

The measures of pseudorandomness and the NIST tests

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Abstract

A few years ago new quantitative measures of pseudorandomness of binary sequences have been introduced. Since that these measures have been studied in many papers and many constructions have been given along these lines. In this paper the connection between the new measures and the NIST tests is analyzed. It is shown that finite binary sequences possessing strong pseudorandom properties in terms of these new measures usually also pass or nearly pass most of the NIST tests.

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1 Introduction

The National Institute of Standards and Technology (=NIST) of the US issued the document [27] which we refer to as “NIST tests”. We quote the introduction of this documentum: “The need for random and pseudorandom numbers arises in many cryptographic applications. For example, common cryptosystem...” [e.g., the Vernam cipher] “...employs keys that must be generated in

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a random fashion... This document discusses the randomness testing of random numbers and pseudorandom number generators that may be used for many purposes including cryptographic, modeling and simulation applications. The focus of this document is on those applications where randomness is required for cryptographic purposes. A set of statistical tests for randomness is described in this document.” The NIST tests is a package consisting of 15 tests, and in each of these 15 cases one has to compute the value of a certain statistics composed from the elements of the given sequence. Then we have to check whether this value is close enough to the expected value of this statistics for a random binary sequence. If, say, we want to check the quality of a PRBG (=pseudorandom bit generator; an algorithm generating a long bit sequence from a short random one called “seed”), then this can be done by testing several bit sequences generated from random seeds by the PRBG; if these sequences pass the NIST tests then the PRBG is suitable for further consideration. As the NIST tests writes: “These tests may be useful as a first step in determining whether or not a generator is suitable for a particular cryptographic application. However, no set of statistical tests can absolutely certify a generator as appropriate for usage in a particular application, i.e., statistical testing cannot serve as a substitute for cryptanalysis.” The weak point of this “first step” by using the NIST tests is that they are of a posteriori type, i.e., we do not have any a priori control of the pseudorandom quality of the output sequences of the PRBG so that we do not know anything about the output sequences not tested by the NIST tests.

Thus one might like to replace this a posteriori type testing based on the NIST tests with a method for a priori testing (called “theoretical testing” by Knuth) of all the output sequences of the PRBG (which seems to be a too optimistic goal) or at least to combine and complete the NIST tests by a method of this type (this is a more realistic goal). In 1997 Mauduit and the third author [18] made a significant step in this direction: they introduced certain measures of pseudorandomness, and they presented an example for binary sequence which possess strong pseudorandom properties in terms of these measures. Since that more than 150 papers have been written in which these measures are studied, further measures are introduced, or further “good” constructions are presented; an excellent survey of these papers is given by Gyarmati [12]. It is a natural question to ask: how is this direction related to the NIST tests? Can one, indeed, complete the a posteriori testing by using these new results? A partial answer was given by the second and third author in [26]: they studied the connection of 3 NIST tests and a further often used test with the measures of pseudorandomness introduced in [18], and they showed that the values of the statistics to be computed in each of these tests can be estimated well by using the measures of pseudorandomness mentioned above, moreover, they also presented numerical calculations to show that a “random” sequence selected from a family of binary sequences constructed by using the Legendre symbol [10, 18] passes all the NIST tests (of 2005). In this paper our goal is to continue that work in the following direction: we will study the connection between 3 further NIST tests and our measures of pseudorandomness. (The 8 remaining NIST tests are too complicated to study their theoretical connection with the measures of pseudorandomness.) Moreover, we will present further numerical calculations to show that a “random” sequence selected from two other families constructed by using other principles also passes or “almost passes” all the NIST tests.

2 The measures of pseudorandomness

First we recall a few definitions and facts from [18] and other related papers that we will need in this paper.

Consider a finite binary sequence

$$E_N = (e_1, \dots, e_N) \in \{-1, +1\}^N. \quad (1)$$

(Note that in the analysis in some of the NIST tests the bit sequences are also transformed into sequences consisting of -1 and +1.) Then the *well-distribution measure* of E_N is defined as

$$W(E_N) = \max_{a,b,t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|, \quad (2)$$

where the maximum is taken over all $a, b, t \in \mathbb{N}$ such that $1 \leq a \leq a + (t-1)b \leq N$, while the *correlation measure of order k* of E_N is defined as

$$C_k(E_N) = \max_{M,D} \left| \sum_{n=1}^M e_{n+d_1} e_{n+d_2} \dots e_{n+d_k} \right|, \quad (3)$$

where the maximum is taken over all $D = (d_1, \dots, d_k)$ and M such that $0 \leq d_1 < \dots < d_k \leq N-M$. Then the sequence is considered as a “good” pseudorandom sequence if both these measures $W(E_N)$ and $C_k(E_N)$ (at least for “small” k) are “small” in terms of N (in particular, both are $o(N)$ as $N \rightarrow \infty$). Indeed, it is shown in [4] that for a “truly random” $E_N \in \{-1, +1\}^N$ both $W(E_N)$ and, for fixed k , $C_k(E_N)$ are of order of magnitude $N^{1/2}$ with probability “near 1” (see also [2] and [14]). Thus for “really good” pseudorandom sequences we expect the measures (2) and (3) to be not much greater than $N^{1/2}$. In [18] a combination of the well-distribution and correlation measures was also introduced: the *combined pseudorandom measure of order k* of the sequence E_N in (1) is defined as

$$Q_k(E_N) = \max_{a,b,t,D} \left| \sum_{j=0}^{t-1} e_{a+jb+d_1} e_{a+jb+d_2} \dots e_{a+jb+d_k} \right|$$

where the maximum is taken over all $a, b, t \in \mathbb{N}$ and k -tuples $D = (d_1, d_2, \dots, d_k)$ of non-negative integers $d_1 < d_2 < \dots < d_k$ such that all the subscripts $a+jb+d_i$ belong to $\{1, 2, \dots, N\}$. (Clearly, we have $W(E_N) = Q_1(E_N)$ and $C_k(E_N) \leq Q_k(E_N)$ for $k \geq 2$.) We will also need the definition of normality measures also introduced in [18]. The *normality measure of order k* of the sequence E_N of form (1) is defined as

$$N_k(E_N) = \max_{X \in \{-1, +1\}^k} \max_{0 < M \leq N+1-k} \left| |\{n : 0 \leq n \leq M, (e_{n+1}, \dots, e_{n+k}) = X\}| - \frac{M}{2^k} \right|. \quad (4)$$

It was also shown in [18] (see Proposition 1 and its proof there) that for all N , E_N and $k < N$ we

have

$$N_k(E_N) \leq \frac{1}{2^k} \sum_{t=1}^k \binom{k}{t} C_t(E_N) \leq \max_{1 \leq t \leq k} C_t(E_N). \quad (5)$$

Thus if $C_t(E_N)$ is small for all $t \leq k$, then $N_k(E_N)$ is also small.

3 Three principles for constructing large families of binary sequences with strong pseudorandom properties

It is well known that the Legendre polynomial has many pseudorandom properties [6, 7]. It was shown in [18] that the Legendre symbol also possesses strong pseudorandom properties in terms of the pseudorandom measures described in Section 2: if p is an odd prime, we write $N = p - 1$ and

$$E_N = (e_1, \dots, e_N) \quad \text{with } e_n = \left(\frac{n}{p}\right) \text{ for } n = 1, \dots, N,$$

then we have

$$W(E_N) \ll N^{1/2} \log N \quad \text{and} \quad C_k(E_N) \ll kN^{1/2} \log N$$

for all $k < N$ (where \ll is Vinogradov's notation: $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; in both cases the implicit constants can be computed explicitly (and are relatively small constants)).

Goubin, Mauduit and Sárközy [10] studied the generalization of this construction with $f(n)$ in place of n (where $f(x) \in \mathbb{F}_p[x]$). Their results can be combined in the following way:

Theorem A. *If p is a prime number, $f(x) \in \mathbb{F}_p[x]$ (\mathbb{F}_p being the field of the modulo p residue classes) has degree $k(> 0)$, $f(x)$ has no multiple zero in $\overline{\mathbb{F}_p}$ (= the algebraic closure of \mathbb{F}_p), and the binary sequence $E_p = (e_1, \dots, e_p)$ is defined by*

$$e_n = \begin{cases} \left(\frac{f(n)}{p}\right) & \text{for } (f(n), p) = 1 \\ +1 & \text{for } p \mid f(n), \end{cases} \quad (6)$$

then we have

$$W(E_p) < 10kp^{1/2} \log p.$$

Moreover, assume that also $\ell \in \mathbb{N}$, and one of the following assumptions holds:

- (i) $\ell = 2$;
- (ii) $\ell < p$, and 2 is a primitive root modulo p ;
- (iii) $(4k)^\ell < p$.

Then we also have

$$C_\ell(E_p) < 10k\ell p^{1/2} \log p.$$

The second principle is to utilize the fact that the multiplicative inverse modulo p is distributed in a random way in $(0, p)$. Denote the least non-negative residue of n modulo p by $r_p(n)$, and if the prime p is fixed, then denote the multiplicative inverse of a modulo p by a^{-1} (so that $a \cdot a^{-1} \equiv 1 \pmod{p}$). The following theorem (here we present the result in a slightly simplified form) was proved in [20].

Theorem B. Assume that p is an odd prime number, $f(x) \in \mathbb{F}_p[x]$ has degree k with $0 < k < p$ and no multiple zero in $\overline{\mathbb{F}_p}$. Define the binary sequence $E_p = (e_1, \dots, e_p)$ by

$$e_n = \begin{cases} +1 & \text{if } (f(n), p) = 1, r_p(f(n)^{-1}) < \frac{p}{2} \\ -1 & \text{if either } (f(n), p) = 1, r_p(f(n)^{-1}) > \frac{p}{2} \text{ or } p \mid f(n). \end{cases} \quad (7)$$

Then we have

$$W(E_p) \ll kp^{1/2}(\log p)^2.$$

Moreover, if $\ell \in \mathbb{N}$, $2 \leq \ell \leq \frac{p}{2k}$ and $f(x) \in \mathbb{F}_p[x]$ is of the form $f(x) = (x + a_1)(x + a_2) \cdots (x + a_k)$ with $a_1, \dots, a_k \in \mathbb{F}_p$ ($a_i \neq a_j$ for $i \neq j$) then we also have

$$C_\ell(E_p) \ll k\ell p^{1/2}(\log p)^{\ell+1}. \quad (8)$$

For further related results see also [5, 15, 16]. For example Liu [15] gave another (and simpler) condition to control the correlation measure.

Theorem C. Assume that p is an odd prime number, $f(x) \in \mathbb{F}_p[x]$ is a polynomial of degree $(0 <)k(< p)$ such that 0 is its unique zero in \mathbb{F}_p . If the sequence E_N is defined as in Theorem B and $\ell < p$, then (8) also holds.

The third construction is based on elliptic curves. Let $p > 3$ be a prime number and let \mathbf{E} be an elliptic curve over \mathbb{F}_p defined by the Weierstrass equation

$$y^2 = x^3 + Ax + B$$

with coefficients $A, B \in \mathbb{F}_p$ and non-zero discriminant (see [31]). The \mathbb{F}_p -rational points $\mathbf{E}(\mathbb{F}_p)$ of \mathbf{E} form an Abelian group with the point in infinity \mathcal{O} as the neutral element, where the group operation is denoted by \oplus . For a rational point $R \in \mathbf{E}(\mathbb{F}_p)$, a multiple of R is defined by $nR = \bigoplus_{i=1}^n R$. Let $\mathbb{F}_p(\mathbf{E})$ be the function field of \mathbf{E} over \mathbb{F}_p and as usual for $f \in \mathbb{F}_p(\mathbf{E})$ let $\deg f$ denote of the degree of f in $\mathbb{F}_p(\mathbf{E})$, see [31]. For example, for the coordinate functions we have $\deg x = 2$ and $\deg y = 3$.

Let $G \in \mathbf{E}(\mathbb{F}_p)$ be of order T and $f \in \mathbb{F}_p(\mathbf{E})$. Define the binary sequence $E_T = (e_1, \dots, e_T)$ by

$$e_n = \begin{cases} \left(\frac{f(nG)}{p} \right) & \text{if } (f(nG), p) = 1, \\ +1 & \text{otherwise.} \end{cases} \quad (9)$$

The first author studied the pseudorandomness of this sequence [22]. His results can be combined in the following way:

Theorem E. Let $G \in \mathbf{E}(\mathbb{F}_p)$ be a generator of $\mathbf{E}(\mathbb{F}_p)$ of prime order T . Let $f \in \mathbb{F}_p(\mathbf{E})$ which is not a perfect square in $\overline{\mathbb{F}_p}(\mathbf{E})$ with degree $k = \deg f$. Then

$$W(E_T) \leq 6kp^{1/2} \log T.$$

Moreover, assume that also $\ell \in \mathbb{N}$, and one of the conditions (i), (ii), (iii) of Theorem A holds with

p replaced by T . Then

$$C_\ell(E_T) < 2\ell k p^{1/2} \log T.$$

4 The “frequency test within a block”

First we will study the connection of this test (which appears as Section 2.2 in [27]) with the measures of pseudorandomness described in Section 2. We quote [27]: “The focus of this test is to determine whether the frequency of ones in an M -bit block is approximately $M/2$, as would be expected under an assumption of randomness. For block size $M = 1$, this test degenerates to test 1, the Frequency (Monobit) test” (which was analyzed in [26]).

Let $E_N = (e_1, \dots, e_N) \in \{-1, +1\}^N$ be the sequence to be tested, M the length of each block, and, as [27] writes,

“Partition the input sequence into $t = \lfloor \frac{N}{M} \rfloor$ non-overlapping blocks.”

(Here and later we adjust the notations of [27] to our notation.) The quotation continues:

“Discard any unused bits. Determine the proportion π_i of ones in each block of length M for $1 \leq i \leq t$ ”

Now “Compute the χ^2 statistic

$$X_1 = 4M \sum_{i=1}^t \left(\pi_i - \frac{1}{2} \right)^2 .” \quad (10)$$

Then the sequence E_N passes this test if the value of this statistic is small enough in the sense described in [27]; we skip the technical details.

In 2.2.7 [27] writes: “The block size M should be selected such that

$$M \geq 20, \quad M > N/100 \text{ and } t < 100.” \quad (11)$$

Theorem 1. *Using the notation above and assuming (11), for every $E_N \in \{-1, +1\}^N$ we have*

$$X_1 \leq 2 \cdot 10^4 \frac{W(E_N)^2}{N}. \quad (12)$$

Proof. Clearly we have

$$\begin{aligned} \pi_i &= \frac{|\{e_j : (i-1)M < j \leq iM, e_j = +1\}|}{M} \\ &= \frac{1}{M} \sum_{j=(i-1)M+1}^{iM} \frac{1}{2}(e_j + 1) = \frac{1}{2M} \sum_{j=(i-1)M+1}^{iM} e_j + \frac{1}{2} \end{aligned}$$

whence, using the notation of Section 2,

$$\left| \pi_i - \frac{1}{2} \right| = \frac{1}{2M} \left| \sum_{j=(i-1)M+1}^{iM} e_j \right| \leq \frac{1}{2M} W(E_N)$$

for every $1 \leq i \leq t$. Thus it follows from (10) that

$$X_1 \leq 4M \cdot t \left(\frac{1}{2M} W(E_N) \right)^2 = \frac{2t}{M} W(E_N)^2.$$

By using (11), (12) follows from this. \square

In each of the constructions described in Section 3 the upper bound in inequality (12) is less than a constant multiple of a fixed power of $\log N$, so that this upper bound falls just a little short of the desired $< c$ (with a small positive constant c). In many applications this can be interpreted as a strong tendency towards pseudorandomness which is sufficient for our purposes, while if we have to stick to the threshold bound belonging to the test, then this good upper bound points to the direction that choosing successive random sequences from our family studied we have a good chance to find soon a sequence which also satisfies the stronger inequality prescribed in the test.

5 The “test for the longest run of ones in a block”

This test appears in Section 2.4 of [27]. We quote [27]: “The focus of the test is the longest run of ones within M bit blocks. The purpose of this test is to determine whether the length of the longest run of ones within the tested sequence is consistent with the length of the largest run of ones that would be expected in a random sequence.”. The test to answer this question is carried out in [27] in the following way:

Assume the N, M, t are positive integers with

$$N = Mt, \tag{13}$$

N is the length of the sequence $E_N = (e_1, \dots, e_N) \in \{-1, +1\}^N$ to be tested (again we switch from bit sequences to ± 1 sequences), M is taken from a certain special sequence $8, 128, 10^4, \dots$ (see [27]), E_N is split in t blocks of length M , t and thus also N is large enough in terms of M (in particular, for $M = 8, 128, 10^4$ the number N must be at least $128, 272, 750000$, respectively) and $K (= 3, 5, 6, \dots)$ is certain positive integer assigned to the given M value. The set $\{0, 1, \dots, M\}$ is split in $K + 1$ disjoint parts $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_K$ so that

$$\{0, 1, \dots, M\} = \mathcal{P}_0 \cup \mathcal{P}_1 \cup \dots \cup \mathcal{P}_K, \quad \mathcal{P}_i \cap \mathcal{P}_j = \emptyset \quad \text{for } 0 \leq i < j \leq K, \tag{14}$$

e.g., for $M = 10^4$, $K = 6$ in [27] we have

$$\{0, 1, \dots, 10^4\} = \{0, 1, \dots, 10\} \cup \{11\} \cup \{12\} \cup \{13\} \cup \{14\} \cup \{15\} \cup \{16, 17, \dots, 10^4\}.$$

Then for $i = 0, 1, \dots, K$ we count how many of the t blocks is such that the length of the longest

run of +1's in it belongs to the part \mathcal{P}_i of $\{0, 1, \dots, M\}$; let ν_i denote the number of blocks with this property. Let π_i be the probability of the event that the length of the longest run of +1's in a random sequence of +1 and -1 with length M is i . The test statistic to be computed is a weighted square mean of the deviations of the ν_i 's from their expected values $t\pi_i$:

$$X_2 = \sum_{i=0}^K \frac{(\nu_i - t\pi_i)^2}{t\pi_i}, \quad (15)$$

“which, under the randomness hypothesis, has an approximate χ^2 -distribution with K degrees of freedom”. Here the theoretical values π_i can be replaced by approximating numerical values which for certain pairs M, K are provided in Section 3.4 of [27]. (We remark that for fixed M , the choice of K and the computation of the values approximating π_i is based on the analysis of distribution of the longest run in random walks; see, e.g. [25, Chapter 7].)

We will show that the statistic X_2 in (15) can be estimated in the following way:

Theorem 2. *We have*

$$X_2 \leq \frac{M}{N} \left(\sum_{r=1}^M \binom{M}{r} Q_r(E_N) \right)^2.$$

Note that this estimate gives a good bound for X_2 only if N is large in terms of M ; the first table in [27], pp. 2-8 seems to indicate that this can be assumed.

Proof. We will use the following notations:

For $Z \in \mathbb{N}$, let Φ_Z be the set of the binary sequences

$$F_Z = (f_1, f_2, \dots, f_Z) \in \{-1, +1\}^Z,$$

for such a sequence F_Z let $\psi(F_Z)$ denote the length of the longest run of +1's in F_Z , and for $j \in \mathbb{N}$, $jM \leq Z$, write $F_Z^{(j, M)} = (f_{(j-1)M+1}, f_{(j-1)M+2}, \dots, f_{jM})$.

Then for $i = 0, 1, \dots, K$, by the definition of ν_i and (13) we have

$$\nu_i = \sum_{\substack{1 \leq j \leq t \\ \psi(F_N^{(j, M)}) \in \mathcal{P}_i}} 1, \quad (16)$$

and the expectation of ν_i choosing any $F_N \in \Phi_N$ with equal probability $1/2^N$ is

$$\mathbb{E}(\nu_i) = \mathbb{E} \left(\sum_{\substack{1 \leq j \leq t \\ \psi(F_N^{(j, M)}) \in \mathcal{P}_i}} 1 \right) = t \cdot \mathbb{E} \left(\sum_{\substack{G \in \Phi_M \\ \psi(G) \in \mathcal{P}_i}} 1 \right) = t\pi_i. \quad (17)$$

Let $G^{(1)}, G^{(2)}, \dots, G^{(\gamma_i)}$ be the sets G counted in the last sum, and write $\mathcal{G}_i = \{G^{(1)}, G^{(2)}, \dots, G^{(\gamma_i)}\}$ and $G^{(j)} = (g_1^{(j)}, g_2^{(j)}, \dots, g_M^{(j)})$. Each of these sets $G^{(j)}$ contributes by 1 to this sum, and they are to be selected with probability $1/2^M$ uniformly. Thus it follows from (17) that

$$\mathbb{E}(\nu_i) = t\pi_i = t \frac{|\mathcal{G}_i|}{2^M} = \frac{t}{2^M} \gamma_i. \quad (18)$$

Moreover, it follows from (14) that each of the 2^M sets $G \in \Phi_M$ is counted in exactly one \mathcal{G}_i with weight 1, thus we have

$$\sum_{i=0}^K |\mathcal{G}_i| = \sum_{i=0}^K \gamma_i = 2^M. \quad (19)$$

Now we will estimate ν_i for $0 \leq i \leq K$. Consider a subset $G^{(\ell)} = (g_1^{(\ell)}, g_2^{(\ell)}, \dots, g_M^{(\ell)}) \in \mathcal{G}_i$. Then for $j = 1, 2, \dots, t$ clearly we have

$$\prod_{x=1}^M \frac{1 + e_{(j-1)M+x} g_x^{(\ell)}}{2} = \begin{cases} 1 & \text{if } E_N^{(j,M)} = G^{(\ell)}, \\ 0 & \text{if } E_N^{(j,M)} \neq G^{(\ell)}, \end{cases}$$

whence

$$\sum_{\ell=1}^{\gamma_i} \prod_{x=1}^M \frac{1 + e_{(j-1)M+x} g_x^{(\ell)}}{2} = \begin{cases} 1 & \text{if } E_N^{(j,M)} \in \{G^{(1)}, G^{(2)}, \dots, G^{(\gamma_i)}\} = \mathcal{G}_i, \\ 0 & \text{if } E_N^{(j,M)} \notin \mathcal{G}_i, \end{cases}$$

so that by (16) we have

$$\begin{aligned} \nu_i &= \sum_{\substack{1 \leq j \leq t \\ \psi(F_N^{(j,M)}) \in \mathcal{P}_i}} 1 = \sum_{1 \leq j \leq t} \sum_{E_N^{(j,M)} \in \mathcal{G}_i} 1 = \sum_{j=1}^t \sum_{\ell=1}^{\gamma_i} \prod_{x=1}^M \frac{1 + e_{(j-1)M+x} g_x^{(\ell)}}{2} \\ &= \sum_{\ell=1}^{\gamma_i} \sum_{j=1}^t \left(\frac{1}{2^M} + \frac{1}{2^M} \sum_{r=1}^M \sum_{1 \leq x_1 < \dots < x_r \leq M} g_{x_1}^{(\ell)} \dots g_{x_r}^{(\ell)} e_{(j-1)M+x_1} \dots e_{(j-1)M+x_r} \right) \\ &= \frac{t}{2^M} \gamma_i + \frac{1}{2^M} \sum_{\ell=1}^{\gamma_i} \left(\sum_{r=1}^M \sum_{1 \leq x_1 < \dots < x_r \leq M} g_{x_1}^{(\ell)} \dots g_{x_r}^{(\ell)} \sum_{j=1}^t e_{(j-1)M+x_1} \dots e_{(j-1)M+x_r} \right). \quad (20) \end{aligned}$$

It follows from (18) and (20) that

$$\begin{aligned} |\nu_i - t\pi_i| &= \frac{1}{2^M} \left| \sum_{\ell=1}^{\gamma_i} \left(\sum_{r=1}^M \sum_{1 \leq x_1 < \dots < x_r \leq M} g_{x_1}^{(\ell)} \dots g_{x_r}^{(\ell)} \sum_{j=1}^t e_{(j-1)M+x_1} \dots e_{(j-1)M+x_r} \right) \right| \\ &\leq \frac{1}{2^M} \sum_{\ell=1}^{\gamma_i} \sum_{r=1}^M \sum_{1 \leq x_1 < \dots < x_r \leq M} |g_{x_1}^{(\ell)} \dots g_{x_r}^{(\ell)}| \left| \sum_{j=1}^t e_{(j-1)M+x_1} \dots e_{(j-1)M+x_r} \right| \\ &= \frac{\gamma_i}{2^M} \sum_{r=1}^M \binom{M}{r} Q_r(E_N) = \pi_i \sum_{r=1}^M \binom{M}{r} Q_r(E_N). \quad (21) \end{aligned}$$

By (13), (18), (19) and (21) we have

$$\begin{aligned} X_2 &= \sum_{i=0}^K \frac{(\nu_i - t\pi_i)^2}{t\pi_i} \leq \sum_{i=0}^K \frac{\pi_i}{t} \left(\sum_{r=1}^M \binom{M}{r} Q_r(E_N) \right)^2 \\ &= \frac{1}{t} \left(\sum_{r=1}^M \binom{M}{r} Q_r(E_N) \right)^2 \sum_{i=0}^K \pi_i = \frac{M}{N} \left(\sum_{r=1}^M \binom{M}{r} Q_r(E_N) \right)^2 \end{aligned}$$

which completes the proof of the theorem. \square

We remark that in Theorem 2 the statistic X_2 is estimated in terms of the combined pseudorandom measure Q_k , while in the most important constructions studied in Theorems A, B, C and E only the measures W and C_k were estimated, and no estimates are known for the measures Q_k (and the situation is similar in most of the other constructions). However, this gap can be bridged easily, since in most cases the estimate of Q_k can be reduced easily to the estimate of C_k . For example, in case of the Legendre symbol construction (6) studied in Theorem A, we can show that if the sequence E_p is defined by (6) in Theorem A and we assume that all the assumptions in the theorem hold, then we have

$$Q_k(E_p) \leq C_k(E_p) + 2k,$$

and in case of the two other constructions similar results could be proved.

6 The “linear complexity test”

The *linear complexity* $L(\tilde{E}_N)$ of a *bit* sequence $\tilde{E}_N = (\tilde{e}_1, \dots, \tilde{e}_N) \in \{0, 1\}^N$ is defined as the length L of a shortest linear recurrence relation (linear feedback shift register – LFSR)

$$\tilde{e}_{n+L} \equiv c_{L-1}\tilde{e}_{n+L-1} + \dots + c_1\tilde{e}_{n+1} + c_0\tilde{e}_n \pmod{2}, \quad 1 \leq n \leq N - L$$

where $c_0, \dots, c_{L-1} \in \{0, 1\}$, that \tilde{E}_N satisfies, with the convention that $L(\tilde{E}_N) = 0$ if $\tilde{E}_N = (0, \dots, 0)$, and $L(\tilde{E}_N) = N$ if $\tilde{E}_N = (0, \dots, 0, 1)$. For binary sequence E_N of form (1) we also define the linear complexity as $L(E_N) = L(\tilde{E}_N)$ with $\tilde{e}_n = (1 + e_n)/2$.

The linear complexity is a measure for the unpredictability of a sequence. A large linear complexity is necessary (but not sufficient) for cryptographic applications. The *linear complexity test* appears as Section 2.10 in [27]. We quote: “The purpose of this test is to determine whether or not the sequence is complex enough to be considered random. Random sequences are characterized by longer LFSRs. An LFSR that is too short implies non-randomness.”

Brandstätter and Winterhof [3] showed that a small correlation measure implies large linear complexity:

$$L(E_N) \geq N - \max_{1 \leq k \leq L(E_N)+1} C_k(E_N). \quad (22)$$

This result provides a lower bound for the linear complexity of sequences generated by using the Legendre symbol (6) and elliptic curves (9). Namely, if E_p is a sequence generated by (6) using a squarefree polynomial $f(x) \in \mathbb{F}_p[x]$ of degree k and 2 is a primitive root modulo p , then Theorem A and (22) imply that

$$p \leq L(E_p) + \max_{1 \leq k \leq L(E_p)+1} C_k(E_p) \ll kLp^{1/2} \log p,$$

so that

$$L(E_p) \gg \frac{p^{1/2}}{k \log p}. \quad (23)$$

Similarly, if the sequence E_T is generated by (9) using a squarefree function $f(x, y) \in \mathbb{F}_p[\mathbf{E}]$

with degree k and 2 is a primitive root modulo T , then Theorem E and (22) imply that

$$L(E_T) \gg \frac{p^{1/2}}{k \log T}.$$

By the celebrated Hasse-Weil Theorem (see e.g. [31], Theorem 4.2) we have $|p + 1 - |\mathbf{E}(\mathbb{F}_p)|| \leq 2p^{1/2}$, thus

$$L(E_T) \gg \frac{T^{1/2}}{k \log T}. \quad (24)$$

In practice, the bounds (23) and (24) are sometimes sufficient. However, the linear complexity of a truly random binary sequence of length N is around $N/2$, thus in more demanding applications one may have to show that the linear complexity of the given sequence is near $N/2$ (or at least it is $\gg N$); the linear complexity of the sequence can be determined by using the well-known Berlekamp-Massey algorithm [17].

In even more demanding cases one may need an even more precise study of the complexity properties of the sequence. In [27] this is done by the ‘‘linear complexity test’’ described in [27, p. 2–24]. This test requires a more controlled distribution of the linear complexity of the sequences. Namely, it compares the linear complexity within blocks of length M to the expected value of the linear complexity

$$\mu_M = \frac{M}{2} + \frac{4 + r_2(M)}{18}$$

(here again $r_2(M)$ is the non-negative remainder of M modulo 2).

We quote: ‘‘Partition the N -bit sequence into t independent blocks of M bits, where $N = t \cdot M$.’’ Then ‘‘determine the linear complexity L_i of each of the t blocks ($i = 1, \dots, t$)’’. ‘‘For each substring, calculate a value of T_i where

$$T_i = (-1)^M \cdot (L_i - \mu_M) + \frac{2}{9}.$$

Define the intervals:

$$\begin{aligned} I_0 &= (-\infty, -2, 5], \\ I_j &= (-2.5 + j - 1, -2.5 + j], \quad j = 1, \dots, 5, \\ I_6 &= (2.5, \infty), \end{aligned}$$

and put $v_j = |\{i : T_i \in I_j, i = 1, \dots, t\}|$.

Finally, we define the statistic

$$X_3 = \sum_{j=0}^6 \frac{(v_j - t \cdot \pi_j)^2}{t \cdot \pi_j},$$

where π_j ($i = 0, \dots, 6$) are the probabilities for the classes I_j :

$$\pi_j = P \left((-1)^M \cdot (L(E_M) - \mu_M) + \frac{2}{9} \in I_j \right),$$

where E_M is chosen uniformly from $\{-1, +1\}^M$. The acceptance of the sequence depends on the value of the statistic X_3 : one has to compute the ‘‘ P -value’’ defined in [27], p. 2-25, (7) and if the ‘‘ P -value’’ is ≥ 0.01 , then the sequence passes the test.

If the sequence E_N to be tested possesses strong pseudorandom properties in terms of the measures described in Section 2, and the length M of the blocks is much smaller than the length N of the sequence (say, we have $M = o(\log N)$), then one could give a reasonable upper bound for the statistic X_3 by the method used in Section 5 (although here even more work and computation would be needed). However, according to “input size recommendation” in [27, Section 2.10.7], M must be very large ($500 \leq M \leq 5000$) so that to have $M = o(\log N)$, N must be huge (say, $N > 10^{10000}$), thus we will not present the details here. This, of course, does not mean that shorter sequences with good pseudorandom properties fail this test and, indeed, the numerical examples in Section 8 will show that sequences of this type tend to pass this test, but we cannot show that this is necessarily so.

Remark 3. As mentioned previously, small linear complexity implies non-randomness. However, recent results show that there are many sequences whose linear complexity is very near to its expected value but which also have some cryptographic weakness: Winterhof and the first author provided a large class of highly predictable sequences whose linear complexity is close to its mean [24]. A simple way to eliminate such sequences is to consider also the *expansion complexity* of the sequences defined in [8, 23].

7 Discrete Fourier Transform (Spectral) Test

The “NIST tests” writes: “The purpose of this test is to detect periodic features (i.e., repetitive patterns that are near each other) in the tested sequence that would indicate a deviation from the assumption of randomness”.

This is one of the tests which are too complicated to estimate the statistic to be studied by using our measures of pseudorandomness. Instead, we will do the following: we show that the goal of the test described above can be also achieved by using our measures of pseudorandomness and, indeed, the combined pseudorandom measure of order k described above is especially suitable for this. Take the following example:

Example 7.1. Consider the 4-tuple $+1, -1, -1, +1$, and repeat it $M = 500000$ times. Then letting $N = 4M = 2000000$, we get a binary sequence $E_N = (e_1, e_2, \dots, e_N)$ with $e_{4k-3} = +1$, $e_{4k-2} = -1$, $e_{4k-1} = -1$, $e_{4k} = +1$ for $k = 1, 2, \dots, M$. This sequence is periodic with period 4. Its combined pseudorandom measure of order 4 can be estimated in the following way:

$$Q_4(E_N) \geq \left| \sum_{j=0}^{M-1} e_{4j+1} e_{4j+2} e_{4j+3} e_{4j+4} \right| = \left| \sum_{j=0}^{M-1} 1 \right| = M = \frac{N}{4},$$

so that this measure is big, is as large as $\frac{1}{4}$ times the length of the sequence, much larger than the optimal $\asymp N^{1/2}$. This fact is reflected in the periodicity, thus the sequence is far from being of pseudorandom nature.

Of course, if a sequence is not completely periodic but is only almost periodic with period k , than its Q_k measure is still large.

Applying the Discrete Fourier Transform Test for testing the sequence E_N defined above with 20 samples of length 100000, we find that it fails this test (strongly).

So that both approaches point out the periodic nature of E_N , thus *it fails both tests*.

Now let us study a more complicated example.

Example 7.2. Consider two especially important special sequences: the Rudin-Shapiro sequence (defined by $(-1)^{\sum_i \varepsilon_i(n)\varepsilon_{i+1}(n)}$ where ε_i denotes the i -th binary digit of n) and the Thue-Morse sequence (defined by $(-1)^{\sum_i \varepsilon_i(n)}$). It is known [19] that in both cases if we take a sequence of length N then its correlation measure of order 2 is very large, it is $\gg N$ which is again much larger than the optimal $\asymp N^{1/2}$ so that in terms of the measures of pseudo randomness described above they are both far from being of pseudorandom nature.

But what about the Discrete Fourier Transform Test, do these sequences also fail this test? First consider the Rudin-Shapiro sequence:

```

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RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES
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C1  C2  C3  C4  C5  C6  C7  C8  C9  C10  P-VALUE  PROPORTION  STATISTICAL TEST
-----
  3   3   0   2   3   3   1   2   3   0  0.637119    20/20      Frequency
  4   0   0   0   3   2   0   0   0  11  0.000000 *   16/20     * BlockFrequency
  2   3   5   2   1   2   1   1   1   2  0.637119    20/20      CumulativeSums
  2   1   6   2   0   3   2   1   1   2  0.213309    20/20      CumulativeSums
  0   0   0   0   0   0   0   0   0   0  20  0.000000 *   20/20      Runs
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * LongestRun
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * Rank
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * FFT
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * Universal
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * ApproximateEntropy
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * Serial
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * Serial
20   0   0   0   0   0   0   0   0   0  0  0.000000 *    0/20     * LinearComplexity

```

Figure 1: Results of 13 NIST tests for the Rudin-Shapiro sequence

(Here and in some further tables an asterisk indicates after the column “ P -value” that the P -values belonging to the sequence studied and the test named in the last column are non-uniform, while after column “proportion” it denotes that many or all of these sequences fail the test in question.) Now consider the Thue-Morse sequence:

RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES													
C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	P-VALUE	*	PROPORTION	STATISTICAL TEST
0	0	0	0	0	0	0	0	0	20	0.000000	*	20/20	Frequency
0	0	0	0	0	0	0	0	0	20	0.000000	*	20/20	BlockFrequency
0	0	0	0	0	0	0	0	0	20	0.000000	*	20/20	CumulativeSums
0	0	0	0	0	0	0	0	0	20	0.000000	*	20/20	CumulativeSums
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* Runs
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* LongestRun
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* Rank
19	0	0	0	0	0	0	0	1	0	0.000000	*	2/20	* FFT
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* Universal
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* ApproximateEntropy
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* Serial
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* Serial
20	0	0	0	0	0	0	0	0	0	0.000000	*	0/20	* LinearComplexity

Figure 2: Results of 13 NIST tests for the Thue-Morse sequence

So that both sequences fail this test. The Fourier transform of the Rudin-Shapiro polynomial, whose maximum modulus is very close to its L^2 norm, and this is a very unusual property (see [29]). Similarly, the Fourier transform of the Thue-Morse sequence has a very small L^1 norm (see [9]). This explains why they fail the Discrete Fourier Transform test which detects that they are far from the DFT of a random sequence.

Our examples show that both approaches can be used effectively for detecting some sort of periodicity.

8 Numerical calculation

In the previous sections we showed that those binary sequences $E_N \in \{-1, +1\}^N$ whose pseudo-random measures $W(E_N)$ and $C_k(E_N)$ are small, also have strong pseudorandom properties in terms of the NIST tests *a priori*, i.e. they provably pass or “almost pass” most of the NIST tests. In this section we test sequences constructed by principles described in Section 3 *a posteriori*. The examples show that these sequences typically pass the NIST tests, even if we can only prove a slightly weaker pseudorandomness.

We use “Statistical Test Suite for random and pseudorandom number generators for cryptographic application” (**sts-1.4**) from the National Institute of Standards and Technology (NIST). We chose the following parameters for the test suite.

In order to save space we omit the results of the non-overlapping template matching (**NonOverlappingTemplate**), the random excursions (**RandomExcursions**) and the random excursions variant tests (**RandomExcursionsVariant**).

The results of the tests are given in Figures 4, 5 and 6. Columns **C1** up to **C10** correspond to

BlockFrequency	$M = 128$
OverlappingTemplate	$m = 9$
ApproximateEntropy	$m = 10$
LinearComplexity	$M = 500$

Figure 3: Parameter choices for NIST test suite

the frequency specific to the test. Then the P-VALUE is the result of the application of a χ^2 -test, and PROPORTION is the proportion of sequences that pass the test.

8.1 Sequences generated using the Legendre symbol

We constructed 20 sequences with length $p = 10^5 + 3$ by (6) with the first 20 squarefree polynomial of degree 31 with respect to the lexicographic ordering: $f_i(x) = x^{31} + i$ ($i = 1, \dots, 20$). Since 2 is a primitive root modulo $p = 10^5 + 3$, Theorem A implies that all the sequences $E_p(i)$ generated with $f_i(x)$ ($i = 1, \dots, 20$) have small well-distribution and correlation measures.

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RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES
-----
C1  C2  C3  C4  C5  C6  C7  C8  C9  C10  P-VALUE  PROPORTION  STATISTICAL TEST
-----
 0   0   0   0   0   0   0   0   0   20  0.000000 *  20/20  Frequency
 1   4   2   2   2   2   3   3   1   0  0.739918  20/20  BlockFrequency
 0   1   0   0   0   1   1   4   7   6  0.000199  20/20  CumulativeSums
 0   1   0   0   0   1   1   4   7   6  0.000199  20/20  CumulativeSums
 4   4   1   1   3   1   1   3   0   2  0.437274  20/20  Runs
 1   2   2   1   5   1   2   3   1   2  0.637119  20/20  LongestRun
 2   1   0   3   1   0   4   2   5   2  0.213309  20/20  Rank
 0   0   0   0   0  10   0  10   0   0  0.000000 *  20/20  FFT
 2   6   0   0   1   6   2   1   2   0  0.006196  19/20  OverlappingTemplate
 0  20   0   0   0   0   0   0   0   0  0.000000 *  20/20  Universal
 1   1   6   1   1   2   2   1   4   1  0.162606  20/20  ApproximateEntropy
 1   2   1   1   3   2   2   5   1   2  0.637119  20/20  Serial
 0   4   1   3   1   5   0   2   0   4  0.066882  20/20  Serial
 1   4   1   2   2   3   0   1   3   3  0.637119  20/20  LinearComplexity
-----

```

Figure 4: Results of 14 NIST tests for sequences generated by using the Legendre symbol

8.2 Sequences generated using the multiplicative inverse

We took $p = 2 \cdot 10^5 + 3$ and considered the polynomials

$$f_i(x) = x \cdot \prod_{j=15(i-1)+1}^{15i} (x^2 + j^2), \quad i = 1, \dots, 20.$$

Since $2 \cdot 10^5 + 3 \equiv 3 \pmod{4}$, -1 is quadratic non-residue, and the least non-negative remainders of $-j^2$ ($j = 1, \dots, 300$) modulo p , $r_p(-j^2)$, are also quadratic non-residues. Then these polynomials satisfy the conditions of Theorem C, thus they have small well-distribution and correlation measures. However the sequences generated by (7) with the polynomials $f_i(x)$ have a non-trivial symmetry. Namely, $f_i(-x) = -f_i(x)$, so $e_n = -e_{p-n}$ for all $1 \leq n < p$ if the sequence $E_p = (e_0, \dots, e_{p-1}) \in \{-1, +1\}^p$ is generated such a way. (For tools to detect such symmetries see [11]). To avoid this phenomenon we just considered the first half of the sequences: $E(i)_{(p+1)/2} = \{e_0(i), \dots, e_{(p-1)/2}(i)\}$, where $e_n(i)$ ($0 \leq n < p/2$) is defined by the rule (7). In this way we obtained 20 sequences of length 100002.

RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES

C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	P-VALUE	PROPORTION	STATISTICAL TEST
3	2	1	3	0	2	3	4	1	1	0.637119	20/20	Frequency
5	2	0	1	3	4	2	1	1	1	0.275709	19/20	BlockFrequency
3	1	0	2	2	1	3	0	3	5	0.275709	20/20	CumulativeSums
3	2	2	1	4	0	3	1	4	0	0.350485	20/20	CumulativeSums
2	3	2	5	3	1	1	2	0	1	0.437274	20/20	Runs
2	1	1	2	5	2	3	0	2	2	0.534146	20/20	LongestRun
2	3	3	3	0	0	2	1	4	2	0.534146	20/20	Rank
2	2	1	5	3	3	1	0	2	1	0.437274	20/20	FFT
4	1	1	1	3	0	2	4	1	3	0.437274	20/20	OverlappingTemplate
0	0	0	0	0	0	20	0	0	0	0.000000 *	20/20	Universal
5	4	1	2	2	2	1	1	2	0	0.350485	20/20	ApproximateEntropy
1	2	1	2	1	0	2	7	4	0	0.017912	20/20	Serial
2	1	2	0	3	3	3	2	0	4	0.534146	19/20	Serial
1	3	3	0	3	2	1	4	1	2	0.637119	20/20	LinearComplexity

Figure 5: Results of 14 NIST tests for sequences generated by using the multiplicative inverse

8.3 Sequences generated using elliptic curves

In order to generate sequences with elliptic curves we chose pseudorandom curves and points following the NIST recommendation (FIPS 186-3). We took the prime $p = 10^5 + 3$ and a pseudorandom elliptic curve of the form

$$y^2 = x^3 - 3x + b$$

over \mathbb{F}_p with the additional restriction, that the number T of the \mathbb{F}_p -rational points is prime and 2 is a primitive root modulo T . Then we selected a pseudorandom point P on the curve.

Our parameters were the following:

$$\mathbf{E} : y^2 = x^3 - 3x + 74439 \quad \text{over } \mathbb{F}_{10^5+3}.$$

Its cardinality T is 100523. The point was $P = (85611, 76395)$. We took the functions $f_i(x, y) = x^{31} + x + y + i$ ($i = 0, \dots, 19$). Since 2 is a primitive root modulo T , and the functions $f_i(x, y)$ ($i = 0, \dots, 19$) are not perfect squares, Theorem E implies, that the well-distribution and correlation measures of sequences generated by the polynomials $f_i(x, y)$ ($i = 0, \dots, 19$) are small.

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RESULTS FOR THE UNIFORMITY OF P-VALUES AND THE PROPORTION OF PASSING SEQUENCES
-----
C1  C2  C3  C4  C5  C6  C7  C8  C9  C10  P-VALUE  PROPORTION  STATISTICAL TEST
-----
 4   1   3   0   1   2   2   2   2   3  0.739918   20/20      Frequency
 0   3   4   0   3   2   2   1   2   3  0.534146   20/20      BlockFrequency
 1   5   2   1   3   1   0   2   3   2  0.437274   20/20      CumulativeSums
 4   0   2   4   2   1   3   1   2   1  0.534146   20/20      CumulativeSums
 4   2   4   3   2   0   2   1   1   1  0.534146   20/20      Runs
 5   3   1   3   4   0   0   3   1   0  0.090936   20/20      LongestRun
 3   4   1   0   2   1   1   1   4   3  0.437274   19/20      Rank
 3   3   2   2   4   1   1   1   1   2  0.834308   20/20      FFT
 4   1   1   2   1   1   6   2   0   2  0.122325   20/20      OverlappingTemplate
 0   0   0   0   0   0   0   0   0   0  0.000000 * 20/20      Universal
 5   1   2   1   0   5   3   1   1   1  0.122325   20/20      ApproximateEntropy
 1   2   2   3   4   0   1   3   2   2  0.739918   19/20      Serial
 1   1   4   2   1   2   4   1   2   2  0.739918   19/20      Serial
 1   3   3   6   0   2   2   1   1   1  0.162606   20/20      LinearComplexity
-----

```

Figure 6: Results of 14 NIST tests for sequences generated by using the elliptic curves

Summarizing: we have considered altogether 60 binary sequences which have been proved to possess good pseudorandom properties in terms of the pseudorandom measures described in Section 2, and we tested them by 14 NIST tests. 834 times out of 840 the sequences passed the test so that the NIST tests confirmed the good pseudorandom quality of the sequence.

9 Conclusion Remarks

The tables in Section 8 show that the sequences “good” in terms of the measures defined in Section 2 are usually also “good” in terms of the NIST tests. Does this mean that we may eliminate the NIST tests, replace them by estimating the pseudorandom measures described above? Certainly not: both methods have advantages and disadvantages. The greatest advantage of using the measures of pseudorandomness described above is that at least for certain special sequences they enable us to provide “a priori”, “theoretical” testing without any further computations. An other advantage of this method is that in many constructions (like the ones described in Section 3) we can give a good upper bound (simultaneously by just a single computation) for the correlation measure of order k of a sequence of length N for every k with $k < N^c$ (say, with $c = 1/4$), and

since it is known [4] that the correlation measures of order k , resp. ℓ are independent if $k \nmid \ell$ and $\ell \nmid k$, thus by estimating the correlation measures whose order is less than N^c , we test $N^{c'}$ (with $c' < c$) independent pseudorandom properties of the sequence, while in the NIST tests only 15 properties are tested. On the other hand, the disadvantage of this approach is that in most cases it is very difficult to estimate these measures, e.g. there are no algorithms for estimating correlation measure of high order. On the other hand, NIST provides good and fast algorithms for performing these tests, while its disadvantage is that it can not be used for “a priori”, “theoretical” testing.

In Section 2 we described only the most important measures of pseudorandomness and in Section 3 we presented only three constructions for sequences having good pseudorandom properties in terms of these measures. There are also other measures of pseudorandomness and many further constructions; a survey of these measures and constructions; a survey of these measures and constructions is presented in [12]. In this paper we have been focusing on studying pseudorandom properties of *single binary sequences*. However, as we mentioned in Section 1, if our goal is to test the quality of a pseudorandom generator, then it is not enough to restrict ourselves to testing single sequences; one also has to continue the work by using the tools of cryptanalysis for testing the *family of the sequences* generated by the given algorithm. Tools for helping this work also have been introduced (in the spirit of the measures described in Section 2): family complexity [1], cross-correlation measure [13], distance minimum and avalanche effect [30], etc., and in each of these cases constructions have been presented for families possessing good pseudorandom properties in terms of these measures. A survey of this type of papers is presented in [28].

References

- [1] R. Ahlswede, L. H. Khachatrian, C. Mauduit and A. Sárközy, A complexity measure for families of binary sequences, *Period. Math. Hungar.* 46 (2003), 107–118.
- [2] N. Alon, Y. Kohayakawa, C. Mauduit, C. G. Moreira and V. Rödl, Measures of pseudorandomness for finite sequences: typical values, *Proc. Lond. Math. Soc.* 95 (2007) 778–812
- [3] N. Brandstätter, A. Winterhof, Linear complexity profile of binary sequences with small correlation measure. *Period. Math. Hungar.* 52 (2006), no. 2, 1–8.
- [4] J. Cassaigne, C. Mauduit, A. Sárközy, On finite pseudorandom binary sequences. VII. The measures of pseudorandomness, *Acta Arith.* 103 (2002), no. 2, 97–118.
- [5] Z. Chen, Z. Lin, Modified constructions of binary sequences using multiplicative inverse, *Appl. Math. J. Chinese Univ. Ser. B* 23 (2008), no. 4, 490–500.
- [6] T. W. Cusick et al., *Stream Ciphers and Number Theory*, North Holland, 1998, Math. Library, vol. 55.
- [7] I. Damgård, On the randomness of Legendre and Jacobi sequences, *Advances in cryptology-CRYPTO’88* (Santa Barbara, CA, 1988), 163–172, *Lecture Notes in Comput. Sci.*, 403, Springer, Berlin, 1990.
- [8] C. Diem, On the use of expansion series for stream ciphers. *LMS J. Comput. Math.* 15 (2012), 326–340.

- [9] E. Fouvry and C. Mauduit, Sommes des chiffres et nombres presque premiers, *Math. Annalen* 305 (1996), 571-599.
- [10] L. Goubin, C. Mauduit, A. Sárközy, Construction of large families of pseudorandom binary sequences, *J. Number Theory* 106 (2004), no. 1, 56–69.
- [11] K. Gyarmati, On a pseudorandom property of binary sequences, *Ramanujan J.* 8 (2004), 289-302.
- [12] K. Gyarmati, Measures of pseudorandomness, *Finite fields and their applications*, 43–64, Radon Ser. Comput. Appl. Math., 11, De Gruyter, Berlin, 2013.
- [13] K. Gyarmati, C. Mauduit and A. Sárközy, The cross-correlation measure for families of binary sequences, in: *Applied Algebra and Number Theory*, eds. G. Larcher et al., Cambridge Univ. Press, 2014, pp. 126-143.
- [14] Y. Kohayakawa, C. Mauduit, C. G. Moreira, V. Rödl, Measures of pseudorandomness for finite sequences: minimum and typical values, *Proceedings of WORDS’03*, 159–169, TUCS Gen. Publ., 27, Turku Cent. Comput. Sci., Turku, 2003.
- [15] H. Liu, Large families of pseudorandom binary sequences and lattices by using the multiplicative inverse, *Acta Arith.* 159 (2013), no. 2, 123–131.
- [16] H. Liu, J. Gao, A note on large families of pseudorandom binary sequences and lattices. *JISE J. Inf. Sci. Eng.* 30 (2014), no. 5, 1635–1654.
- [17] J. L. Massey, Shift-register synthesis and BCH decoding. *IEEE Trans. Information Theory* IT-15 1969 122–127.
- [18] C. Mauduit, A. Sárközy, On finite pseudorandom binary sequences I: Measures of pseudorandomness, the Legendre symbol, *Acta Arith.* **82** (1997) 365–377.
- [19] C. Mauduit, A. Sárközy, On finite pseudorandom binary sequences II. (The Champernowne, Rudin-Shapiro and Thue-Morse sequences. A further construction.), *J. Number Theory* **73** (1998), 256–276.
- [20] C. Mauduit, A. Sárközy, Construction of pseudorandom binary sequences by using the multiplicative inverse. *Acta Math. Hungar.* 108 (2005), no. 3, 239–252.
- [21] A. J. Menezes, P. C. van Oorschot, S. A. Vanstone, *Handbook of applied cryptography*. CRC Press Series on Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 1997.
- [22] L. Mérai, Construction of pseudorandom binary sequences over elliptic curves using multiplicative characters, *Publ. Math. Debrecen*, 80 (2012) no. 1-2, 199–213.
- [23] L. Mérai, H. Niederreiter, A. Winterhof, Expansion complexity and linear complexity of sequences over finite fields, *Cryptogr. Commun.* 9 (2017), no. 4, 501-509.
- [24] L. Mérai, A. Winterhof, On the pseudorandomness of automatic sequences, to appear in *J. Number Theory*.

- [25] P. Révész, Random Walk in Random and Non-Random Environments, World Scientific, Singapore, 1990.
- [26] J. Rivat, A. Sárközy, On pseudorandom sequences and their application, General theory of information transfer and combinatorics, 343–361, Lecture Notes in Comput. Sci., 4123, Springer, Berlin, 2006.
- [27] A. Rukhin et al., NIST Special Publication 800-22, Revision 1.a, A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications, <https://www.nist.gov/publications/statistical-test-suite-random-and-pseudorandom-number-generators-cryptographic>
- [28] A. Sárközy, On pseudorandomness of families of binary sequences, J. Discrete Applied Math. 216(3) (2017), 670-676.
- [29] H. S. Shapiro, Extremal Problems for Polynomials and Power Series, M. S. Thesis, M. I. T., 1951.
- [30] V. Tóth, Collision and avalanche effect in families of pseudorandom binary sequences, Period. Math. Hungar. 59 (2009), 1-8.
- [31] L. C. Washington, Elliptic Curves: Number Theory and Cryptography, 2nd edition. Chapman & Hall/CRC Press, 2008.