The foundations of $(2n, k)$ -manifolds

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Abstract

In the focus of our paper is a system of axioms that serves as a basis for introducing structural data for $(2n, k)$ -manifolds M^{2n} , where M^{2n} is a smooth, compact $2n$ -dimensional manifold with a smooth effective action of the k -dimensional torus T^k . In terms of these data a construction of the model space \mathfrak{E} with an action of the torus T^k is given, such that there exists a T^k -equivariant homeomorphism $\mathfrak{E} \to M^{2n}$. This homeomorphism induces a homeomorphism $\mathfrak{E}/T^k \to$ M^{2n}/T^k . The number $d = n - k$ is called the complexity of an $(2n, k)$ -manifold. Our theory comprises toric geometry and toric topology, where $d = 0$. It is shown that the class of homogeneous spaces G/H of compact Lie groups, where rk $G = \text{rk } H$, contains $(2n, k)$ -manifolds that have non zero complexity. The results are demonstrated on the complex Grassmann manifolds $G_{k+1,q}$ with an effective action of the torus T^k .^{[1](#page-0-0)}

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1 Introduction

The goal of this paper is to extend the issues from [\[5\]](#page-47-0), specify the axioms and present the new results of our theory of $(2n, k)$ - manifolds. It is about a wide class of $2n$ dimensional smooth, compact, oriented manifolds with an effective action of the compact torus T^k having only isolated fixed points. We propose the tools for an effective description of the equivariant structure of such manifolds as well as the structure of their orbit spaces.

This class contains toric and quasitoric manifolds M^{2n} , whose effective description in toric topology (see [\[4\]](#page-47-1)) is given by the combinatorial data (P^n, Λ) , where P^n is a *n*-dimensional simple polytope and Λ is a characteristic function from the set of facets of the polytope P^n to the lattice \mathbb{Z}^n that satisfies Davis-Januszkiewicz (*) con-dition [\[9\]](#page-48-0). In this case, a $(2n, n)$ -manifold is obtained, while the orbit space M^{2n}/T^n is homeomorphic to the polytope P^n . In this paper are described the key examples of the manifolds M^{2n} with an effective action of the torus T^k which form a basis for the theory of $(2n, k)$ -manifolds.

Any $(2n, k)$ -manifold is equipped with the so called almost moment map $\mu : M^{2n} \to$ \mathbb{R}^k , whose image is a convex polytope P^k . The polytope P^k does not need to be simple and the orbit space M^{2n}/T^k for $k < n$ is not homeomorphic to P^k in general. For the Grassmann manifolds $G_{k+1,q}$, the polytope P^k is the hypersimplex $\Delta_{k+1,q}$ and for the complex flag manifolds F_{k+1} , the polytope P^k is the permutahedron Pe^k . Note that $\Delta_{k+1,q}$ is a simple polytope only for $q = 1$ or $q = k$, while Pe^k is a simply polytope for any k .

One of the main tools, which we introduce, is a family of admissible polytopes P_{σ} which are spanned by some subsets σ of vertices of the polytope P^k . In the quasitoric case the family of admissible polytopes coincides with the family of the faces of the simple polytope P^k including the polytope P^k . In the case of Grassmann manifolds $G_{k+1,q}$ our admissible polytopes coincide with the admissible polytopes from the paper [\[14\]](#page-48-1) in which a number of results about these polytopes are obtained .

To each admissible polytope P_{σ} corresponds a T^{k} -invariant subspace $W_{\sigma} \subset M^{2n}$, called the stratum, and a subtorus $T_{\sigma} \subseteq T^k$ which acts *trivially* on W_{σ} such that the torus $T^{\sigma} = T^{k}/T_{\sigma}$, $\dim T^{\sigma} = \dim P_{\sigma}$ acts freely on W_{σ} . The polytope P^{k} is considered to be an admissible polytope and the corresponding stratum W is called the main stratum. One of our axioms requires that $\mu(W_{\sigma}) = \stackrel{\circ}{P_{\sigma}}$ and that the restriction of the induced almost moment map $\hat{\mu}$ to W_{σ}/T^{σ} is a projection of a fiber bundle with the base \hat{P}_{σ} . This implies that $W_{\sigma}/T^{\sigma} \cong \stackrel{\circ}{P}_{\sigma} \times F_{\sigma}$, for some topological space F_{σ} , which is called the space of parameters of the stratum W_{σ} . In the case of Grassmann manifolds, our strata W_{σ} coincide with the strata of Gel'fand, MacPherson, Goresky and Serganova, which they introduced using the action of the algebraic torus $(\mathbb{C}^*)^{k+1}$ on $G_{k+1,q}$. In this case, the subtorus $(\mathbb{C}^*)^{\sigma}$ acts freely on W_{σ} and $F_{\sigma} = W_{\sigma}/(\mathbb{C}^*)^{\sigma}$. Note that in general case, an action of the compact torus T^k on a $(2n, k)$ -manifold M^{2n} does not extend to an effective action of the algebraic torus $(\mathbb{C}^*)^k$.

The union of all admissible polytopes P_{σ} is the polytope P^k and to each point $x \in P^k$

we assign the cortége $\sigma(x) = \{P_{\sigma} : x \in P_{\sigma}\}\)$. We assume that μ is a smooth map and obtain that if dim $P_{\sigma} = k$ for any $P_{\sigma} \in \sigma(x)$ then x is a regular value for μ .

Towards our goal to describe the equivariant topology of an $(2n, k)$ -manifold M^{2n} and its orbit space M^{2n}/T^k , we introduce structural data and obtain a *model* for the space M^{2n}/T^k in terms of these structural data. Our structural data consist of the *virtual* spaces of parameters \tilde{F}_{σ} together with the continuous projections $p_{\sigma} : \tilde{F}_{\sigma} \to F_{\sigma}$, and the *universal* space of parameters $\mathcal F$ together with the embeddings $I_{\sigma} : \tilde{F}_{\sigma} \to \mathcal F$. Note that for the main stratum, the virtual space of parameters \tilde{F} coincides with the space of

parameters F. We use the fact that the orbit space of the main stratum $W/T^k \cong \stackrel{\circ}{P^k} \times F$

is a dense set in M^{2n}/T^k . It is required that the compactification of $\overrightarrow{P^k} \times \overrightarrow{F}$, which corresponds to the compactification $\overline{W/T^k} = M^{2n}/T^k$, is realized by the topology of the *universal* space of parameters F , which is a compactification of the space of parameters F of the main stratum such that $\mathcal{F} = \cup_{\sigma} I_{\sigma}(\tilde{F}_{\sigma})$. We realized in detail this approach in [\[6\]](#page-47-2) for the Grassmann manifold $G_{4,2}$ and in [\[7\]](#page-47-3) for the Grassmann manifold $G_{5,2}$. In general case, we obtain the orbit spaces M^{2n}/T^k as a quotient

space of the union $\cup_{\sigma} (P_{\sigma}^{\circ} \times \tilde{F}_{\sigma})$ by an equivalence relation which is defined in terms of the structural maps $I_{\sigma} : \tilde{F}_{\sigma} \to \mathcal{F}$ and $p_{\sigma} : \tilde{F}_{\sigma} \to F_{\sigma}$.

The complexity of an $(2n, k)$ -manifold is defined to be the number $d = n - k$. Our definition of the complexity of an action generalizes the definition of the complexity of an algebraic torus action $(\mathbb{C}^*)^k$ in algebraic geometry and symplectic geometry. The complexity 1 torus actions, under some appropriate assumptions, are widely studied in algebraic and symplectic geometry (see [\[22,](#page-48-2) [18\]](#page-48-3)). In the recent paper [\[2\]](#page-47-4), inspired by our work [\[6,](#page-47-2) [7\]](#page-47-3), the approach is described for solving the classification problem of complexity 1 torus actions in terms of equivariant topology. The problem of the torus actions of complexity 2 or more is still not well understood and, in the literature, it is considered to be quite difficult. Our theory leads to the results in this direction. In the papers [\[6,](#page-47-2) [7\]](#page-47-3) the orbit spaces $\mathbb{C}P^5/T^4$ and $G_{5,2}/T^5$ of the complexity 2 actions on an (10, 3) and (12, 4)-manifolds, respectively are described in detail.

The axioms and methods of this paper rely on the well known results on the algebraic torus action on Grassmann manifolds [\[13,](#page-48-4) [14,](#page-48-1) [15,](#page-48-5) [16\]](#page-48-6), as well as on our results on the orbit spaces of the compact torus action on Grassmann manifolds [\[6,](#page-47-2) [7\]](#page-47-3). We emphasize the Grassmann manifolds $G_{k+1,2}$ with an effective action of the torus T^k of complexity $k - 2$, $k \ge 2$, since many papers have been devoted to them in the recent time, due to their connection with the moduli spaces of curves (see, for example, [\[16,](#page-48-6) [17,](#page-48-7) [20\]](#page-48-8)).

2 The key examples of manifolds with torus actions

In this section we provide the key examples and the basic facts, which served as a starting point for establishing axioms for the theory of $(2n, k)$ -manifolds.

2.1 Quasitoric manifolds

A quasitoric manifold is a topological analog of a non-singular projective toric variety from algebraic geometry. We follow [\[4\]](#page-47-1) and recall that a quasitoric manifold is a smooth, closed manifold M^{2n} equipped with a smooth action of the torus T^n such that:

- the action of the torus T^n on M^{2n} is locally standard;
- the orbit space M^{2n}/T^n is diffeomorphic to a simple polytope P^n as a manifold with corners.

The second condition gives that there exists a smooth map $\mu : M^{2n} \to P^n$ that is constant on $Tⁿ$ -orbits and maps an p-dimensional orbit to an interior point of some p-dimensional face of P^n . In follows that $\mu^{-1}(\stackrel{\circ}{P^n})$, is a dense set in M^{2n} , the action of the torus T^n is free on $\mu^{-1}(\stackrel{\circ}{P^n})$ and the vertices of the polytope P^n correspond to the fixed points of $Tⁿ$ -action. We recall the notion of the characteristic map and the characteristic matrix for a quasitoric manifold M^{2n} which, together with the combinatorics of the polytope $Pⁿ$, determine an equivariant topology and cohomology of a manifold M^{2n} .

Let $\{F_1, \ldots, F_m\}$ be the set of all facets of P^n . The stationary subgroups $T(F_i)$ for the faces F_i are one-dimensional connected subgroups in T^n and they can be written as $T(F_i) = (e^{2\pi\sqrt{-1}\lambda_{1i}\varphi}, \ldots, e^{2pi\\sqrt{-1}\lambda_{ni}\varphi})$, where $\varphi \in \mathbb{R}$ and $\lambda_i = (\lambda_{1i}, \ldots, \lambda_{ni}) \in$ \mathbb{Z}^n . Denote by $S(T^n)$ the set of all connected subgroups of the torus T^n . The characteristic map $l: \{F_i\} \to S(T^n)$, which is defined by $l: F_i \to T(F_i)$, can be described using the characteristic matrix Λ whose columns are the integer vectors λ_i , $1 \leq i \leq m$ which satisfy the following condition: if the intersection $F = F_{i_1} \cap \ldots \cap F_{i_n}$ is a vertex of the polytope P^n , then the vectors $\lambda_{i_1}, \ldots, \lambda_{i_n}$ form a basis for \mathbb{Z}^n . Due to Davis-Januszkiewicz theorem (see [\[9\]](#page-48-0)), the matrix Λ and the combinatorics of the polytope P^n determine together the cohomology of M^{2n} .

Let **F** be the partially ordered set of all faces for P^n . The points from $\mu^{-1}(F) \subset M^{2n}$ have the same stabilizer for any $F \in \mathbf{F}$, so the characteristic map extends to the map $\mathbf{F} \to S(T^n)$, which to each face F assigns the stationary subgroup of the set $\mu^{-1}(F)$. More precisely, the face $F = F_{i_1} \cap \ldots \cap F_{i_k}$ maps to the image of the subgroup $T(F_{i_1}) \times \cdots \times T(F_{i_k}) \subset T^n$. The map $\mathbf{F} \to \mathcal{S}(T^n)$ is completely determined by the matrix Λ and it is denoted by Λ as well. A quasitoric manifold M^{2n} can be recovered, up to diffeomorphism, using the characteristic pair $(Pⁿ, \Lambda)$. In other words, it can be constructed a model for M^{2n} by:

$$
M \cong (T^n \times P^n) / \approx, \ (t_1, p_1) \approx (t_2, p_2) \text{ if and only if } p_1 = p_2, \ t_1 t_2^{-1} \in \Lambda(F(p_1)),
$$
\n(1)

where $F(p_1)$ is the smallest face of the polytope P^n that contains p_1 .

In this case, for any point $p \in P^n$, the cortége $\sigma(p) = \{P_{\sigma} : p \in P_{\sigma}\}$ consists of one polytope, that is the face $F(p)$.

2.2 Complex Grassmann manifolds $G_{k+1,q}$

The complex Grassmann manifold $G_{k+1,q}$ consists of all q-dimensional complex subspaces in the complex vector space \mathbb{C}^{k+1} . The canonical action of the torus T^{k+1} on \mathbb{C}^{k+1} , considered in the canonical basis, induces the action of the torus T^{k+1} on the manifold $G_{k+1,q}$. This action is not effective, as the diagonal subgroup $\Delta =$ $\{(t, \ldots, t), t \in S^1\}$ acts trivially on $G_{k+1,q}$. The torus $T^k = T^{k+1}/H$ acts effectively on $G_{k+1,q}$.

We recall some classical constructions on the complex Grassmann manifolds. After fixing a basis in an q-dimensional subspace $L \subset \mathbb{C}^{k+1}$, this subspace can be represented by the $q \times (k + 1)$ matrix $A(L)$ such that rank $A(L) = q$. For any subset $J \subset \{1, \ldots, k+1\}$ consisting of q elements, $|J| = q$, denote by $A_J(L)$ the matrix of dimension $q \times q$ given by the columns of the matrix A_L that are indexed by J. We will assume that the set of all subsets $J = \{j_1 < j_2 \ldots < j_q\} \subset \{1, \ldots, k+1\}$ is ordered lexicographically. Using this ordering define the vector

$$
P(A(L)) = (PJ(A(L))) = (\det A_J(L)),
$$

whose coordinates are called the Plücker coordinates of a point $L \in G_{k+1,q}$.

The Plücker coordinates depend on a fixed basis for L and they are, up to constant, uniquely defined. More precisely, two bases f_1, \ldots, f_q and e_1, \ldots, e_q for a subspace L are related by $f_j = \sum_{i=1}^q \alpha_j^i e_i, 1 \le j \le q$. It implies that the Plucker coordinates for L in these two bases are related by

$$
P_f^J(L) = \left(\sum_{\sigma \in S_k} \text{sign}(\sigma) \alpha_{\sigma(1)}^1 \cdots \alpha_{\sigma(k)}^k\right) P_e^J(L) = \det(\alpha),\tag{2}
$$

where $\alpha = (\alpha_j^i)$ is a transition matrix between these two bases. In this way the Plücker coordinates produce an embedding of the Grassmann manifold $G_{k+1,q}$ into $\mathbb{C}P^{N-1}$, where $N = \binom{k+1}{q}$.

The Plücker coordinates define the smooth atlas $\{(M_I, u_I)\}\$ on $G_{k+1,q}$, where I runs through all q-element subsets of the set $\{1, \ldots, k+1\}$, as follows. Here $M_I = \{L \in$ $G_{k+1,q}: P^{I}(L) \neq 0$ and the coordinate map $u_I: M_I \to \mathbb{C}^{q(k+1-q)}$ is defined by the Plücker coordinates $P^{J}(L)$, $J = (I \setminus \{i_p\}) \cup \{j_s\}, i_p \in I, j_s \in \{1, ..., k+1\} \setminus I$, in such a basis of a subspace L that the $(q \times q)$ -dimensional sub-matrix of the matrix $A(L)$ whose columns are indexed by I is an identity matrix.

Let us consider the action of T^{k+1} on $\mathbb{C}P^{N-1}$, which is given by the composition of the q-th exterior power representation $T^{k+1} \to T^N$ and the standard action of T^N on $\mathbb{C}P^{N-1}$. The standard moment map $\mathbb{C}P^{N-1} \to \mathbb{R}^{k+1}$ for such an action of the torus T^{k+1} induces the moment map μ : $G_{k+1,q} \to \mathbb{R}^{k+1}$ (see [\[21\]](#page-48-9)), which is defined by

$$
\mu(L) = \frac{\sum_{J} |P^{J}(L)|^2 \delta_{J}}{\sum_{J} |P^{J}(L)|^2},
$$
\n(3)

where $\delta_J \in \mathbb{R}^{k+1}$ are the vectors whose coordinates are given by

$$
(\delta_J)_i = 1, \quad i \in J, \quad (\delta_J)_i = 0, \quad i \notin J,
$$

and J runs through the q-element subsets of $\{1, \ldots, k+1\}$.

The map μ is T^{k+1} -invariant and the image of μ is, by its definition, a convex hull over the points δ_J . The convex polytope obtained in this way is known as the hypersimplex $\Delta_{k+1,q}$, see [\[23\]](#page-48-10). In particular, $\Delta_{4,2}$ is the octahedron.

Recall that the Grassmann manifold $G_{k+1,q}$ admits as well an action of the algebraic torus $(\mathbb{C}^*)^{k+1}$, which is induced by the coordinate wise action of $(\mathbb{C}^*)^{k+1}$ on \mathbb{C}^{k+1} . The orbit $(\mathbb{C}^*)^{k+1} \cdot L$ is a smooth submanifold in $G_{k+1,q}$ for any point $L \in G_{k+1,q}$. The stationary subgroup $(\mathbb{C}^*)_L^{k+1}$ of a point L is a toral subgroup in $(\mathbb{C}^*)^{k+1}$ and the algebraic torus $(\mathbb{C}^*)^L = (\mathbb{C}^*)^{k+1} / (\mathbb{C}^*)_L^{k+1}$ acts freely on the orbit $(\mathbb{C}^*)^{k+1} \cdot L$.

Moreover, $\mu((\mathbb{C}^*)^{k+1} \cdot L) = P_L$, where P_L is a convex polytope spanned by the vertices δ_J of the hypersimplex $\Delta_{n,k}$ indexed by those J such that $P^{J}(L) \neq 0$.

3 Definition of $(2n, k)$ -manifolds

We assume the following to be given:

- a smooth, closed, oriented manifold M^{2n} ;
- a smooth, effective action θ of the torus T^k on M^{2n} , where $1 \leq k \leq n$, such that the stabilizer of any point is a connected subgroup of T^k ;
- a smooth, θ -equivariant map $\mu : M^{2n} \to \mathbb{R}^k$, whose image is a k-dimensional convex polytope P^k , where \mathbb{R}^k is considered with the trivial T^k - action. We assume $\mu : M^{2n} \to \mathbb{R}^k$ to be an open map.

The map μ we call an *almost moment map* for the given T^k -action on M^{2n} .

We say that the triple (M^{2n}, θ, μ) is an $(2n, k)$ -manifold if it satisfies the six axioms which we formulate below.

3.1 A smooth manifold structure.

Axiom 1. *There exists a smooth atlas* $\mathfrak{M} = \{(M_i, \varphi_i)\}_{i \in I}$ *, where* M_i *are an open subsets in* M^{2n} *and* $\varphi_i : M_i \to \mathbb{R}^{2n}$ *are coordinate homeomorphisms. Any chart* M_i *is* T^k -invariant, contains exactly one fixed point x_i with $\varphi_i(x_i) = (0, \ldots, 0)$, such that $x_i \neq x_j$ for $i \neq j$, The closure of any chart M_i is the whole manifold M^{2n} .

Any atlas that satisfies Axiom [1](#page-6-2) has finitely many charts, since M^{2n} is a compact manifold. It implies:

Corollary 3.1. *The action of* T^k *on* M^{2n} *has finitely many isolated fixed points.*

By m denote the number of fixed points for T^k -action on M^{2n} . We enumerate as $(M_1, \varphi_1), \ldots, (M_m, \varphi_m)$ the charts given by Axiom [1](#page-6-2) The sets $Y_i = M^{2n} \setminus M_i$ are closed in M^{2n} and T^k -invariant, by their definition.

There is a standard concept of the boundary for a subset Y of a topological space X in general topology. There is also a concept of the boundary of a manifold, manifold with corners and, in that context the boundary of a convex polytope, used in algebraic topology and differential geometry. In all these cases ∂ is a standard notation for the boundary. It is not difficult to realize that all these concepts are *not always* appropriate for some purposes in the theory of $(2n, k)$ -manifolds. Therefore, we introduce the new notion of the boundary and the corresponding symbol. The boundary $\bar{\partial}$ of a subset Y in a topological space X is the set $\overline{\partial}Y = \overline{Y} \setminus Y$, where \overline{Y} is the closure of a set Y in a space X. Note that if Y is an open set in X, then $\overline{\partial}Y = \partial Y$, where $\partial Y = \overline{Y} \cap \overline{X \setminus Y}$ is the standard boundary as defined in general topology. In the sequel, set $\bar{\partial}Y \subset X$ is called the $\bar{\partial}$ −boundary of a set Y as well.

Therefore, the fact that the above defined sets M_i are dense, open sets in M^{2n} , implies that $Y_i = \partial M_i = \overline{\partial} M_i$.

For any $\sigma = \{i_1, \ldots, i_l\} \subseteq [1, m]$ let us consider the set :

$$
W_{\sigma} = M_{i_1} \cap \cdots \cap M_{i_l} \cap Y_{i_{l+1}} \cap \cdots \cap Y_{i_m},
$$

where $\{i_{l+1}, \ldots, i_m\} = [1, m] \setminus \{i_1, \ldots, i_l\}.$

Definition 3.2. *The non-empty set* W_{σ} *is said to be a stratum. The index set* $\sigma \subset [1, m]$ *of a stratum* $W_{\sigma7}$ *is said to be an admissible set.*

Lemma 3.3. *The strata* W_{σ} are T^k -invariant, pairwise disjoint and their union is the *whole manifold* M^{2n} *.*

Proof. Any stratum W_{σ} is T^k -invariant since the sets M_i and Y_i are T^k - invariant. Further, if $W_{\sigma_1} \neq W_{\sigma_2}$ then $\sigma_1 \neq \sigma_2$ and, thus, we can assume that there exists $i \in \sigma_1$ such that $i \notin \sigma_2$, $1 \leq i \leq m$. Therefore, for $x \in W_{\sigma_1}$ it follows that $x \in M_i$ and, thus, $x \notin Y_i$, which gives that $x \notin W_{\sigma_2}$. Similarly, if $x \in W_{\sigma_2}$ then $x \in Y_i$ and thus $x \notin M_i$, which implies $x \notin W_{\sigma_1}$. The union of all strata is the manifold M^{2n} , since the charts cover M^{2n} . Note that it is uniquely defined an admissible set $\sigma = \sigma(x) = \{i \in [1, m] | x \in M_i\}$ for any point $x \in M^{2n}$. \Box

This Lemma together with the fact that the T^k -action on M^{2n} is continuous implies that the closure $\overline{W_{\sigma}}$ is a T^k -invariant set for any stratum W_{σ} .

Example 3.4. *The set* $W_{[1,m]} = M_1 \cap \cdots \cap M_m$ *is non-empty and, thus, it is a stratum* and the set $\sigma = [1,m]$ is an admissible set. The set $W_{[1,m]}$ is an open dense set in M^{2n} $since, by Axiom 1$ $since, by Axiom 1$ any chart M_i , $1 \leq i \leq m$ is an open, dense set in M^{2n} . The stratum $W_{[1,m]}$ is called the main stratum and it is further denoted by $W = W_{[1,m]}$.

Example 3.5. *The set* $W_{\{i\}} = M_i \cap \left(\bigcap_{j \neq i} Y_j\right)$ *is a stratum and the set* $\sigma = \{i\}$ *is an admissible set for any* $1 \leq i \leq m$. It follows from the observation that the set W_i *is non-empty since, by Axiom [1](#page-6-2) the fixed point* x_i *which belongs to* M_i *also belongs to all* Y_i , $j \neq i$, $1 \leq j \leq m$.

Remark 3.6*.* Since $M^{2n} = M_1 \cup \cdots \cup M_m$, we see that $W_{\emptyset} = Y_1 \cap \cdots \cap Y_m = \emptyset$, which implies that W_{\emptyset} is not a stratum and \emptyset is not an admissible set.

Remark 3.7. A stratum W_{σ} different from the main stratum W is not an open set in M^{2n} . It follows from the observations that in this case there exists a chart M_i such that $M_i \cap W_{\sigma} = \emptyset$ and that M_i is a dense set in M^{2n} .

Lemma 3.8. *The boundary* $\overline{\partial}W_{\sigma}$ *of a stratum* W_{σ} *is contained in the union of the strata* $W_{\tilde{\sigma}}$ *, where* $\tilde{\sigma}$ *are the admissible sets such that* $\tilde{\sigma} \subset \sigma$ *.*

Proof. Let $W_{\sigma} = M_{i_1} \cap \cdots \cap M_{i_l} \cap Y_{i_{l+1}} \cap \ldots \cap Y_{i_m}$. Then the boundary ∂W_{σ} is contained in the union of the sets $\begin{pmatrix} p \\ \bigcap \end{pmatrix}$ $\bigcap_{q=1}^{r} \bar{\partial} M_{i_{j_q}}$ \setminus ∩ $\sqrt{ }$ \bigcap $s \neq j_1,...,j_p$ M_{i_s} \setminus ∩ $\begin{pmatrix} m \\ n \end{pmatrix}$ $\bigcap_{j=l+1} Y_{i_j}$! = $\left(\begin{array}{c}p\\p\end{array}\right)$ $\bigcap_{q=1} Y_{i_{j_q}}$! ∩ $\sqrt{ }$ \cap $s \neq j_1,...,j_p$ M_{i_s} ! ∩ $\begin{pmatrix} m \\ n \end{pmatrix}$ $\bigcap_{j=l+1} Y_{i_j}$ \setminus , where $j_1 < \ldots < j_p$ and $1 \le p \le l$. Such nonempty sets give $W_{\tilde{\sigma}}$, where $\tilde{\sigma} = \sigma \setminus \{i_{j_1}, \dots i_{j_p}\}\$. Hence, $\bar{\partial}W_{\sigma} \subseteq \cup W_{\tilde{\sigma}}$, where $\tilde{\sigma}$ goes through all proper admissible subsets of σ .

4 Almost moment map, strata and admissible polytopes

Axiom 2. The map μ is a bijection between the set of fixed points and the set of vertices *of the polytope* P^k .

Since an k-dimensional polytope has at least $k + 1$ vertices, it follows:

Corollary 4.1. The number of fixed points for T^k -action on M^{2n} is not less then $k+1$.

Let $S(P^k)$ be a family of all convex polytopes that are spanned by the vertices of the polytope P^k . By $\mathfrak S$ denote the set of all admissible sets. Using the almost moment map μ , we define the map $s : \mathfrak{S} \to S(P^k)$ as follows. Put $v_i = \mu(x_i)$, $1 \le i \le m$ and let $\sigma = \{i_1, \ldots, i_l\} \subseteq [1, m]$. By Axiom [1,](#page-6-2) for any $i_j \in \sigma$ there exists a unique fixed point $x_{i_j} \in M_{i_j}$. We put

 $s(\sigma) = P_{\sigma}$, where $P_{\sigma} = \text{convhull}(v_{i_1}, \dots, v_{i_l}).$

Definition 4.2. The polytope $P_{\sigma} \in S(P^k)$ is said to be an admissible polytope if it is *in the image of the map* $s : \mathfrak{S} \to S(P^k)$ *.*

Remark 4.3. Since $W_{\emptyset} = \emptyset$, it follows that \emptyset is not an admissible polytope.

Example 4.4. *The polytope* P^k *is an admissible polytope. The set* $\sigma = \{1, \ldots, m\}$ *is an admissible set as it is shown in Example* [3.4.](#page-7-0) *In addition, by Axiom* [2,](#page-8-1) $s(\sigma)$ *is a convex hull of all vertices of* P^k *, which implies that* $s(\sigma) = P^k$ *.*

Example 4.5. Any vertex v_i of P^k is an admissible polytope. To see that, by Axiom [2,](#page-8-1) take the fixed point x_i such that $\mu(x_i) = v_i$. Then the set $\sigma = \{i\}$ is an admissible set *as it is shown in Example* [3.5](#page-7-1) *and* $s(\sigma) = v_i$.

Definition 4.6. *The set of all admissible polytopes is said to be pure if any admissible polytope of the dimension* $\leq k - 1$ *is a face of some admissible polytope of the dimension* k*.*

Example 4.7. *The set of admissible polytopes of a quasitoric manifold is a pure set, Moreover it follows from [\[6\]](#page-47-2) and [\[7\]](#page-47-3) (Proposition 9, Propositions 11-15 and Corollary* 18) that the set of admissible polytopes for the Grassmann manifolds $G_{4,2}$ and $G_{5,2}$ *are pure sets as well.*

Remark 4.8. For a general $(2n, k)$ -manifold, two admissible polytopes P_{σ_1} and P_{σ_2} may have nonempty intersection $\overset{\circ}{P}_{\sigma_1} \cap \overset{\circ}{P}_{\sigma_2}$. For example, one can verify this in the case of complex Grassmann manifold $G_{4,2}$ which is an $(8, 3)$ -manifold, see [\[6\]](#page-47-2).

Definition 4.9. The point $p \in P^k$ is said to be an exceptional point if $p \in \stackrel{\circ}{P}_{\sigma_1} \cap \stackrel{\circ}{P}_{\sigma_2}$ for some different admissible polytopes $P_{\sigma_1}, P_{\sigma_2}$. Otherwise, it is said to be simple.

In this way, the set of exceptional points $S \subseteq P^k$ is defined.

5 Stabilizers for the torus action on the strata

By $S(T^k)$ denote, as above, the set of all connected subgroups of the torus T^k . Note that a connected subgroup of the torus T^k is a torus. Let us consider a function χ : $M^{2n} \to S(T^k)$ which to any point x assigns its stabilizer $\chi(x)$ regarded to the given T^k -action on M^{2n} . It follows from the set-up assumptions that $\chi(x)$ is a connected subgroup of the torus T^k . We assume the following to be satisfied:

Axiom 3. *The characteristic function* χ *is constant on any stratum* W_{σ} *.*

Using Axiom [3,](#page-9-2) the torus $T^{\sigma} = T^{k}/\chi(W_{\sigma})$ can be defined for any stratum W_{σ} .

Corollary 5.1. *The torus* T^{σ} *acts freely on* W_{σ} *, which gives the principal bundle*

$$
T^{\sigma} \to W_{\sigma} \to W_{\sigma}/T^{\sigma}.
$$
 (4)

It is shown in [\[7\]](#page-47-3) (Remark 3) that, in the case of Grassmann manifolds, the notion of the strata as defined in [\[14\]](#page-48-1), coincides with our notion of the strata. In addition, one verifies [\[7\]](#page-47-3) (Proposition 4) that the characteristic function is constant on any stratum of the Grassmann manifolds.

6 Orbit spaces of the strata

By its definition the almost moment map $\mu : M^{2n} \to P^k$ is T^k - invariant. Therefore, it induces the map $\hat{\mu} : M^{2n}/T^k \to P^k$.

Axiom 4. *The almost moment map* µ*:*

- *a*) maps a stratum W_{σ} onto $\stackrel{\circ}{P_{\sigma}}$,
- *b*) induces the fiber bundle $\widehat{\mu}_{\sigma} : W_{\sigma}/T^{\sigma} \to \stackrel{\circ}{P_{\sigma}},$
- *c*) dim $P_{\sigma} = \dim T^{\sigma}$.

An immediate consequence of this Axiom is:

Corollary 6.1. *It holds*

- $\widehat{\mu}(W/T^k) = P^k$ for the main stratum $W = M_1 \cap \cdots \cap M_m$,
- $\mu(W_{\{i\}}) = \{v_i\}$, where v_i is a vertex and $W_{\{i\}} = M_i \cap \left(\bigcap_{j \neq i}$ Y_j .

Remark 6.2. Note that Axiom [4](#page-9-3) does not require a fiber bundle $\hat{\mu}_{\sigma} : W_{\sigma}/T^{\sigma} \to \stackrel{\circ}{P_{\sigma}}^{\rho}$ to be smooth, since there is no argument to claim that, in general, a stratum W_{σ} is a smooth submanifold in M^{2n} . The main stratum being open is of course a smooth submanifold, but for the other strata it does not have to be the case. Even for the Grassmann manifolds, the differential geometry of the strata can be very complicated, see [\[19\]](#page-48-11).

By $[F_{\sigma}]$ denote the homeomorphic type of a fiber for the fiber bundle $\hat{\mu}_{\sigma}: W_{\sigma}/T^{\sigma} \to \stackrel{\circ}{P_{\sigma}}$.

Definition 6.3. *The space* F_{σ} *of a homeomorphic type* $[F_{\sigma}]$ *is called the space of parameters of a stratum* W_{σ} .

Since \hat{P}_{σ} is contractible, for the fiber bundle $W_{\sigma}/T^{\sigma} \to \hat{P}_{\sigma}$ we conclude that the following holds:

Corollary 6.4. *The fiber bundle* $\widehat{\mu}_{\sigma}$: $W_{\sigma}/T^{\sigma} \rightarrow P_{\sigma}$ is isomorphic to the trivial bundle. *That is* W_{σ}/T^{σ} *is homeomorphic to* $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$ *by the fiber wise homeomorphism*

Definition 6.5. *For any* $\sigma \in \mathfrak{S}$ *, we fix the space* F_{σ} *and the trivialization* h_{σ} : $W_{\sigma}/T^{\sigma} \rightarrow P_{\sigma}^{\circ} \times F_{\sigma}$ as structural data of $(2n, k)$ -manifolds.

Let $\overline{W_{\sigma}/T^{\sigma}}$ denote the closure of W_{σ}/T^{σ} . It is a compact subset in M^{2n}/T^k since we assume M^{2n} to be a compact manifold. We obtain:

Corollary 6.6. $\widehat{\mu}(W_{\sigma}/T^{\sigma}) = P_{\sigma}$.

Proof. It holds $P_{\sigma \subset_{\sigma}} \widehat{\mu}(W_{\sigma}/T^{\sigma}) \subseteq P_{\sigma}$ since $\widehat{\mu} : M^{2n}/T^k \to P^k$ is continuous and $\hat{\mu}(W_{\sigma}/T^{\sigma}) = \stackrel{\circ}{P_{\sigma}}$. Furthermore, since $\hat{\mu}(W_{\sigma}/T^k)$ is a compact set, it follows $\widehat{\mu}(W_{\sigma}/T^{\sigma}) = P_{\sigma}.$

The trivialization $h_{\sigma}: W_{\sigma}/T^{\sigma} \to \stackrel{\circ}{P}_{\sigma} \times F_{\sigma}$ induces the projection $\xi_{\sigma}: W_{\sigma}/T^{\sigma} \to F_{\sigma}$. For any point $c_{\sigma} \in F_{\sigma}$, define a subspace $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ of W_{σ} by

$$
W_{\sigma}[\xi_{\sigma}, c_{\sigma}] = (\pi_{\sigma}^{-1} \circ \xi_{\sigma}^{-1})(c_{\sigma}), \tag{5}
$$

where $\pi_{\sigma}: W_{\sigma} \to W_{\sigma}/T^{\sigma}$ is a projection.

Definition 6.7. *The space* $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ *is said to be the leaf of a stratum* W_{σ} *given by the trivialization* h_{σ} *.*

Note that, by its definition, a leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ is invariant under the action of the torus T^k and $W_{\sigma} = \bigcup_{c_{\sigma} \in F_{\sigma}} W_{\sigma}[\xi_{\sigma}, c_{\sigma}].$

The definition of a leaf also implies:

Lemma 6.8. Let $\hat{\mu}_{\xi_{\sigma},c_{\sigma}}$ denote the restriction of the map $\hat{\mu}$: $M^{2n}/T^k \to P^k$ to $W_{\sigma}[\xi_{\sigma}, c_{\sigma}] / T^{\sigma}$. Then the map $\widehat{\mu}_{\xi_{\sigma}, c_{\sigma}} : W_{\sigma}[\xi_{\sigma}, c_{\sigma}] / T^{\sigma} \rightarrow \stackrel{\circ}{P_{\sigma}}$ is a homeomorphism for *any* $c_{\sigma} \in F_{\sigma}$ *.*

Moreover, we obtain:

Lemma 6.9. $\widehat{\mu}(\overline{W_{\sigma}[\xi_{\sigma}, c_{\sigma}]/T^{\sigma}}) = P_{\sigma}$.

Proof. It follows from Lemma [6.8](#page-11-0) that $\widehat{\mu}(\overline{W_{\sigma}[\xi_{\sigma}, c_{\sigma}]/T^{\sigma}}) \subseteq P_{\sigma}$, since $\widehat{\mu}$ is a continuous map. On the other hand, $\overline{W_{\sigma}[\xi_{\sigma}, c_{\sigma}]} / \overline{T^{\sigma}}$ is a closed subset in the compact space M^{2n}/T^k and, thus, it is a compact set as well. It implies that $\hat{\mu}(\overline{W_{\sigma}}[\xi_{\sigma}, c_{\sigma}]/T^{\sigma})$ is a compact set in P_{σ} which contains the interior of P_{σ} , which further implies that $\widehat{\mu}(W_{\sigma}[\xi_{\sigma}, c_{\sigma}]/T^{\sigma}) = P_{\sigma}.$ □

Suppose it is given an 2n-dimensional manifold M^{2n} with an effective action of the algebraic torus $(\mathbb{C}^*)^k$. Assume that the induced action of the compact torus $T^k \subset$ $({\mathbb C}^*)^k$ on M^{2n} satisfies Axioms 1-4 and the following folds:

- (1) Any stratum W_{σ} is $(\mathbb{C}^*)^k$ invariant;
- (2) The free action of the torus T^{σ} on W_{σ} extends to a free action of the algebraic torus $(\mathbb{C}^*)^{\sigma}$ on W_{σ} ;
- (3) The projections $\hat{\mu}: W_{\sigma}/T^{\sigma} \to \stackrel{\circ}{P}_{\sigma}$ and $\hat{\pi}: W_{\sigma}/T^{\sigma} \to W_{\sigma}/(\mathbb{C}^{*})^{\sigma}$ define the homeomorphism $h_{\sigma} = (\hat{\mu}, \hat{\pi}) : W_{\sigma}/T^{\sigma} \to \stackrel{\circ}{P}_{\sigma} \times W_{\sigma}/(\mathbb{C}^{*})^{\sigma}$.

Then, the space of parameters F_{σ} of a stratum W_{σ} can be identified with $F_{\sigma} \cong$ $W_{\sigma}/(\mathbb{C}^*)^{\sigma}.$

Definition 6.10. We say than an action of the compact torus T^k on an $(2n, k)$ -manifold M^{2n} extends to the compatible action of the algebraic torus $(\mathbb{C}^*)^k$ if it is defined an action of the algebraic torus $(\mathbb{C}^*)^k$ on M^{2n} , which satisfies the conditions 1-3 given *above.*

We use the description of the strata as spaces consisting of leafs to formulate the properties which allow to describe the gluing of the strata. Recall that we have fixed the projection $\xi_{\sigma}: W_{\sigma}/T^{\sigma} \to F_{\sigma}$ as a part of our structural data.

Axiom 5. *For any leaf* $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ *of* W_{σ} *it holds:*

- *a*) it is a smooth submanifold in M^{2n} and the induced map $\mu_{\xi_\sigma,c_\sigma} : W_\sigma[\xi_\sigma,c_\sigma] \to \stackrel{\circ}{P}_\sigma$ *is a smooth fiber bundle,*
- *b)* its boundary $\partial W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ *is the union of leafs* $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$ *for exactly one* $c_{\bar{\sigma}} \in$ $F_{\overline{\sigma}}$ *, where* $P_{\overline{\sigma}}$ *runs through some admissible faces for* P_{σ} *and* $\sigma \in \mathfrak{S}$ *,*
- *c)* the map $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}, \eta_{\sigma,\bar{\sigma}}(c_{\sigma}) = c_{\bar{\sigma}}$ given by b) is a continuous map.

Remark 6.11*.* Axiom [5,](#page-12-0) as it will be seen in Section [2.2,](#page-5-0) is motivated by the results of Atiyah, Guillemin-Sternberg and Gel'fand-MacPherson about $(\mathbb{C}^*)^n$ -action on the complex Grassmann manifolds $G_{n,k}$.

We deduce the following important consequence of the statement *b*) of Axiom [5.](#page-12-0)

Proposition 6.12. *A face of any admissible polytope is an admissible polytope.*

Proof. Let us fix some face $P_{\tilde{\sigma}}$ of an admissible polytopes P_{σ} and let us consider a point $p \in \stackrel{\circ}{P_{\sigma}}$. By Lemma [6.9,](#page-11-1) we see that $\hat{\mu}^{-1}(p) \cap \bar{\partial}W_{\sigma}[\xi_{\sigma}, c_{\sigma}]/T^k \neq \emptyset$. Therefore, there exists a point x from $\bar{\partial}$ -boundary of the leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ such that $\pi(x) \in \hat{\mu}^{-1}(p)$.
Le invalies that $y(x) = x$ Dy Ariam 5, we have that y belanguite some last W_{σ} $[\xi_{\sigma}, \xi_{\sigma}]$ It implies that $\mu(x) = p$. By Axiom [5,](#page-12-0) we have that x belongs to some leaf $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$, where $P_{\bar{\sigma}}$ is a face of P_{σ} . This implies that $P_{\bar{\sigma}}$ is an admissible polytope and that $\mu(x) = p \in \stackrel{\circ}{P_{\bar{\sigma}}}$. Therefore, $p \in \stackrel{\circ}{P_{\bar{\sigma}}} \cap \stackrel{\circ}{P_{\bar{\sigma}}}$, and, since $P_{\bar{\sigma}}$ and $P_{\tilde{\sigma}}$ are faces of the same polytope P_{σ} , it follows that $P_{\tilde{\sigma}} = P_{\bar{\sigma}}$. Therefore, $P_{\tilde{\sigma}}$ is an admissible polytope. □

Remark 6.13. In the case of the canonical action of the algebraic torus $(\mathbb{C}^*)^{k+1}$ on $G_{k+1,q}$, this result is obtained in [\[1,](#page-47-5) [14\]](#page-48-1).

Remark 6.14*.* Note that combining Axiom [5](#page-12-0) and the proof of Proposition [6.12](#page-12-1) we obtain that if $P_{\bar{\sigma}}$ is a face of P_{σ} then there exists a leaf $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\tilde{\sigma}}]$ which is contained in the $\bar{\partial}$ -boundary of the leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$.

It follows that the condition *b)* of Axiom [5](#page-12-0) can be strengthen:

Corollary 6.15. *For any* $c_{\sigma} \in F_{\sigma}$, the boundary $\bar{\partial}W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ of a leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ of *a stratum* W_{σ} *is the union of leafs* $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$ *for exactly one* $c_{\bar{\sigma}} \in F_{\bar{\sigma}}$ *, where* $P_{\bar{\sigma}}$ *runs through* all *faces for* P_{σ} *.*

The statement *a)* of Axiom [5](#page-12-0) combining with Corollary [6.15](#page-12-2) directly implies that Lemma [6.9](#page-11-1) can be strengthen:

Corollary 6.16. *The map* $\widehat{\mu}: \overline{W_{\sigma}[\xi_{\sigma}, c_{\sigma}]/T^{\sigma}} \to P_{\sigma}$ *is a homeomorphism.*

Let $P_{\bar{\sigma}}$ be a face of P_{σ} . Then Corollary [6.15](#page-12-2) implies an existence of the map

$$
\eta_{\sigma,\bar{\sigma}}: F_{\sigma} \to F_{\bar{\sigma}} \tag{6}
$$

defined by: $\eta_{\sigma,\bar{\sigma}}(c_{\sigma})$ is a unique point $c_{\bar{\sigma}}$ from $F_{\bar{\sigma}}$ such that $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}] \subset \bar{\partial}W_{\sigma}[\xi_{\sigma}, c_{\sigma}].$ The statement *c*) of Axiom [5](#page-12-0) states that the map $\eta_{\sigma,\bar{\sigma}}$ is a continuous map.

Let now $P_{\bar{\sigma}}$ be a face of $P_{\bar{\sigma}}$ and $P_{\bar{\sigma}}$ be a face of P_{σ} . Then Axiom [5](#page-12-0) states an existence of the projections $\xi_{\bar{\sigma}}$: $W_{\bar{\sigma}}/T^k \to F_{\bar{\sigma}}$ and ξ_{σ} : $W_{\sigma}/T^k \to F_{\sigma}$ such that

$$
W_{\bar{\sigma}}[\xi_{\bar{\sigma}},c_{\bar{\sigma}}] \subset \bar{\partial}W_{\bar{\sigma}}[\xi_{\bar{\sigma}},c_{\bar{\sigma}}] \subset \overline{W_{\bar{\sigma}}[\xi_{\bar{\sigma}},c_{\bar{\sigma}}]}, \ \ W_{\bar{\sigma}}[\xi_{\bar{\sigma}},c_{\bar{\sigma}}] \subset \bar{\partial}W_{\sigma}[\xi_{\sigma},c_{\sigma}] \subset \overline{W_{\sigma}[\xi_{\sigma},c_{\sigma}]}
$$

Since the leafs are disjoint, it follows that

$$
W_{\bar{\sigma}}[\xi_{\bar{\sigma}},c_{\bar{\sigma}}] \subset \overline{W_{\sigma}[\xi_{\sigma},c_{\sigma}]}\setminus W_{\sigma}[\xi_{\sigma},c_{\sigma}] = \overline{\partial}W_{\sigma}[\xi_{\sigma},c_{\sigma}].
$$

Altogether this implies:

Corollary 6.17. For any pair $P_{\bar{\sigma}} \subset P_{\sigma}$ there exists the map $\eta_{\sigma,\sigma'} : F_{\sigma} \to F_{\bar{\sigma}}$ such *that if* $P_{\bar{\sigma}} \subset P_{\bar{\sigma}} \subset P_{\sigma}$ *then* $\eta_{\bar{\sigma}, \bar{\bar{\sigma}}} \circ \eta_{\sigma, \bar{\sigma}} = \eta_{\sigma, \bar{\bar{\sigma}}}$ *.*

7 On singular and regular values of the almost moment map

We characterize the singular and regular values of the almost moment map $\mu : M^{2n} \to$ P^k .

Definition 7.1. *The cortége* $\sigma(x)$ *of a point* $x \in P^k$ *is a set of admissible polytopes defined by:*

$$
\sigma(x) = \{P_{\sigma} \in P_{\mathfrak{S}} : x \in \stackrel{\circ}{P}_{\sigma}\}.
$$

Obviously, $\sigma(x) \neq \emptyset$ for any point $x \in P^k$, since $\cup_{\sigma \in \mathfrak{S}} \stackrel{\circ}{P_{\sigma}} = P^k$. Moreover, $P^k \in$ $\sigma(x)$ for any point $x \in \stackrel{\circ}{P^k}$.

Definition 7.2. *The point* $x \in P^k$ *is said to be a regular point if* $\dim P_\sigma = k$ *for all* $P_{\sigma} \in \sigma(x)$.

Note that if $x \in P^k$ is a regular point then $x \in P^k$. By P^k_r we denote the set of regular points in P^k .

Remark 7.3. Note that the set of regular points in \hat{P}^k is a non-empty set and moreover it is a dense set in P^k . This follows from the fact that there are finitely many admissible polytopes, so the union of admissible polytopes of the dimension less then k has the dimension less then k.

For a given $(2n, k)$ -manifold M^{2n} , let Z denote the union of all open admissible polytopes \hat{P}_{σ} whose dimension is $\lt k$. The following immediately holds:

Lemma 7.4. The set $\hat{P^k} \setminus (Z \cap \hat{P^k})$ coincides with P_r^k . In particular P_r^k is an open *set in* P ^k *which has finitely many connected components.*

Example 7.5. For the Grassmann manifold $G_{4,2}$, the set $P_r^3 \subset \overset{\circ}{\Delta}_{4,2}$ is the complement to the union of three open diagonal squares $\stackrel{\circ}{P}_{12,34}$, $\stackrel{\circ}{P}_{13,24}$, $\stackrel{\circ}{P}_{14,23}$ in $\stackrel{\circ}{\Delta}_{4,2}$ (see [\[6\]](#page-47-2)).

For the Grassmann manifold $G_{5,2}$, the set $P_r^4 \subset \stackrel{\circ}{\Delta}_{5,2}$ is the complement to the union of of ten open prisms $\stackrel{\circ}{P_i}$, $1 \leq i \leq 10$ in $\stackrel{\circ}{\Delta}_{5,2}$ (see [\[7\]](#page-47-3),Proposition 9).

Recall that x is a regular value of the almost moment map $\mu : M^{2n} \to P^k$ if and only if any $y \in \mu^{-1}(x)$ is a regular point, that is the differential of μ at y has rank equal to k.

Theorem 7.6. *If* $x \in P_r^k$ *then* x *is a regular value for the almost moment map* μ : $M^{2n} \to P^k$.

Proof. Let $x \in P^k$ be a regular point. It holds that $\mu^{-1}(x) \subset \cup_{\sigma} W_{\sigma}$, where the union goes over all admissible sets σ such that $P_{\sigma} \in \sigma(x)$. Now, if $y \in \mu^{-1}(x)$ W_{σ} , then y belongs to a unique leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$. The differential of the map $\mu_{\xi_{\sigma}, c_{\sigma}}$: $W_{\sigma}[\xi_{\sigma}, c_{\sigma}] \to \stackrel{\circ}{P}_{\sigma}$ is an epimorphism, according to Axiom [5.](#page-12-0) Since $P_{\sigma} \in \sigma(x)$, it follows that the rank of the differential of $\mu_{\xi_{\sigma},c_{\sigma}}$ at y is equal to $\dim P_{\sigma}=k$, which proves the statement. П

Note that the above proof implies that any point of the main stratum W is a regular point of the almost moment map.

Corollary 7.7. For any point $x \in \overrightarrow{P}^k$ the preimage $M_x^{2n-k} = \mu^{-1}(x)$ is a closed *submanifold in* M²ⁿ *of the dimension* 2n−k*. The torus* T ^k *acts freely on the manifold* M_{x}^{2n-k} and the orbit space M_{x}^{2n-k}/T^{k} is a smooth manifold of the dimension $2n-2k$, *which can be identified with some compactification of the space of parameters* F *of the main stratum.*

From Remark [7.3,](#page-13-1) it follows the well-known fact:

Corollary 7.8. The set of regular values of the almost moment map $\mu : M^{2n} \to P^k$ is *a dense set in* P k *.*

Remark 7.9*.* From our axioms, in general case, does not follow that the set of regular values of the almost moment map μ coincides with the set P_r^k . Nevertheless, in the case of Grassmann manifolds $G_{k+1,q}$, it is proved in [\[7\]](#page-47-3) that these two sets coincide.

We can push up this further.

Lemma 7.10. *The set* $M_r^{2n} = \mu^{-1}(P_r^k)$ *is a dense submanifold in* M^{2n} *with a free* action of the torus T^k . The orbit space M_r^{2n}/T^k is a smooth manifold which is dense in M^{2n}/T^k *. Moreover, the maps* $\mu : M_r^{2n} \to P_r^k$ and $\hat{\mu} : M_r^{2n} \to P_r^k$ are projections *of the smooth fiber bundles.*

Proof. The first statement follows from the observations that P_r^k is an open set in P^k and that, by Axiom [4,](#page-9-3) T^k acts freely on M_r^{2n} .

The second statement follows from the observation that the almost moment map μ : $M_r^{2n} \to P_r^k$ is, by Theorem [7.6,](#page-14-0) a submersion and moreover, being open, μ is a proper map. Since μ is T^k -invariant for the free T^k -action on M_r^{2n} , it follows that the same holds for the induced map $\hat{\mu}: M_r^{2n}/T^k \to P_r^k$. Then, classical Ehressmann's fibration theorem implies that the maps μ and $\hat{\mu}$ produce the stated fiber bundles.

Let $P_{r,1}^k, \ldots, P_{r,s}^k$ be the connected components of the open set P_r^k and $M_{r,i}^{2n} = \mu^{-1}(P_{r,i}^k)$, $1 \leq i \leq s$. Then, the maps $\mu : M_{r,i}^{2n} \to P_{r,i}^k$ define smooth fiber bundles with connected base. In this way, we obtain:

Corollary 7.11. *The manifolds* M_x^{2n-k} *and* M_y^{2n-k} *are diffeomorphic for any* $x, y \in$ $P^k_{r,i}, 1 \leq i \leq s.$

7.1 The main stratum and its orbit space

As noted in Example [3.4](#page-7-0) the main stratum W is a dense set in M^{2n} , which implies that W/T^k is a dense set in the orbit space M^{2n}/T^k . Moreover, $\hat{\mu}(W/T^k) = \stackrel{\circ}{P^k}$ and $W/T^k \cong P^k \times F$, where F denotes the space of parameters of the main stratum W, as introduced by Definition [6.3.](#page-10-0) It follows that there exists such a compactification of the

product $\stackrel{\circ}{P^{k}} \times F$ that is homeomorphic to M^{2n}/T^{k} .

The first result in this direction is the following.

Proposition 7.12. If the space F is a point then the orbit space M^{2n}/T^k is homeo*morphic to the polytope* P^k .

Proof. If F is a point, then W consists of one leaf. Thus, Corollary [6.16](#page-13-2) implies that the closure $\overline{W/T^k}$ is homeomorphic to the polytope P^k , which further implies that M^{2n}/T^k is homeomorphic to the polytope P^k . \Box

Example 7.13. *If* M^{2n} *is a quasitoric manifold, then F is a point.*

Assume now that the space of parameters F of the main stratum is not a point. Let $x \in \stackrel{\circ}{P^k}$ and put $\mathfrak{S}(x) = \{ \sigma \in \mathfrak{S} | x \in \stackrel{\circ}{P_{\sigma}} \}$, that is $\sigma \in \mathfrak{S}(x)$ if and only if $P_{\sigma} \in \sigma(x)$. Then $\hat{\mu}^{-1}(x)$ is a closed set in M^{2n}/T^k and

$$
\widehat{\mu}^{-1}(x) = \bigcup_{\sigma \in \mathfrak{S}(x)} \{ y \in W_{\sigma}/T^k : \widehat{\mu}(y) = x \}.
$$

Since the subspace $\{y \in W_{\sigma}/T^k : \hat{\mu}(y) = x\}$ is homeomorphic to F_{σ} , we introduce a translation is a subspace of the subspace σ topology on the union $\bigcup_{\sigma \in \mathfrak{S}(x)} F_{\sigma}$ such that the space obtained as this union becomes homeomorphic to $\hat{\mu}^{-1}(x)$. Let $\overline{F_x}$ denote the closure of the space $F_x = \hat{\mu}^{-1}(x) \cap$ $W/T^k \cong F$. Since $\overline{W/T^k} = M^{2n}/T^k$, it follows that

$$
\overline{F_x} = \widehat{\mu}^{-1}(x) \cong \cup_{\sigma \in \mathfrak{S}(x)} F_{\sigma},\tag{7}
$$

for any $x \in P^{\kappa}$. From Corollary [7.7](#page-14-1) and Corollary [7.11](#page-15-1) it follows:

Corollary 7.14. *If* $x \in P_r^k$ *then* $\overline{F_x} = M_x^{2n-k}/T^k$ *is a smooth manifold in* M^{2n}/T^k *. Moreover, the manifolds* $\overline{F_x}$ *and* $\overline{F_y}$ *are homeomorphic for any* $x, y \in P_{r,i}^k$, $1 \leq i \leq s$.

Remark 7.15. We want to note that although a space F_x is homeomorphic to the space of parameters F for any $x \in P^k$, there is no argument to claim that, in general, their compactifications $\overline{F_x}$ are homeomorphic. But, Corollary [7.14](#page-16-0) implies that the set of manifolds $\overline{F_x}$, $x \in P_r^k$, is finite, up to diffeomorphism.

We first point the following:

Lemma 7.16. *The space of parameters* F *of the main stratum is a compact space if* and only if all points from $\stackrel{\circ}{P^{k}}$ are simple.

Proof. If all points from $\overrightarrow{P^k}$ are simple, then the condition $P_{\sigma} \cap \overrightarrow{P^k} \neq \emptyset$ implies that $P_{\sigma} = P^k$. This means that $\hat{\mu}^{-1}(x) \cong F$ for any point $x \in \stackrel{\circ}{P^k}$. Therefore, F is a compact space. In opposite direction, if F is a compact space, from the fact that $F \cong F_x \subseteq \hat{\mu}^{-1}(x)$ for any point $x \in \stackrel{\circ}{P^k}$, it follows that $F \cong \hat{\mu}^{-1}(x)$, which implies that all points from \overrightarrow{P}^k are simple. \Box

In the case when the spaces $\overline{F_x}$, $x \in \overline{P}^k$ are all homeomorphic and all points from the boundary of the polytope P^k are simple, we provide an explicit topological description of the orbit space M^{2n}/T^k .

Let P_{σ_0} be a face of the polytope P^k and by $\mathfrak{S}(\sigma_0)$ denote the set of those admissible sets $\sigma \in \mathfrak{S}$ for which P_{σ_0} is a face of the polytope P_{σ} and P_{σ} is not a face of the polytope P^k .

Proposition 7.17. Assume that all points from ∂P^k are simple, that a space $\overline{F_x}$ is h omeomorphic to the space \bar{F} *for all* $x \in \stackrel{\circ}{P^{k}}$. Then for any $\sigma_0 \in \mathfrak{S}$, such that P_{σ_0} is a *face of the polytope* P^k , *there exists a point* $x \in \overset{\circ}{P^k}$ such that $\hat{\mu}^{-1}(x) \cong \cup_{\sigma \in \mathfrak{S}(\sigma_0)} F_{\sigma}$ *and, thus*

$$
\cup_{\sigma\in\mathfrak{S}(\sigma_0)}F_{\sigma}\cong\overline{F}.
$$

Proof. We first note that the assumption that all points from ∂P^k are simple implies that if P_{σ_0} is not a face of some admissible polytope P_{σ} then $\partial P_{\sigma} \cap P_{\sigma_0} = \emptyset$. Otherwise

we would have a point $z \in \partial P_{\sigma} \cap P_{\sigma_0}$ meaning that z belongs to some face $P_{\bar{\sigma}}$ of P_{σ} , which is an admissible polytope. Since z is a simple point, it follows that $P_{\bar{\sigma}} = P_{\sigma_0}$. Therefore, since there are only finitely many admissible polytopes, we see that there exists a neighborhood U of P_{σ_0} in P^k such that $U \cap P_{\sigma} = \emptyset$ for any admissible polytope P_{σ} for which P_{σ_0} is not a face of P_{σ} . Further $U \cap (\bigcap_{\sigma \in \mathfrak{S}(\sigma_0)} P_{\sigma}) \neq \emptyset$ as P_{σ_0} belongs to all polytopes from $\mathfrak{S}(\sigma_0)$. For a point $x \in U \cap (\bigcap_{\sigma \in \mathfrak{S}(\sigma_0)} P_{\sigma})$, we obtain

$$
\overline{F} \cong \widehat{\mu}^{-1}(x) \cong \cup_{\sigma \in \mathfrak{S}(\sigma_0)} F_{\sigma}, \text{ since } x \in \stackrel{\circ}{P^k}.
$$

By $\stackrel{\circ}{P_{\mathfrak{S}}}$ denote the set of interiors of all admissible polytopes that are not proper faces of the polytope P^k .

Theorem 7.18. Assume that all points from ∂P^k are simple and that a space $\widehat{\mu}^{-1}(x) \subset$ M^{2n}/T^k is homeomorphic to the space \bar{F} for all $x \in \stackrel{\circ}{P^k}$. Assume also that the set $\stackrel{\circ}{P_{\mathfrak{S}}}$ can be divided into subsets $\stackrel{\circ}{P_{\mathfrak{S}_1}}, \ldots, \stackrel{\circ}{P_{\mathfrak{S}_l}}$ such that the areas from $\stackrel{\circ}{P_{\mathfrak{S}_i}}$ give the *polytopal decomposition for the area* $\sum_{i=1}^{\infty} P^k$ *and* $F_{\sigma} \cong F_{\mathfrak{S}_i}$ *for any* $\sigma \in \mathfrak{S}_i$ *, where* $1 \leq i \leq l$. Then the orbit space M^{2n}/T^k is homeomorphic to the quotient space $P^k\times \bar{F}/\approx$, where

$$
(x, f_1) \approx (y, f_2) \Leftrightarrow x = y \in P_{\sigma_0} \subset \partial P^k \text{ and } \eta_{\sigma_1, \sigma_0}(f_1) = \eta_{\sigma_2, \sigma_0}(f_2).
$$

Here $\sigma_1, \sigma_2 \in \mathfrak{S}(\sigma_0)$ *are such that* $f_1 \in F_{\sigma_1}$, $f_2 \in F_{\sigma_2}$ *and the map* $\eta_{\sigma_i, \sigma_0}$ *is defined by the formula* [\(6\)](#page-13-3)*.*

Proof. Put $W_{\mathfrak{S}_i}/T^k = \hat{\mu}^{-1}(P^k) \cap (\cup_{\sigma \in \mathfrak{S}_i} W_{\sigma}/T^k)$. Since $\hat{\mu}_{\sigma} : W_{\sigma}/T^k \to \stackrel{\circ}{P}_{\sigma}$ is a fiber bundle and, by the assumption, all F_{σ} for $\sigma \in \mathfrak{S}_i$ are homeomorphic to the space $F_{\mathfrak{S}_i}$, it follows that $\hat{\mu}: W_{\mathfrak{S}_i} / T^k \to \overset{\circ}{P^k}$ is a fiber bundle with a fiber $F_{\mathfrak{S}_i}$. Since $\overset{\circ}{P^k}$ is contractible, it follows that $W_{\mathfrak{S}_i}/T^k \cong \mathbb{P}^k \times F_{\mathfrak{S}_i}$. Since $\overline{F} \cong \hat{\mu}^{-1}(x)$, for $x \in \mathbb{P}^k$, it follows that $\hat{\mu}^{-1}(\hat{P}^k) = \bigcup_{i=1}^l W_{\mathfrak{S}_i} \cong \hat{P}^k \times (\bigcup_{i=1}^l F_{\mathfrak{S}_i}) \cong \hat{P}^k \times \bar{F}$, where the last homeomorphism holds according to [\(7\)](#page-16-1).

Let P_{σ_0} be a face of the polytope P^k . It is an admissible polytope. Since any point from ∂P^k is simple, it follows that $W_{\sigma_0}/T^k = \hat{\mu}^{-1}(\stackrel{\circ}{P}_{\sigma_0})$ and, thus, the projection $\hat{\mu}^{-1}$: $\hat{\mu}^{-1}(\hat{P}_{\sigma_0}) \to \hat{P}_{\sigma_0}$ is a fiber bundle with the fiber F_{σ_0} . It implies that the space $\hat{\mu}^{-1}(\hat{P}_{\sigma_0})$) is homeomorphic to the direct product $\stackrel{\circ}{P}_{\sigma_0} \times F_{\sigma_0}$. We also have that $F_{\sigma_0} \subseteq \overline{F}$ since $\hat{\mu}^{-1}(\hat{P^k})$ is an everywhere dense set in M^{2n}/T^k and \overline{F} is a closed set. Combining this and Proposition [7.17,](#page-16-2) we obtain $F_{\sigma_0} \subseteq \bigcup_{\sigma \in \mathfrak{S}(\sigma_0)} F_{\sigma}$. Therefore, the family of maps $\eta_{\sigma,\sigma_0}: F_{\sigma} \to F_{\sigma_0}$ produces the map $\eta: F \to F_{\sigma_0}$ that is surjective. It implies that $\hat{P}_{\sigma_0} \times F_{\sigma_0}$ is homeomorphic to a quotient of the space $\hat{P}_{\sigma_0} \times \overline{F}$ by the equivalence relation $(x, c_1) \approx (x, c_2)$ if and only if $\eta(c_1) = \eta(c_2)$, that is $\eta_{\sigma_1, \sigma_0}(c_1) = \eta_{\sigma_2, \sigma_0}(c_2)$, where $c_1 \in F_{\sigma_1}, c_2 \in F_{\sigma_2}$ and $\sigma_2, \sigma_2 \in \mathfrak{S}$. As the faces of P^k are the only admissible polytopes from ∂P^k , this proves the statement. \Box

Theorem 7.19. *Assume that*

- *1)* all points from the boundary ∂P^k are simple;
- 2) a space $\widehat{\mu}^{-1}(x) \subset M^{2n}/T^k$ is homeomorphic to the space \overline{F} for any $x \in \stackrel{\circ}{P^k};$
- 3) the set $\overrightarrow{P}_{\mathfrak{S}}$ can be divided into subsets $\overrightarrow{P}_{\mathfrak{S}_1}, \ldots, \overrightarrow{P}_{\mathfrak{S}_l}$ such that the areas from $\stackrel{\circ}{P_{\mathfrak{S}_i}}$, $1 \leq i \leq l$ give the polytopal decomposition for the area $\stackrel{\circ}{P^k};$
- 4) $F_{\sigma} \cong F_{\mathfrak{S}_i}$ for any $\sigma \in \mathfrak{S}_i$, where $1 \leq i \leq l$.

If F_{σ_0} is a point for any $\sigma_0 \in \mathfrak{S}$ such that P_{σ_0} is a face of the polytope P^k then:

$$
M^{2n}/T^k \cong S^{k-1} * \bar{F}.
$$

Proof. Recall that the joint $X * Y$ for topological spaces X and Y is defined to be a quotient space of the product $CX \times Y$ by the equivalence relation $(x_1, 0, y_1) \approx$ $(x_2, 0, y_2) \Leftrightarrow x_1 = x_2$, where CX is a cone over X. In our case Theorem [7.18](#page-17-0) gives that

$$
M^{2n}/T^k\cong P^k\times \overline{F}/\approx, \ \ (x,f_1)\approx (y,f_2)\Leftrightarrow x=y\in \partial P^k.
$$

We identify the polytope P^k with the closed disc $\overline{D^k}$ and further identify $\overline{D^k}$ with the cone CS^{k-1} over its boundary S^{k-1} . It implies that

$$
M^{2n}/T^k \cong CS^{k-1} \times \overline{F}/\approx, \ (x,0,f_1) \approx (x,0,f_2),
$$

which proves the statement.

8 Complex of admissible polytopes

Let $P_{\mathfrak{S}}$ denote the family of all admissible polytopes. Combining Example [4.4,](#page-8-2) Example [4.5](#page-8-3) and Proposition [6.12](#page-12-1) we obtain:

- $P^k \in P_{\mathfrak{S}}$ and $v \in P_{\mathfrak{S}}$ for any vertex v .
- If $P_{\sigma} \in P_{\mathfrak{S}}$ and $P_{\bar{\sigma}}$ is a face of the polytope P_{σ} , then $P_{\bar{\sigma}} \in P_{\mathfrak{S}}$.

Let us consider the complex $C(M^{2n}, P^k)$, which is obtained a formal disjoint union of the interiors of all admissible polytopes:

$$
C(M^{2n}, P^k) = \bigcup_{P_{\sigma} \in P_{\mathfrak{S}}} \stackrel{\circ}{P_{\sigma}}.
$$
 (8)

By the defintion [\(8\)](#page-18-1), there is a bijection between the set of cells in $C(M^{2n}, P^k)$ and the set of admissible polytopes $P_{\mathfrak{S}}$.

Lemma 8.1. *There is the canonical map* $\hat{\pi}$: $C(M^{2n}, P^k) \rightarrow P^k$.

 \Box

Proof. For any polytope P'_{σ} C_{σ} in $C(M^{2n}, P^k)$ there is a unique admissible polytope $P_{\sigma} \in P_{\mathfrak{S}}$ such that $\overset{\circ}{P}_{\sigma}$ corresponds to the polytope P_{σ}' p'_{σ} and we define $\widehat{\pi}(P'_{\sigma})$ p'_{σ}) = $\stackrel{\circ}{P}_{\sigma}$.

Corollary 8.2. For any $P_{\sigma} \in P_{\mathfrak{S}}$ there exists a unique polytope P_{σ} ['] σ' in $C(M^{2n}, P^k)$ such that the map $\hat{\pi}$: $P'_{\sigma} \rightarrow P_{\sigma}$ is a homeomorphism.

Corollary 8.3. *The canonical map* $\hat{\pi}$: $C(M^{2n}, P^k) \rightarrow P^k$ *is a quotient map.*

Proof. Define an equivalence relation \approx on $C(M^{2n}, P^k)$ by $x \approx y$ if and only if $\hat{\pi}(x) = \hat{\pi}(y)$. It is obvious that $P^k = C(M^{2n}, P^k) / \approx$ and that $\hat{\pi} : C(M^{2n}, P^k) \to$
 P^k is a surprise the sumpliment P^k is a quotient map. \Box

Corollary 8.4. *The canonical map* $\hat{\pi}$: $C(M^{2n}, P^k) \to P^k$ *is a bijection if and only if the only admissible polytopes are the whole* P ^k *and its faces.*

Example 8.5. *For a quasitoric manifold* M^{2n} *the canonical map* $\hat{\pi}$: $C(M^{2n}, P^n) \rightarrow P^n$ P ⁿ *is a bijection.*

For the almost moment map $\mu : M^{2n} \to P^k$ we prove the following result:

Lemma 8.6. *There is the canonical map* $f : M^{2n} \to C(M^{2n}, P^k)$ *such that* $\mu = \hat{\pi} \circ f$ *.*

Proof. For any point $x \in M^{2n}$ there exists a unique stratum W_{σ} such that $x \in W_{\sigma}$. Then $\mu(x) \in P_{\sigma}^{\circ}$ and by Corollary [8.2](#page-19-0) there exists a unique polytope $P_{\sigma}^{'} \subseteq C(M^{2n}, P^k)$ such that $\hat{\pi}$: $P'_{\sigma} \rightarrow P_{\sigma}$ is a homeomorphism. It implies that there exists a unique $y \in P'_\sigma$ \mathcal{F}' such that $\hat{\pi}(y) = \mu(x)$. In this way, the map $f : M^{2n} \to C(M^{2n}, P^k)$ is defined by $f(x) = y$. \Box

For the induced map $\hat{\mu} : M^{2n}/T^k \to P^k$ we obtain:

Corollary 8.7. *There is the canonical map* \hat{f} : $M^{2n}/T^k \rightarrow C(M^{2n}, P^k)$ *such that* $\widehat{\mu} = \widehat{\pi} \circ \widehat{f}$.

We further assume that $P_{\mathfrak{S}}$ is partially ordered by the inclusion of admissible polytopes. It is defined the boundary operator d which to each admissible polytope P_{σ} assigns the disjoint union of of all its faces. The operator d induces the operator d_C on the cells of the complex $C(M^{2n}, P^k)$ by

$$
d_C(P_\sigma^{'})=(dP_\sigma)_C,
$$

where $(dP_{\sigma})_C \subset C(M^{2n}, P^k)$ corresponds to dP_{σ} by the bijection between $P_{\mathfrak{S}}$ and $C(M^{2n}, P^k)$.

It is important to emphasize the following. One can try to ask if it is possible to define similarly the boundary operator d on the set of all strata $W_{\mathfrak{S}}$ by $d(W_{\sigma}) = \overline{W_{\sigma}} \setminus W_{\sigma}$. The answer is negative in general. Namely, as it will be shown in Section [18,](#page-45-0) the Grassmann manifold $G_{7,3}$ is an example of a manifold that belongs to our class, but

which contains a stratum W_{σ} whose boundary is not the union of the strata. Therefore, the operator d is not defined for $G_{7,3}$. This example is taken from [\[14\]](#page-48-1).

We still want to note that there are important examples of $(2n, k)$ -manifolds for which the operator \tilde{d} is well defined. It is proved in the paper [\[14\]](#page-48-1) that the operator \tilde{d} is well defined for the manifolds $G_{k+1,2}$ and $G_{6,3}$. In the paper [\[6\]](#page-47-2), the cases of Grassmann manifold $G_{4,2}$ and complex projective space $\mathbb{C}P^5$ with the canonical action of T^4 are studied in detail. In these cases the operator \tilde{d} is defined on the complex of the strata $W_{\mathfrak{S}}$. Moreover, the canonical map $f : M^{2n} \to C(M^{2n}, P^k)$ gives a mapping from the complex of strata $W_{\mathfrak{S}}$ to the complex $C(M^{2n}, P^k)$. More precisely:

Lemma 8.8. *If the operator* $\tilde{d}(W_{\sigma}) = \overline{W_{\sigma}} - W_{\sigma}$ *is defined on the set of strata* $W_{\mathfrak{S}}$ *then the map* $f : M^{2n} \to C(M^{2n}, P^k)$ *induces the map* $f_{\mathfrak{S}} : W_{\mathfrak{S}} \to C(M^{2n}, P^k)$ *which is a bijective map between these complexes and commutes with the boundary operators.*

Proof. Let W_{σ} be a stratum. Then $\mu(W_{\sigma}) = \rho_{\sigma}^2$ and Lemma [8.6](#page-19-1) implies that $\hat{\pi}(f(W_{\sigma})) = \rho_{\sigma}^2$. By Corollary [6.4](#page-10-1) we conclude that $f(W_{\sigma}) = P_{\sigma}'$ σ , which is a cell of the complex $C(M^{2n}, P^k)$. Therefore, the map $f_{\mathfrak{S}} : W_{\mathfrak{S}} \to C(M^{2n}, P^k)$ is defined by $f_{\mathfrak{S}}(W_{\sigma}) =$ $f(W_{\sigma})$. The map $f_{\mathfrak{S}}$ is a bijection because of the above stated bijection between the set of strata $W_{\mathfrak{S}}$ and the set of admissible polytopes $P_{\mathfrak{S}}$. Moreover, it follows from Corollary [6.15](#page-12-2) that $d \circ f_{\mathfrak{S}} = f_{\mathfrak{S}} \circ \tilde{d}$ which means that $f_{\mathfrak{S}}$ commutes with the boundary operators. □

8.1 CW-topology on $C(M^{2n}, P^k)$

The complex of all admissible polytopes $C(M^{2n}, P^k)$ can be naturally endowed with a topology such that $C(M^{2n}, P^k)$ becomes CW-complex, which we denote by $CW(M^{2n}, P^k)$.

- The cells of these complex are the open polytopes $\overrightarrow{P}_{\sigma}$ for $P_{\sigma} \in P_{\mathfrak{S}}$.
- The characteristic function on the boundary of the cells, which defines their attaching, is defined by the operator d_C . The skeletons are defined inductively by the dimension of the cells. The definition of the operator d_C verifies that the cell axiom of CW-complex is satisfied.
- According to the axioms of CW-complex, it is defined on $CW(M^{2n}, P^k)$ the weak topology compatible with the cell decomposition.

Then the following is satisfied:

Lemma 8.9. *The canonical map* $\hat{\pi}$: $CW(M^{2n}, P^k) \rightarrow P^k$

- *is a continuous map;*
- \bullet is a cell map for the standard cell decomposition of the polytope P^k if and only *if* $\hat{\pi}$ *is a homeomorphism.*

Proof. To prove that the map $\hat{\pi}$ is continuous, it is enough to notice that if $U \subseteq P^k$ is a algoed set in P^k then $U \cap P$ is a algoed set in P for any political P continuous probation closed set in P^k then $U \cap P$ is a closed set in P, for any polytope P over some subsets of vertices of the polytope P^k . This will be true as well for the admissible polytopes, which implies that $\hat{\pi}^{-1}(U)$ is a closed set in $CW(M^{2n}, P^k)$. The second statement follows directly from Corollary [8.4.](#page-19-2) П

Example 8.10. *For a quasitoric manifold the canonical map* $\hat{\pi}$: $CW(M^{2n}, P^n) \rightarrow$
 P^n is a homogeneously we P ⁿ *is a homeomorphism.*

Proposition 8.11. *If the set of admissible polytopes* $P_{\mathfrak{S}}$ *contains a k-dimensional poly*tope different from P^k , the canonical map $f : M^{2n} \to CW(M^{2n}, P^k)$ is not a con*tinuous map.*

Proof. Let $P_{\sigma} \in P_{\mathfrak{S}}$, $\dim P_{\sigma} = k$ and $P_{\sigma} \neq P^k$. Then P'_{σ} $p'_{\sigma} \cong \stackrel{\circ}{P_{\sigma}}$ is an open set in $CW(M^{2n}, P^k)$. Let us consider the corresponding stratum W_{σ} , that is $\mu(W_{\sigma}) = \stackrel{\circ}{P_{\sigma}}$. Since $\mu = \hat{\pi} \circ f$, it follows that $f^{-1}(P'_\sigma)$ \mathcal{O}'_{σ} = W_{σ} , which is, by Remark [3.7,](#page-8-4) not an open set in M^{2n} . \Box

Remark 8.12*.* Note that the same assumption as in Proposition [8.11](#page-21-1) leads that the canonical map $\hat{f}: M^{2n}/T^k \to CW(M^{2n}, P^k)$ is not continuous as well.

Example 8.13. *We demonstrate Proposition [8.11](#page-21-1) in the case of Grassmann manifold* G4,² *endowed with the canonical action of* T 4 *. Following [\[6\]](#page-47-2), any four sided* p yramid P in $\Delta_{4,2}$ is an admissible polytope, which implies that $P^{'} \cong P^{'} P$ is an open *set in* $CW(G_{4,2}, \Delta_{4,2})$ *. On the other hand* $f^{-1}(P')$ *is an* $(\mathbb{C}^*)^3$ -orbit in the eight*dimensional manifold* G4,2*, so it can not be an open set.*

8.2 Quotient topology on $C(M^{2n}, P^k)$

As Proposition [8.11](#page-21-1) points, the CW-topology on $C(M^{2n}, P^k)$ is not compatible with the topology of M^{2n} . Therefore, the CW-topology is not quite appropriate for the description of a topology of the orbit space M^{2n}/T^k .

We define another topology on the complex $C(M^{2n}, P^k)$ such that the space $CQ(M^{2n}, P^k)$ = $C(M^{2n}, P^k)$ becomes a quotient space of M^{2n} by the canonical map $f : M^{2n} \to$ $C(M^{2n}, P^k)$. More precisely, we consider a subset $U \subseteq C(M^{2n}, P^k)$ to be open if and only if the subset $f^{-1}(U) \subseteq M^{2n}$ is open. This is equivalent to say that $CQ(M^{2n}, P^k)$ is a quotient space of M^{2n}/T^k by the canonical map $\hat{f}: M^{2n}/T^k \to$ $C(M^{2n}, P^k)$.

Lemma 8.14. *The maps* $\hat{\pi}: CQ(M^{2n}, P^k) \to P^k$, $f: M^{2n} \to CQ(M^{2n}, P^k)$ and $\widehat{f}: M^{2n}/T^k \to CQ(M^{2n}, P^k)$ are continuous, canonical maps.

• Axiom [5](#page-12-0) implies that a face of any polytope $P' \in CQ(M^{2n}, P^k)$ is contained in the boundary of P' regarded to the quotient topology.

• Note also that $P^{k'} = P^k$ is a dense set in $CQ(M^{2n}, P^k)$ since $f^{-1}(P^{k'}) = W$, the main stratum which is a dense set in M^{2n} .

Lemma 8.15. *If the set of admissible polytopes consists of* P ^k *and its faces, that is if the canonical map map* $\hat{\pi}$: $C(M^{2n}, P^k) \rightarrow P^k$ *is a bijection, then the space* COM^{2n}, P^k *is a Unumber space* $CQ(M^{2n}, P^k)$ is a Hausdorff topological space.

Lemma 8.16. *If the set of admissible polytopes* $P_{\mathfrak{S}}$ *contains a polytope* P_{σ} *such that* $\stackrel{\circ}{P}_{\sigma}\subset \stackrel{\circ}{P^{k}}$, then the space $CQ(M^{2n}, P^{k})$ is not a Hausdorff topological space.

 $(\mathbf{p}'_{\sigma})\subset \stackrel{\circ}{P^{k}}$ *Proof.* Let P_{σ} be an admissible polytope as stated in the formulation. Then $\hat{\pi}(P_{\sigma}')$ and for any point $x \in P_\sigma^{\prime} \subset CQ(M^{2n}, P^k)$ there exists a point $y \in P^{k'} \subset CQ(M^{2n}, P^k)$ such that $\hat{\pi}(x) = \hat{\pi}(y)$. Let further V be an open set in $CQ(M^{2n}, P^k)$ containing the point x. Then $\hat{f}^{-1}(V)$ is an open set in M^{2n}/T^k and it contains all points from W_{σ}/T^k which map to $\hat{\pi}(x)$ by the map $\hat{\mu}$. On the other hand, by [\(7\)](#page-16-1) there exists a point $m \in \hat{\mu}^{-1}(\hat{\pi}(x)) \cap W/T^k$ such that $m \in \hat{f}^{-1}(V)$. Note that $\hat{f}(\hat{\mu}^{-1}(\hat{\pi}(x)) \cap W/T^k) = y$ which implies that $y \in V$. Thus every neighborhood of the point x in $CQ(M^{2n}, P^k)$ contains a point $y \in P^{k'}$. \Box

Corollary 8.17. If the set of admissible polytopes $P_{\mathfrak{S}}$ is a pure set and contains a polytope different from P^k and its faces then the space $CQ(M^{2n}, P^k)$ is not a Hausdorff *topological space.*

Let us discuss the relation between the CW-topology and the CQ-topology on $C(M^{2n}, P^k)$.

Lemma 8.18. *If the canonical map* $\hat{\pi}$: $C(M^{2n}, P^k) \rightarrow P^k$ *is a bijection then the CW-topology and the CQ-topology on* $C(M^{2n}, P^k)$ *coincide.*

Example 8.19. *For a quasitoric manifold* M^{2n} *these two topologies on* $C(M^{2n}, P^n)$ = P ⁿ *coincide.*

As a direct consequence of Proposition [8.11,](#page-21-1) we also deduce the following:

Lemma 8.20. If the set of admissible polytopes $P_{\mathfrak{S}}$ contains a k-dimensional polytope different from P^k , then there exists a set which is open in $CW(M^{2n}, P^k)$, but which is *not open in* $CQ(M^{2n}, P^k)$ *.*

Example [8.13](#page-21-2) demonstrates the situation described in this Lemma. The interior of any four-sided pyramid is an open set in $CW(G_{4,2}, \Delta_{4,2})$, but it is not an open set in $CQ(G_{4,2}, \Delta_{4,2}).$

Therefore, in general, the sets in $C(M^{2n}, P^k)$ which are open in the CW-topology are not necessarily open in the CQ-topology.

The inverse inclusion does not hold as well, in general the sets in $C(M^{2n}, P^k)$ which are open in the quotient topology are not necessarily open in the CW-topology. We demonstrate this in the case of Grassmann manifold $G_{4,2}$.

8.2.1 An example of a closed set in $CQ(G_{4,2}, \Delta_{4,2})$ that is not closed in $CW(G_{4,2}, \Delta_{4,2})$

We follow the notation and the methods from [\[6\]](#page-47-2). Let us consider the set $\mathcal C$ in $G_{4,2}$ given by the matrices

$$
\mathcal{C} = \left(\begin{array}{rrr} 1 & 0 & c & 1 \\ 0 & 1 & 1 & 1 \end{array} \right), \ c \neq 0, 1.
$$

This set belongs to the main stratum since all its points have all non-zero Plücker coordinates. We obtain the closure of C by attaching the limit points when $c \to 0, 1, \infty$.

• When $c \to 0$, we obtain the point

$$
C_0 = \left(\begin{array}{rrr} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right).
$$

• When $c \to 1$, we obtain the point

$$
C_1 = \left(\begin{array}{rrr} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right).
$$

• When $c \to \infty$, in order to see which point is obtaining, we can proceed as follows. Since the set C belongs to all charts, it follows that it can be written down in the local coordinates of the chart M_{23} . The points in the charts M_{12} and M_{23} are uniquely expressed by the matrices:

$$
\left(\begin{array}{ccc} 1 & 0 & z_1 & z_2 \\ 0 & 1 & z_3 & z_4 \end{array}\right) \ \ \text{and} \ \ \left(\begin{array}{ccc} w_1 & 1 & 0 & w_2 \\ w_3 & 0 & 1 & w_4 \end{array}\right).
$$

Therefore, the transition map, on the intersection of these charts, from the coordinates (z_1, z_2, z_3, z_4) in the chart M_{12} to the coordinates (w_1, w_2, w_3, w_4) in the chart M_{23} is given by the formulas

$$
w_1 = -\frac{z_3}{z_1}, w_2 = z_4 - \frac{z_2 z_3}{z_1}, w_3 = \frac{1}{z_1}, w_4 = \frac{z_2}{z_1}.
$$

This implies that the set C writes in the chart M_{23} as

$$
\mathcal{C} = \begin{pmatrix} -\frac{1}{c} & 1 & 0 & 1 - \frac{1}{c} \\ \frac{1}{c} & 0 & 1 & \frac{1}{c} \end{pmatrix}, \ c \neq 0, 1,
$$

and, when $c \to \infty$, we obtain the point

$$
C_{\infty} = \left(\begin{array}{rrr} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right).
$$

Let us consider the closed set $\overline{C} = C \cup \{C_0, C_1, C_\infty\}$. Its image by the canonical map $f: G_{4,2} \to C(G_{4,2}, \Delta_{4,2})$ will be as follows:

- $f(C) \subset \triangle_{4,2}$, since C belongs to the main stratum.
- $f(C_0) \in \stackrel{\circ}{P_{23}}$, where P_{23} is a four sided pyramid that does not contain the vertex δ_{23} .
- $f(C_1) \in \stackrel{\circ}{P_{34}}$, where P_{34} is a four sided pyramid that does not contain the vertex $\delta_{34}.$
- $f(C_{\infty}) \in (\delta_{23}, \delta_{34})$, where $(\delta_{23}, \delta_{34})$ is an edge with the vertices δ_{23} and δ_{34} .

Therefore, the set $f(\overline{C})$ is not a closed set in $CW(G_{4,2}, \Delta_{4,2})$ since $f(C) \cap \Delta_{4,2} =$ $\mu(C) \cup \mu({C_{\infty}})$ which it is not a closed set in $\Delta_{4,2}$, the points $\mu(C_0), \mu(C_1)$ from the closure of $\mu(C)$ are missing. These points belong to the open cells \hat{P}_{23} and \hat{P}_{34} , respectively.

On the other hand, $f(\overline{C})$ is obviously a closed set in the quotient topology $CQ(G_{4,2}, \Delta_{4,2})$.

Therefore, the CW topology and the CQ topology on $C(G_{4,2}, \Delta_{4,2})$ are essentially different: for each of these two topologies there is a set which is open in one topology, but which is not open in the other.

Note that, in a general case when the operator \tilde{d} is defined on the set of all strata $W_{\mathfrak{S}}$, it can be also defined a weak topology on $W_{\mathfrak{S}}$ such that the boundary of any stratum is given by the operator \tilde{d} . In this case $A \subseteq W_{\mathfrak{S}}$ is closed if and only if its intersection with the closure of any stratum is closed. It is obvious that this topology coincide with the topology induced from M^{2n} , since the topology on strata is induced from the topology of a manifold M^{2n} .

8.3 Induced partial ordering on $CQ(M^{2n}, P^k)$

There is a canonical way [\[8\]](#page-47-6) to introduce a preorder on $CQ(M^{2n}, P^k)$ using the quotient topology. More precisely for $x, y \in CQ(M^{2n}, P^k)$ one defines

$$
x \leq y
$$
 if and only if $x \in \overline{y}$,

where \overline{y} denotes the closure of y in the quotient topology. Note that on a general Hausdorff topological space this preorder becomes trivial, which means that $x \leq y$ implies $x = y$. Since $\hat{\pi}$: $CQ(M^{2n}, P^k) \to P^k$ is a continuous map, it follows:

$$
x \leq y
$$
 implies $\widehat{\pi}(x) = \widehat{\pi}(y)$.

Obviously if the set of all admissible polytopes consists of P^k and its faces this preorder will be trivial because in this case, by Lemma [8.15,](#page-22-0) $CQ(M^{2n}, P^k)$ is a Hausdorff space. If this is not the case from the definition of the preorder, it directly follows:

Lemma 8.21. *If* $x \in P^{k'}$ then there is no $y \in CQ(M^{2n}, P^k)$, $y \neq x$ such that $x \leq y$. *On the other hand, if* $P_{\sigma} \neq P^k$ *is an admissible polytope that is not a face of* P^k *then for any* $x \in P'_\sigma$ g'_{σ} there exists a unique $y \in P^{k'}$ such that $x \leq y$.

Recall that, in a partially ordered set (X, \leq) , an upper set is defined to be a subset U of X such that if $x \in U$ and $x \leq y$ then $y \in U$ and, accordingly, it is defined a lower set. It is a standard fact that, regarded to the specialization preorder on a topological space X , every open set in X is an upper set and every closed set in X is a lower set. Recall also that a topological space X is said to be an P. S. Alexandrov space if the intersection of any family of open sets is an open set. Alexandrov topologies on X are in one-to-one correspondence with preorders on X meaning that X is an Alexandrov space if and only if every upper set regarded to the specialization preorder is an open set. In particular, it implies that a Hausdorff topological space is an Alexandrov space if and only it is a discrete space.

As for the space $CQ(M^{2n}, P^k)$, for simplicity, we consider the case when the set of all admissible polytopes $P_{\mathfrak{S}}$ is a pure set. If $P_{\mathfrak{S}}$ consists only of P^k and its faces, it follows from Lemma [8.15](#page-22-0) that $CQ(M^{2n}, P^k)$ is a Hausdorff topological space, which is obviously not discrete, so it is not an Alexandrov space. We prove that the same holds in general:

Lemma 8.22. *Assume that the set of admissible polytopes is a pure set and contains a polytope different from* P^k *and its faces. Then the space* $CQ(M^{2n}, P^k)$ *is not a Hausdorff topological space and it is not an Alexandrov space as well.*

Proof. Since $P_{\mathfrak{S}}$ contains a polytope P_{σ} different from P^k and its faces, it follows that there exists a face $P_{\bar{\sigma}}$ of P_{σ} such that $\stackrel{\circ}{P_{\bar{\sigma}}} \subset \stackrel{\circ}{P_{\kappa}}^k$. Let $V \subset CQ(M^{2n}, P^k)$ be the smallest, under inclusion, upper set that contains $P^{k'}$ and $P'_{\bar{e}}$ $p'_{\bar{\sigma}}$. Then P'_{σ} $p'_{\sigma} \not\subset V$ since there is no $x \in P'_{\bar{\sigma}}$ $y'_{\overline{\sigma}}$ such that $x \leq y$ for some $y \in P'_{\sigma}$ $\hat{\pi}_{\sigma}$ as $\hat{\pi}(x) \neq \hat{\pi}(y)$. The set V is not open in $CQ(M^{2n}, P^k)$. Namely $f^{-1}(V)$ contains the stratum over $\bigcap_{\sigma=0}^{\infty}$ but it does not contain the stratum over \hat{P}_{σ} , while from Axiom [5](#page-12-0) it follows that there exist points in the stratum over $\stackrel{\circ}{P_{\sigma}}$ which are in the closure of the stratum over $\stackrel{\circ}{P_{\sigma}}$. \Box

Example 8.23. *For the sake of clearness we provide an explicit example of a set which is an upper set in* $CQ(G_{4,2}, \Delta_{4,2})$ *, but which is not open. Let* $V \subset CQ(G_{4,2}, \Delta_{4,2})$ *is* given as the union of Δ' ₂ $\sum_{4,2}$ and $P_{12,34}^{'}$, where $P_{12,34}$ is a square in $\Delta_{4,2}$ which does not *contain the vertices* δ_{12} *and* δ_{34} *, see* [\[6\]](#page-47-2)*. Then V is not an open set in* $CQ(G_{4,2}, \Delta_{4,2})$ since $f^{-1}(V)$ in not open in $G_{4,2}$. Namely, $f^{-1}(V)$ consists of the main stratum and of the four-dimensional $(\mathbb{C}^*)^3$ -orbit which maps to the square $\overset{\circ}{P}_{12,34}$. On the one hand, *this orbit is contained in the closures of the two six-dimensional* (C ∗) 3 *-orbits, which map to the pyramids* P_{12} *and* P_{34} *by the moment map. Therefore, the set* $f^{-1}(V)$ *is not open in* $G_{4,2}$ *, which implies that the set* V *is not open in* $CQ(G_{4,2}, \Delta_{4,2})$ *. On the other hand, it is obvious that* V *in an upper set since for any* $x \in V$ *if* $x \leq y$ *it follows that* $x = y$ *or* $x \in P'_{12,34}$ *,* $y \in \Delta'_{4,2}$ *and* $\hat{\pi}(x) = \hat{\pi}(y)$ *.*

9 The space $\mathfrak{E}(M^{2n}, P^k)$

Using results from previous sections, we can define the set $\mathfrak{E}(M^{2n}, P^k)$ over the complex $C(M^{2n}, P^k)$ by:

$$
\mathfrak{E}(M^{2n}, P^k) = \{(x, y) \in C(M^{2n}, P^k) \times M^{2n} : x \in P'_{\sigma}, y \in W_{\sigma}, \hat{\pi}(x) = \mu(y)\}.
$$
\n(9)

A topology on $\mathfrak{E}(M^{2n}, P^k)$ is defined as the induced topology by the embedding $\mathfrak{E}(M^{2n}, P^k) \to CQ(M^{2n}, P^k) \times M^{2n}$. In this case the topology on $CQ(M^{2n}, P^k)$ can be obtained as a quotient topology defined by the map $p: \mathfrak{E}(M^{2n}, P^{k}) \rightarrow CQ(M^{2n}, P^{k}).$

Lemma 9.1. *The space* $\mathfrak{E}(M^{2n}, P^k)$ *is a compact Hausdorff topological space.*

Proof. The canonical map $f : M^{2n} \to CQ(M^{2n}, P^k)$ is a continuous, surjective map. The space $\mathfrak{E}(M^{2n}, P^k)$ can be identified with the graph of the map f. Since the manifold M^{2n} is a Hausdorff topological space, it follows that $\mathfrak{E}(M^{2n}, P^k)$ is a Hausdorff space as well. As for the compactness let us consider a covering $E_i, i \in I$ of the space $\mathfrak{E}(M^{2n}, P^k)$ by an open sets. Without loss of generality, we may assume that $E_i = (U_i \times V_i) \cap \mathfrak{E}(M^{2n}, P^k)$ for $i \in I$, where U_i and V_i are open sets in $CQ(M^{2n}, P^k)$ and M^{2n} respectively. Note that for any point $y \in M^{2n}$ there exists a point $x \in CQ(M^{2n}, P^k)$ such that $(x, y) \in \mathfrak{E}(M^{2n}, P^k)$, which implies that $V_i, i \in I$ is an open covering for M^{2n} . In addition, for any point $x \in CQ(M^{2n}, P^k)$, there exists a point $y \in M^{2n}$ such that $(x, y) \in \mathfrak{E}(M^{2n}, P^k)$, which implies that $U_i, i \in I$ is an open covering for $CQ(M^{2n}, P^k)$. The manifold M^{2n} is assumed to be a compact space, which implies that its quotient space $CQ(M^{2n}, P^k)$ is a compact space as well. Therefore, there are finite sub-coverings U_1, \ldots, U_s for $CQ(M^{2n}, P^k)$ and V_1, \ldots, V_l for M^{2n}/T^k , which implies that $X_{ij}=(U_i\times V_j)\cap \mathfrak{E}(M^{2n}, P^k)$ is a finite sub-covering of E_i , $i \in I$ for $\mathfrak{E}(M^{2n}, P^k)$. \Box

There are two natural projections:

$$
G_1: \mathfrak{E}(M^{2n}, P^k) \to CQ(M^{2n}, P^k) \text{ and } G_2: \mathfrak{E}(M^{2n}, P^k) \to M^{2n}.
$$
 (10)

- The maps G_1 and G_2 are obviously surjective. For the map G_1 , this follows from the observation that for any point $x \in P'_\sigma \subset CQ(M^{2n}, P^k)$ there exists a point $y \in W_{\sigma} \subset M^{2n}$, such that $\hat{\pi}(x) = \mu(y)$. This is because $\mu(W_{\sigma}) = \stackrel{\circ}{P_{\sigma}} = \hat{\pi}(P'_{\sigma})$ $\binom{p'}{\sigma}$, which implies that $(x, y) \in \mathfrak{E}(M^{2n}, P^k)$. As for the surjectivity of the map G_2 , for any point $y \in W_{\sigma} \subset M^{2n}$ and any point $x \in P_{\sigma}'$ such that $\hat{\pi}(x) = \mu(y)$, we see that $(x, y) \in \mathfrak{E}(M^{2n}, P^k)$.
- The maps G_1 and G_2 are obviously continuous. It follows from the fact that $G_1^{-1}(U) = (U \times M^{2n}) \cap \mathfrak{E}(M^{2n}, P^k)$ and $G_2^{-1}(V) = (CQ(M^{2n}, P^k) \times V) \cap$ $\mathfrak{E}(M^{2n}, P^k)$ are open sets in $\mathfrak{E}(M^{2n}, P^k)$ for open sets $U \subseteq CQ(M^{2n}, P^k)$ and $V \subseteq M^{2n}$.

Lemma 9.2. *The map* G_2 *is injective, while the map* G_1 *is not injective.*

Proof. The map G_2 is injective since the condition $G_2(x_1, y) = G_2(x_2, y)$ implies σ^{\prime} . But, $\hat{\pi}$: $P_{\sigma}^{\prime} \rightarrow P_{\sigma}^{\circ}$ is a homeomorthat $\hat{\pi}(x_1) = \hat{\pi}(x_2) = \hat{\mu}(y)$ and $x_1, x_2 \in P'_\emptyset$
which is in this part of the map G is a phism, which implies $x_1 = x_2$. The map G_1 is not injective since for any y_1, y_2 which belong to the same non-trivial T^k -orbit of a stratum W_{σ} , there exists a point $x \in P'_{\sigma}$ σ such that $\hat{\pi}(x) = \mu(y_1) = \mu(y_2)$ and $(x, y_1), (x, y_2) \in \mathfrak{E}(M^{2n}, P^k)$. It implies that $G_1(x, y_1) = G_1(x, y_2) = x.$ п

Altogether this leads to the following key result:

Theorem 9.3. The space $\mathfrak{E}(M^{2n}, P^k)$ is homeomorphic to the space M^{2n} . The home*omorphism* G_2 : $\mathfrak{E}(M^{2n}, P^k) \to M^{2n}$ *is given by the map* $G_2(x, y) = y$ *.*

Proof. The map G_2 : $\mathfrak{E}(M^{2n}, P^k) \to M^{2n}$ is a continuous bijection. Since $\mathfrak{E}(M^{2n}, P^k)$ is a compact space and M^{2n} is Hausdorff, it follows that elementary topology arguments lead that G_2 is a homeomorphism. \Box

We define an action of the torus T^k on $CQ(M^n, P^k) \times M^{2n}$ using the given T^k -action on M^{2n} . Since the strata as well as the almost moment map μ are invariant for this torus action, it follows that this action induces an action of the torus T^k on $\mathfrak{E}(M^{2n}, P^k)$. We obtain

$$
\mathfrak{E}(M^{2n}, P^k)/T^k = \{(x, y) \in CQ(M^n, P^k) \times M^{2n}/T^k : x \in P'_{\sigma}, y \in W_{\sigma}/T^k, \widehat{\pi}(x) = \widehat{\mu}(y)\}.
$$

Remark 9.4. Since $W_{\sigma}/T^k \cong \stackrel{\circ}{P}_{\sigma} \times F_{\sigma}$ and P'_{σ} $\phi_{\sigma} \cong \stackrel{\circ}{P}_{\sigma}$, it follows that the points from the set $\mathfrak{E}(M^{2n}, P^k)/T^k$ can be represented as the pairs (x, c_{σ}) , where $x \in P'_{\sigma}$ σ^{\prime} and $c_{\sigma} \in F_{\sigma}$.

From Lemma [9.1](#page-26-1) it follows

Corollary 9.5. *The space* $\mathfrak{E}(M^{2n}, P^{k})/T^{k}$ *is a compact Hausdorff topological space.*

Note that the maps G_1 and G_2 defined by [\(10\)](#page-26-2) are T^k -equivariant, where $CQ(M^{2n}, P^k)$ is considered to be with the trivial T^k -action. Therefore, they induce the maps of the corresponding orbit spaces \widehat{G}_1 : $\mathfrak{E}(M^{2n}, P^k)/T^k \to CQ(M^n, P^k)$ and \widehat{G}_2 : $\mathfrak{E}(M^{2n}, P^k)/T^k \to M^{2n}/T^k$.

Combining the T^k -equaivariance of the map \widehat{G}_2 and Theorem [9.3](#page-27-0) we obtain :

Theorem 9.6. *The map* $\widehat{G_2}$: $\mathfrak{E}(M^{2n}, P^k)/T^k \to M^{2n}/T^k$ given by $\widehat{G_2}(x, y) = y$ is *a homeomorphism.*

The map $\widehat{G_1}$ has the following important feature:

Proposition 9.7. *The space* $\widehat{G_1}^{-1}(P_o)$ $g_{\sigma}^{'}$) is homeomorphic to the space $P_{\sigma}^{'}\times F_{\sigma}$, that is *to the space* $\stackrel{\circ}{P_{\sigma}}\times F_{\sigma}$ *for any* $\sigma \in \mathfrak{S}$ *.*

Proof. For any point $x \in P_\sigma' \in CQ(M^n, P^k)$ we have that $\widehat{G_1}^{-1}(x) = \{(x, y) | y \in$ W_{σ}/T^k , $\hat{\mu}(y) = \pi(x)$. It follows from Corollary [6.4](#page-10-1) that the space $\widehat{G_1}^{-1}(x)$ is homeomorphic to the space F_{σ} . Then Corollary [8.2](#page-19-0) implies that the space $\widehat{G}_1^{-1}(P'_{\sigma})$ σ) is homeomorphic to the space $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$. \Box

Remark 9.8. We want to emphasize that, according to Proposition [9.7,](#page-27-1) the space $\mathfrak{E}(M^{2n}, P^k)/T^k$, that is the orbit space M^{2n}/T^k , is the union of the trivial fiber bundles $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$, where σ runs through the set G. Then, according to Axiom [5,](#page-12-0) the gluing of fibers F_{σ} and $F_{\bar{\sigma}}$ for $\stackrel{\circ}{P_{\sigma}}$ and $\stackrel{\circ}{P_{\sigma}} \subset \partial \stackrel{\circ}{P_{\sigma}}$ respectively, is given by the map $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$.

10 The gluing of the orbit spaces of strata in $\mathfrak{E}(M^{2n}, P^{k})/T^{k}$

In the previous section we proved that the space $\mathfrak{E}(M^{2n}, P^{k})/T^{k}$, that is the orbit space M^{2n}/T^k , is the union of total spaces of the trivial fiber bundles $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$, where $\sigma \in \mathfrak{S}$. In this section we want to describe how the trivial bundles $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$ are glued together, or, in other words, to describe the $\bar{\partial}$ -boundary of $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$. Recall that the $\bar{\partial}$ boundary of $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$ is homeomorphic to the $\bar{\partial}$ - boundary of the orbit space W_{σ}/T^k . Note that $\overline{W_{\sigma}/T^k} \cong \overline{W_{\sigma}}/T^k$, which implies that $\bar{\partial}(W_{\sigma}/T^k) \cong \bar{\partial}W_{\sigma}/T^k$. Lemma [3.8](#page-8-5) immediately implies:

Corollary 10.1. *There is an embedding* $\bar{\partial}(\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}) \subset \cup_{\tilde{\sigma}\subset\sigma} \stackrel{\circ}{P_{\tilde{\sigma}}} \times F_{\tilde{\sigma}}$ *, where* $\tilde{\sigma}$ *runs through all admissible subsets of the set* σ*.*

Remark 10.2. We want to point that for a general $(2n, k)$ -manifold, there might exist an admissible subsets $\tilde{\sigma}$ of a admissible set σ for which the polytope $P_{\tilde{\sigma}}$ is not a face of the polytope P_{σ} . Furthermore, $P_{\tilde{\sigma}}$ might not belong to the boundary of the polytope P_{σ} . Therefore, we *specially* denote by $\bar{\sigma}$ those admissible subsets of a admissible set σ for which $P_{\bar{\sigma}}$ is a face of the polytope P_{σ} .

Note that, as we will demonstrate in Section [18,](#page-45-0) the $\bar{\partial}$ -boundary of the orbit space W_{σ}/T^k can not be, in general, represented as a union of the orbit spaces of some other strata. This implies that, in general, the $\bar{\partial}$ -boundary of the space $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$ in $\mathfrak{E}(M^{2n}, P^k)/T^k$ can not be represented as a union of total spaces of some other trivial fiber bundles.

As for the boundary of the polytopes in $CQ(M^{2n}, P^k)$ we have the following:

Lemma 10.3. *If* $P'_{\tilde{\sigma}} \subset \partial P'_{\sigma}$ for some any P'_{σ} $\sigma, P_{\tilde{\sigma}}' \subset CQ(M^{2n}, P^k)$ then $\tilde{\sigma} \subset \sigma$.

Proof. If $P'_{\tilde{\sigma}} \subset \partial P'_{\sigma}$ it follows that $f^{-1}(P'_{\tilde{\sigma}})$ $g_{\tilde{\sigma}}'$) $\subset f^{-1}(\partial P_{\sigma}')$, where $f : M^{2n} \to$ $CQ(M^{2n}, P^k)$ is a quotient map. It implies that $f^{-1}(\partial P'_\sigma) = \bar{\partial} f^{-1}(P'_\sigma)$ $(\sigma)^{\prime}_{\sigma})=\bar{\partial}W_{\sigma}\subseteq$ $\cup_{\tilde{\sigma}\subset \sigma} W_{\tilde{\sigma}}$ and, thus, $f^{-1}(P'_{\tilde{\sigma}})$ $\sigma(\check{\sigma}) = W_{\tilde{\sigma}}$ for some $\tilde{\sigma} \subset \sigma$.

Remark 10.4. Recall that we already remarked that for any $P'_{\bar{\sigma}}$ $P'_{\bar{\sigma}}, P'_{\sigma} \subset CQ(M^{2n}, P^k),$ such that $P_{\bar{\sigma}}$ is a face of the polytope P_{σ} , we have that $P_{\bar{\sigma}}' \subset \partial P_{\sigma}^{'}$ in $CQ(M^{2n}, P^k)$.

At the end of this section we derive some results under the additional assumption: if $P'_{\tilde{\sigma}}$ $p'_\sigma, P'_\sigma\subset CQ(M^{2n}, P^k)$ and $\partial P'_\sigma\cap P'_\sigma$ $P_{\tilde{\sigma}}' \neq \emptyset$ then $P_{\tilde{\sigma}}' \subset \partial P_{\sigma}'$.

Then Lemma [10.3](#page-28-1) implies:

Corollary 10.5. *The boundary of any area* $P'_{\sigma} \subset CQ(M^{2n}, P^k)$ *is the union of some areas* P ′ $\sigma_{\tilde{\sigma}}$ such that $\tilde{\sigma} \subset \sigma$.

From previous results we also have the following direct consequence:

Corollary 10.6. For any point $x \in P'_\sigma$ p'_{σ} it holds $\overline{x} = \{y \in P'_{\tilde{\sigma}}\}$ $P_{\tilde{\sigma}}' \mid P_{\tilde{\sigma}}' \subset \partial P_{\sigma}', \hat{\pi}(y) =$ $\widehat{\pi}(x)\}.$

The closure of the space of parameters F_{σ} for P_{σ}' σ_{σ} can be described as follows:

Lemma 10.7. *For any point* $x \in P'_\sigma$ σ *it holds:*

$$
\overline{x \times F_{\sigma}} \subseteq \bigcup_{y \in \overline{x}} (y \times F_{\tilde{\sigma}}).
$$

Proof. Let $(y, c) \in \overline{x \times F_\sigma}$. Then $\widehat{G}_2(y, c) = a \in M^{2n}/T^k$ is such a point that $a \in \overline{W_{\sigma}}/T^k$ and $\widehat{f}(a) = y \in P_{\widetilde{\sigma}}'$ p'_{σ} , where $P'_{\sigma} \subset \partial P'_{\sigma}$ and $y \in \overline{x}$. \Box

Proposition 10.8. *There is an embedding* $\bar{\partial}(\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}) \subset \cup \stackrel{\circ}{P_{\sigma}} \times F_{\tilde{\sigma}}$, where $\tilde{\sigma}$ *runs through all admissible sets such that* $P'_{\tilde{\sigma}} \subset \partial P'_{\sigma}$

Proof. Let, as before, $\hat{f} : M^{2n}/T^k \to CQ(M^{2n}, P^k)$ be a quotient map and let a point $a \in \mathfrak{E}(M^{2n}, P^k)/T^k$ belongs to the $\bar{\partial}$ -boundary of the space $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$. This means that the point $b = \hat{F}^{-1}(a)$ belongs to the $\bar{\partial}$ -boundary of the space W_{σ}/T^k . It implies that $\hat{f}(b)$ belongs to the boundary of the polytope P'_{σ} O_{σ}' in $CQ(M^{2n}, P^k)$.

We can say more, that is which points form the union \cup $\bigcirc_{\sigma}^{\circ} \times F_{\tilde{\sigma}}$ given by Proposi-tion [10.8](#page-29-0) are for sure contained in the $\bar{\partial}$ -boundary of the space $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$.

Let $\bar{\sigma} \subset \sigma$ be such a subset that $P'_{\bar{\sigma}}$ p'_{σ} is a face of the polytope P'_{σ} η_{σ} and let $\eta_{\sigma,\bar{\sigma}}: F_{\sigma} \to F_{\bar{\sigma}}$ is a map introduced by [\(6\)](#page-13-3). Put $F_{\sigma,\bar{\sigma}} = \eta_{\sigma,\bar{\sigma}}(F_{\sigma})$.

Lemma 10.9. *For any* $\bar{\sigma} \subset \sigma$ *, such that* $P'_{\bar{\sigma}}$ $p'_{\bar{\sigma}}$ is a face of the polytope P'_{σ} \int_{σ}^{\prime} , there is an \vec{P}_{σ} $\propto F_{\sigma,\bar{\sigma}} \subset \bar{\partial}(\stackrel{\circ}{P}_{\sigma} \times F_{\sigma}).$

Proof. If $P_{\bar{\sigma}} \times c_{\bar{\sigma}} \subset P_{\bar{\sigma}} \times F_{\sigma,\bar{\sigma}}$ then there exists a point $c_{\sigma} \in F_{\sigma}$ such that $\eta_{\sigma,\bar{\sigma}}(c_{\sigma}) =$ $c_{\bar{\sigma}}$. It means that the $\bar{\partial}$ -boundary of the leaf $W_{\bar{\xi}_{\sigma},c_{\sigma}}$ contains the leaf $W_{\bar{\xi}_{\sigma},c_{\sigma}}$. It implies that $W_{[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]} / T^k \subset \bar{\partial} W_{[\xi_{\sigma}, c_{\sigma}]} / T^k$, thus $\stackrel{\circ}{P_{\bar{\sigma}}} \times c_{\bar{\sigma}} \subset \bar{\partial} (\stackrel{\circ}{P_{\sigma}} \times c_{\sigma}).$ \Box As for the points from $\stackrel{\circ}{P_{\tilde{\sigma}}} \times F_{\tilde{\sigma}}$, where $P'_{\tilde{\sigma}} \subset \partial P'_{\sigma}$, we prove the following:

Lemma 10.10. Let $y \in P'_\sigma$ where $P'_\sigma \subset \partial P'_\sigma$. Then there exists a point $c_{\tilde{\sigma}} \in F_{\tilde{\sigma}}$ such *that* $(y, c_{\tilde{\sigma}})$ *belongs to the* $\bar{\partial}$ *-boundary of the space* $\stackrel{\circ}{P_{\sigma}} \times F_{\sigma}$.

Proof. If $P'_{\tilde{\sigma}} \subset \partial P'_{\sigma}$ then $\hat{f}^{-1}(P'_{\tilde{\sigma}})$ $\overline{\partial}_{\tilde{\sigma}}^{\prime}\big) \cap \overline{\partial}_{\tilde{\sigma}}\widehat{f}^{-1}(P_{\sigma}')$ \mathcal{O}'_{σ}) \neq \emptyset . It implies that $W_{\tilde{\sigma}}/T^{\sigma} \cap$ $\bar{\partial}W_{\sigma}/T^k \neq \emptyset$ and, moreover, that $P_{\tilde{\sigma}}' \subset \tilde{f}(W_{\tilde{\sigma}}/T^k \cap \bar{\partial}W_{\sigma}/T^k)$, which means that for $\sigma_{\tilde{\sigma}}'$ there exists a point $c_{\tilde{\sigma}} \in F_{\tilde{\sigma}}$ such that $(y, c_{\tilde{\sigma}}) \in \partial(\stackrel{\circ}{P_{\sigma}} \times F_{\sigma})$. any point $y \in P'_{\tilde{\sigma}}$ \Box

11 A universal space of parameters

In the theory of $(2n, k)$ -manifolds there is an effect, for which we found an example in [\[14\]](#page-48-1). It is about that there exists an $(2n, k)$ -manifold M^{2n} and strata $W_{\sigma}, W_{\sigma'} \subset$ M^{2n} such that $W_{\sigma'} \cap \bar{\partial}W_{\sigma} \neq \emptyset$, but $W_{\sigma'} \not\subset \bar{\partial}W_{\sigma}$. The realization of this effect we elaborate in Section [18](#page-45-0) for the case $M^{24} = G_{7,3}$. Note that according to [\[14\]](#page-48-1) this effect does not appear in the case of $(4(n-2), n-1)$ -manifolds $G_{n,2}$, although they are manifolds of the complexity $n-3$. The manifolds $G_{n,2}$ and the orbits spaces $G_{n,2}/T^n$ are in the focus of attention due to the paper of Kapranov [\[16\]](#page-48-6). The considered effect (we call it Gel'fand-Serganova effect) shows that the description of the equivariant structure of $(2n, k)$ -manifolds is a quite difficult problem. In [\[7\]](#page-47-3) we proposed an approach for the solution of this problem for the Grassmann manifolds that is based on the notion of a universal space of parameters. This new notion we formalize for $(2n, k)$ -manifolds by the following axiom .

We recall that we use the following notation. For an admissible set σ we denote by $\bar{\sigma} \subset \sigma$ such an admissible set $\bar{\sigma}$ that $P_{\bar{\sigma}}$ is a face of P_{σ} . We denote by F_{σ} the space of parameters of a stratum W_{σ} and by F_i the space of parameters of the stratum W_i that consists of i -th fixed point.

Axiom 6. *There exists a topological space* \mathcal{F} *, for any* $\sigma \in \mathfrak{S}$ *there exist topological* spaces $\tilde{F_{\sigma}}$ and continuous inclusions $I_{\sigma} : \tilde{F}_{\sigma} \to \mathcal{F}$ such that

a) $\tilde{F} = F$ *the space of parameters of the main stratum and* \tilde{F} *is a compactification* $of I(F)$,

b)
$$
I_{\sigma}(\tilde{F}_{\sigma}) \subset I_{\bar{\sigma}}(\tilde{F}_{\bar{\sigma}})
$$
 and $\mathcal{F} = \bigcup_{i=1}^{m} I_i(\tilde{F}_i)$,

- *c)* for any $\sigma \in \mathfrak{S}$ there exist continuous projections $p_{\sigma} : \tilde{F}_{\sigma} \to F_{\sigma}$ such that $p_{\bar{\sigma}} \circ \tilde{\eta}_{\sigma,\bar{\sigma}} = \eta_{\sigma,\bar{\sigma}} \circ p_{\sigma}$, where $\tilde{\eta}_{\sigma,\bar{\sigma}} : \tilde{F}_{\sigma} \to \tilde{F}_{\bar{\sigma}}$ is an inclusion given by the *condition b).*
- *e*) the map $H : \mathcal{E} = \cup_{\sigma} P_{\sigma}' \times \tilde{F}_{\sigma} \to \mathfrak{E}(M^{2n}, P^{k})/T^{k}$ defined by $H(x_{\sigma}, \tilde{c}_{\sigma}) =$ $(x_{\sigma}, p_{\sigma}(\tilde{c}_{\sigma}))$ *is a continuous map, where a topology on* $\mathcal E$ *is induced by the em* $bedding \mathcal{E} \rightarrow CQ(M^{2n}, P^k) \times \mathcal{F}.$

Definition 11.1. *The space* F *is said to be the universal space of parameters and the spaces* \tilde{F}_{σ} , $\sigma \in \mathfrak{S}$ *are said to be the virtual spaces of parameters.*

It follows from Axiom [6](#page-30-1) that the orbit space M^{2n}/T^k can be described in terms of the structural elements of $(2n, k)$ -manifolds defined by our six axioms.

Theorem 11.2. For any $(2n, k)$ -manifold M^{2n} the orbit space M^{2n}/T^k is homeomor*phic to a quotient space of the space* $\mathcal E$ *by an equivalence relation* \approx *such that the map* H defines the homeomorphism $\hat{H}:\mathcal{E}/\thickapprox \to \mathfrak{E}(M^{2n}, P^{k})/T^{k}.$

Proof. The orbit space M^{2n}/T^k is homeomorphic to $\mathfrak{E}(M^{2n}, P^k)/T^k$ and the map $H: \mathcal{E} = \bigcup_{\sigma} P_{\sigma}^{'} \times \tilde{F}_{\sigma} \to \mathfrak{E}(M^{2n}, P^{k})/T^{k}$ is surjective. Since $\mathfrak{E}(M^{2n}, P^{k})/T^{k}$ is a Hausdorff space, the statement follows. \Box

Remark 11.3. It follows from the condition c) of Axiom [6](#page-30-1) that the inclusion $\tilde{\eta}_{\sigma,\bar{\sigma}}$: $\tilde{F}_{\sigma} \to \tilde{F}_{\bar{\sigma}}$ is a lifting of the map $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$. We elaborate this more closely. Let $\tilde{c}_\sigma \in \tilde{F}_\sigma$ and let us consider the leaf $W_{[\xi_\sigma, p_\sigma(\tilde{c}_\sigma)]}$, where p_σ is given by the condition c) of Axiom [6.](#page-30-1) Then, by Axiom [5,](#page-12-0) there exists a unique leaf $W_{\{\xi_{\bar{\sigma}}, c_{\bar{\sigma}}\}}$ that belongs to the $\bar{\partial}$ - boundary of $W_{\left[\xi_{\sigma},p_{\sigma}(\tilde{c})\right]}$. Let $y \in W_{\left[\xi_{\tilde{\sigma}},c_{\tilde{\sigma}}\right]}/T^{\tilde{\sigma}}$ and (y_n) a sequence of points from $W_{\left[\xi_{\sigma}, p_{\sigma}(\tilde{c})\right]}/T^{\sigma}$ that converges to the point y. By Theorem [11.2,](#page-31-0) we have that $y = (x, [\tilde{c}_{\bar{\sigma}}]_{p_{\bar{\sigma}}}) \in P'_{\bar{\sigma}} \times \tilde{F}_{\bar{\sigma}}/p_{\bar{\sigma}}$ and $y_n = (x_n, [\tilde{c}_{\sigma}]_{p_{\sigma}}) \in P'_{\sigma} \times \tilde{F}_{\sigma}/p_{\sigma}$, and y_n converges to y in the topology of \mathcal{E}/H . Since $\eta_{\sigma,\bar{\sigma}}(p_{\sigma}(\tilde{c}_{\sigma})) = c_{\bar{\sigma}}$, it follows that the condition c) of Axiom [6](#page-30-1) implies that $\tilde{\eta}_{\sigma,\bar{\sigma}}(\tilde{c}_{\sigma}) \in p_{\bar{\sigma}}^{-1}(c_{\bar{\sigma}})$

It immediately also follows:

Corollary 11.4. If a polytope P_{σ} is a face of the polytope P^k then $I(F) \subset I_{\sigma}(\tilde{F}_{\sigma})$. *Moreover,*

$$
\partial P^k \times I(F) \subset \cup_{\sigma} \stackrel{\circ}{P}_{\sigma} \times I_{\sigma}(\tilde{F}_{\sigma}),
$$

where σ runs through \textbf{all} admissible sets such that P_σ is a face of the polytope P^k .

From previous constructions it follows that, for $(2n, k)$ -manifolds that satisfy Theo-rem [7.18](#page-17-0) or Theorem [7.19,](#page-18-2) it holds $\mathcal{F} = \bar{F}$, where \bar{F} is a notation used in these theorems. Moreover, in the case of such manifolds M^{2n} , for any admissible polytope

 $\hat{P}_{\sigma} \subset \hat{P}^k$ the virtual space of parameters \tilde{F}_{σ} coincides with its space of parameters F_{σ} .

The manifolds $G_{4,2}$, $\mathbb{C}P^5$ and F_3 are examples of manifolds that satisfy Theorems [7.18,](#page-17-0) [7.19.](#page-18-2) It is proved in [\[6\]](#page-47-2) that the universal spaces of parameters for $G_{4,2}$ and $\mathbb{C}P^{5}$ are the manifolds $\mathbb{C}P^1$ and $\mathbb{C}P^2$, respectively. Proposition [15.2](#page-41-0) of the current paper proves that the universal space of parameters for F_3 is $\mathbb{C}P^1$. The first non-trivial example in this direction is the Grassmann manifold $G_{5,2}$.

In [\[7\]](#page-47-3) (Corollary 21) it is proved that the universal space of parameters for $G_{5,2}$ can be taken to be to the blow up of $\mathbb{C}P^2$ at four points. Moreover, the manifold $G_{5,2}$ provides an example such that $\mathcal{F} \neq \overline{F}_x$ for all points $x \in \overset{\circ}{\Delta}_{5,2}$ and that virtual spaces of parameters \tilde{F}_{σ} of strata are, in general, wider then spaces of parameters F_{σ} , that is the projections $p_{\sigma} : \tilde{F}_{\sigma} \to F_{\sigma}$ are not identity maps.

As it is shown in the paper [\[7\]](#page-47-3), the spaces of parameters of the strata in $G_{5,2}$ over the pyramids $K_{ij}(7) \subset \Delta_{5,2}$, $1 \leq i < j \leq 5$ consist of a point (Corollary 12), while their virtual spaces of parameters are homeomorphic to $\mathbb{C}P¹$ (Theorem 11, Lemma 27), Recall that the pyramid $K_{ij}(7)$ is a convex hull of the points δ_{kl} , $kl \neq ij$ and $1 \leq k < l \leq 5$.

We will discuss the case of Grassmann manifold $G_{5,2}$ in more details in Subsection [14.1](#page-34-1)

12 Quasitoric manifolds M^{2n} as $(2n, n)$ -manifolds

As it is presented in Subsection [2.1,](#page-4-0) a quasitoric manifold M^{2n} is equipped with a smooth action of the torus T^n and a smooth T^n -invariant map $\mu : M^{2n} \to P^n$ which is induced by the projection $\pi : M^{2n} \to P^n$.

Theorem 12.1. A quasitoric manifold has a structure of $(2n, n)$ -manifold.

Proof. Let $P_v \subset P^n$ denote the complement to the union of those faces of P^n which *do not contain* the vertex v. The set P_v is an open subset in P^n and $M_v = \mu^{-1}(P_v)$ is an open subset in M^{2n} . The set M_v is T^n -invariant, it contains exactly one fixed point x_v and $\mu(x_v) = v$. Moreover, the set M_v is a dense set in the manifold M^{2n} . It follows from the description of a model for a quasitoric manifold, see [\(1\)](#page-4-1), that M_v is homeomorphic to the quotient space $(T^n \times P_v) / \approx$, which is further homeomorphic to the space $(T^n \times \mathbb{R}^n_+) \cong \mathbb{C}^n$. We take the sets $M_i = M_{v_i}$ as charts for a quasitoric manifold M^{2n} , where i runs through the list of all vertices of the polytope P^n . In this way we obtain that Axiom [1](#page-6-2) and Axiom [2](#page-8-1) are satisfied.

Admissible polytopes P_{σ} are the faces of the polytope P^{n} and P^{n} itself. The strata W_{σ} are indexed by the sets σ that run through the set of vertices of all faces for the polytope P^n . It directly follows that $\hat{\mu}: W_{\sigma} \to \stackrel{\circ}{P_{\sigma}}$ and $W_{\sigma}/T^n \cong \stackrel{\circ}{P_{\sigma}}$, so Axiom 3 and Axiom [4](#page-9-3) are also satisfied.

We see that any stratum W_σ consists of one leaf and its ∂-boundary is the union of the strata over the faces of the corresponding admissible polytope P_{σ} , so Axiom [5](#page-12-0) is satisfied as well. Axiom [6](#page-30-1) is obviously satisfied, the universal space of parameters can be taken to be a point, since for all strata the spaces of parameters are points. □

13 $(2n, 1)$ -manifolds

We first observe the following:

Proposition 13.1. Any $(2n, 1)$ -manifold is homeomorphic to the standard sphere S^{2n}

Proof. By definition, for an $(2n, 1)$ -manifold M^{2n} there exists an almost moment map $\mu : M^{2n} \to [-1,1]$ which is S^1 -invariant. Then Axiom 2 implies that the action of

 S^1 on M^{2n} has exactly two fixed points A_1 , A_2 and Axiom 1 implies that M^{2n} has an atlas consisting of two charts $(M_1, u_1), (M_2, u_2)$ each of them containing exactly one fixed point. By Axiom [3](#page-9-2) and Axiom [2,](#page-8-1) we have that $A_1 = W_1 = M_1 \cap Y_2$ and $A_2 = W_2 = Y_1 \cap M_2$, where $Y_i = M^{2n} \setminus M_i$, $i = 1, 2$. It implies that $M_1 = M^{2n} \setminus A_2$ and $M_2 = M^{2n} \setminus A_1$. Since $u_i : M_i \to \mathbb{R}^{2n}$, $i = 1, 2$ are homeomorphisms, it follows that M^{2n} is the one-point compactification of \mathbb{R}^{2n} and hence, it is homeomorphic to the sphere S^{2n} . □

The vice verse is true as well:

Theorem 13.2. The standard sphere S^{2n} has a structures of an $(2n, 1)$ -manifold for *any* n*.*

Proof. Represent the sphere S^{2n} as the hypersurface

$$
|z_1|^2 + \dots + |z_n|^2 + r^2 = 1 \text{ in } \mathbb{R}^{2n+1} \cong \mathbb{C}^n \times \mathbb{R}
$$

and let us consider the action of the circle S^1 on S^{2n} defined by

$$
t(z_1,...,z_n,r)=(t^{\epsilon_1}z_1,...,t^{\epsilon_n}z_n,r),
$$
 where $\epsilon_k=\pm 1$.

The fixed point for this action are $A_1 = (0, \ldots, 0, 1)$ and $A_{-1} = (0, \ldots, 0, -1)$. An almost moment map we define by

$$
\mu: S^{2n} \to P = [-1,1], \ \mu(z_1,\ldots,z_n,r) = r.
$$

It is straightforward to see that Axiom [2](#page-8-1) is satisfied.

Let us consider the atlas consisting of two charts $(M_1, u_1), (M_{-1}, u_{-1})$, where

$$
M_1 = \{(z, r), r \neq 1\} \text{ and } u_1(z, r) = \frac{1}{1 - r} z,
$$

$$
M_{-1}\{(z, r), r \neq -1\} \to \mathbb{C}^n \text{ and } u_{-1}(z, r) = \frac{1}{1 + r} z.
$$

The charts M_1 and M_{-1} are S^1 -equivariant, each of them contains exactly one fixed point and $\overline{M_1} = \overline{M_{-1}} = S^{2n}$, so Axiom [1](#page-6-2) is satisfied. The only non point stratum is the main stratum $W_{\{-1,1\}} = M_1 \cap M_{-1} = \{(z,r) \in S^{2n} | r \neq -1,1\}.$ The induced map $\hat{\mu}: W_{\{-1,1\}}/\hat{S}^1 \to (-1,1)$ is a fiber bundle with the fiber $\mathbb{C}P^{n-1}$, so Axiom [4](#page-9-3) is satisfied and the orbit space $W_{\{-1,1\}}/S^1$ is homeomorphic to the trivial bundle $\mathbb{C}P^{n-1} \times (-1,1)$. The circle S^1 acts freely on $W_{\{-1,1\}}$, so the projection π : $W_{\{-1,1\}} \to W_{\{-1,1\}}/S^1$ is a fiber bundle. Therefore, the leaf $W_{\{-1,1\}}[\xi, c]$ defined by [\(5\)](#page-12-0) is given as $S^1 \cdot c \times (-1,1)$, where $\xi : W_{\{-1,1\}}/S^1 \to \mathbb{C}P^{n-1}$ is a fixed projection and $c \in \mathbb{C}P^{n-1}$. It implies that $\overline{\partial}$ -boundary of $W_{\{-1,1\}}[\xi, c]$ consists of the two fixed points A_1 and A_{-1} , which are leafs over the vertices of the interval P. In this way we see that Axiom [5](#page-12-0) is satisfied as well. Axiom [6](#page-30-1) is obviously satisfied. The universal space of parameters can be taken to be $\mathbb{C}P^{n-1}$. It coincides with the virtual spaces of parameters for all strata. □

Since the only admissible polytopes for $P^1 = [-1,1]$ are P^1 and its faces $-1, 1$, it follows that Theorem [7.19](#page-18-2) can be directly applied:

Theorem 13.3. *It holds*

$$
S^{2n}/T^1 = \partial P * \mathbb{C}P^{n-1}
$$
, where $P = [-1, 1]$.

14 Complex Grassmann manifolds $G_{k+1,q}$ as $(2n, k)$ -manifolds with $n = q(k + 1 - q)$

According to Subsection [2.2,](#page-5-0) the complex Grassmann manifold $G_{k+1,q}$ is canonically endowed with an effective action of the torus T^k and the smooth T^k -invariant moment map $\mu: G_{k+1,q} \to \mathbb{R}^{k+1}$, whose image is the hypersimplex $\Delta_{n,k}$.

Using the Plücker coordinates and the corresponding atlas, as defined in Subsection [2.2,](#page-5-0) it is not difficult to prove:

Proposition 14.1. The manifold $G_{k+1,q}$ has a structure that satisfies the first five ax*ioms of* $(2n, k)$ *-manifolds, where* $n = q(k + 1 - q)$ *.*

The detailed proof that Axioms 1 - 4 are satisfied can be found in [\[7\]](#page-47-3), Section 2. We provide here the verification of Axiom [5.](#page-12-0)

14.1 Proof of Axiom [5](#page-12-0)

(In order to avoid the confusion with indices , we use in this subsection the notation $z_{i,j}$ along with the common notation z_{ij} .) It follows from the definition of a leaf and the description of strata for $G_{k+1,q}$ (see [\[7\]](#page-47-3), Subsection 3.1), that any leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ is a $(\mathbb{C}^*)^{\sigma}$ -orbit of a point from W_{σ} . It immediately implies that any leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ is a smooth submanifold in $G_{k+1,q}$. On the other hand for any stratum W_{σ} the closure of the $(\mathbb{C}^*)^{\sigma}$ -orbit of a point from W_{σ} is a toric manifold. By the result of [\[1\]](#page-47-5) the complement to this $(\mathbb{C}^*)^{\sigma}$ -orbit in this toric manifold consists of $(\mathbb{C}^*)^{\sigma}$ -orbits of smaller dimensions and the moment map gives a bijection between these orbits and the faces of the polytope P_{σ} . Moreover, the induced moment map gives a diffeomorphism between $\overline{W_{\sigma}[\xi_{\sigma}, c_{\sigma}]/T^{\sigma}}$ and P_{σ} as manifolds with corners. In this way we verify that the first and the second conditions of Axiom [5](#page-12-0) are satisfied for the Grassmann manifolds $G_{k+1,q}$.

More precisely, we proved:

Lemma 14.2. Let $P_{\bar{\sigma}}$ be a face of the polytope P_{σ} . Then the map $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$ is *defined by* $\eta_{\sigma,\bar{\sigma}}(c_\sigma) = c_{\bar{\sigma}}$ *such that the leaf* $W_{\bar{\sigma}}[\xi_{\bar{\sigma}},c_{\bar{\sigma}}]$ *is a unique* $(\mathbb{C}^*)^{\bar{\sigma}}$ -orbit over $P_{\bar{\sigma}}$ *that belongs to the* $\bar{\partial}$ -*boundary of the* $(\mathbb{C}^*)^{\sigma}$ -*orbit* $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ *over* P_{σ} *.*

We verify that the third condition of Axiom [5](#page-12-0) is satisfied. In order to do that, we prove the following result:

Proposition 14.3. *The map* $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$ *is a continuous map for any admissible sets* $(\sigma, \bar{\sigma})$ *such that* $P_{\bar{\sigma}}$ *is a face of the polytope* P_{σ} *.*

Proof. Since the polytope P_{σ} is a face of the polytope P_{σ} , it follows that they have a common vertex, so there always exists a chart M_I such that both W_{σ} and $W_{\bar{\sigma}}$ belong to this chart. Therefore, the proof that the map $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$ is continuous can be realized using the local coordinates of such a fixed chart.

We proceed with the proof through the several steps. We first show (according to the paper [\[7\]](#page-47-3)), that the space of parameters of a stratum W_{σ} in $G_{k+1,q}$ can be embedded into $(\mathbb{C}P_B^1)^{N-l}$, where $\mathbb{C}P_B^1 = \mathbb{C}P^1 \setminus B$, $B = \{(0:1), (1:0)\}$ and $N = (q-1)(k-1)$ $q, 0 \leq l \leq N$.

Recall that we proved in [\[7\]](#page-47-3) (Proposition 1) that the space of parameters of the main stratum for the Grassmann manifolds $G_{k+1,2}$ can be embedded into $({\mathbb{C}}P^1_A)^{k-2}$, where $\mathbb{C}P_A^1 = \mathbb{C}P^1 \setminus A$, $A = \{(0:1), (1:0), (1:1)\}$. This embedding decomposes by the embedding $F \to (\mathbb{C}P^1)^n/PGL(2,\mathbb{C})$ (see also [\[16\]](#page-48-6)) and the proof does not use the charts of the manifold $G_{k+1,2}$.

We construct now an analogue embedding for the space of parameters of the main stratum W of an arbitrary Grassmann manifold $G_{k+1,q}$, $q \geq 1$, using the coordinates in a chart. The main stratum belongs to the intersection of all charts (M_I, u_I) . Fix the first chart (M_I, u_I) , where $I = \{1, \ldots, q\}$ with the local coordinates z_{ij} , $1 \leq i \leq q$, $1 \leq j \leq k+1-q$. The action of the algebraic torus $(\mathbb{C}^*)^{k+1}$ on $G_{k+1,q}$ induces $(\mathbb{C}^*)^{k+1}$ -action on $\mathbb{C}^{q(k+1-q)}$, which is, according to [\[7\]](#page-47-3) (Section 3), given by the representation

$$
(t_1, \ldots, t_{k+1}) \to (\frac{t_{q+1}}{t_1}, \ldots, \frac{t_{k+1}}{t_1}, \ldots, \frac{t_{q+1}}{t_q}, \ldots, \frac{t_{k+1}}{t_q}).
$$

Let $\tau_j = \frac{t_{q+j}}{t_1}$ $\frac{q+j}{t_1}, 1 \le j \le k+1-q$ and $\tau_{k+i-q} = \frac{t_{q+1}}{t_i}$ $\frac{q+1}{t_i}$, $2 \leq i \leq q$. We obtain an effective action of the torus $(\mathbb{C}^*)^k$ on $\mathbb{C}^{q(k+1-q)}$, which is given as the composition of the representation

$$
(\tau_1, ..., \tau_k) \to (\tau_{1,1}, \tau_{1,2}, ..., \tau_{q,k+1-q}),
$$
 where $\tau_{i,j} = \frac{\tau_i \tau_j}{\tau_1}$, (11)

and the standard action of the torus $(\mathbb{C}^*)^{q(k+1-q)}$ on $\mathbb{C}^{q(k+1-q)}$.

Lemma 14.4. *The points of the* $(\mathbb{C}^*)^k$ -*orbit of a point* $\mathbf{a} = (a_{1,1}, \ldots, a_{q,k+1-q})$ *of the main stratum satisfy the equations*

$$
c'_{i,j}z_{1,1}z_{i,j} = c_{i,j}z_{i,1}z_{1,j}, \ \ 2 \leq i \leq q, \ 2 \leq j \leq k+1-q,\tag{12}
$$

where $c'_{i,j} = a_{i,1}a_{1,j}$ and $c_{i,j} = a_{1,1}a_{i,j}$.

Proof. If a point $(z_{1,1},...,z_{q,k+1-q})$ belongs to the $(\mathbb{C}^*)^k$ -orbit of a point $\mathbf{a} = (a_{1,1},...,a_{q,k+1-q})$ then $z_{i,j} = \tau_{i,j} a_{i,j}$. It follows that $c'_{i,j} z_{1,1} z_{i,j} = \tau_{1,1} \tau_{i,j} a_{i,1} a_{1,j} a_{1,1} a_{i,j}$, while $c_{i,j} z_{i,1} z_{1,j} =$ $\tau_{i,1}\tau_{1,j}a_{1,1}a_{i,j}a_{i,1}a_{1,j}$. Since $\tau_{i,j} = \frac{\tau_{i}\tau_{j}}{\tau_{i,j}}$ $\frac{i\tau_j}{\tau_1}$, the statement follows from [\(11\)](#page-35-0).

We find useful to note the following:

Lemma 14.5. *Let* $(a_{1,1},...,a_{q,k+1-q})$ *be the local coordinates of a point* $L \in G_{k+1q}$ *in a chart* M_I *. Then*

$$
a_{p,s} = P^{\hat{I}}(L), \quad \hat{I} = (I \setminus \{s\}) \cup \{p\}.
$$
 (13)

Proof. Let the matrix $A(L)$ represents an element L in the chart M_I . Let us consider the submatrix $A_i(L)$ of this matrix. Since the submatrix $A_i(L)$ is an identity matrix and $\hat{I} = (I \setminus \{s\}) \cup \{p\}$, it follows that $a_{p,s} = P^{\hat{I}}(L)$. \Box

The Plücker coordinates of a point $L \in G_{k+1,q}$, up to common multiply, do not depend on the choice of a chart. All Plücker coordinates of all points from the main stratum W are non-zero, so it follows from Lemma [14.5](#page-36-0) that all coordinates $a_{i,j}$ are non-zero. Moreover, all (2×2) -minors of the matrix $A(L) = (a_{i,j})_{1 \leq i \leq q, 1 \leq j \leq k+1-q}$ are nonzero. It implies that $(c_{i,j} : c'_{i,j}) \notin A = \{(1 : 0), (0 : 1), (1 : 1)\}$. Therefore, this shows that the main stratum W, written in the local coordinates of M_I , belongs to the family of algebraic manifolds which are given by the system [\(12\)](#page-35-1), where $(c_{i,j} : c'_{i,j}) \in \mathbb{C}P^1_A$.

Lemma 14.6. Let the coordinates of a point **b** = $(b_{1,1}, \ldots, b_{q,k+1-q})$ of the main *stratum satisfy the equations* [\(12\)](#page-35-1) *of a point* **a**. Then the point **b** *belongs* to the $(\mathbb{C}^*)^k$ *orbit of the point* a*.*

Proof. Let $b_{i,1} = \tau_{i,1}a_{i,1}$, $1 \le i \le q$ and $b_{1,j} = \tau_{1,j}a_{1,j}$, $2 \le j \le k+1-q$, for some point $(\tau_{1,1}, \ldots \tau_{q,1}, \tau_{1,2}, \ldots \tau_{1,k+1-q}) \in (\mathbb{C}^*)^k$. From [\(12\)](#page-35-1) it follows that

$$
b_{i,2} = \frac{c_{i,2}b_{i,1}b_{1,2}}{c'_{i,2}b_{1,1}} = \frac{a_{1,1}a_{i,2}\tau_{i,1}a_{i,1}\tau_{1,2}a_{1,2}}{a_{i,1}a_{1,2}\tau_{1,1}a_{1,1}} = \frac{\tau_{1,2}\tau_{i,1}}{\tau_{1,1}}a_{i,2} = \tau_{i,2}a_{i,2},
$$

where $\tau_{i.2}$ is as in [\(11\)](#page-35-0).

In the same way, we obtain $b_{i,j} = \tau_{i,j} a_{i,j}$, which proves the statement.

 \Box

Altogether, we obtain

Proposition 14.7. *The map* $f : W \to (\mathbb{C}P_A^1)^N$ *, where* $N = (q-1)(k-q)$ *given in the local coordinates of a chart* M_I *by*

$$
f(a_{1,1},\ldots,a_{q,k+1-q}) =
$$

$$
((c_{2,2}:c'_{2,2}),\ldots,(c_{2,k+1-q}:c'_{2,k+1-q}),\ldots,(c_{q,2}:c'_{q,2}),\ldots,(c_{q,k+1-q}:c'_{q,k+1-q})),
$$

$$
c_{i,j} = a_{1,1}a_{i,j}, \quad c'_{i,j} = a_{i,1}a_{1,j},
$$

is $(\mathbb{C}^*)^k$ -invariant, where $(\mathbb{C}P_A^1)^N$ *is considered with the trivial* $(\mathbb{C}^*)^k$ -action. More*over, it induces an embedding of the space of parameters* $F = W/(\mathbb{C}^*)^k$ *of the main stratum into* $({\mathbb{C}}P^1_A)^N$.

Applying the same argument as for the main stratum we obtain as well analogue result in the following case:

Corollary 14.8. Let W_{σ} be a stratum such that $W_{\sigma} \subset M_I$ and $u_I(W_{\sigma}) \subset \{z =$ $(z_{i,j}) \in C^{q(k+1-q)} \mid z_{i,j} \neq 0$ for all (i,j) *}. Then the space of parameters* F_{σ} of the *stratum* W_{σ} *can be embedded into* $(\mathbb{C}P_B^1)^N$, $N = (q-1)(k-q)$ *as above.*

Remark 14.9. The condition that all local coordinates in a fixed chart M_I of all points of a stratum W_{σ} are non-zero is the property of a fixed chart. Precisely, if we consider the Grassmann manifold $G_{4,2}$ then the stratum W_{34} defined by the condition that $T^{ij}(W_{34}) = 0$ if and only if $(i, j) = 34$, belongs to the intersections of the charts M_{12} and M_{13} ". It is easy to check [\[6\]](#page-47-2) that all local coordinates for all points of the stratum W_{34} in the chart M_{12} are non-zero, while in the chart M_{13} all points of this stratum have one zero coordinate.

The result similar to that in Proposition [14.7](#page-36-1) holds for any stratum whose space of parameters is not a point. We first recall that the definition of the strata as well as results of [\[7\]](#page-47-3) (Subsection 3.1) imply:

Lemma 14.10. Let W_{σ} be a stratum and let M_I be a chart on the Grassmann manifold $G_{k+1,q}$ *. Then:*

- *The stratum* W_{σ} *belongs to the chart* M_I *if and only if* $I \in \sigma$ *, that is if and only* $if P^I(W_{\sigma}) \neq 0,$
- Let $W_{\sigma} \subset M_I$, $L_0 \in W_{\sigma}$, $u_I(L_0) = (z_{1,1}(L_0), \ldots, z_{q,k+1-q}(L_0))$ and $z_{i,j}(L_0) =$ 0 *for some* (i, j) *. Then* $z_{i,j}(L) = 0$ *for any point* $L \in W_{\sigma}$ *,*
- Let $J \subset \{(1,1), \ldots, (q, k+1-q)\}$ be such a subset of the set of indices for the *coordinates in a chart* M_I *, that* $z_{i,j}(L_0) \neq 0$ *if and only if* $(i, j) \in J$ *for some point* $L_0 \in W_{\sigma} \subset M_I$ *. Then*

$$
u_I(W_\sigma) \subset \mathbb{C}^J = \{ (z_{1,1}, \ldots, z_{q,k+1-q}) \in \mathbb{C}^{q(k+1-q)} \mid z_{i,j} \neq 0 \Longleftrightarrow (i,j) \in J \}.
$$

Here and further by the symbol \mathbb{C}^{J} we denote the linear space of maps $\{J \to \mathbb{C}\}.$ In an analogous way as for the main stratum we prove:

Lemma 14.11. *The space of parameters* $F_{\sigma} \neq pt$ *of a stratum* W_{σ} *can be embedded into* $({\mathbb{C}}P_B^1)^g$ for some $1 \le g \le N$.

Proof. Let us consider a stratum $W_{\sigma} \subset M_I$ and assume that

$$
u_I(W_\sigma) \subset \mathbb{C}^J = \{ (z_{1,1}, \ldots, z_{q,k+1-q}) | z_{i,j} \neq 0, (i,j) \in J, z_{i,j} = 0, (i,j) \notin J \},\
$$

where $J = \{(i_1, j_1), \ldots, (i_l, j_l)\} \subseteq \{(1, 1), \ldots, (q, k + 1 - q)\}.$

The algebraic torus $(\mathbb{C}^*)^{\sigma} = (\mathbb{C}^*)^s \subset (\mathbb{C}^*)^l$, $s \leq l$ is defined to be a torus of the maximal dimension that acts freely on the stratum W_{σ} . There is a representation $(\mathbb{C}^*)^s \to (\mathbb{C}^*)^l$ obtained in an analogous way as the representation [\(11\)](#page-35-0). Namely, we take the representation [\(11\)](#page-35-0) composed with the projection $(\mathbb{C}^*)^{q(k+1-q)} \to (\mathbb{C}^*)^l$ on the coordinate subspace \mathbb{C}^{J} and the canonical action of $(\mathbb{C}^{*})^{l}$ on \mathbb{C}^{l} .

Let $\tau_{i_1,j_1}, \ldots, \tau_{i_l,j_l}$ be coordinates on the torus $(\mathbb{C}^*)^s$. Without loss of generality, the coordinates $\tau_{i_1, j_1}, \ldots, \tau_{i_s, j_s}$ can be taken as coordinates for the torus $(\mathbb{C}^*)^s$. Then the representation of $(\mathbb{C}^*)^s \to (\mathbb{C}^*)^l$ writes in these coordinates as an identity on the coordinates $\tau_{i_1,j_1}, \ldots, \tau_{i_s,j_s}$ and it is given by $\tau_{ip,j_p} = \tau_{i_1,j_1}^{\varepsilon_{1p}}$ $\tau_{i_1,j_1}^{\varepsilon_{1p}} \cdots \tau_{i_s,j}^{\varepsilon_{sp}}$ $\sum_{i_s,j_s}^{\varepsilon_{sp}}$ for $s+1\leq p\leq$ l, where $\varepsilon_{rp} = 0, 1, -1$ for any $1 \le r \le s$. Therefore, as in the proof of Lemma [14.4,](#page-35-2) we conclude that the points from the stratum W_{σ} , written in the local coordinates of the chart M_I , satisfy the following system of equations:

$$
\begin{cases} c'_{i_p,j_p} z_{i_p,j_p} \prod_{\varepsilon_{rp}=-1} z_{i_r,j_r} = c_{i_p,j_p} \prod_{\varepsilon_{rp}=1} z_{i_r,j_r}, \ s+1 \le p \le l, \ 1 \le r \le s, \\ z_{i,j} = 0, \ (i,j) \notin \{(i_1,j_1), \dots, (i_l,j_l)\}, \end{cases} \tag{14}
$$

where $c_{i,j}, c'_{i,j} \neq 0$. In this way, an embedding of the space of parameters F_{σ} into $({\mathbb C} P^1_B)^{l-s}$ is defined . □

Remark 14.12*.* The embedding described in Lemma [14.11](#page-37-0) can be presented more explicitly in an analogous way as it is done in Proposition [14.7.](#page-36-1) Namely according to the system of equations [\(14\)](#page-38-0) there is the map $f_{\sigma}: W_{\sigma} \to (\mathbb{C}P_B^1)^{l-s}$, which is in the local coordinates of the chart M_I given by

$$
f_{\sigma}(b_{i_1,j_1},\ldots,b_{i_l,j_l}) = ((c_{i_p,j_p} : c'_{i_p,j_p}))_{s+1 \le p \le l},
$$

$$
c_{i_p,j_p} = b_{i_p,j_p} \prod_{\varepsilon_{rp}=-1} b_{i_r,j_r}, \quad c'_{i_p,j_p} = \prod_{\varepsilon_{rp}=1} b_{i_r,j_r}, \quad 1 \le r \le s.
$$

This map is invariant for the considered $(\mathbb{C}^*)^s$ -action, so it induces an embedding of the space of parameters $F_{\sigma} = W_{\sigma}/(\mathbb{C}^*)^{\sigma}$ into $(\mathbb{C}P_B^1)^{l-s}$.

We finally describe the map $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$.

Lemma 14.13. Let a polytope $P_{\bar{\sigma}}$ is a face of the polytope P_{σ} . Let us consider the *above constructed embeddings of the spaces of parameters* $F_{\bar{\sigma}} \subset (\mathbb{C}P_B^1)^r$ *and* $F_{\sigma} \subset$ $({\mathbb{C}} P_B^1)^b$. Then $r < b$ and the space $({\mathbb{C}} P_B^1)^r$ is a factor in the product $({\mathbb{C}} P_B^1)^b$. The *map* $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$ *is given by the projection* $({\mathbb{C}}P_B^1)^b \to ({\mathbb{C}}P_B^1)^r$.

Proof. Let us consider the leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ of the stratum W_{σ} . Let $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$ be a unique leaf of $W_{\bar{\sigma}}$ which is contained in the $\bar{\partial}$ -boundary of $W[\xi_{\sigma}, c_{\sigma}]$. The coordinates of the points from the leaf $W_{\sigma}[\xi_{\sigma}, c_{\sigma}]$ satisfy equations [\(14\)](#page-38-0) for a fixed point of the space $(\mathbb{C}P^1_A)^{l-s}$. It follows that the coordinates of the points from the leaf $W_{\bar{\sigma}}[\xi_{\bar{\sigma}}, c_{\bar{\sigma}}]$ in a fixed chart satisfy the following system:

$$
\begin{cases}\nc'_{i_p,j_p} z_{i_p,j_p} \prod_{\varepsilon_{rp}=-1} z_{i_r,j_r} = c_{i_p,j_p} \prod_{\varepsilon_{rp}=1} z_{i_r,j_r}, \ p \in V, \ V \subset \{s+1,\ldots,l\}, \\
z_{i_p,j_p} = 0 \text{ for } p \in \{s+1,\ldots,l\} \setminus V, \\
z_{i,j} = 0, \ (i,j) \notin \{(i_1,j_1),\ldots,(i_l,j_l)\}.\n\end{cases} \tag{15}
$$

In this way, we see that the space of parameters $F_{\bar{\sigma}}$ of the stratum $W_{\bar{\sigma}}$ can be embedded into $(\mathbb{C}P_B^1)^r$, where $r = |V| < b$, while the space of parameters F_{σ} of the stratum

 W_{σ} is embedded into $({\mathbb{C}}P_B^1)^b$, $b = l - s$. Moreover, the space $({\mathbb{C}}P_B^1)^r$ contains the space $(\mathbb{C}P_B^1)^b$ as its coordinate subspace, with the coordinates indexed by $i_p j_p$, where $p \in V$. Therefore, the coordinate projection $(\mathbb{C}P_B^1)^b \to (\mathbb{C}P_B^1)^r$ gives the map $\eta_{\sigma,\bar{\sigma}}: F_{\sigma} \to F_{\bar{\sigma}}$ □

Corollary 14.14. *The map* $\eta_{\sigma,\bar{\sigma}} : F_{\sigma} \to F_{\bar{\sigma}}$ *is a continuous map.*

In this way we completed the proof of Proposition [14.3.](#page-35-3)

 \Box

Remark 14.15. The strata W_{σ} coincide with the matroid strata defined in [\[15\]](#page-48-5). Moreover, in the papers [\[14\]](#page-48-1) and [\[15\]](#page-48-5), see also [\[6\]](#page-47-2) for the summary, the three more stratifications of the Grassmann manifolds $G_{k+1,q}$ are defined: by "the soft Schubert cells", by the moment map and by the arrangements of planes, and it is showed that all these four stratifications are equivalent.

Remark 14.16. The closure of the strata W_{σ} on the Grassmann manifolds are known in the literature as matroid varieties. These varieties do not behave nicely regarded to the matroids assigned to them. In that context these varieties were recently studied in [\[11\]](#page-48-12), [\[12\]](#page-48-13).

14.2 Axiom [6](#page-30-1) for the manifolds $G_{k+1,q}$

In order to verify Axiom [6](#page-30-1) we need to introduce an universal space of parameters F , virtual spaces of parameters \tilde{F}_{σ} , continuous embeddings $I_{\sigma} : \tilde{F}_{\sigma} \to \mathcal{F}$ and projections $\tilde{F}_{\sigma} \to F_{\sigma}$, where σ runs through the admissible sets. We also need to prove that:

- 1) F is a compactification of the space of parameters $F = \tilde{F}$ of the main stratum and $\mathcal{F} = \cup_{\sigma} I_{\sigma}(\tilde{F}_{\sigma}),$
- 2) The map $H: \mathcal{E} = \bigcup_{\sigma} P_{\sigma}' \times \tilde{F}_{\sigma} \to \mathfrak{E}(G_{k+1,q}, \Delta_{k+1,q})/T^k$ is a continuous map.

At this moment we have proved Axiom [6](#page-30-1) for the Grassmannians $G_{4,2}$ and $G_{5,2}$. These results are described in detail in [\[6\]](#page-47-2) and [\[7\]](#page-47-3) according to the theory presented above. They turned out to be non trivial and already have been further developed in the papers of several authors. We formulate these results precisely.

In the case of Grassmann manifold $G_{4,2}$:

- (1) $F = \tilde{F} = \mathbb{C} \setminus \{0, 1\};$
- (2) $\mathcal{F} = \mathbb{C}P^1;$
- (3) $F_{\sigma} = \tilde{F}_{\sigma} = pt$ for the strata W_{σ} such that $\Pr_{\sigma}^{\circ} \subset \triangle_{4,2}^{\circ}$; $F_{\sigma} = pt$, $\tilde{F}_{\sigma} = \mathbb{C}P^1$, for the strata W_{σ} such that $P_{\sigma} \subset \partial \Delta_{4,2}$.

Following the approach described above (see the details in [\[6\]](#page-47-2)), the embeddings I_{σ} : $\tilde{F}_{\sigma} \rightarrow \mathcal{F}$ can be described using the notion of coordinates in a chart. We do it in

the chart M_{12} . The strata W_{σ} such that $W_{\sigma} \subset M_{12}$ and $\Pr_{\sigma} \subset \Lambda_{4,2}$ are indexed by the admissible sets $\sigma_{ij} = \{12, 13, 14, 23, 24, 34\} \setminus \{ij\}$, $ij \neq 12$, and $\sigma_{13,24} =$ $\{12, 23, 14, 34\}, \sigma_{14, 23} = \{12, 23, 24, 34\}.$ For them it holds $I_{\sigma_{34}}(\tilde{F}_{\sigma_{34}}) = (1 : 1),$ $I_{\sigma_{14}}(\tilde{F}_{\sigma_{14}}) = I_{\sigma_{23}}(\tilde{F}_{\sigma_{23}}) = (0 : 1), I_{\sigma_{13}}(\tilde{F}_{\sigma_{13}}) = I_{\sigma_{24}}(\tilde{F}_{\sigma_{24}}) = (1 : 0)$ and $I_{\sigma_{14,24}}(\tilde{F}_{\sigma_{14,23}}) = (0:1), I_{\sigma_{13,24}}(\tilde{F}_{\sigma_{13,24}}) = (1:0)$. There are also the strata W_{σ} that do not belong to the chart M_{12} but for which $\Pr_{\sigma}^{\circ} \subset \Lambda_{4,2}^{\circ}$. These are the strata W_{σ_1} , $\sigma_1 = \{13, 14, 23, 24, 34\}$ and W_{σ_2} , $\sigma_2 = \{13, 14, 23, 24\}$. The strata W_{σ_1} and W_{σ_2} belong to the chart M_{13} . In the paper [\[6\]](#page-47-2), see proof of Proposition 4, it is showed that the transition map from the coordinates of the chart M_{13} to the coordinates of the chart M_{12} induces a homeomorphism of the space of parameters of the main stratum $F = \mathbb{C} \setminus \{0, 1\}$ that is given by the involution $c \to \frac{c}{c-1}$. This homeomorphism extends to a homeomorphism of the universal space of parameters $\mathbb{C}P^1$. The strata W_{σ_1} and W_{σ_2} in the chart M_{13} are limits of the main stratum W, when the parameter c of the main stratum, written in the coordinates of the chart M_{13} , tends to ∞ . It follows that in the chart M_{12} we have that $I_{\sigma_1}(\tilde{F}_{\sigma_1}) = I_{\sigma_2}(\tilde{F}_{\sigma_2}) = (1:1)$. The admissible polytopes P_{σ} for the other strata such that $W_{\sigma} \subset M_{13} \setminus (M_{12} \cap M_{13})$ belong to the boundary $\partial \Delta_{4,2}$. For them I_{σ} : $\tilde{F}_{\sigma} \cong \mathbb{C}P^1 \to \mathbb{C}P^1$ is given by the map $c \to \frac{c}{c-1}$. In an analogous way we describe these embeddings in the coordinates of the chart M_{12} for the strata which does not belong to the union of these two charts.

In the case of Grassmann manifold $G_{5,2}$:

- (1) $F = \tilde{F} = \{((c_1 : c'_1))$ c'_1), $(c_2 : c'_2)$ c'_2), $(c_3 : c'_3)$ S_3)) $\in (\mathbb{C}P_A^1)^3 | c_1 c_2'$ $c_2'c_3 = c_1'$ $\big\{c_2c_3'\big\};$
- (2) $\mathcal F$ is given as the blowup at the point $((1:1), (1:1), (1:1))$ of the compact non singular surface $\{((c_1 : c'_1), (c_2 : c'_2), (c_3 : c'_3)) \in (\mathbb{C}P^1)^3 | c_1c'_2c_3 = c'_1c_2c'_3\};\$
- (3) An explicit description of the virtual spaces of parameters \tilde{F}_{σ} for all admissible polytopes and the spaces of parameters F_{σ} for all strata W_{σ} , as well as the corresponding embeddings I_{σ} : $\tilde{F}_{\sigma} \to \mathcal{F}$ and projections p_{σ} : $\tilde{F}_{\sigma} \to F_{\sigma}$ is given in [\[6\]](#page-47-2).

Together with Proposition [14.1,](#page-34-2) we get

Theorem 14.17. *The complex Grassmann manifolds* $G_{4,2}$ *and* $G_{5,2}$ *have a canonical structure of* (8, 3) *and* (10, 4)*-manifolds respectively.*

Remark 14.18*.* We believe that the approach developed in [\[7\]](#page-47-3), in the case $G_{5,2}$, for finding virtual spaces of parameters \tilde{F}_{σ} and an universal space of parameters $\mathcal{F} =$ $\cup_{\sigma} I_{\sigma}(\tilde{F}_{\sigma})$ together with the results of the current paper brings the proof of Axiom [6](#page-30-1) for all Grassmann manifolds $G_{k+1,q}$.

15 The complex manifold of complete flags

The complete complex flag manifold F_{k+1} consists of flags of k complex subspaces $L = (L_1 \subset L_2 \subset ... \subset L_k)$ in \mathbb{C}^{k+1} . It is a homogeneous space, which can be represented by $F_{k+1} = U(k+1)/T^{k+1}$. As in the case of complex Grassmann manifolds the canonical action of the torus T^{k+1} on \mathbb{C}^{k+1} induces an effective action of the torus T^k on the manifold F_{k+1} . This action extends to an action of the corresponding algebraic torus. The moment map $\mu: F_{k+1} \to \mathbb{R}^{k+1}$ is defined by

$$
\mu(L) = \mu_1(L_1) + \mu_2(L_2) + \ldots + \mu_k(L_k), \tag{16}
$$

where $L = (L_1 \subset L_2 \subset \ldots \subset L_k)$ and

$$
\mu_i(L_i) = \frac{\sum_J |P^J(L_i)|^2 \delta_J}{\sum_J |P^J(L_i)|^2},
$$

where $J \subset \{1, ..., k + 1\}, |J| = i$.

The image $\mu_i(L_i)$ is the hypersimplex $\Delta_{k+1,i}$ since it is the image by the moment map of the Grassmann manifold $G_{k+1,i}$. Therefore, the image of the map μ is the Minkowski sum of the hypersimplices $\Delta_{k+1,i}$, $1 \leq i \leq k$, which is known to be the standard permutahedron Pe^k . Recall that the standard permutahedron Pe^k is a convex hull over the set of $(k + 1)!$ points given by $\sigma(0, 1, 2, \ldots k)$, where σ runs through the symmetric group S_{k+1} .

The algebraic torus $(\mathbb{C}^*)^{k+1}$ acts on F_{k+1} . In an analogous way as for the complex Grassmann manifolds we prove:

Theorem 15.1. *The complete complex flag manifold* F_{k+1} *satisfies the first five axioms of an* $(2n, k)$ *-manifold, where* $2n = k(k + 1)$ *.*

We provide the proof in detail for the case $k = 2$ and, furthermore, we show that F_3 satisfies the sixth axiom as well. For the manifold F_{k+1} the atlas required by Axiom [1](#page-6-2) can be constructed as follows:

The charts are the sets $M_{i_1,i_1i_2,...,i_1...i_k} = \{L = (L_1 \subset L_2 \subset ... \subset L_k) \in$ F_{k+1} | $P^{i_1}(L_1) \neq 0, P^{i_1 i_2}(L_2) \neq 0, \ldots, P^{i_1 \ldots i_k}(L_k) \neq 0, 1 \leq i_1, \ldots, i_k \leq$ $k + 1, i_p \neq i_q, p \neq q, 1 \leq p, q \leq k$. Then any point $L \in M_{i_1, i_1 i_2, ..., i_1 ... i_k}$ can be represented by the matrix A_L such that $a_{i,j} = 1$ and $a_{i,p} = 0$, $j + 1 \le p \le k$. The coordinate map $u_{i_1,i_1i_2,...,i_1...i_k}$: $M_{i_1,i_1i_2,...,i_1...i_k} \to \mathbb{C}^{\frac{k(k-1)}{2}}$ is given by

 $u_{i_1,i_1i_2,...,i_1...i_k}(L) = (a_{ij}), \ i \neq i_p \text{ and } j < p, \ 1 \leq p \leq k.$

Proposition 15.2. *The manifold* F_3 *has a structure of an* (6, 2)*-manifold and* $F_3/T^2 \cong$ $S^1 \ast \mathbb{C}P^1 \cong S^4$

Proof. We consider the atlas for F_3 whose charts are given by

$$
M_{i,ij} = \{ L = (L_1 \subset L_2) \in F_3 \mid P^i(L_1) \neq 0, P^{ij}(L_2) \neq 0 \}, 1 \leq i, j \leq 3, i \neq j,
$$

and the homeomorphisms $u_{i,ij}: M_{i,ij} \to \mathbb{C}^3$ are defined as above. Any point $L \in$ $M_{i,ij}$ can be represented by the matrix $A_L = (a_{sl}), 1 \le s \le 3, 1 \le l \le 2$ such that $a_{i1} = 1, a_{i2} = 0$ and $a_{i2} = 1$. Then $u_{i,ij}(L) = (a_{sl})$, where $sl \neq i1, i2, j2$. This atlas is invariant under the canonical action of the algebraic torus $(\mathbb{C}^*)^3$.

Let us consider the chart $M_{1,12}$. Any point $L \in M_{1,12}$ represents by the matrix

$$
\begin{pmatrix} 1 & 0 \\ a_1 & 1 \\ a_2 & a_3 \end{pmatrix}
$$
 and $u_{1,12}(L) = (a_1, a_2, a_3)$.

The moment map [\(16\)](#page-41-1) in this chart writes as

$$
\mu(L) = \frac{1}{1+|a_1|^2+|a_2|^2}((1,0,0)+|a_1|^2(0,1,0)+|a_2|^2(0,0,1))+
$$

+
$$
\frac{1}{1+|a_3|^2+|a_1a_3-a_2|^2}((1,1,0)+|a_3|^2(1,0,1)+|a_1a_3-a_2|^2(0,1,1)).
$$

The action of the algebraic torus $(\mathbb{C}^*)^2$ on $\mathbb{C}^3 = u_{1,12}(M_{1,12})$, induced by the action of $(\mathbb{C}^*)^3$ on F_3 , is, in the chart $M_{1,12}$, given by

$$
(t_1, t_2) \cdot (a_1, a_2, a_3) = (t_1a_1, t_2a_2, \frac{t_2}{t_1}a_3).
$$

Thus, the orbits for this action are as follows:

- (1) if $a_i \neq 0, 1 \leq i \leq 3$ and $a_1 a_3 a_2 \neq 0$ it is the hypersurface $\frac{z_2 z_3}{z_1} = c$, where $c \in \mathbb{C} \setminus \{0, 1\};$
- (2) if all $a_i \neq 0$ and $a_1a_3 a_2 = 0$ it is the hypersurface $\frac{z_2z_3}{z_1} = 1$;
- (3) if $a_i = 0$ and $a_s, a_l \neq 0$ it is the coordinate subspace \mathbb{C}_{ls} ;
- (4) if $a_i, a_s = 0$ and $a_i \neq 0$ it is the coordinates axis \mathbb{C}_i ;
- (5) if all $a_i = 0$ it is the point $(0, 0, 0)$.

Note that the orbits given by (1) form the main stratum and they are parametrized by $c \in \mathbb{C} \setminus \{0,1\}$. All other strata consist of one orbit. Since the main stratum is an everywhere dense set in F_3 , it follows that all other orbits can be parametrized using this parametrization of the main stratum.

The admissible polytopes for the chart $M_{1,12}$ are:

- (1) the hexagon Pe^2 ;
- (2) the quadrilateral Q over the vertices $(2, 1, 0), (2, 0, 1), (1, 2, 0), (1, 0, 2)$;
- (3) the three quadrilaterals Q_i , $1 \le i \le 3$ defined by the vertices: $Q_1 (2, 1, 0), (2, 0, 1), (1, 0, 2), (0, 1, 2)$; $Q_2-(2, 1, 0), (2, 0, 1), (1, 2, 0), (0, 2, 1); Q_3-(2, 1, 0), (1, 2, 0), (0, 2, 1), (0, 1, 2);$
- (4) the three segments I_i , $1 \le i \le 3$ defined by the vertices: $I_1 (2, 1, 0), (0, 1, 2)$; $I_2 - (2, 1, 0), (2, 0, 1), I_3 - (2, 1, 0), (1, 2, 0);$
- (5) the vertex $(2, 1, 0)$.

We see that the orbit over the square Q can be parametrized by the point $c = 1$, the orbits over Q_1 by the point $c = \infty$, while the orbits over Q_2 and Q_3 by the point $c = 0$. Also the orbits over the intervals I_1 and I_2 can be parametrized by any point $c \in \mathbb{C} \cup \{\infty\}$, while the orbit over I_3 by the point $c = 0$ and every fixed point by any point $c \in \mathbb{C} \cup \{\infty\}$. Note that Q_2 , Q_3 and I_3 glue together to give the interior of the polytope Pe^2 and the orbits over them are all parametrized by the point $c = 0$.

By considering the other charts it is easy to generalize this case and conclude that all admissible polytopes are given by the hexagon and its faces and six quadrilaterals in the hexagon and their faces. The quadrilateral complementary to Q , as well as its edge that belongs to the interior of the hexagon, can be parametrized by the point $c = 1$, while the quadrilateral complementary to Q_1 as well as its edge that belongs to the interior of hexagon can be parametrized by the point $c = \infty$. In this way we prove that $\hat{\mu}^{-1}(x) \cong \mathbb{C}P^1$ for all points x from the interior of the hexagon. It follows from Theorem [7.19](#page-18-2) that $F_3/T^3 \cong (\partial Pe^2) * \mathbb{C}P^1$.

Remark 15.3*.* We believe that the approach developed for finding an universal space of parameters and virtual spaces of parameters in the case of Grassmann manifolds, can be in an analogous way applied to the complete flag manifolds as well.

16 The orbit spaces of some key examples

Let us consider the Grassmann manifold $G_{4,2}$ of the complex two-dimensional subspaces in \mathbb{C}^4 as an example of (8, 3)-manifold over the hypersimplex $\Delta_{4,2}$. As it is remarked in Section 14.2, all points $x \in \partial \Delta_{4,2}$ are simple and $\hat{\mu}^{-1}(x)$ is homeomorphic to $\mathbb{C}P^1$ for all points $x \in \overset{\circ}{\Delta}_{4,2}$. Theorem [7.19](#page-18-2) implies that the orbit space $G_{4,2}/T^4$ is homeomorphic to the space $(\partial \Delta_{4,2}) * \mathbb{C}P^1$. This statement is one of the key result of the paper [\[6\]](#page-47-2).

The action of T^4 on the complex projective space $\mathbb{C}P^5$ is also studied in detail in [\[6\]](#page-47-2). This action is given as the composition of the second symmetric power representation Λ^2 : $T^4 \to T^6$ and the canonical action of the torus T^6 on $\mathbb{C}P^6$. This action is not effective, but it induces an effective action of the torus $T³$, for which the Plücker map $P: G_{4,2} \to \mathbb{C}P^5$ is an equivariant map. As a result, we obtain on $\mathbb{C}P^5$ a structure of (10,3)-manifold with the almost moment map $\mu : \mathbb{C}P^5 \to \Delta_{4,2}$. The space of parameters F of the main stratum can be identified with $\{(c_1, c_2) \in \mathbb{C}^2 | c_1 c_2 \neq 0\}$ 0}. All points $x \in \partial \Delta_{4,2}$ are simple and $\hat{\mu}^{-1}(x)$ is homeomorphic to the complex projective plane $\mathbb{C}P^2$ for any point $x \in \stackrel{\circ}{\Delta}_{4,2}$. Applying again Theorem [7.19](#page-18-2) we obtain that $\mathbb{C}P^5/T^4$ is homeomorphic to the space $(\partial \Delta_{4,2}) \ast \mathbb{C}P^2$. Moreover, the Plücker embedding induces the embedding $\hat{\mathcal{P}}$: $G_{4,2}/T^3 \rightarrow \mathbb{C}P^5/T^3$, which, by the stated homeomorphisms, produces the embedding $(\partial \Delta_{4,2}) * \mathbb{C}P^1 \to (\partial \Delta_{4,2}) * \mathbb{C}P^2$, whose restriction to $\partial \Delta_{4,2}$ is given by an identity map, while its restriction to $\mathbb{C}P^1$ is given by an inclusion in $\mathbb{C}P^2$. This is one more key result of the paper [\[6\]](#page-47-2).

 \Box

Let us consider now the Grassmann manifold $G_{5,2}$ of the complex two-dimensional subspaces in \mathbb{C}^5 as an example of (12, 4)-manifold over the hypersimplex $\Delta_{5,2}$. In this case Theorem [7.19](#page-18-2) does not apply, since not all points from $\partial \Delta_{5,2}$ are simple. More precisely, the points which belong to the interiors of the octahedra are singular. Nevertheless, in this case, we can describe the orbit space $G_{5,2}/T^5$ by providing an explicit constructions of the universal space of parameters, virtual spaces of parameters as well as the construction of the corresponding projection and embedding maps. This is realized in the paper [\[7\]](#page-47-3) and it is proved, as a key result, that $G_{5,2}/T^5$ is homotopy equivalent to the wedge $(S^2 * \mathbb{C}P^1) \vee (S^3 * \mathbb{C}P^2)$. Note that $S^2 \cong \partial \Delta_{4,2}$, $S^3 \cong \partial \Delta_{5,2}$ and $S^2 * \mathbb{C}P^1$ is homeomorphic to the orbit space $G_{4,2}/T^4$.

17 Examples for the construction of virtual spaces of parameters for $G_{k+1,q}$.

The construction of virtual spaces of parameters should use the fact that the main stratum is an open, dense set in an $(2n, k)$ -manifold M^{2n} . The idea for introducing these spaces as well to call them as virtual can be illustrated by our work on the description of the orbit space $G_{5,2}/T^5$. see [\[7\]](#page-47-3).

We start by considering the fixed points in an $(2n, k)$ -manifold M^{2n} . For a vertex v of the polytope P^k let $P^v_{\sigma} = \{P_{\sigma} \in P_{\sigma} | v \in P_{\sigma}\}\$. We introduce a partial ordering on the set P_{σ}^v by : $P_{\sigma_1} < P_{\sigma_2}$ if and only if P_{σ_1} is a face of the polytope P_{σ_2} . In particular $v < P_{\sigma}$ for any $P_{\sigma} \in P_{\sigma}^v$. Proposition [11.3](#page-31-1) implies that if $P_{\sigma_1} < P_{\sigma_2}$ then $I_{\sigma_2}(\tilde{F}_{\sigma_2}) \subset I_{\sigma_1}(\tilde{F}_{\sigma_1})$ and , in particular, $I_{\sigma}(\tilde{F}_{\sigma}) \subset I_i(\tilde{F}_i)$, for any $P_{\sigma} \in P_{\sigma}^v$, where by \tilde{F}_i is denoted the virtual space of parameters of the *i*-th vertex v_i .

In the case of Grassmann manifold $G_{k+1,q}$ due to an action of the symmetric group S_{k+1} , we obtain

Lemma 17.1. The spaces \tilde{F}_i and \tilde{F}_j are homeomorphic for any two vertices v_i, v_j of $\Delta_{k+1,q}$

In order to illustrate more closely the idea for introducing virtual spaces of parameters as well an universal space of parameters, let us consider the stratum W_{σ} , $\sigma =$ $\{12, 13, 14, 15, 24, 34, 45\}$ in $G_{5,2}$. In the local coordinates z_{ij}^{12} of the chart M_{12} this stratum is given by the equations: $z_{11}^{12} = z_{31}^{12} = 0$. Following [\[6\]](#page-47-2), let $(c_{i,12} : c'_{i,12})$, $1 \leq i \leq 3$ be such coordinates in $({\mathbb{C}}P^1)^3$ that the main stratum in the local coordinates of the chart M_{12} is given by the system of equations $c'_{1,12}z_{11}^{12}z_{22}^{12} = c_{1,12}z_{21}^{12}z_{12}^{12}$, $c'_{2,12}z_{11}^{12}z_{32}^{12} = c_{2,12}z_{31}^{12}z_{12}^{12}, c'_{3,12}z_{21}^{12}z_{32}^{12} = c_{3,12}z_{31}^{12}z_{22}^{12}$. The condition that $z_{11}^{12}, z_{31}^{12} \rightarrow$ 0 implies that in \overline{F}_{12} we have that $(c_{1,12} : c'_{1,12}) = (0 : 1), (c_{3,12} : c'_{3,12}) = (1 : 0),$ while the limit of the points, $(c_{2,12} : c'_{2,12})$ in \bar{F}_{12} is *not defined*. Here \bar{F}_{12} is the closure of the space of parameters F_{12} of the main stratum in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ considered in the chart M_{12} . We define the virtual space of parameters to be $\tilde{F}_{\sigma,12} =$ $\{((0: 1), (c_{2,12}: c'_{2,12}), (1: 0)), (c_{2,12}: c'_{2,12}) \in \mathbb{C}P^1\} \subset \overline{F}_{12}$. It contains all non defined points $(c_{2,12} : c'_{2,12}) \in \mathbb{C}P^1$. Thus, the virtual space of parameters $\tilde{F}_{\sigma,12}$

resolves the singularities corresponding to uncertainties when the points from the main stratum converge to the points of the stratum W_{σ} .

The situation is the same if consider the charts M_{13} , M_{15} , M_{24} , M_{34} , M_{45} which contain this stratum. Precisely, in the local coordinates of the chart M_{13} this stratum is given by the equations $z_{11}^{13} = z_{31}^{13} = 0$ and the virtual space of parameters is given by $\widetilde{F}_{\sigma,13} = \{((0:1), (c_{2,13} : c'_{2,13}), (1:0)), (c_{2,13} : c'_{2,13}) \in \mathbb{C}P^1\} \subset \overline{F}_{13}$. In the local coordinate of the chart M_{15} , this stratum is defined by $z_{11}^{15} = z_{21}^{15} = 0$ and as the virtual space of parameters we obtain $\tilde{F}_{\sigma,15} = \{((c_{1,15} : c'_{1,15}), (0 : 1), (0 : 1)), (c_{1,15} : c'_{1,15})\}$ $c_{1,15}') \in \mathbb{C}P^1$ $\subset \bar{F}_{15}$. In the chart M_{24} the stratum W_{σ} is given by $z_{22}^{24} = z_{32}^{24} = 0$ and the virtual space of parameters is given by $\tilde{F}_{\sigma,24} = \{((0 : 1), (0 : 1), (c_{3,24} :$ $c'_{3,24}$), $(c_{3,24}: c'_{3,24}) \in \mathbb{C}P^1$ $\subset \overline{F}_{24}$. Further, in the local coordinates of the chart M_{34} this stratum is defined by $z_{22}^{34} = z_{32}^{34} = 0$ and the virtual space of parameters is given by $\tilde{F}_{\sigma,34} = \{((0:1), (0:1), (c_{1,34} : c'_{3,34})), (c_{3,34} : c'_{3,34}) \in \mathbb{C}P^1\} \subset \bar{F}_{34}$. In the local coordinates of the chart M_{45} this stratum is defined by $z_{21}^{45} = z_{31}^{45} = 0$ and the virtual space of parameters is given by $\tilde{F}_{\sigma,45} = \{((1:0), (1:0), (c_{3,45} :$ $c'_{3,45})$, $(c_{3,45} : c'_{3,45}) \in \mathbb{C}P^1$ $\subset \overline{F}_{45}$.

This stratum belongs to the chart M_{14} as well, in this chart all coordinates of its points are non-zero and it is given by the equations: $z_{11}^{14}z_{22}^{14} = z_{21}^{14}z_{12}^{14}$, $z_{11}^{14}z_{32}^{14} =$ $z_{31}^{14}z_{12}^{14}$, $z_{21}^{14}z_{32}^{14}$ = $z_{31}^{14}z_{22}^{14}$. These equations imply that the virtual space of parameters $\tilde{F}_{\sigma,14}$, considering it as a subset in the closure \bar{F}_{14} , is given by the point ((1 : 1), $(1:1)$, $(1:1)$ \in \overline{F}_{14} . According to Axiom [6,](#page-30-1) a virtual space of parameters \overline{F}_{σ} is defined by the stratum W_{σ} and its construction should not depend on the choice of the charts that contains W_{σ} . The virtual space of parameters, in the local coordinates of the charts M_{12} , M_{13} , M_{15} , M_{24} , M_{34} , M_{45} , for the stratum we consider here, is homeomorphic to $\mathbb{C}P^1$, while in the chart M_{14} , when approaching the limit point $((1:1), (1:1), (1:1))) \in \overline{F}_{14}$, a singularity does not appear. Therefore, in order to obtain the space $\tilde{F}_{\sigma,14}$ we need to consider the blow up of the space \bar{F}_{14} at the point $((1:1),(1:1),(1:1))).$

18 Gel'fand-Serganova example

According to Lemma [3.8,](#page-8-5) for any stratum W_{σ} there is an inclusion $\bar{\partial}W_{\sigma} \subseteq \cup W_{\overline{\sigma}}$, where $\overline{\sigma}$ runs through all admissible subsets of σ given. We show here that this inclusion is strict in general, meaning that there exist such strata W_{σ} and $W_{\bar{\sigma}}$ for which

$$
\bar{\sigma} \subset \sigma; \ \bar{\partial} W_{\sigma} \cap W_{\bar{\sigma}} \neq \emptyset, \ \text{but} \ W_{\bar{\sigma}} \not\subset \bar{\partial} W_{\sigma}.
$$

Such a pair of strata W_{σ} and $W_{\bar{\sigma}}$ is given in the paper of Gel'fand-Serganova [\[14\]](#page-48-1). The space of parameters of the strata W_{σ} in $G_{7,3}$ is a point, while its ∂-boundary has non empty intersection with the stratum $W_{\bar{\sigma}}$ whose space of parameters has non-zero dimension.

Consider a point $L \in G_{7,3}$ given by the matrix

$$
\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & b_1 & c_1 & d_1 \\ 0 & 1 & 0 & a_2 & 0 & c_2 & d_2 \\ 0 & 0 & 1 & a_3 & b_3 & 0 & d_3 \end{array}\right),
$$

such that

$$
d_1 d_2 d_3 \neq 0, \ d_1 c_2 = d_2 c_1, \ d_1 b_3 = d_3 b_1, \ d_2 a_3 = a_2 d_3, \ a_i, b_i, c_i \neq 0. \tag{17}
$$

Form the condition [\(17\)](#page-46-0) it follows that

$$
a_3b_1c_2 = a_2b_3c_1.
$$

The charts on the Grassmann manifolds are indexed by the non-zero Plücker coordinates (see Subsection [2.2\)](#page-5-0). It implies that the set of points which satisfy [\(17\)](#page-46-0) form the stratum W obtained as the intersection of the sets X_{ijk} , $1 \leq i \leq j \leq k \leq 7$, where $X_{ijk} = Y_{ijk}$ for $ijk = 126, 135, 234, 147, 257, 367$ and $X_{ijk} = M_{ijk}$ for the other indices.

On the other hand, according to Subsection [2.2,](#page-5-0) the $(\mathbb{C}^*)^6$ -orbit of a point $(0, b_1, c_1, d_1, a_2, 0, c_2, d_2, a_3, b_3, 0, d_3)$, in the local coordinates of the chart M_{123} , is given by

$$
(0, \frac{t_5}{t_1}b_1, \frac{t_6}{t_1}c_1, \frac{t_7}{t_1}d_1, \frac{t_4}{t_2}a_2, 0, \frac{t_6}{t_2}c_2, \frac{t_7}{t_2}d_2, \frac{t_4}{t_3}a_3, \frac{t_5}{t_3}b_3, 0, \frac{t_7}{t_3}d_3),
$$

which can be written as

$$
(0, \tau_1 b_1, \tau_2 c_1, \tau_3 d_1, \tau_4 a_2, 0, \tau_5 c_2, \frac{\tau_3 \tau_5}{\tau_2} d_2, \tau_6 a_3, \frac{\tau_1 \tau_5 \tau_6}{\tau_2 \tau_4} b_3, 0, \frac{\tau_3 \tau_5 \tau_6}{\tau_2 \tau_4} d_3). \tag{18}
$$

Therefore, all points of the form [\(18\)](#page-46-1) belong to the one $(\mathbb{C}^*)^6$ -orbit, that is the stratum W consists of this one $(\mathbb{C}^*)^6$ -orbit.

Consider a point $L_1 \in G_{7,3}$ given by the matrix

$$
\left(\begin{array}{cccccc}\n1 & 0 & 0 & 0 & b_1 & c_1 & 0 \\
0 & 1 & 0 & a_2 & 0 & c_2 & 0 \\
0 & 0 & 1 & a_3 & b_3 & 0 & 0\n\end{array}\right),\n\tag{19}
$$

where $a_3b_1c_2 = a_2b_3c_1$. Let W' be a stratum that contains this point. It is the intersection of the sets X_{ijk} where $X_{ijk} = M_{ijk}$ for $ijk = 123, 124, 125, 135, 136, 235, 236, 456$ and $X_{ijk} = Y_{ijk}$ for the other indices. It implies that the stratum W' consists of the points that can be represented by the matrices

$$
\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & b_1 & c_1 & 0 \\ 0 & 1 & 0 & a_2 & 0 & c_2 & 0 \\ 0 & 0 & 1 & a_3 & b_3 & 0 & 0 \end{array}\right),
$$

such that $a_3b_1c_2 \neq -a_2b_3c_1$.

On the one hand, it is obvious that the point L_1 from the stratum W' as well as its $(\mathbb{C}^*)^6$ -orbit, which is given by $(0, \tau_1 b_1, \tau_2 c_1, 0, \tau_3 a_2, 0, \tau_4 c_2, 0, \tau_5 a_3, \frac{\tau_1 \tau_4 \tau_5}{\tau_2 \tau_3} b_3, 0, 0)$ such that $a_3b_1c_2 = a_2b_3c_1$, belong to $\overline{\partial}$ - boundary of the stratum W.

On the other hand, the stratum W' does not coincide with the $(\mathbb{C}^*)^6$ -orbit of a point L₁. Namely, this $(\mathbb{C}^*)^6$ -orbit is contained in W['], but its points satisfy relation $a_3b_1c_2 =$ $a_2b_3c_1$, while the points from W' are defined by the weaker relation $a_3b_1c_2 \neq -a_2b_3c_1$. It implies that

$$
\bar{\partial}W\cap W^{'}\neq\emptyset\;\;\text{and}\;\;W^{'}\not\subset\bar{\partial}W.
$$

Gel'fand-Serganova leads to the following important comment:

Corollary 18.1. *The map* $\eta_{\sigma,\bar{\sigma}}$: $F_{\sigma} \to F_{\bar{\sigma}}$, whose existence is stated by Axiom [5,](#page-12-0) is *not a surjection in general.*

Remark 18.2*.* It follows from the results of the paper of Gel'fand-Serganova [\[14\]](#page-48-1) that the map $\eta_{\sigma,\bar{\sigma}}: F_{\sigma} \to F_{\bar{\sigma}}$ is a surjection for the Grassmann manifolds $G_{k+1,2}$ and $G_{6,3}$.

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