DUALS OF A FRAME IN QUATERNIONIC HILBERT SPACES

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Abstract. Frames in a separable quaternionic Hilbert space were introduced and studied in [17] to have more applications. In this paper, we extend the study of frames in quaternionic Hilbert spaces and introduce different types of duals of a frame in separable quaternionic Hilbert spaces. As an application, we give the orthogonal projection of $\ell^2(\mathfrak{Q})$ onto the range of analysis operator of the given frame, in terms of elements of canonical dual frame and elements of the frame in quaternionic Hilbert space. Finally, we give an expression for the orthogonal projection in terms of operators related to the frame and its canonical dual frame in quaternionic Hilbert space.

1. Introduction

Formally, frames for Hilbert spaces (in particular for $L^2[a,b]$) were introduced way back in 1952 by Duffin and Schaeffer [11] as a tool to study of non-harmonic Fourier series. They defined the following

"A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{H}." \tag{1.1}$$

Moreover, the positive constants A and B, respectively, are called *lower* and *upper* frame bounds for the frame $\{x_n\}_{n\in\mathbb{N}}$. The inequality (1.1) is called the *frame inequality* for the frame $\{x_n\}_{n\in\mathbb{N}}$. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is called a *Bessel sequence* if it satisfies upper frame inequality in (1.1). A frame $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ is said to be

- *tight* if it is possible to choose A, B satisfying inequality (1.1) with $A = B$.
- *Parseval* if it is possible to choose A, B satisfying inequality (1.1) with $A = B = 1$.
- *exact* if removal of any x_n renders the collection $\{x_i\}_{i\neq n}$ no longer a frame for $\mathcal{H}$.

Later, frames were further reintroduced, in 1986 by, Daubechies, Grossmann and Meyer [10], they observed that frames can be used to approximate functions in $L^2(\mathbb{R})$. One can also considered frames as one of the generalizations of orthonormal bases in Hilbert spaces and being redundant frames expansions are more useful and advantageous over basis expansions in a variety of practical applications. Now a days, frames are regarded as one of an important tool to study various areas like representation of signals, characterization of function spaces

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and other fields of applications such as: signal and image processing [4], filter bank theory [3], wireless communications [14] and sigma-delta quantization [2]. For more literature on frame theory, one may refer to [5, 9].

In recent years, many generalizations of frames have been introduced and studied. In 2004, Casazza and Kutyniok [6] defined frames of subspaces (frames of subspaces has many applications in sensor networks and packet encoding), Li and Ogawa [15] introduced the notion of pseudo-frames in Hilbert spaces using two Bessel sequences, Fornasier [12] introduced the notion of bounded quasi-projectors, Christensen and Eldar [8] gave oblique frames. In 2006, Sun [19] introduced generalized frames or g-frames for Hilbert spaces and proved that frames of subspaces (fusion frames), pseudo frames, bounded quasi-projectors and oblique frames are special cases of $g-$frames.

Recently, Khokulan, Thirulogasanthar and Srisatkunarajah [16] introduced and studied frames for finite dimensional quaternionic Hilbert spaces. Sharma and Virender [18] study some different types of dual frames of a given frame in a finite dimensional quaternionic Hilbert space and gave various types of reconstructions with the help of dual frame. Very recently, Sharma and Goel [17] introduced and studied frames for seperable quaternionic Hilbert spaces and Chen, Dang and Qian [7] had studied frames for Hardy spaces in the contexts of the quaternionic space and the Euclidean space in the Clifford algebra.

In this paper, we extend the study of frames in quaternionic Hilbert spaces and introduce different types of duals of a frame in separable quaternionic Hilbert spaces. As an application, we give the orthogonal projection of $\ell^2(\mathfrak{Q})$ onto the range of analysis operator of the given frame, in terms of elements of canonical dual frame and elements of the frame in quaternionic Hilbert spaces. Finally, we give an expression for the orthogonal projection in terms of operators related to the frame and its canonical dual frame in quaternionic Hilbert spaces.

2. Quaternionic Hilbert space

As the quaternions are non-commutative in nature therefore there are two different types of quaternionic Hilbert spaces, the left quaternionic Hilbert space and the right quaternionic Hilbert space depending on positions of quaternions. In this section, we will study some basic notations about the algebra of quaternions, right quaternionic Hilbert spaces and operators on right quaternionic Hilbert spaces.

Throughout this paper, we will denote $\mathfrak{Q}$ to be a non-commutative field of quaternions, I be a non empty countable set of indicies, $V_R(\mathfrak{Q})$ be a separable right quaternionic Hilbert space, by the term "right linear operator", we mean a "right $\mathfrak{Q}$-linear operator" and $\mathfrak{B}(V_R(\mathfrak{Q}))$ denotes the set of all bounded (right $\mathfrak{Q}$-linear) operators of $V_R(\mathfrak{Q})$:

$$\mathfrak{B}(V_R(\mathfrak{Q})) := \{T : V_R(\mathfrak{Q}) \to V_R(\mathfrak{Q}) : \|T\| < \infty\}.$$

The non-commutative field of quaternions $\mathfrak{Q}$ is a four dimensional real algebra with unity. In $\mathfrak{Q}$, 0 denotes the null element and 1 denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by i, j, k. i.e.,

$$\mathfrak{Q} = \{x_0 + x_1 i + x_2 j + x_3 k \ : \ x_0, \ x_1, \ x_2, \ x_3 \in \mathbb{R}\}$$

where $i^2 = j^2 = k^2 = -1$; $ij = -ji = k$; $jk = -kj = i$ and $ki = -ik = j$. For each quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathfrak{Q}$, the conjugate of q is denoted by $\overline{q}$ and is defined as

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k \in \mathfrak{Q}.$$

If $q = x_0 + x_1 i + x_2 j + x_3 k$ is a quaternion, then x_0 is called the real part of q and $x_1 i + x_2 j + x_3 k$ is called the imaginary part of q. The modulus of a quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is defined as

$$|q|= (\overline{q}q)^{1/2} = (q\overline{q})^{1/2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

For every non-zero quaternion $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathfrak{Q}$, there exists a unique inverse q^{-1} in $\mathfrak{Q}$ as

$$q^{-1} = \frac{\overline{q}}{|q|^2} = \frac{x_0 - x_1 i - x_2 j - x_3 k}{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Definition 2.1. A *right quaternionic vector space* $\mathbb{V}_R(\mathfrak{Q})$ is a vector space under right scalar multiplication over the field of quaternionic $\mathfrak{Q}$, i.e.,

$$\begin{aligned} \mathbb{V}_R(\mathfrak{Q}) \times \mathfrak{Q} &\to \mathbb{V}_R(\mathfrak{Q}) \\ (u, q) &\to uq \end{aligned} \tag{2.1}$$

and for each $u, v \in \mathbb{V}_R(\mathfrak{Q})$ and $p, q \in \mathfrak{Q}$, the right scalar multiplication (2.1) satisfying the following properties:

$$(u + v)q = uq + vq$$
$$u(p + q) = up + uq$$
$$v(pq) = (vp)q.$$

Definition 2.2. A *right quaternionic pre-Hilbert space* or *right quaternionic inner product space* $\mathbb{V}_R(\mathfrak{Q})$ is a right quaternionic vector space together with the binary mapping $\langle .|. \rangle : \mathbb{V}_R(\mathfrak{Q}) \times \mathbb{V}_R(\mathfrak{Q}) \to \mathfrak{Q}$ (called the *Hermitian quaternionic inner product*) which satisfies following properties:

(a) $\overline{\langle v_1|v_2\rangle} = \langle v_2|v_1\rangle$ for all $v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$.
(b) $\langle v|v\rangle > 0$ for all $0 \neq v \in V_R(\mathfrak{Q})$.
(c) $\langle v|v_1 + v_2\rangle = \langle v|v_1\rangle + \langle v|v_2\rangle$ for all $v, v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$.
(d) $\langle v|uq\rangle = \langle v|u\rangle q$ for all $v, u \in \mathbb{V}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.

Let $\mathbb{V}_R(\mathfrak{Q})$ be right quaternionic inner product space with the Hermitian inner product $\langle .|. \rangle$. Define the quaternionic norm $\|.\|: \mathbb{V}_R(\mathfrak{Q}) \to \mathbb{R}^+$ on $\mathbb{V}_R(\mathfrak{Q})$ by

$$\|u\|= \sqrt{\langle u|u\rangle},\ u \in \mathbb{V}_R(\mathfrak{Q}). \tag{2.2}$$

Definition 2.3. The right quaternionic pre-Hilbert space is called a *right quaternionic Hilbert space*, if it is complete with respect to the norm (2.2) and is denoted by $V_R(\mathfrak{Q})$.

Theorem 2.4 (The Cauchy-Schwarz Inequality)**.** [13] *If $V_R(\mathfrak{Q})$ is a right quaternionic Hilbert space then*

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle, \quad \textit{for all} \ \ u, v \in V_R(\mathfrak{Q}).$$

Moreover, a norm as defined in (2.2) satisfies the following properties:

(a) $\|uq\|= \|u\||q|$, *for all* $u \in V_R(\mathfrak{Q})$ *and* $q \in \mathfrak{Q}$.
(b) $\|u + v\|\leq \|u\|+\|v\|$, *for all* $u, v \in V_R(\mathfrak{Q})$.
(c) $\|u\|= 0$ *for some* $u \in V_R(\mathfrak{Q})$, *then* $u = 0$.

For the non-commutative field of quaternions $\mathfrak{Q}$, define the quaternionic Hilbert space $\ell_2(\mathfrak{Q})$ by

$$\ell_2(\mathfrak{Q}) = \left\{ \{q_i\}_{i\in\mathbb{N}} \subset \mathfrak{Q} : \sum_{i\in\mathbb{N}} |q_i|^2 < +\infty \right\}$$

under right multiplication by quaternionic scalars together with the quaternionic inner product on $\ell_2(\mathfrak{Q})$ defined as

$$\langle p|q\rangle = \sum_{i\in\mathbb{N}} \overline{p_i}q_i,\ p = \{p_i\}_{i\in\mathbb{N}} \text{ and } q = \{q_i\}_{i\in\mathbb{N}} \in \ell_2(\mathfrak{Q}). \tag{2.3}$$

It is easy to observe that $\ell_2(\mathfrak{Q})$ is a right quaternionic Hilbert space with respect to quaternionic inner product (2.3).

Definition 2.5 ([13]). Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert Space and S be a subset of $V_R(\mathfrak{Q})$. Then, define the set:

- $S^\perp = \{v \in V_R(\mathfrak{Q}) : \langle v|u\rangle = 0\ \forall\ u \in S\}$.
- $\langle S\rangle$ be the right $\mathfrak{Q}$-linear subspace of $V_R(\mathfrak{Q})$ consisting of all finite right $\mathfrak{Q}$-linear combinations of elements of S.

Theorem 2.6 ([13]). *Let $V_R(\mathfrak{Q})$ be a quaternionic Hilbert space and let N be a subset of $V_R(\mathfrak{Q})$ such that, for $z, z' \in N$ such that $\langle z|z'\rangle = 0$ if $z \neq z'$ and $\langle z|z\rangle = 1$. Then the following conditions are equivalent:*

(a) For every $u, v \in V_R(\mathfrak{Q})$, the series $\sum_{z\in N}\langle u|z\rangle\langle z|v\rangle$ converges absolutely and

$$\langle u|v\rangle = \sum_{z\in N}\langle u|z\rangle\langle z|v\rangle.$$

(b) For every $u \in V_R(\mathfrak{Q})$, $\|u\|^2 = \sum_{z\in N} |\langle z|u\rangle|^2$.

(c) $N^\perp = 0$.

(d) $\langle N\rangle$ is dense in H.

Definition 2.7 ([13]). Every quaternionic Hilbert space $V_R(\mathfrak{Q})$ admits a subset N, called *Hilbert basis or orthonormal basis* of $V_R(\mathfrak{Q})$, such that, for $z, z' \in N$, $\langle z|z'\rangle = 0$ if $z \neq z'$ and $\langle z|z\rangle = 1$ and satisfies all the conditions of Theorem 2.6.

Further, if there are two such sets, then they have the same cardinality. Furthermore, if N is a Hilbert basis of $V_R(\mathfrak{Q})$, then for every $u \in V_R(\mathfrak{Q})$ can be uniquely expressed as

$$u = \sum_{z\in N} z\langle z|u\rangle$$

where the series $\sum_{z\in N} z\langle z|u\rangle$ converges absolutely in $V_R(\mathfrak{Q})$.

Definition 2.8 ([1]). Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and T be an operator on $V_R(\mathfrak{Q})$. Then T is said to be

- *right $\mathfrak{Q}$-linear* if $T(v_1\alpha+v_2\beta) = T(v_1)\alpha+T(v_2)\beta$, for all $v_1, v_2 \in V_R(\mathfrak{Q})$ and $\alpha, \beta \in \mathfrak{Q}$.
- *bounded* if there exist $K \geq 0$ such that $\|T(v)\| \leq K\|v\|$, for all $v \in V_R(\mathfrak{Q})$.

Definition 2.9 ([1]). Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and T be an operator on $V_R(\mathfrak{Q})$. Then the *adjoint operator* T^* of T is defined by

$$\langle v|Tu\rangle = \langle T^*v|u\rangle, \text{ for all } u, v \in V_R(\mathfrak{Q}).$$

Further, T is said to be *self-adjoint* if $T = T^*$.

Theorem 2.10 ([1]). *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and S and T be two bounded right $\mathfrak{Q}$-4linear operators on $V_R(\mathfrak{Q})$. Then*

(a) $T + S$ and $TS \in \mathfrak{B}(V_R(\mathfrak{Q}))$. Moreover:

$$\|T + S\| \leq \|T\| + \|S\| \text{ and } \|TS\| \leq \|T\|\|S\|.$$

(b) $\langle Tv|u\rangle = \langle v|T^*u\rangle$.
(c) $(T+S)^* = T^* + S^*$.
(d) $(TS)^* = S^*T^*$.
(e) $(T^*)^* = T$.
(f) $I^* = I$, *where* I *is the identity operator on* $V_R(\mathfrak{Q})$.
(g) If T *is an invertible operator then* $(T^{-1})^* = (T^*)^{-1}$.

3. Frames in quaternionic Hilbert spaces

We begin this section with the following definition of frames in a separable right quaternionic Hilbert space $V_R(\mathfrak{Q})$ defined in [17]:

Definition 3.1. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathbb{N}}$ is said to be a *frame* for $V_R(\mathfrak{Q})$, if there exist two finite constants with $0 < A \leq B$ such that

$$A\|u\|^2 \leq \sum_{i\in\mathbb{N}} |\langle u_i|u\rangle|^2 \leq B\|u\|^2, \text{ for all } u \in V_R(\mathfrak{Q}). \tag{3.4}$$

The positive constants A and B, respectively, are called *lower* and *upper* frame bounds for the frame $\{u_i\}_{i\in\mathbb{N}}$. The inequality (3.4) is called *frame inequality* for the frame $\{u_i\}_{i\in I}$. A sequence $\{u_i\}_{i\in\mathbb{N}}$ is called a *Bessel sequence* for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$ with bound B, if $\{u_i\}_{i\in\mathbb{N}}$ satisfies the right hand side of the inequality (3.4). A frame $\{u_i\}_{i\in\mathbb{N}}$ for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$ is said to be

- *tight*, if it is possible to choose A and B satisfying inequality (3.1) with $A = B$.
- *Parseval frame*, if it is tight with $A = B = 1$.
- *exact*, if it ceases to be a frame whenever anyone of its element is removed.

If $\{u_i\}_{i\in\mathbb{N}}$ is a Bessel sequence for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$. Then, the *(right) synthesis operator* for $\{u_i\}_{i\in\mathbb{N}}$ is a right linear operator $T : \ell_2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ defined by

$$T(\{q_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} u_i q_i, \quad \{q_i\}_{i\in\mathbb{N}} \in \ell_2(\mathfrak{Q}).$$

The adjoint operator T^* of right synthesis operator T is called the *(right) analysis operator*. Further, the analysis operator $T^* : V_R(\mathfrak{Q}) \to \ell_2(\mathfrak{Q})$ is given by

$$T^*(u) = \{\langle u_i|u\rangle\}_{i\in\mathbb{N}}, \ u \in V_R(\mathfrak{Q}).$$

Infact, for $u \in V_R(\mathfrak{Q})$ and $\{q_i\}_{i\in\mathbb{N}} \in \ell_2(\mathfrak{Q})$, we have

$$\begin{aligned}
\langle T^*(u)|\{q_i\}_{i\in\mathbb{N}}\rangle &= \langle u|T(\{q_i\}_{i\in\mathbb{N}})\rangle \\
&= \left\langle u \middle| \sum_{i\in\mathbb{N}} u_i q_i \right\rangle \\
&= \sum_{i\in\mathbb{N}} \langle u|u_i\rangle q_i \\
&= \left\langle \{\langle u_i|u\rangle\}_{i\in\mathbb{N}}, \{q_i\}_{i\in\mathbb{N}} \right\rangle.
\end{aligned}$$

Thus

$$T^*(u) = \{\langle u_i|u\rangle\}_{i\in\mathbb{N}}, \ u \in V_R(\mathfrak{Q}).$$

Theorem 3.2 ([17]). *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$. Then, $\{u_i\}_{i\in\mathbb{N}}$ is a Bessel sequence for $V_R(\mathfrak{Q})$ with bound B if and only if the right linear operator $T : \ell_2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ defined by*

$$T\left(\{q_i\}_{i\in\mathbb{N}}\right) = \sum_{i\in\mathbb{N}} u_i q_i, \ \ \{q_i\}_{i\in\mathbb{N}} \in \ell_2(\mathfrak{Q})$$

is a well defined and bounded operator with $\|T\| \leq \sqrt{B}$.

Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$. Then, the *(right) frame operator* $S : V_R(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ for the frame $\{u_i\}_{i\in\mathbb{N}}$ is a right linear operator given by

$$\begin{aligned} S(u) &= TT^*(u) \\ &= T(\{\langle u_i|u\rangle\}_{i\in\mathbb{N}}) \\ &= \sum_{i\in\mathbb{N}} u_i\langle u_i|u\rangle, \ \ u \in V_R(\mathfrak{Q}). \end{aligned}$$

Theorem 3.3 ([17]). *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively and frame operator S. Then S is positive, bounded, invertible and self adjoint right linear operator on $V_R(\mathfrak{Q})$.*

Theorem 3.4 ([17]). *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively and frame operator S. Then $\{S^{-1}u_i\}_{i\in\mathbb{N}}$ is also a frame for $V_R(\mathfrak{Q})$ with bounds B^{-1} and A^{-1} and right frame operator S^{-1}.*

Theorem 3.5 ([17]). *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $V_R(\mathfrak{Q})$. Then $\{u_i\}_{i\in\mathbb{N}}$ is a frame for $V_R(\mathfrak{Q})$ if and only if the right linear operator $T : \ell_2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$*

$$T(\{q_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} u_i q_i, \quad \{q_i\}_{i\in\mathbb{N}} \in V_R(\mathfrak{Q})$$

is a well-defined and bounded mapping from $\ell_2(\mathfrak{Q})$ onto $V_R(\mathfrak{Q})$.

4. Duals of a frame in quaternionic Hilbert spaces

In view of Theorem 3.4, we have the following definition

Definition 4.1. Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with frame operator S. Then

- $\{S^{-1}(u_i)\}_{i\in\mathbb{N}}$ is called the *canonical dual frame* for the frame $\{u_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$.
- a sequence $\{v_i\}_{i\in\mathbb{N}} \subset V_R(\mathfrak{Q})$ is called an *alternate dual* for the frame $\{u_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$ if it satisfies

$$u = \sum_{i=1}^{\infty} v_i\langle u_i|u\rangle, \quad u \in V_R(\mathfrak{Q}).$$

In view of above definition, one may observe that canonical dual frame for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$ is also an alternate dual frame for $V_R(\mathfrak{Q})$ and an alternate dual of a frame may not be unique. In order to show their existence we give a following example

Example 4.2. Let $N=\{z_i\}_{i\in\mathbb{N}}$ be a Hilbert basis for a right quaternionic Hilbert space $V_R(\mathfrak{Q})$. Then, for each $z_i, z_k \in N$, $i,k\in\mathbb{N}$, we have

$$\langle z_i|z_k\rangle = \begin{cases} 0, & \text{for } i\neq k \\ 1, & \text{for } i=k. \end{cases}$$

Define a sequence $\{u_i\}_{i\in\mathbb{N}}\subset V_R(\mathfrak{Q})$ by

$$u_i = z_j, \quad i\in\{2j-1,2j\}, \quad j=1,2,\cdots.$$

Then $\{u_i\}_{i\in\mathbb{N}}$ is frame for $V_R(\mathfrak{Q})$. Moreover, the canonical dual $\{\widetilde{u}_i\}_{i\in\mathbb{N}}$ of $\{u_i\}_{i\in\mathbb{N}}$ is given by

$$\widetilde{u}_i = \frac{z_j}{2}, \quad i\in\{2j-1,2j\}, \quad j=1,2,\cdots.$$

Next, define sequences $\{v_i\}_{i\in\mathbb{N}}$ and $\{w_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$ by

$$v_{2i-1}=z_i \text{ and } v_{2i}=0, \quad i=1,2,\cdots.$$

and

$$w_{2i-1}=0 \text{ and } w_{2i}=z_i, \quad i=1,2,\cdots.$$

Then $\{v_i\}_{i\in\mathbb{N}}$ and $\{w_i\}_{i\in\mathbb{N}}$ are alternate duals for the frame $\{u_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$. □

In the following result we show that if a Bessel sequence is an alternate dual for a given frame in quaternionic Hilbert space $V_R(\mathfrak{Q})$, then it becomes a frame for $V_R(\mathfrak{Q})$ and the given frame becomes its alternate dual.

Theorem 4.3. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with lower and upper frame bounds A and B, respectively and $\{v_i\}_{i\in\mathbb{N}}$ be a Bessel sequence for $V_R(\mathfrak{Q})$. If $\{v_i\}_{i\in\mathbb{N}}$ is a alternate dual for $\{u_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$, then $\{v_i\}_{i\in\mathbb{N}}$ is also a frame for $V_R(\mathfrak{Q})$. Further, $\{u_i\}_{i\in\mathbb{N}}$ is a alternate dual for $\{v_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$.*

Proof. As $\{u_i\}_{i\in\mathbb{N}}$ is a frame for $V_R(\mathfrak{Q})$, therefore $\{\langle u_i|u\rangle\}_{i\in\mathbb{N}}\in\ell^2(\mathfrak{Q})$, $u\in V_R(\mathfrak{Q})$. Also, $\{v_i\}_{i\in\mathbb{N}}$ is a Bessel sequence, so by Theorem 3.2, $\sum\limits_{i=1}^{\infty} v_i\langle u_i|u\rangle$ exists. So, we have

$$\begin{aligned}
\langle u|v\rangle &= \left\langle u \middle| \sum_{i=1}^{\infty} v_i\langle u_i|v\rangle \right\rangle \\
&= \sum_{i=1}^{\infty}\langle u|v_i\rangle\langle u_i|v\rangle \\
&= \left\langle \sum_{i=1}^{\infty} u_i\overline{\langle u|v_i\rangle} \middle| v \right\rangle \\
&= \left\langle \sum_{i=1}^{\infty} u_i\langle v_i|u\rangle \middle| v \right\rangle, \quad v\in V_R(\mathfrak{Q}).
\end{aligned}$$

This gives $u = \sum\limits_{i=1}^{\infty} u_i \langle v_i | u \rangle, \ u \in V_R(\mathfrak{Q})$. Then we have

$$\begin{aligned}
\|u\|^2 &= \sup_{\|v\|=1} |\langle u|v\rangle|^2 \\
&= \sup_{\|v\|=1} \left| \left\langle \sum_{i=1}^{\infty} u_i \langle v_i|u\rangle \middle| v \right\rangle \right|^2 \\
&\leq \sup_{\|v\|=1} \left| \sum_{i=1}^{\infty} \langle u|v_i\rangle \langle u_i|v\rangle \right|^2 \\
&\leq \sup_{\|v\|=1} \left(\sum_{i=1}^{\infty} |\langle u|v_i\rangle|^2 \right) \left(B\|v\|^2 \right) \\
&= B \sum_{i=1}^{\infty} |\langle v_i|u\rangle|^2, \quad u \in V_R(\mathfrak{Q}).
\end{aligned}$$

Hence, $\{v_i\}_{i\in\mathbb{N}}$ is frame for $V_R(\mathfrak{Q})$ with alternate dual frame $\{u_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$. □

In the next result, we show that among all the representations of an element $u \in V_R(\mathfrak{Q})$ in terms of a frame $\{u_i\}_{i\in\mathbb{N}}$ for $V_R(\mathfrak{Q})$ with coefficient sequence in $\ell^2(\mathfrak{Q})$, the sequence $\{\langle S^{-1}(u_i)|u\rangle\}_{i\in\mathbb{N}} \in \ell^2(\mathfrak{Q})$, has the minimum $\ell^2(\mathfrak{Q})$ - norm. Indeed we have the following:

Theorem 4.4. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with the frame operator S. Fix $\tilde{u} \in V_R(\mathfrak{Q})$, if $\tilde{u} = \sum\limits_{i=1}^{\infty} u_i q_i$, for some quaternion sequence $\{q_i\}_{i\in\mathbb{N}} \in \ell^2(\mathfrak{Q})$ then*

$$\sum_{i=1}^{\infty} |q_i|^2 = \sum_{i=1}^{\infty} |\langle S^{-1}u_i|\tilde{u}\rangle|^2 + \sum_{i=1}^{\infty} |\langle S^{-1}u_i|\tilde{u}\rangle - q_i|^2.$$

In particular, $\{\langle S^{-1}u_i|\tilde{u}\rangle\}_{i\in\mathbb{N}}$ has the minimal $\ell^2(\mathfrak{Q})$-norm among all quaternion sequences $\{q_i\}_{i\in\mathbb{N}} \in \ell^2(\mathfrak{Q})$.

Proof. For each $u \in V_R(\mathfrak{Q})$, we have

$$\begin{aligned}
\langle S^{-1}u|u\rangle &= \left\langle S^{-1}u \middle| \sum_{i=1}^{\infty} u_i \langle S^{-1}u_i|u\rangle \right\rangle \\
&= \sum_{i=1}^{\infty} \langle S^{-1}u|u_i\rangle \langle S^{-1}u_i|u\rangle \\
&= \left\langle \{\langle S^{-1}u_i|u\rangle\}_{i\in\mathbb{N}} | \{\langle S^{-1}u_i|u\rangle\}_{i\in\mathbb{N}} \right\rangle_{\ell^2(\mathfrak{Q})}.
\end{aligned}$$

Also

$$\begin{aligned}\langle S^{-1}\tilde{u}|\tilde{u}\rangle &= \left\langle S^{-1}\tilde{u}\middle|\sum_{i=1}^{\infty} u_i q_i\right\rangle\\ &= \sum_{i=1}^{\infty}\langle\tilde{u}|S^{-1}u_i\rangle q_i\\ &= \left\langle\{\langle S^{-1}u_i|\tilde{u}\rangle\}_{i\in\mathbb{N}}|\{q_i\}_{i\in\mathbb{N}}\right\rangle_{\ell^2(\mathfrak{Q})}.\end{aligned}$$

So, $\{q_i-\langle S^{-1}u_i|\tilde{u}\rangle\}_{i\in\mathbb{N}}$ is orthogonal to $\{\langle S^{-1}u_i|\tilde{u}\rangle\}_{i\in\mathbb{N}}$ in $\ell^2(\mathfrak{Q})$. Therefore, we have

$$\begin{aligned}\|\{q_i\}_{i\in\mathbb{N}}\|^2_{\ell^2(\mathfrak{Q})} &= \|\{q_i-\langle S^{-1}u_i|\tilde{u}\rangle\}+\{\langle S^{-1}u_i|\tilde{u}\rangle\}\|^2_{\ell^2(\mathfrak{Q})}\\ &= \|\{q_i-\langle S^{-1}u_i|\tilde{u}\rangle\}\|^2_{\ell^2(\mathfrak{Q})}+\|\{\langle S^{-1}u_i|\tilde{u}\rangle\}\|^2_{\ell^2(\mathfrak{Q})}.\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{\infty}|q_i|^2 = \sum_{i=1}^{\infty}|\langle S^{-1}u_i|\tilde{u}\rangle|^2+\sum_{i=1}^{\infty}|\langle S^{-1}u_i|\tilde{u}\rangle-q_i|^2.$$ □

Next, we give equivalent conditions for two frames in a quaternionic Hilbert space, where one becomes alternate dual of the other and vice versa, in terms of their corresponding analysis and synthesis operators. More precisely we have :

Theorem 4.5. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space. Let $\{u_i\}_{i\in\mathbb{N}}$ and $\{v_i\}_{i\in\mathbb{N}}$ be the frames for $V_R(\mathfrak{Q})$ with synthesis operators T and U, respectively. Then the following statements are equivalent*

(a). *$\{u_i\}_{i\in\mathbb{N}}$ is an alternate dual for $\{v_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$.*
(b). *$TU^*=\mathcal{I}$.*
(c). *$UT^*=\mathcal{I}$.*
(d). *$(T^*U)^2=T^*U$.*

Proof. $(a)\Rightarrow(b)$ For each $u\in V_R(\mathfrak{Q})$, we have

$$\begin{aligned}u &= \sum_{i=1}^{\infty} v_i\langle u_i|u\rangle\\ &= UT^*(u).\end{aligned}$$

$(b)\Rightarrow(c)$ Straight forward.
$(c)\Rightarrow(d)$ We have,

$$\begin{aligned}\left(T^*U\right)^2 &= T^*(UT^*)U\\ &= T^*\mathcal{I}U\\ &= T^*U.\end{aligned}$$

$(d)\Rightarrow(a)$ For each $\{q_i\}_{i\in\mathbb{N}}\in\ell^2(\mathfrak{Q})$, we have

$$T^*U(\{q_i\}_{i\in\mathbb{N}}) = \left\{\left\langle u_i\middle|\sum_{i=1}^{\infty} v_i q_i\right\rangle\right\}_{i\in\mathbb{N}}. \tag{4.1}$$

This gives

$$UT^*U\left(\{q_i\}_{i\in\mathbb{N}}\right) = \sum_{i=1}^{\infty} v_i \left\langle u_i \middle| \sum_{i=1}^{\infty} v_i q_i \right\rangle.$$

Therefore, we have

$$(T^*U)^2(\{q_i\}_{i\in\mathbb{N}}) = \left\{ \left\langle u_i \middle| \sum_{i=1}^{\infty} v_i \left\langle u_i \middle| \sum_{i=1}^{\infty} v_i q_i \right\rangle \right\rangle \right\}_{i\in\mathbb{N}}. \tag{4.2}$$

So (4.1) and (4.2) gives

$$\sum_{i=1}^{\infty} v_i q_i = \sum_{i=1}^{\infty} v_i \left\langle u_i \middle| \sum_{i=1}^{\infty} v_i q_i \right\rangle.$$

Again since $\{v_i\}_{i\in\mathbb{N}}$ is an frame for $V_R(\mathfrak{Q})$, therefore U is onto. Thus we have

$$v = \sum_{i=1}^{\infty} v_i \langle u_i | v\rangle, \quad v \in V_R(\mathfrak{Q}).$$

Hence $\{v_i\}_{i\in\mathbb{N}}$ is an alternate dual frame for $\{u_i\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$. □

Next, we give a result concerning a relationship between the analysis operator and the canonical dual of a frame in a right quaternionic Hilbert space. Further, an expression for the pseudo inverse of the synthesis operator of a frame in terms of its canonical dual is given.

Theorem 4.6. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with the frame operator S. Let T and T^* be the synthesis operator and the analysis operator, respectively for $\{u_i\}_{i\in\mathbb{N}}$ and $\widetilde{T}$ and $\widetilde{T}^*$ be the synthesis operator and the analysis operator, respectively for the canonical dual frame $\{S^{-1}(u_i)\}_{i\in\mathbb{N}}$. Then,*

(a) range (T^*) = range $(\widetilde{T}^*)$.
(b) the pseudo inverse $(T)^\dagger$ of the synthesis operator T is $\widetilde{T}^*$, i.e.
$$(T)^\dagger u = \{\langle S^{-1}(u_i)|u\rangle\}_{i\in\mathbb{N}}, \quad u \in V_R(\mathfrak{Q}).$$

Proof. (a) For each $u \in V_R(\mathfrak{Q})$, we have

$$\begin{aligned}(\widetilde{T}^*)u &= \{\langle S^{-1}(u_i)|u\rangle\}_{i\in\mathbb{N}} \\ &= \{\langle u_i|S^{-1}u\rangle\}_{i\in\mathbb{N}} \\ &= T^*S^{-1}u.\end{aligned}$$

Since S is a topological isomorphism, it follows that *range* $(T^*) =$ *range* $(\widetilde{T}^*)$.

(b) As $\ker(T)^\perp = range(T^*) = range(\widetilde{T}^*)$. Therefore, $T|_{\ker(T)^\perp}\colon range(T^*) \to V_R(\mathfrak{Q})$ is a topological isomorphism. Therefore the pseudo inverse $(T)^\dagger$ is $\left(T|_{\ker(T)^\perp}\right)^{-1}$. Further,

$$\begin{aligned}\left(T|_{\ker(T)^\perp}\right)\widetilde{T}^*u &= T\widetilde{T}^*u \\ &= \sum_{i=1}^{\infty} u_i\langle S^{-1}u_i|u\rangle \\ &= u, \quad u \in V_R(\mathfrak{Q}).\end{aligned}$$

Furthermore, for each $q \in range\ (T^*) = range\ (\widetilde{T}^*)$, there exist $u \in V_R(\mathfrak{Q})$ such that $q = \widetilde{T}^*u$. So, by Theorem 4.5, we have

$$\begin{aligned}\widetilde{T}^*\left(T|_{\ker(T)^\perp}\right)q &= \widetilde{T}^*T\widetilde{T}^*u \\ &= \widetilde{T}^*u.\end{aligned}$$

Hence $\widetilde{T}^* = \left(T|_{\ker(T)^\perp}\right)^{-1}$. □

Finally in this section, with a given frame and its canonical dual frame for a quaternionic Hilbert space, we characterize frame and its canonical dual frame for a given closed subspace of the quaternionic Hilbert space.

Theorem 4.7. *Let $V_R(\mathfrak{Q})$ be a quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with frame operators S. Let $\mathcal{P}$ be the orthogonal projection of $V_R(\mathfrak{Q})$ onto a closed subspace $\mathcal{M}$ of $V_R(\mathfrak{Q})$. Then*

(a) $\{\mathcal{P}u_i\}_{i\in\mathbb{N}}$ is frame for $\mathcal{M}$ with the same frame bounds as $\{u_i\}_{i\in\mathbb{N}}$ and $\{\mathcal{P}S^{-1}(u_i)\}_{i\in\mathbb{N}}$ is an alternative dual frame for $\{\mathcal{P}u_i\}_{i\in\mathbb{N}}$.

(b) $\{\mathcal{P}S^{-1}(u_i)\}_{i\in\mathbb{N}}$ is canonical dual frame for $\{\mathcal{P}u_i\}_{i\in\mathbb{N}}$ if and only if $\mathcal{P}S = S\mathcal{P}$.

Proof. (a) For each $u \in \mathcal{M}$, we have

$$\sum_{i=1}^{\infty}|\langle\mathcal{P}u_i|u\rangle|^2 = \sum_{i=1}^{\infty}|\langle u_i|u\rangle|^2$$

Therefore, $\{\mathcal{P}u_i\}_{i\in\mathbb{N}}$ is frame for $\mathcal{M}$ with same frame bounds as that of $\{u_i\}_{i\in\mathbb{N}}$ and by the similar argument, $\{\mathcal{P}S^{-1}(u_i)\}_{i\in\mathbb{N}}$ is also a frame for $\mathcal{M}$ with frame bounds inverse of $\{u_i\}_{i\in\mathbb{N}}$. Further, for each $u \in \mathcal{M}$

$$\begin{aligned}\sum_{i=1}^{\infty}\mathcal{P}u_i\langle\mathcal{P}S^{-1}u_i|u\rangle &= \mathcal{P}\left(\sum_{i=1}^{\infty}u_i\langle S^{-1}u_i|\mathcal{P}u\rangle\right) \\ &= u.\end{aligned}$$

Hence, the result follows.

(b) Let V be the frame operator for $\{\mathcal{P}u_i\}_{i\in\mathbb{N}}$ as a frame for $\mathcal{M}$. Then, we have

$$V^{-1}\mathcal{P}(u_i) = \mathcal{P}S^{-1}(u_i), \quad i \in \mathbb{N}.$$

This gives

$$V^{-1}\mathcal{P}(u) = \mathcal{P}S^{-1}(u), \quad u \in V_R(\mathfrak{Q}).$$

Since S and V are topological isomorphisms on $V_R(\mathfrak{Q})$ and $\mathcal{M}$, respectively, therefore we have

$$\begin{aligned}\mathcal{P}S &= VV^{-1}\mathcal{P}S\\ &= V\mathcal{P}S^{-1}S\\ &= V\mathcal{P}.\end{aligned}$$

Since $\mathcal{P}$, S and V are self-adjoint, so we have $S\mathcal{P}=\mathcal{P}V$. Hence, we have $S\mathcal{P}=\mathcal{P}S$.

Conversely, we have

$$\begin{aligned}V\mathcal{P}u &= \sum_{i=1}^{\infty}\mathcal{P}u_i\langle\mathcal{P}u_i|\mathcal{P}u\rangle\\ &= \mathcal{P}\bigg(\sum_{i=1}^{\infty}u_i\langle u_i|\mathcal{P}^2u\rangle\bigg)\\ &= \mathcal{P}\bigg(\sum_{i=1}^{\infty}u_i\langle u_i|\mathcal{P}u\rangle\bigg)\\ &= \mathcal{P}S\mathcal{P}u\\ &= S\mathcal{P}u, \quad u\in V_R(\mathfrak{Q}).\end{aligned}$$

Therefore, for each $u\in\mathcal{M}$, we have

$$\begin{aligned}Vu &= V\mathcal{P}u\\ &= S\mathcal{P}u\\ &= Su.\end{aligned}$$

This gives $V=S|_{\mathcal{M}}$. Thus S maps $\mathcal{M}$ bijectively onto itself. Therefore we have

$$\begin{aligned}V^{-1}\mathcal{P}(u_i) &= (S|_{\mathcal{M}})^{-1}\mathcal{P}u_i\\ &= S^{-1}\mathcal{P}(u_i)\\ &= \mathcal{P}S^{-1}(u_i).\end{aligned}$$

Hence the canonical dual frame for $\{\mathcal{P}u_i\}_{i\in\mathbb{N}}$ is $\{\mathcal{P}S^{-1}(u_i)\}_{i\in\mathbb{N}}$ in $V_R(\mathfrak{Q})$. □

5. Application

In this section, as an application of the canonical dual of a given frame, we give the orthogonal projection of $\ell^2(\mathfrak{Q})$ onto the range of the analysis operator of the frame, in terms of the elements of the frame and its canonical dual. Apart from this, we also give an expression for the orthogonal projection in terms of operators related to the frame and its canonical dual in a right quaternionic Hilbert space.

Theorem 5.1. *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a frame for $V_R(\mathfrak{Q})$ with the synthesis operator T and the frame operator S. Let $\tilde{T}$ and $\tilde{S}$ be the synthesis operator and the frame operator for the canonical dual frame $\{S^{-1}u_i\}_{i\in\mathbb{N}}$ of $\{u_i\}_{i\in\mathbb{N}}$. Then the orthogonal projection Q of $\ell^2(\mathfrak{Q})$ onto the range of T^* is given by*

$$Q\Big(\{q_i\}_{i\in\mathbb{N}}\Big)=\bigg\{\bigg\langle u_j\bigg|\sum_{i=1}^{\infty}S^{-1}u_iq_i\bigg\rangle\bigg\}_{j\in\mathbb{N}}.$$

Also,

$$Q = T^*\tilde{T} = T^*S^{-1}T.$$

Proof. For $q = \{q_i\}_{i\in\mathbb{N}} \in \ell^2(\mathfrak{Q})$,

$$\begin{aligned} u &= S^{-1}Su \\ &= \sum_{i=1}^{\infty} S^{-1}u_i\langle u_i|u\rangle, \quad u \in V_R(\mathfrak{Q}). \end{aligned}$$

Moreover, we have $T(q) = \sum_{i=1}^{\infty} u_i q_i$ and $T^*u = \{\langle u_i|u\rangle\}_{i\in\mathbb{N}}$. It is sufficient to show that Q is the identity on *range* T^* and is zero on $(range\ T^*)^{\perp} = \ker T$. By definition

$$\begin{aligned} Q(T^*u) &= Q\left(\{\langle u_i|u\rangle\}_{i\in\mathbb{N}}\right) \\ &= \left\{\left\langle u_j \middle| \sum_{i=1}^{\infty} S^{-1}u_i\langle u_i|u\rangle \right\rangle\right\}_{j\in\mathbb{N}} \\ &= \{\langle u_j|u\rangle\}_{j\in\mathbb{N}} \\ &= T^*u, \quad u \in V_R(\mathfrak{Q}). \end{aligned}$$

Again, for $q = \{q_i\}_{j\in\mathbb{N}} \in (range\ T^*)^{\perp} = \ker\ T$

$$\begin{aligned} Q(q) &= Q\bigg(\{q_i\}_{i\in\mathbb{N}}\bigg) \\ &= \left\{\left\langle u_j \middle| \sum_{i=1}^{\infty} S^{-1}u_i q_i \right\rangle\right\}_{j\in\mathbb{N}} \\ &= \left\{\left\langle u_j \middle| S^{-1}\sum_{i=1}^{\infty} u_i q_i \right\rangle\right\}_{j\in\mathbb{N}} \\ &= \left\{\langle u_j|S^{-1}T(q)\rangle\right\}_{j\in\mathbb{N}} \\ &= 0. \end{aligned}$$

Thus, Q is the orthogonal projection of $\ell^2(\mathfrak{Q})$ onto the range of T^*. Further, we have

$$\begin{aligned} T^*S^{-1}T(q) &= T^*\tilde{T}(q) \\ &= T^*\left(\sum_{i=1}^{\infty} S^{-1}u_i q_i\right) \\ &= \left\{\left\langle u_j \middle| \sum_{i=1}^{\infty} S^{-1}u_i q_i \right\rangle\right\}_{j\in\mathbb{N}} \\ &= Q(q). \end{aligned}$$

This completes the proof. □

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