)}$. Let A_0 be the generator of C_0 semigroup $\{S'(t)\}_{t\ge 0}$ then A_0 is the part of A in $\overline{D(A)}$ defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},\\ A_0x = Ax, \text{ for } x \in \overline{D(A_0)}, \end{cases}$$

we also have $||S'(t)||_X \leq \overline{M}e^{\delta t}$, $t \geq 0$, where \overline{M} and δ are the constants considered in Definition 6.

Lemma 3 ([27]) Let S(t) be a uniformly continuous semigroup of bounded linear operators. Then $t \longrightarrow S(t)$ is differentiable in norm and

$$S'(t) = AS(t) = S(t)A.$$

3 Pseudo almost periodicity, almost periodicity, preliminaries and hypotheses

It follows that the solution $x(\cdot)$ of (1)-(3) is from the space BM_h and thus one may adopt and introduce the concept of almost periodicity for sequences and continuous functions.

Definition 7 ([5], [16]) Let $f \in BC(\mathbb{R}, X)$. We say that f is almost periodic or uniformly almost periodic (u.a.p), when the following property is satisfied:

$$\forall \varepsilon > 0, \exists l_{\varepsilon} > 0, \forall \alpha \in \mathbb{R}, \exists \delta \in [\alpha, \alpha + l_{\varepsilon}), \|f(\cdot + \delta) - f(\cdot)\|_X < \varepsilon.$$

Definition 8 Let W a Banach space, and Ω a non empty subset of the Banach space $Y \times Z$. We denote by $P_c(X)$ the set of the compact subset of $Y \times Z$. Let a mapping $F \in C^0(J \times Y \times Z, W)$; $F : (t, x) \longrightarrow F(t, x)$. We say that F is uniformly almost periodic in t for $(y, z) \in \Omega$ when

$$\forall \epsilon > 0, \forall K \in P_c(Y \times Z), \exists l = l(\epsilon, K), \ \forall \alpha \in \mathbb{R}, \ \exists \tau \in [\alpha, \alpha + l),$$

such that

$$\sup_{t \in \mathbb{R}} \sup_{(y,z) \in K} \|F(t+\tau, y, z) - F(t, y, z)\| \le \epsilon.$$

Definition 9 ([32]) The set of sequences $\{\tau_{k+j} - \tau_k\}, k, j \in \mathbb{Z}$ is said to be uniformly almost periodic if for arbitrary $\epsilon > 0$ there exists a relatively dense set of ϵ -almost periods common for any sequences.

Definition 10 ([14]) A sequence $I : \mathbb{Z} \longrightarrow X$ is called almost periodic sequence, when for each $\epsilon > 0$, $\exists N_{\epsilon} \in \mathbb{N}$ such that among any N_{ϵ} consecutive integers there exists an integer τ with the property

$$\forall n \in \mathbb{Z}, \ \|I_{n+\tau} - I_n\|_X \le \epsilon$$

In other words, $\widehat{T}(I, \epsilon) = \{\tau \in \mathbb{Z} : \forall n \in \mathbb{Z} \mid ||I_{n+\tau} - I_n||_X \leq \epsilon\}$ is relatively dense in \mathbb{Z} . The class of all almost periodic sequences is denoted by APS.

Definition 11 ([32]) The function $x \in BM'_h$ is said to be piecewise almost periodic if

 (a_1) the set of sequences $\{\tau_{k+j} - \tau_k\}, k, j \in \mathbb{Z}, \{\tau_k\} \in \mathbf{T}$ is uniformly almost periodic.

 (a_2) for any $\epsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $x(\cdot)$ and satisfy the inequality $|t' - t''| < \delta$, then $|x(t') - x(t'')| < \epsilon$.

(a₃) for any $\epsilon > 0$ there exists a relatively dense set K such that if $\tau \in K$, then $|x(t+\tau) - x(t)| < \epsilon$ for all $t \in \mathbb{R}$ satisfying $|t - t_k| > \epsilon$, $k \in \mathbb{Z}$.

In other words, a Bohr almost periodic function is a continuous function which possesses very much almost periods i.e. for all $\epsilon > 0$ the set

$$T(f,\epsilon) = \{\tau \in \mathbb{Z} : \|f(\cdot + \tau) - f(\cdot)\|_X \le \epsilon\}$$

is relatively dense in \mathbb{Z} . The elements of $T(f, \epsilon)$ are called ϵ -periods. We denote by $AP(\mathbb{R}, X)$ the set of the Bohr a.p. functions from \mathbb{R} to X. It is well-known that the set $AP(\mathbb{R}, X)$ is a Banach space with the supremum norm. We refer the reader to ([5], [13], [18]) for the basic theory of almost periodic functions and their applications. Besides, the concept of pseudo almost periodicity (pap) was introduced by Zhang (see for example [37]) in the early nineties. It is a natural generalization of the classical almost periodicity. Define the class of functions $PAP_0(\mathbb{R}, X)$ as follows.

$$PAP_0(\mathbb{R}, X) = \left\{ f \in BC(\mathbb{R}, X) \mid \lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^T \|f(t)\|_X dt = 0 \right\}$$

Definition 12 A function $f \in BC(\mathbb{R}, X)$ is called pseudo almost periodic if it can be expressed as

$$f = f_1 + f_2,$$

where $f_1 \in AP(\mathbb{R}, X)$ $(AP(\mathbb{R} \times \Omega, X))$ and $f_2 \in PAP_0(\mathbb{R}, X)$ $(PAP_0(\mathbb{R} \times \Omega, X))$.

The collection of such functions will be denoted by $PAP(\mathbb{R}, X)$ (resp $PAP(\mathbb{R} \times X, X)$). Besides, we have a similar definition for pseudo almost periodic sequence.

Definition 13 A sequence $I : \mathbb{Z} \longrightarrow X$ is said to be in AP_0S , when it satisfies the ergodicity condition

$$\lim_{p \to +\infty} \frac{1}{2p} \sum_{i=-p}^{p} \|I(i)\|_{X} = 0.$$

Remark 1

Notice that

- 1. A sequence vanishing at infinity is a AP_0S sequence.
- 2. The sequence $(I(n))_{n \in \mathbb{Z}}$ defined by

$$I(n) = \begin{cases} 1, & \text{if } n = 2^k, \\ 0, & \text{if } n \neq 2^k, \end{cases}$$

is an example of AP_0S sequence which not vanishing at infinity. 3. For $k \in \mathbb{N}$, the sequence $(I(n))_{n \in \mathbb{Z}}$ defined by

$$I(n) = \begin{cases} 1, & \text{if } n = 2^{k^2} \\ 0, & \text{if } n \neq 2^{k^2} \end{cases}$$

is an example of an unbounded AP_0S sequence.

Definition 14 We define the space of pseudo almost periodic sequences by

$$PAPS = APS \oplus AP_0S.$$

Remark 2 If $f,g \in PAP(\mathbb{R} \times \Omega, \mathbb{R}^n)$, then $f \pm g \in PAP(\mathbb{R} \times \Omega, \mathbb{R}^n)$ and $f \times g \in PAP(\mathbb{R} \times \Omega, \mathbb{R}^n)$.

For more details about the properties of pseudo almost periodic sequences we refer the reader to ([14], [34]).

Now, we introduce necessary fundamental properties of the almost periodic concept, which will be used later.

Lemma 4 ([29]) Let the set of sequences $\{\tau_{j+k} - \tau_k\}_{k \in \mathbb{Z}}, j \in \mathbb{Z}$ be uniformly almost periodic. Then for each p > 0 there exists a positive integer N such that on each interval of length p no more than N elements of the sequence $\{\tau_k\}$, *i.e.*,

$$i(s,t) \le N(t-s) + N,$$

where i(s,t) is the number of points τ_k in the interval (s,t)

Lemma 5 ([13]) Let the parametric function $t \to \varphi(t, x(\cdot))$ is almost periodic function, for all $x \in X \subseteq \mathbb{R}^n$. Then the function $t \to \int_0^t \varphi(t, x(s)) ds$ is almost periodic.

Lemma 6 ([3], [4]) If $\{x(n)\}_{n \in \mathbb{Z}}$ is a AP_0S sequence, then there exists a function $w \in PAP_0(\mathbb{R}, X)$ such that $w(n) = x(n), n \in \mathbb{Z}$.

For some preliminary results on almost periodic functions, we refer the reader to ([11], [18] and [35]).

Definition 15 (See [28]) An operator F form Banach space X to a Banach space Y is called a condensing operator if it is continuous and for every bounded subset B of X the inequality $\chi[f(B)] < \chi(B)$ holds, where $\chi(\cdot)$ denotes the measure of noncompactness.

Theorem 2 (Sadovskii [28]) Let F is a condensing operator on a Banach space X. If $FD \subseteq D$ for a convex, bounded and closed subset D of X, then F has a fixed point in D.

Definition 16 The function $x \in BM'_h$ is said to be piecewise pseudo almost periodic solution of (1)-(3) if x is pseudo almost periodic function and satisfy;

(i)
$$x(t) = \phi(t), t \in (-\infty, 0];$$

(ii)
$$\left[x(s) - g\left(s, x_s, \int_0^s h(s, \tau, x_\tau) d\tau\right)\right] ds \in D(A)$$
 on J ;

(iii)

$$\begin{cases} x(t) = S'(t) \left[\phi(0) - g(0, \phi, 0)\right] + g\left(t, x_t, \int_0^t h(t, s, x_s) ds\right) \\ + \lim_{\lambda \to +\infty} \int_0^t S' \left(t - s\right) \mathcal{B}_{\lambda} f\left(s, x_s, \int_0^s k(t, \tau, x_{\tau}) d\tau\right) ds \\ + \sum_{0 < t_k < t} S' \left(t - t_k\right) \delta(t - t_k) I_k\left(x(t_k^-)\right), \ t \in J, \text{ where } \mathcal{B}_{\lambda} = \lambda R(\lambda, A). \end{cases}$$

$$\tag{4}$$

Lemma 7 ([29], [33]) Suppose the following conditions hold.

(A₁) The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k, j \in \mathbb{Z}$, $\{\tau_k\} \in \mathbf{T}$, is almost periodic and there exists $\theta > 0$ such that

$$\inf_k \{\tau_k^1\} = \theta > 0.$$

(A₂) The functions f and g are almost periodic and locally Hölder continuous with points of discontinuous at the moments $t = \tau_k$, $k \in \mathbb{Z}$, at which it is continuous from the left.

(A₃) The sequence b_k , $k \in \mathbb{Z}$, is almost periodic. Then for each $\epsilon > 0$ there exist ϵ_1 , $0 < \epsilon_1 < \epsilon$ and relatively dense sets T of real numbers and Q of whole numbers such that the following relations fulfilled:

 $\begin{aligned} (i)\|f(t+\tau,\cdot,\cdot) - f(t,\cdot,\cdot)\|_X &<\epsilon, \ t \in \mathbb{R}, \ \tau \in T, \ \|t-\tau_k\| > \epsilon, \ k \in \mathbb{Z}; \\ (ii)\|g(t+\tau,\cdot,\cdot) - g(t,\cdot,\cdot)\|_X &<\epsilon, \ t \in \mathbb{R}, \ \tau \in T, \ \|t-\tau_k\| > \epsilon, \ k \in \mathbb{Z}; \\ (iii)\|I_{k+q} - I_k\|_X &<\epsilon, \ q \in Q, \ k \in \mathbb{Z}; \\ (iv)\|\tau_{k+q} - \tau_k\| &<\epsilon_1, \ q \in Q, \ \tau \in T, \ k \in \mathbb{Z}. \end{aligned}$

The solutions of the functional impulsive system of integro-differential equations of the form (1)-(3) are characterized in the following way:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ S'(t) \left[\phi(0) - g(0, \phi, 0)\right] + g\left(t, x_t, \int_0^t h(t, s, x_s) ds\right) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t S'\left(t - s\right) \mathcal{B}_{\lambda} f\left(s, x_s, \int_0^s k(t, \tau, x_{\tau}) d\tau\right) ds \\ + \sum_{0 < t_k < t} S'\left(t - t_k\right) \delta(t - t_k) I_k\left(x(t_k^-)\right), \quad t \in J, \text{ where } \mathcal{B}_{\lambda} = \lambda R(\lambda, A), \end{cases}$$
(5)

where

$$\delta(t - t_k) = \begin{cases} 1, & \text{if } t = t_k, \\ 0, & \text{if } t \neq t_k. \end{cases}$$

Now, we are in a position to state the existence theorem. Our first theorem is based on the Sadovskii's theorem.

4 Existence of pseudo almost periodic solution of impulsive integro-differential systems with infinite Delay

Lemma 8 The functional $\xi : s \longrightarrow S'(t-s) B_{\lambda} f(s, x_s, \int_0^s k(t, \tau, x_{\tau}) d\tau)$ is integrable on [0, t).

Proof.

We have, from the hypothesis (H_2) :

$$\begin{split} \|\xi(s)\| &= \left\| S'\left(t-s\right) \mathcal{B}_{\lambda}f(s,x_{s},\int_{0}^{s}k(t,\tau,x_{\tau})d\tau)\right\| \\ &= \left\| AS\left(t-s\right) \mathcal{B}_{\lambda}f(s,x_{s},\int_{0}^{s}k(t,\tau,x_{\tau})d\tau)\right\| \\ &\leq \frac{\overline{M}\lambda}{\lambda-\delta} \left\| (-A)^{1-\beta}S(t-s)\right\| \left\{ \left\| (-A)^{\beta}f\left(s,x_{s},\int_{0}^{s}k(t,\tau,x_{\tau})d\tau\right) - (-A)^{\beta}f(s,0,0)\right| \\ &+ c_{2} \right\} \\ &\leq \frac{\overline{M}\lambda}{\lambda-\delta} \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\{ F_{1}\|x_{s}\|_{\mathcal{BM}_{h}} \\ &+ c_{2} \right\} \\ &\leq \frac{\overline{M}\lambda}{\lambda-\delta} \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \left\{ F_{1}(\|x_{0}\|_{\mathcal{BM}_{h}} + l\sup_{r\in[0,s]}\|x(r)\|_{X}) + c_{2} \right\}. \end{split}$$

Thus from Bochner's theorem, it follows that $s \longrightarrow S'(t-s) B_{\lambda} f(s, x_s, \int_0^s k(t, \tau, x_{\tau}) d\tau)$ is integrable on [0, t). This achieve the proof of Lemma 8.

Define the operator $\Gamma : \operatorname{BM}_h'' \longrightarrow \operatorname{BM}_h''$ by

$$\Gamma(x)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ -S'(t)[\phi(0) - g(0, \phi, 0)] + g\left(t, x_t, \int_0^t h(t, s, x_s) ds\right) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t S'(t - s) \mathcal{B}_{\lambda} f(s, x_s, \int_0^s k(t, \tau, x_{\tau}) d\tau) ds \\ + \sum_{0 < t_k < t} S'(t - t_k) \,\delta(t - t_k) I_k\left(x(t_k^-)\right), & t \in J, \end{cases}$$

The operator Γ has a fixed point $x(\cdot)$. This fixed point is then PPAP solution of the system (1)-(3).

We show the following

Lemma 9 For u pseudo almost periodic the operator $\Gamma(u)$ is pseudo almost periodic.

Proof.

First, since $\alpha_k \in L^1(J, [0, +\infty))$, by hypothesis (H_5) using the Hille-Yosida condition with n = 1, (we have, $||B_\lambda||_X = ||\lambda(\lambda I - A)^{-1}||_X \leq \overline{\frac{M\lambda}{\lambda - \delta}}$), and by

hypothesis (H_2) we observe that Γ is bounded.

$$\begin{split} \| \Gamma(x)(t) \|_{X} &\leq \| S'(t) \|_{X} \| \phi(0) - g(0,\phi,0) \|_{X} + \left\| g\left(t,x_{t}, \int_{0}^{t} h(t,s,x_{s}) ds\right) \right\|_{X} \\ &+ \lim_{\lambda \to +\infty} \frac{\overline{M}\lambda}{\lambda - \delta} \left\| \int_{0}^{t} S'(t-s) f\left(s,x_{s}, \int_{0}^{s} k(t,\tau,x_{\tau}) d\tau\right) ds \right\|_{X} \\ &+ \left\| \sum_{0 < t_{k} < t} S'(t-t_{k}) I_{k} \left(x(t_{k}^{-})\right) \right\|_{X} \\ &\leq M \| \phi(0) - g(0,\phi,0) \|_{X} + \left\| g\left(t,x_{t}, \int_{0}^{t} h(t,s,x_{s}) ds\right) - g(0,0,0) \right\|_{X} + \| g(0,0,0) \|_{X} \\ &+ \lim_{\lambda \to +\infty} \frac{M\overline{M}\lambda}{\lambda - \delta} \left\| \int_{0}^{t} f\left(s,x_{s}, \int_{0}^{s} k(t,\tau,x_{\tau}) d\tau\right) ds \right\|_{X} \\ &+ \left\| \sum_{0 < t_{k} < t} S'(t-t_{k}) I_{k} \left(x(t_{k}^{-})\right) \right\|_{X} \\ &\leq M \| \phi(0) - g(0,\phi,0) \|_{X} + G_{1} \| x_{t} \|_{BM_{h}} + G_{2} p(t) \| x_{t} \|_{BM_{h}} + \| g(0,0,0) \|_{X} \\ &+ M\overline{M} \int_{0}^{t} \sup_{\| x_{s} \|_{C^{1}} \| \int_{0}^{s} k(t,\tau,x_{\tau}) d\tau \| < k} \left\| f\left(s,x_{s}, \int_{0}^{s} k(t,\tau,x_{\tau}) d\tau\right) \right\|_{X} ds \\ &+ \left\| \sum_{0 < t_{k} < t} S'\left(t-t_{k}\right) I_{k} \left(x(t_{k}^{-})\right) \right\|_{X} \\ &\leq \| \phi(0) - g(0,\phi,0) \| + (G_{2}\bar{p} + G_{1}) \left(\| x_{0} \|_{BM_{h}} + \sup_{s \in [0,b]} \| x(s) \| \right) + \| g(0,0,0) \|_{X} \\ &+ M\overline{M} \int_{0}^{b} \alpha_{k}(s) ds + M \left\| \sum_{0 < t_{k} < b} L_{k} \left(x(t_{k})\right) \right\|_{X} \end{aligned}$$

Hence $\Gamma(x)$ is bounded.

Now we show that $\Gamma(x)(t)$ is pseudo almost periodic with respect to $t \in \mathbb{R}$. Note that the functions f, g, h and k are pseudo almost periodic. By the fact that the sum and product of two functions pseudo almost periodic are pseudo almost periodic and by Lemma 5, we have

$$t \longrightarrow -S'(t)[\phi(0) - g(0, \phi, 0)] + g\left(t, x_t, \int_0^t h(t, s, x_s)ds\right) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t S'(t-s) \operatorname{B}_{\lambda} f\left(s, x_s, \int_0^s k(t, \tau, x_{\tau})d\tau\right) ds,$$

is pseudo almost periodic.

Let us prove that the function

$$z: t \longrightarrow \sum_{0 < t_k < t} S'\left(t - t_k\right) \delta(t - t_k) I_k\left(x(t_k^-)\right),$$

belongs to $PAP(\mathbb{R}, X)$. Because I_k is pseudo almost periodic sequence, one have $I_k(x(t_k^-)) = I_k = I_k^1 + I_k^2$ where $I_k^1 \in AP(\mathbb{R}, X)$ and $I_k^2 \in AP_0S(\mathbb{R}, X)$. Consequently,

$$z(t) = \sum_{0 < t_k < t} S'(t - t_k) \,\delta(t - t_k) I_k^1 + \sum_{0 < t_k < t} S'(t - t_k) \,\delta(t - t_k) I_k^2$$

= $z_1(t) + z_2(t)$.

Since I_k^1 is almost periodic, then for a fixed $\epsilon>0$ the set

$$\widehat{T}(I_k^1,\epsilon) = \left\{ \tau \in \mathbb{R} : \ \|I_k^1(t+\tau) - I_k^1(t)\|_X \le \epsilon \ \forall n \in \mathbb{Z} \right\}$$

is relatively dense in \mathbb{Z} . That is there exists a positive integer l_{ϵ} such that any interval with the length l_{ϵ} contains at least one point of $\widehat{T}(I_k^1, \epsilon)$. Now, let $\epsilon > 0, \tau \in T, q \in Q$ where the T and Q are defined as in Lemma 7. Then

$$\begin{split} \|z_{1}(t+\tau) - z_{1}(t)\|_{\alpha} &= \left\| (-A)^{-\alpha} (z_{1}(t+\tau) - z_{1}(t)) \right\|_{X} \\ &= \left\| \left(-A \right)^{-\alpha} \left(\sum_{0 < t_{k} < t+\tau} S'(t+\tau-t_{k}) \delta(t-t_{k}) I_{k}^{1} \right) \right\|_{X} \\ &- \sum_{0 < t_{k} < t} S'(t-t_{k}) \delta(t-t_{k}) I_{k}^{1} \right) \right\|_{X} \\ &\leq \sum_{0 < t_{k} < t} \left\| (-A)^{-\alpha} S'(t-t_{k}) \right\|_{X} \left\| I_{k+q}^{1} - I_{k}^{1} \right\|_{X} \\ &\leq \sum_{0 < t_{k} < t} \left\| (-A)^{1-\alpha} S(t-t_{k}) \right\|_{X} \left\| I_{k+q}^{1} - I_{k}^{1} \right\|_{X} \\ &\leq \sum_{0 < t_{k} < t} C_{1-\alpha}(t-t_{k})^{1-\alpha} \exp(-\lambda(t-t_{k})) \left\| I_{k+q}^{1} - I_{k}^{1} \right\|_{X} \\ &\leq \epsilon \sum_{0 < t_{k} < t} C_{1-\alpha}(t-t_{k})^{1-\alpha} \exp(-\lambda(t-t_{k})) \\ &\leq \epsilon \left[\sum_{0 < t-t_{k} < 1} C_{1-\alpha}(t-t_{k})^{1-\alpha} \exp(-\lambda(t-t_{k})) \right] \\ &+ \sum_{j=1}^{\infty} \sum_{j < t-t_{k} < j+1} C_{1-\alpha}(t-t_{k})^{1-\alpha} \exp(-\lambda(t-t_{k})) \\ &+ \sum_{j=1}^{\infty} \sum_{j < t-t_{k} < j+1} C_{1-\alpha}(t-t_{k})^{1-\alpha} \exp(-\lambda(t-t_{k})) \\ &\leq \epsilon C_{1-\alpha} N \left(\frac{m^{1-\alpha}}{\exp(-\lambda)} + \frac{1}{\exp(\lambda) - 1} \right), \end{split}$$

where $||t - t_k|| > \epsilon$ and $m = \min\{t - t_k, 0 < t - t_k \le 1\}.$

Now, we shall prove that

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|z_2(s)\|_{\alpha} \, ds = 0.$$

By Lemma 6 there exists $w \in PAP_0(\mathbb{R}, X)$ such that $w(k) = I^2(k) = I_k^2$, $k \in \mathbb{Z}$.

$$\begin{split} \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \|z_2(s)\|_{\alpha} \, ds &= \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \left\| \sum_{0 < t_k < t} S'(t-t_k) I_k^2 \right\|_{\alpha} \, ds \\ &= \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \left\| \sum_{0 < t_k < t} (-A)^{-\alpha} S'(t-t_k) I_k^2 \right\|_{\alpha} \, ds \\ &\leq \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \left\| \sum_{0 < t_k < t} (-A)^{1-\alpha} S(t-t_k) \right\|_{\chi} \|I_k^2\|_{\chi} \, ds \\ &\leq \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} \left\| \sum_{0 < t_k < t} C_{1-\alpha}(t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \right\|_{\chi} \|I_k^2\|_{\chi} \, ds \\ &\leq \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \\ &\times \|w(k)\|_{\chi} \, ds \\ &+ \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} \sum_{j < t-t_k \le j+1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \\ &\times \|w(t)I(t-k)\|_{\chi} \, ds \\ &+ \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \\ &\times \|w(t)I(t-k)\|_{\chi} \, ds \\ &+ \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \\ &\times \|w(t)I(t-k)\|_{\chi} \, ds \\ &+ \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \\ &\times \|w(t)I(t-k)\|_{\chi} \, ds \\ &\leq \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \|w(t)\|_{\chi} \, ds \\ &+ \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} \sum_{j < t-t_k \le j+1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \|w(t)\|_{\chi} \, ds \\ &\leq \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \sum_{j \ge 1} \sum_{j < t-t_k \le j+1} (t-t_k)^{1-\alpha} \exp(-\lambda(t-t_k)) \|w(t)\|_{\chi} \, ds \end{aligned}$$

$$\leq \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} N \frac{m^{1-\alpha}}{\exp(-\lambda)} \|w(t)\|_X ds$$
$$+ \lim_{r \to +\infty} \frac{C_{1-\alpha}}{2r} \int_{-r}^{r} \frac{N}{\exp(\lambda) - 1} \|w(t)\|_X ds = 0.$$

This achieve the proof of Lemma 9.

Now, for $\phi \in BM_h$ define $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ S'(t)\phi(0), & \text{if } t \in J = [0, b] \end{cases}$$

then $\tilde{\phi} \in BM'_h$. Let $x(t) = y(t) + \tilde{\phi}(t), -\infty < t \le b$. Observe that $x(\cdot)$ satisfies (5) if and only if y satisfies $y_0 = 0$ and

$$y(t) = -S'(t)g(0,\phi,0) + g\left(t, y_t, \int_0^t h(t,s,y_s)ds\right) + \lim_{\lambda \to +\infty} \int_0^t S'(t-s) B_{\lambda}f(s,y_s, \int_0^s k(t,\tau,y_{\tau})d\tau)ds + \sum_{0 < t_k < t} S'(t-t_k) \,\delta(t-t_k) I_k\left(y(t_k^-)\right), \ t \in J,$$

Let $BM''_h = \{y \in BM'_h : y_0 = 0 \in BM_h\}$, then for any $y \in BM''_h$, we have

$$\|y\|_{b} = \|y_{0}\|_{\mathrm{BM}_{h}} + \sup\{\|y(s)\|_{X} : 0 \le s \le b\},\$$

thus $(\mathbf{BM}''_h, \|\cdot\|_b)$ is a Banach space. Let q > 0 and define

$$B_q = \{ y \in BM''_h : \|y\|_b \le q \}.$$

Then for each q > 0 the set B_q is clearly bounded, closed convex set in BM''_h . Further, using the Lemma 1, for any $y \in B_q$ we have

$$\begin{aligned} \|y_{t} + \tilde{\phi}_{t}\|_{\mathrm{BM}_{h}} &\leq \|y_{t}\|_{\mathrm{BM}_{h}} + \|\tilde{\phi}_{t}\|_{\mathrm{BM}_{h}} \\ &\leq \|y_{0}\|_{\mathrm{BM}_{h}} + l \sup_{s \in [0,t]} \|(s)\| + \|\tilde{\phi}_{0}\|_{\mathrm{BM}_{h}} + l \sup_{s \in [0,t]} \|\tilde{\phi}(s)\| \\ &\leq lq + \|\tilde{\phi}_{0}\|_{\mathrm{BM}_{h}} + l \sup_{s \in [0,b]} \|T(s)\phi(0)\| \\ &\leq l(q + M\|\phi(0)\|_{X}) + \|\phi\|_{\mathrm{BM}_{h}} = q'. \end{aligned}$$
(6)

Further by using Lemma 1 and (6), for each $t \in J$ we have

$$||y(t) + \tilde{\phi}(t)||_X \le l^{-1} ||y_t - \tilde{\phi}_t||_{\mathrm{BM}_h} \le l^{-1} q'.$$

This give,

$$\sup_{t \in J} \|y(t) + \tilde{\phi}(t)\| \le l^{-1}q'.$$
(7)

By using hypothesis (H_5) and (7), for each (k = 1, ..., m) we obtain

$$\|I_k(y(t_k^-) + \tilde{\phi}(t_k^-))\|_X \le L_k(\|y(t_k^-) + \tilde{\phi}(t_k^-)\|_X) \le L_k\left(\sup_{t \in J} \|y(t) + \tilde{\phi}(t)\|\right) \le L_k(l^{-1}q').$$
(8)

Define the operator $\psi : \operatorname{BM}_h'' \longrightarrow \operatorname{BM}_h''$ by

$$\psi(y)(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ -S'(t)g(0, \phi, 0) + g\left(t, y_t, \int_0^t h(t, s, y_s)ds\right) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t S'(t-s) B_\lambda f(s, y_s, \int_0^s k(t, \tau, x_\tau)d\tau)ds \\ + \sum_{0 < t_k < t} S'(t-t_k) \,\delta(t-t_k) I_k\left(x(t_k^-)\right), & t \in J, \end{cases}$$

it's clear that the operator Γ has a fixed point if and only if ψ has a fixed point. Thus, we prove that ψ has a fixed point.

Now, we prove some necessary lemmas:

Lemma 10 $\psi(B_q) \subseteq B_q$ for some q > 0.

Proof.

We claim that there exists a positive integer q such that $\psi(B_q) \subseteq B_q$. If this is not true, then for each positive integer q, there is a function $y^q \in B_q$, but $\psi(y^q) \notin B_q$, that is, $\|\psi(y^q(t))\|_X > q$ for some $t(q) \in J$, where t(q)denotes t depending on q. However, on the other hand, using (H_1) - (H_4) and the inequality (8) we have,

$$\begin{split} q &< \|\psi(y^{q})(t)\|_{X} \\ &\leq \|S'(t)\|_{X} \left(\|g(0,\phi,0) - g(t,0,0)\|_{X} + \|g(t,0,0)\|_{X}\right) \\ &+ \left\|g\left(t,y_{t}^{q} + \widetilde{\phi}_{t}, \int_{0}^{t} h(t,s,y_{s}^{q} + \widetilde{\phi}_{s})ds\right) - g(t,0,0)\right\|_{X} + \|g(t,0,0)\|_{X} \\ &+ \left\|\lim_{\lambda \longrightarrow +\infty} \int_{0}^{t} S'(t-s)B_{\lambda}f\left(s,y_{s}^{q} + \widetilde{\phi}_{s}, \int_{0}^{s} k(t,\tau,y_{\tau}^{q} + \widetilde{\phi}_{\tau})d\tau\right)ds\right\|_{X} \\ &+ \sum_{0 < t_{k} < t} \|S'(t-t_{k})\|_{X} \left\|I_{k}\left(y^{q}(t_{k}^{-}) + \widetilde{\phi}(t_{k}^{-})\right)\right\|_{X} \\ &\leq M\left(G_{1}\|\phi\|_{\mathrm{BM}_{h}} + \sup_{t \in J}\|g(t,0,0)\|\right) + G_{1}\left\|y_{t}^{q} + \widetilde{\phi}_{t}\right\|_{\mathrm{BM}_{h}} \\ &+ G_{2}\left\|\int_{0}^{t} h(t,s,y_{s}^{q} + \widetilde{\phi}_{s})ds\right\|_{X} + \sup_{t \in J}\|g(t,0,0)\| \\ &+ \left\|\lim_{\lambda \longrightarrow +\infty} \int_{0}^{t} S'(t-s)B_{\lambda}f\left(s,y_{s}^{q} + \widetilde{\phi}_{s},\int_{0}^{s} k(t,\tau,y_{\tau}^{q} + \widetilde{\phi}_{\tau})d\tau\right)ds\right\|_{X} + M\sum_{k=1}^{m} L_{k}(l^{-1}q') \end{split}$$

Using the Hille-Yosida condition with n = 1, we have,

$$||B_{\lambda}||_{X} = ||\lambda(\lambda I - A)^{-1}||_{X} \le \frac{M\lambda}{\lambda - \delta}.$$

Further $B_{\lambda}x \longrightarrow x$ as $\lambda \longrightarrow +\infty$ for all $x \in \overline{D(A)}$,

$$\lim_{\lambda \to +\infty} \|B_{\lambda}x\|_{X} \le \overline{M} \|x\|_{X} \text{ and } \lim_{\lambda \to +\infty} \|B_{\lambda}\|_{X} \le \overline{M}.$$
 (9)

From (9) and hypothesis (H_3) , for each $t \in J$, we have

$$\sup\left\{\left\|y_{t}+\widetilde{\phi}_{t}\right\|_{\mathrm{BM}_{h}},\left\|\int_{0}^{t}h\left(t,s,y_{s}+\widetilde{\phi}_{s}\right)ds\right\|_{X},\left\|\int_{0}^{b}k\left(t,s,y_{s}+\widetilde{\phi}_{s}\right)ds\right\|_{X}\right\}$$
$$\leq q'\max_{t\in J}\{1,p(t),q(t)\}=q'\overline{\omega}.$$
(10)

Taking $G_3 = \sup_{t \in J} ||g(t, 0, 0)||$, using inequalities (9), (10) and the hypothesis (H_4) - (H_5) , from (9), we obtain

$$q < M (G_1 \|\phi\|_{BM_h} + G_3) + (G_1 + G_2)q'\overline{\omega} + G_3 + M\overline{M} \int_0^b \alpha_{q'\overline{\omega}}(s)ds + M \sum_{k=1}^m L_k(l^{-1}q').$$

Therefore

$$1 < \frac{M(G_1 \|\phi\|_{\mathrm{BM}_h} + G_3)}{q} + (G_1 + G_2)\overline{\omega}\frac{q'}{q} + \frac{M\overline{M}q'\overline{\omega}}{q}\frac{1}{q'\overline{\omega}}\int_0^b \alpha_{q'\overline{\omega}}(s)ds + M\frac{l^{-1}q'}{q}\sum_{k=1}^m \frac{L_k(l^{-1}q')}{l^{-1}q'}.$$
(11)

Noting that $\lim_{q \to +\infty} \frac{q'}{q} = l$, therefore $q' \to +\infty$ as $q \to +\infty$. Taking lower limit from (11) and from hypothesis (H_4) - (H_5) we get

$$(G_1 + G_2)\overline{\omega}l + M\overline{M}\overline{\omega}l\mu + M\sum_{k=1}^m \lambda_k \ge 1.$$

This contradicts the hypothesis of Theorem 3. Hence for some positive integer $q, \psi(B_q) \subseteq B_q$.

Lemma 11 (Decomposition of ψ [28]). The completely continuous operators, contractions and sum of these two operators are condensing operators. Thus to prove that the ψ is condensing operator we decompose ψ as $\psi = \psi_1 + \psi_2$, and prove that the operator ψ_1 is a contraction while ψ_2 is a completely continuous operator.

The operator ψ_1, ψ_2 on B_q , respectively are defined by

$$\psi_1 y(t) = -S'(t)g(0,\phi,0) + g\left(t, y_t + \widetilde{\phi}_t, \int_0^t h\left(t, s, y_s + \widetilde{\phi}_s\right) ds\right), \ t \in J, \ (12)$$

$$\psi_2 y(t) = \lim_{\lambda \to +\infty} \int_0^t S'(t-s) B_\lambda f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(t, \tau, y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds + \sum_{0 < t_k < t} S'(t-t_k) \delta(t-t_k) I_k\left(y(t_k^-) + \widetilde{\phi}(t_k^-)\right), \quad t \in J.$$
(13)

Lemma 12 ψ_1 is a contraction.

Proof.

Let any $u, v \in B_q$ and $t \in J$. Then using the hypotheses (H_1) and Lemma 1, we have

$$\begin{aligned} \|\psi_{1}u(t) - \psi_{1}v(t)\|_{X} &\leq \left\|g\left(t, u_{t} + \widetilde{\phi}_{t}, \int_{0}^{t} h\left(t, s, u_{s} + \widetilde{\phi}_{s}\right) ds\right)\right\|_{X} \\ &- g\left(t, v_{t} + \widetilde{\phi}_{t}, \int_{0}^{t} h\left(t, s, v_{s} + \widetilde{\phi}_{s}\right) ds\right)\right\|_{X} \\ &\leq G_{1}\|u_{t} - v_{t}\|_{\mathrm{BM}_{h}} + G_{2}\int_{0}^{t}\left\|h\left(t, s, u_{s} + \widetilde{\phi}_{s}\right) - h\left(t, s, v_{s} + \widetilde{\phi}_{s}\right)\right\|_{X} ds \\ &\leq (G_{1} + bG_{2}H) \|u_{t} - v_{t}\|_{\mathrm{BM}_{h}} \\ &\leq (G_{1} + bG_{2}H) \left(\|u_{0}\|_{\mathrm{BM}_{h}} + \|v_{0}\|_{\mathrm{BM}_{h}} + l \sup_{s \in [0,t]}\|u(s) - v(s)\|\right). \end{aligned}$$

Since $||u_0||_{BM_h} = ||v_0||_{BM_h} = 0$, we have

 $\|\psi_1 u(t) - \psi_1 v(t)\|_X \le (G_1 + bG_2 H) l \sup_{s \in [0,t]} \|u(s) - v(s)\| \le (G_1 + bG_2 H) l \|u - v\|_b.$

Therefore ψ_1 is a contraction on B_q .

Lemma 13 ψ_2 maps B_q in to an equicontinuous family.

Proof.

Let any $y \in B_q$ and $t_1, t_2 \in (-\infty, 0]$. We prove the equicontinuity for these case $0 < t_1 < t_2 \leq b$, as the equicontinuity for these cases $t_2 < t_1 \leq 0$ and $t_1 < 0 < t_2 \leq b$ are obvious.

For $0 < t_1 < t_2 \le b$ and $\epsilon > 0$ sufficiently small, using the hypothesis (H_4) and the conditions (6), (8) and (10) we obtain

$$\begin{split} \|\psi_{2}(y)(t_{1}) - \psi_{2}(y)(t_{2})\|_{X} &\leq \left\| \lim_{\lambda \to +\infty} \int_{0}^{t_{1}-\epsilon} \left(S'(t_{2}-s) - S'(t_{1}-s) \right) B_{\lambda} \right. \\ &\times f\left(s, y_{s} + \tilde{\phi}_{s}, \int_{0}^{s} k\left(t, \tau, y_{\tau} + \tilde{\phi}_{\tau} \right) d\tau \right) ds \right\|_{X} \\ &+ \left\| \lim_{\lambda \to +\infty} \int_{t_{1}-\epsilon}^{t_{1}} \left(S'(t_{2}-s) - S'(t_{1}-s) \right) \right. \\ &\times B_{\lambda} f\left(s, y_{s} + \tilde{\phi}_{s}, \int_{0}^{s} k\left(t, \tau, y_{\tau} + \tilde{\phi}_{\tau} \right) d\tau \right) ds \right\|_{X} \\ &+ \left\| \lim_{\lambda \to +\infty} \int_{t_{1}}^{t_{2}} S'(t_{2}-s) B_{\lambda} f\left(s, y_{s} + \tilde{\phi}_{s}, \int_{0}^{s} k\left(t, \tau, y_{\tau} + \tilde{\phi}_{\tau} \right) d\tau \right) ds \right\|_{X} \\ &+ \left\| \sum_{0 < t_{k} < t_{1}} \left(S'(t_{1} - t_{k}) - S'(t_{2} - t_{k}) \right) I_{k} \left(y(t_{k}^{-}) + \tilde{\phi}(t_{k}^{-}) \right) \right\|_{X} \\ &+ \left\| \sum_{t_{1} < t_{k} < t_{2}} S'(t_{2} - t_{k}) I_{k} \left(y(t_{k}^{-}) + \tilde{\phi}(t_{k}^{-}) \right) \right\|_{X} \\ &\leq \lim_{\lambda \to +\infty} \left(\int_{0}^{t_{1}-\epsilon} \left\| \left(S'(t_{2} - s) - S'(t_{1} - s) \right) \right\|_{X} \left\| B_{\lambda} \right\|_{X} \alpha_{q'\overline{\omega}}(s) ds \\ &+ \int_{t_{1}}^{t_{2}} \left\| S'(t_{2} - s) \right\|_{X} \left\| B_{\lambda} \right\|_{X} \alpha_{q'\overline{\omega}}(s) ds \\ &+ \int_{t_{1}}^{t_{2}} \left\| S'(t_{2} - s) \right\|_{X} \left\| B_{\lambda} \right\|_{X} \alpha_{q'\overline{\omega}}(s) ds \\ &+ \int_{0 < t_{k} < t_{1}} \left(S'(t_{1} - t_{k}) - S'(t_{2} - t_{k}) \right) L_{k}(l^{-1}q') \\ &+ \sum_{t_{1} < t_{k} < t_{2}} S'(t_{2} - t_{k}) L_{k}(l^{-1}q'). \end{split}$$

Since S'(t) is compact for t > 0 and hence continuous in the uniform operator topology, $\alpha_{q'\overline{\omega}} \in L^1$, and the right hand side of is independent of $y \in B_q$, we obtain $\|\psi_2(y)(t_1) - \psi_2(y)(t_2)\| \longrightarrow 0$ as $(t_1 - t_2) \longrightarrow 0$ with $\epsilon > 0$ sufficiently small. This prove that ψ_2 maps B_q into an equicontinuous family of functions.

Lemma 14 ψ_2 maps B_q into a precompact set in X.

Proof.

Let $0 < t \le b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$,

we define

$$\begin{aligned} (\psi_2 y) \left(t\right) &= \lim_{\lambda \longrightarrow +\infty} \int_0^{t-\epsilon} S'(t-s) B_\lambda f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(t, \tau, y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds \\ &+ \sum_{0 < t_k < t} S'(t-t_k) \delta(t-t_k) I_k\left(y(t_k^-) + \widetilde{\phi}(t_k^-)\right) \\ &= S'(\epsilon) \lim_{\lambda \longrightarrow +\infty} \int_0^{t-\epsilon} S'(t-\epsilon-s) \\ &\times B_\lambda f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(t, \tau, y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds \\ &+ \sum_{0 < t_k < t} S'(t-t_k) \delta(t-t_k) I_k\left(y(t_k^-) + \widetilde{\phi}(t_k^-)\right). \end{aligned}$$

Note that,

$$\left\|\lim_{\lambda \to +\infty} \int_0^{t-\epsilon} S'(t-\epsilon-s) B_{\lambda} f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k(s, \tau, y_\tau + \widetilde{\phi}_\tau) d\tau\right) ds\right\|_X \leq M \overline{M} \int_0^{t-\epsilon} \alpha_{q'\overline{\omega}}(s) ds,$$
(14)

Using the estimations (13), (14) and by the compactness of S'(t) (t > 0), we obtain that the set $Z_{\epsilon}(t) = \{(\psi_{2\epsilon}y)(t) : y \in B_q\}$ is relative compact in X for every ϵ , $0 < \epsilon < t$. Thus, for every $q \in B_q$, we have

$$\begin{aligned} \|(\psi_{2}y)(t) - (\psi_{2\epsilon}y)(t)\|_{X} &\leq \left\|\lim_{\lambda \longrightarrow +\infty} \int_{t-\epsilon}^{t} S'(t-s) B_{\lambda} f\left(s, y_{s} + \widetilde{\phi}_{s}, \int_{0}^{t} k\left(t, \tau, y_{\tau} + \widetilde{\phi}_{\tau}\right) d\tau\right) ds\right\|_{X} \\ &\leq \int_{t-\epsilon}^{t} M \overline{M} \alpha_{q'\overline{\omega}}(s) ds. \end{aligned}$$

Therefore $\|(\psi_2 y)(t) - (\psi_{2\epsilon} y)(t)\| \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$, and hence there are precompact sets arbitrarily close to the set $\{(\psi_{2\epsilon} y)(t) : y \in B_q\}$, thus this set is also relative compact in X.

Lemma 15 $\psi_2 : BM''_h \longrightarrow BM''_h$ is continuous.

Proof. Let $\{y^{(n)}\}_{n=0}^{+\infty} \subseteq BM_h''$, with $y^{(n)} \longrightarrow y$ in BM_h'' . Therefore there exists q > 0 such that $\|y^{(n)}(t)\|_X \leq q$ for all $n \in \mathbb{N}$ and $t \in J$, then $y^{(n)}, y \in B_q$. Using Lemma 1, $\|y_t^{(n)} + \widetilde{\phi}_t\|_{BM_h} \leq q'$, for all $t \in J$, and by using the hypothesis (H_6) , we obtain

(i) For each k, (k = 1, ..., m), I_k is continuous,

(ii)
$$f\left(t, y_t^{(n)} + \widetilde{\phi}_t, \int_0^t k\left(t, s, y_s^{(n)} + \widetilde{\phi}_s\right) ds\right) \longrightarrow f\left(t, y_t + \widetilde{\phi}_t, \int_0^t k\left(t, s, y_s + \widetilde{\phi}_s\right) ds\right).$$

By the inequality

$$\left\| f\left(t, y_t^{(n)} + \widetilde{\phi}_t, \int_0^t k\left(t, s, y_s^{(n)} + \widetilde{\phi}_s\right) ds \right) - f\left(t, y_t + \widetilde{\phi}_t, \int_0^t k\left(t, s, y_s + \widetilde{\phi}_s\right) ds \right) \right\|_X \le 2\alpha_{q'\overline{\omega}}(t)$$

and using the dominated convergence theorem,

$$\begin{split} \left\| \left(\psi_{2\epsilon} y^{(n)} \right) (t) - \left(\psi_{2\epsilon} y \right) (t) \right\|_{X} &= \left\| \lim_{\lambda \longrightarrow +\infty} \int_{0}^{t} S'(t-s) B_{\lambda} \left[f\left(s, y_{s}^{(n)} + \widetilde{\phi}_{s}, \int_{0}^{s} k\left(t, \tau, y_{\tau}^{(n)} + \widetilde{\phi}_{\tau} \right) d\tau \right) \right. \\ &- \left. f\left(s, y_{s} + \widetilde{\phi}_{s}, \int_{0}^{s} k\left(t, \tau, y_{\tau} + \widetilde{\phi}_{\tau} \right) \right) \right] \right\|_{X} \\ &+ \left\| \sum_{0 < t_{k} < t} S'(t-t_{k}) \left[\delta(t-t_{k}) I_{k}(y^{(n)}(t_{k}^{-}) + \widetilde{\phi}(t_{k}^{-})) \right] \right\|_{X} \\ &\leq M \overline{M} \int_{0}^{t} \left\| f\left(t, y_{t}^{(n)} + \widetilde{\phi}_{t}, \int_{0}^{s} k\left(t, \sigma, y_{\sigma}^{(n)} + \widetilde{\phi}_{\sigma} \right) d\sigma \right) \right. \\ &- \left. f\left(t, y_{t} + \widetilde{\phi}_{t}, \int_{0}^{s} k\left(\sigma, y_{\sigma} + \widetilde{\phi}_{\sigma} \right) d\sigma \right) \right\|_{X} ds \\ &+ M \sum_{0 < t_{k} < t} \left\| I_{k} \left(y^{(n)}(t_{k}^{-}) + \widetilde{\phi}(t_{k}^{-}) \right) - I_{k} \left(y(t_{k}^{-}) + \widetilde{\phi}(t_{k}^{-}) \right) \right\|_{X}, \end{split}$$

thus

$$\left\| \left(\psi_{2\epsilon} y^{(n)} \right) (t) - \left(\psi_{2\epsilon} y \right) (t) \right\|_{X} \longrightarrow 0 \text{ as } n \longrightarrow +\infty$$

Therefore, $\|\psi_2 y^{(n)} - \psi_2 y\|_b = \sup_{t \in J} \|\psi_2 y^{(n)}(t) - \psi_2 y(t)\|_X \longrightarrow 0$, and hence ψ_2 is continuous.

Theorem 3 Assume that assumptions (H_1) - (H_6) are fulfilled, $\phi \in BM_h$ and $\phi(0) - g(0, \phi, 0) \in \overline{D(A)}$. Then (1)-(3) has at least one piecewise pseudo almost periodic solution on $(-\infty, b]$ provided that

$$K := l(G_1 + bG_2H) < 1,$$

$$(G_1 + G_2)\overline{\omega}l + M\overline{M}\overline{\omega}l\mu + M\sum_{k=1}^m \frac{L_k(\sigma)}{\sigma} < 1, \text{ where } \overline{\omega} = \max\{1, p(t), q(t)\}$$

Proof.

By using Sadovskii's fixed point theorem we prove that the operator \varGamma defined by

$$\Gamma(x)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ S'(t) \left[\phi(0) - g(0, \phi, 0)\right] + g\left(t, x_t, \int_0^t h(t, s, x_s) ds\right) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t S' \left(t - s\right) \mathcal{B}_{\lambda} f(s, x_s, \int_0^s k(t, \tau, x_{\tau}) d\tau) ds \\ + \sum_{0 < t_k < t} S' \left(t - t_k\right) \delta(t - t_k) I_k\left(x(t_k^-)\right), & t \in J, \end{cases}$$

has fixed point x(.). This fixed point is then a PPAP solution of the system (1)-(3).

For $\phi \in BM_h$, define $\widetilde{\phi}$ by

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), \ t \in (-\infty, 0], \\ S'(t)\phi(0), \ t \in J = [0, b] \end{cases}$$

then $\tilde{\phi} \in BM'_h$. Let $x(t) = y(t) + \tilde{\phi}(t), -\infty < t \leq b$. We remark that the shape of x satisfies (5) if and only if y satisfies $y_0 = 0$ and

$$y(t) = -S'(t)g(0,\phi,0) + g\left(t, y_t + \widetilde{\phi}(t), \int_0^t h\left(t, s, y_s + \widetilde{\phi}_s\right) ds\right) \\ + \lim_{\lambda \longrightarrow +\infty} \int_0^t S'(t-s)B_{\lambda}f\left(s, y_s + \widetilde{\phi}_s, \int_0^s k\left(t, \tau, y_\tau + \widetilde{\phi}_\tau\right) d\tau\right) ds \\ + \sum_{0 < t < t_k} S'(t-t_k)\delta(t-t_k)I_k\left(y(t_k^-) + \widetilde{\phi}(t_k^-)\right), \ t \in J.$$

From Lemma 9-Lemma 15 and the Arzela-Ascoli theorem it follows that $\psi_2 : B_q \longrightarrow B_q$ is completely continuous, thus, we have proved that ψ is condensing operator on B_q . Using the Sadovskii fixed point theorem, ψ has a fixed point $\theta(.)$ in B_q . Let $x(t) = \theta(t) + \tilde{\phi}(t), t \in (-r, b]$, for all r > 0. Then x is an PPAP solution of (1)-(3). This achieve the proof of Theorem 3.

5 Application

Let us consider the following partial neutral mixed integro differential equation with infinite delays, presented by the system:

$$\frac{d}{dt} \left[z(t,x) - G\left(t, \int_{-\infty}^{t} P_1(s-t)z(s,x)ds, \int_0^t \int_{-\infty}^s P_2(s,x,\tau-s)Q_1(z(\tau,x))d\tau ds\right) \right] \\
= \frac{\partial^2}{\partial x^2} \left[z(t,x) - G\left(t, \int_{-\infty}^t P_1(s-t)z(s,x)ds, \int_0^t \int_{-\infty}^s P_2(s,x,\tau-s)Q_1(z(\tau,x))d\tau ds\right) \right] \\
+ F\left(t, \int_{-\infty}^0 P_3(s-t)z(s,x)ds, \int_0^t \int_{-\infty}^s P_4(s,z,\tau-s)Q_2(z(\tau,x))d\tau ds\right), \quad (15) \\
\text{if } (t,x) \in [0,b] \setminus t_k \times [0,\pi],$$

$$z(t,0) = z(t,\pi) = 0, \ 0 \le t \le b,$$
(16)

$$z(t_k^+, x) - z(t_k^-, x) = I_k(z(t_k^-, x)), \text{ if } t = t_k \text{ and } x \in [0, \pi],$$

$$z(t, x) = \phi(t, x), t \in [-r, 0], x \in [0, \pi], r > 0, k = 1, ..., m,$$
(17)

where $F: [0, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and the functions $P_2, P_4: \Delta \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and strongly mesurable functions.

Let $X = L^2[0,\pi]$ with the norm $\|\cdot\|_{L^2}$, define the operator

$$A: D(A) \subset X \longrightarrow X$$
 by $Aw = w''$,

with domain

 $D(A) = \{ w \in X : w, w' \text{ are absolutely continuous } w'' \in X, w(0) = w(\pi) = 0 \},\$ then we have

 $\overline{D(A)} = \{w \in X : w, w' \text{ are absolutely continuous } w(0) = w(\pi) = 0\} \neq X.$ It is well known that (see [15]) A satisfies the following properties:

- (i) $\rho(A) \supseteq (0, +\infty)$.
- (ii) $\|(\lambda I A)^{-1}\| \leq \frac{1}{\lambda}$, for all $\lambda > 0$.

Then $Aw = \sum_{n=1}^{+\infty} n^2 \langle w, w_n \rangle \langle w_n, w_n \rangle \langle w_$

 $\sqrt{\frac{2}{\Pi}}\sin(nx)$, n = 1, 2, ... is the orthogonal set of eigen vectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup S(t), $t \ge 0$, and is given by $S(t)w = \sum_{n=1}^{+\infty} e^{-n^2t} < w, w_n > w_n, w \in X$. Since the analytic semigroup $\{S'(t)\}_{t\ge 0}$ is compact [27], there exists constants M > 0 such that $\|S'(t)\| \le M$.

Further for every $w \in X$, $(-A)^{-\frac{1}{2}}w = \sum_{n=1}^{+\infty} \frac{1}{n} < w, w_n > w_n$, and $\|(-A)\|^{-\frac{1}{2}} = 1.$

The operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}} = \sum_{n=1}^{+\infty} n < w, w_n > w_n,$$

with the domain $D((-A)^{\frac{1}{2}}) = \left\{ w \in X : \sum_{n=1}^{+\infty} n < w, w_n > w_n \in X \right\}$. It follows that $||S(t)|| \le 1$. Let $h(s) = \exp(2s)$, s < 0; then $l = \int_{-\infty}^{0} h(s) ds = \frac{1}{2}$, and define

$$\|\phi\|_{\mathrm{BM}_h} = \int_{-\infty}^0 h(s) \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t, \phi) \in [0, b] \times BM_h$, where $\phi(\theta)(x) = \phi(\theta, x), (\theta, x) \in (-\infty, 0] \times$ $[0, \pi]$. Set $\chi(t)(a) = \chi(t,a)$

$$z(t)(y) = z(t,y);$$

$$k(t,s,\psi)(y) = \int_{-\infty}^{s} P_4(t,y,\theta)Q_2(\psi(\theta)y)d\theta;$$

$$h(t,s,\psi)(y) = \int_{-\infty}^{s} P_2(t,y,\theta)Q_1(\psi(\theta)y)d\theta;$$

$$g\left(t,\psi,\int_0^t h(t,s,\psi)ds\right)(y) = G\left(t,\int_{-\infty}^0 P_1(\theta)\psi(\theta)(y)d\theta,\int_0^t h(t,s,\psi)(y)ds\right);$$

$$f\left(t,\psi,\int_0^t k(t,s,\psi)ds\right)(y) = F\left(t,\int_{-\infty}^0 P_3(\theta)\phi(\theta)(y)d\theta,\int_0^t k(t,s,\psi)(y)ds\right).$$

The above partial differential system (15)-(18) can be formulated as an abstract form as the system (1)-(3). Suppose further that:

 (a_1) The function $P_2(t, y, \theta)$ is continuous in $[0, b] \times [0, \pi] \times (-\infty, 0]$ and $P_2(t, y, \theta) \ge 0, \int_0^t \int_{-\infty}^s P_2(t, y, \theta) d\theta ds = \tilde{p}_2(t, y) < \infty.$

 (a_2) The function $P_4(t, y, \theta)$ is continuous in $[0, b] \times [0, \pi] \times (-\infty, 0]$ and $P_4(t, y, \theta) \ge 0, \int_0^t \int_{-\infty}^s P_4(t, y, \theta) d\theta ds = \tilde{p}_4(t, y) < \infty.$

(a₃) The function $Q_i(\cdot)$ is continuous and for $(\theta, y) \in (-\infty, 0] \times [0, \pi]$,

$$0 \leq Q_i(\psi(\theta)y) \leq \vartheta_i\left(\int_{-\infty}^0 e^{2s} \|\psi(s,\cdot)\|_{L^2} ds\right), \ i=1,2,$$

where $\vartheta_i: [0, +\infty) \longrightarrow (0, +\infty)$ is a continuous and nondecreasing function.

Now we can see that

$$\begin{split} \left\| \int_{0}^{t} h(t,s,\phi)(y) ds \right\|_{L^{2}} &= \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{2}(t,y,\theta) Q(\phi(\theta)(y)) d\theta ds \right)^{2} dy \right]^{\frac{1}{2}} \\ &\leq \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{2}(t,y,\theta) \vartheta_{1} \left(\int_{-\infty}^{0} e^{2s} \|\phi(s)(\cdot)\|_{L^{2}} ds \right) d\theta ds \right)^{2} dy \right]^{\frac{1}{2}} \\ &\leq \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{2}(t,y,\theta) \vartheta_{1} \left(\int_{-\infty}^{0} e^{2s} \sup_{s \in [\theta,0]} \|\phi(s)\|_{L^{2}} ds \right) d\theta ds \right)^{2} dy \right]^{\frac{1}{2}} \\ &\leq \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{2}(t,y,\theta) d\theta ds \right)^{2} dy \right]^{\frac{1}{2}} \vartheta_{1}(\|\phi\|_{\mathrm{BM}_{h}}) \\ &\leq \left[\int_{0}^{\pi} (\tilde{p}_{2}(t,y))^{2} dy \right]^{\frac{1}{2}} \vartheta_{1}(\|\phi\|_{\mathrm{BM}_{h}}) \triangleq p_{2}(t) \vartheta_{1}(\|\phi\|_{\mathrm{BM}_{h}}), \end{split}$$

and

$$\begin{split} \|k(t,s,\phi)(y)\|_{L^{2}} &= \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{4}(t,y,\theta)Q(\phi(\theta)(y))d\theta ds\right)^{2} dy\right]^{\frac{1}{2}} \\ &\leq \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{4}(t,y,\theta)\vartheta_{2} \left(\int_{-\infty}^{0} e^{2s} \|\phi(s)(\cdot)\|_{L^{2}} ds\right) d\theta ds\right)^{2} dy\right]^{\frac{1}{2}} \\ &\leq \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{4}(t,y,\theta)\vartheta_{2} \left(\int_{-\infty}^{0} e^{2s} \sup_{s\in[\theta,0]} \|\phi(s)\|_{L^{2}} ds\right) d\theta ds\right)^{2} dy\right]^{\frac{1}{2}} \\ &\leq \left[\int_{0}^{\pi} \left(\int_{0}^{t} \int_{-\infty}^{s} P_{4}(t,y,\theta) d\theta ds\right)^{2} dy\right]^{\frac{1}{2}} \vartheta_{2}(\|\phi\|_{\mathrm{BM}_{h}}) \\ &\leq \left[\int_{0}^{\pi} \left(\tilde{p}_{4}(t,y)\right)^{2} dy\right]^{\frac{1}{2}} \vartheta_{2}(\|\phi\|_{\mathrm{BM}_{h}}) \triangleq p_{4}(t)\vartheta_{2}(\|\phi\|_{\mathrm{BM}_{h}}). \end{split}$$

By imposing suitable conditions on the above defined pseudo almost periodic functions F, G and I_k , to verify the assumptions of Theorem 3, we conclude that the system (1)-(3) has at least one PPAP solution.

6 Conclusion

We have considered an impulsive integro-differential system with infinite delays. This research serves as a first step for the application of the Sadovskii's theorem in investigation of the existence of PPAP solutions for such systems. After introducing an abstract phase space BM_h (in Section 2). Some result are improved and generalized. The model investigated in this work can be regarded as a natural continuation of the models studied in ([1], [6], [7], [8], [12], [19], [22], [24], [31], [32]). An example is also included to illustrate the importance of the results obtained. The demonstrated techniques can be applied in studying of qualitative properties of many practical problems in diverse domain. Besides, this techniques can been extended to investigate the controllability (see [30]) of the first order impulsive functional integro-differential with infinite delay. In the best of our knowledge, we can applied this method for the model investigated in ([39], [40], [41], [42]).

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