SOME RING-THEORETIC PROPERTIES OF THE RING OF $\mathcal{R}L_\tau$

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ABSTRACT. The aim of this article is to survey ring-theoretic properties of Kasch, the regularity and the injectivity of the ring of real-continuous functions on a topoframe L_τ, i.e., $\mathcal{R}L_\tau$. In order to study these properties, the concept of P-spaces and extremally disconnected spaces are extend to topoframes. For a P-topoframe L_τ, the ring $\mathcal{R}L_\tau$ is $\aleph_0$-Kasch ring. P- topoframes are characterized in terms of ring-theoretic properties of the regularity and injectivity of the ring of real-continuous functions on a topoframe. It follows from these characterizations that the ring $\mathcal{R}L_\tau$ is regular if and only if it is $\aleph_0$-selfinjective. For a completely regular topoframe L_τ, we show that $\mathcal{R}L_\tau$ is a Bear ring if and only if it is a CS-ring if and only if L_τ is extremally disconnected and also prove that it is selfinjective ring if and only if L_τ is an extremally disconnected P-topoframe.

1. INTRODUCTION

The principal tool to be used is *modified pointfree topology*, i.e., *topoframe*, first introduced in [7] as follows.

A topoframe, denoted by L_τ, is a pair (L, τ) consisting of a frame L and a subframe τ all of whose elements are complemented elements in L.

The elements of τ are called open elements in L and τ and the elements of $\tau' = \{t' \mid t \in \tau\}$ are called closed elements in L. Also in [7], authors introduced the concept of a τ-*real-continuous function* on a frame L (or a *real-continuous function* on L_τ) and showed the set of all real-continuous functions on L_τ, denoted by $\mathcal{R}L_\tau$, is an f-ring is actually a generalization of the ring $C(X)$ of all real-valued continuous functions on a completely regular Hausdorff space X.

Recall from [1] that the frame $\mathcal{L}(\mathbb{R})$ of reals is obtained by taking the ordered pairs (p, q) of rational numbers as generators. Now, *the real-valued continuous functions* on L are the homomorphisms $\mathcal{L}(\mathbb{R}) \to L$. The ring $\mathcal{R}L$ of all frame homomorphisms from $\mathcal{L}(\mathbb{R})$ to L, i.e., $\mathcal{R}L = \mathbf{Frm}(\mathcal{L}(\mathbb{R}), L)$, is an f-ring (see [1]. For a topoframe L_τ, the ring $\mathcal{R}L_\tau$ is isomorphic to a sub-f-ring of $\mathcal{R}\tau$ (see [7] for details).

Recall from [13] that a *real-valued function* on a frame L is a homomorphism $f : \mathcal{P}(\mathbb{R}) \to L$ and let $\mathcal{F}_\mathcal{P}L$ be the ring of all real-valued functions on a frame L,

i.e., $\mathcal{F}_{\mathcal{P}}L = \mathbf{Frm}(\mathcal{P}(\mathbb{R}), L)$. For a topoframe L_τ, it is proved in [7] that $\mathcal{R}L_\tau$ is a sub-f-ring of $\mathcal{F}_{\mathcal{P}}L$.

By a *reduced ring* we mean a ring without nonzero nilpotent elements. In [11, 12] for a reduced ring A, some internal conditions on A are equivalent to self-injectivity ($\aleph_0$-selfinjective) of A are provided. Since $\mathcal{R}L_\tau$ is always a reduced ring, we can use these conditions to investigate the injectivity of the ring $\mathcal{R}L_\tau$. Using these conditions, authors in [5] shown that for a space X, the ring $C(X)$ is $\aleph_0$-selfinjective if and only if X is a P-space. They proved also that $C(X)$ is self-injective if and only if X is an extremally disconnected P-space. We are going to extend these results to the more general setting of modified pointfree topology.

This article is composed of three original parts which are partly discrete but for introduction and preliminaries. In the following, these parts are clearly explained.

In Section 3, for a element a in τ with $a' \in \tau'$, we make a idempotent element f_a of $\mathcal{R}L_\tau$ such that $z(f_a) = a'$ and calculate the multiplication $f f_a$ for a element $f \in \mathcal{R}L_\tau$ in Proposition 1. This proposition enables us to prove that if $f \in \mathcal{R}L_\tau$ and $z(f) \in \tau$, then f is a zerodivisor element and the principal ideal (f) of $\mathcal{R}L_\tau$ is a nonessential ideal (see Proposition 2). Therefore, by this proposition, we can conclude that $\mathcal{R}L_\tau$ has no proper regular ideal (Corollary 1). Finally, the concept of P-spaces (or P-frames) is extend to topoframes and in Theorem 1, it is proved that the ring $\mathcal{R}L_\tau$ is a $\aleph_0$-Kasch ring whenever L_τ is a P-topoframe.

The regularity of the ring $\mathcal{R}L_\tau$ is examined in Section 4. We prove in alone theorem of this section that $\mathcal{R}L_\tau$ is a regular ring if and only if L_τ is P-topoframe, see Theorem 2.

Eventually, the injectivity of the ring $\mathcal{R}L_\tau$ is discussed in the last section. First, in the Theorem 3, we show that the topoframe L_τ is a P-topoframe if and only if the ring $\mathcal{R}L_\tau$ is $\aleph_0$-selfinjective. Also it is proved that if L is an extremally disconnected frame and L_τ is a P-topoframe then $\mathcal{R}L_\tau$ is a selfinjective ring, but the converse is not true, see Proposition 5. Afterwards, we define extremally disconnected topoframes and show for a topoframe L_τ, τ is a extremally disconnected frame if and only if L_τ is extremally disconnected (Proposition 4). Also for a completely regular topoframe L_τ, in Proposition 5, it is proved that L_τ is extremally disconnected if and only if $\mathcal{R}L_\tau$ is a Bear ring if and only if it is a CS-ring. Finally we characterize extremally disconnected P-topoframes in terms of ring-theoretic properties of the ring $\mathcal{R}L_\tau$ and show L_τ is an extremally disconnected P-topoframe if and only if $\mathcal{R}L_\tau$ is a self injective ring if and only if $\mathcal{R}L_\tau$ is a Baer regular ring if and only if $\mathcal{R}L_\tau$ is a continuous regular ring if and only if $\mathcal{R}L_\tau$ is a complete regular ring, whenever L_τ is a completely regular topoframe, see Theorem 4.

2. PRELIMINARIES

A good reference to the basic terms and notations in frames is [15]. For undefined terms and notations see [1] on pointfree functions rings, and see [9] on $C(X)$. Also [7, 17] are valuable references on topoframes and the ring of real-continuous functions on a topoframe. Here we recall a few facts about frames, topoframes and their rings of real-continuous functions that will be linked for our discussion.

A *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. The top element and the bottom element of L are denoted by $\top_L$ and $\bot_L$ respectively; omitting the subscripts if no bewilderment may happen. Throughout this context L will denote a frame and topoframe (L, τ) is denoted by L_τ. $\mathfrak{O}X$ is the frame of open subsets of a topological space X.

A *frame homomorphism* (or *frame map*) is a map $f : L \to M$ between two frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

The *pseudocomplement* of an element $x \in L$ is denoted by x^*. General properties of pseudocomplement can be found in [15]. Here we emphasize some of them.

(1) $a \le a^{**}$ and if $a \le b$, then $b^* \le a^*$.
(2) $(\bigvee_{i \in I} a_i)^* = \bigwedge_{i \in I} a_i^*$, the first De Morgan law.
(3) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

An element x of L is *complemented* whenever $x \vee x^* = \top$ and in this case $a' = a^*$.

The homomorphism $\eta : \mathcal{L}(\mathbb{R}) \to O\mathbb{R}$ given by $(p, q) \mapsto]\!]p, q[\![$ is an isomorphism, where $]\!]p, q[\![:= \{x \in \mathbb{R} : p < x < q\}$. Now, a *$\tau$-real-continuous function* on L (or a *real-continuous function* on L_τ) is a frame-homomorphism $f : \mathcal{P}(\mathbb{R}) \to L$ such that for all $p, q \in \mathbb{Q}$, $f(]\!]p, q[\![) \in \tau$. The ring $\mathcal{R}L_\tau$ has as its elements real-continuous functions on L_τ with operations determined by the operations of $\mathbb{R}$ viewed as an f-ring as follows (see [13]).

For $\diamond \in \{+, ., \wedge, \vee\}$ and $f, g \in \mathcal{R}L_\tau$,

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) \mid Y \diamond Z \subseteq X\},$$

where $Y \diamond Z = \{y \diamond z \mid y \in Y, z \in Z\}$.

For any $f \in \mathcal{R}L_\tau$ and $X \subseteq \mathbb{R}$, $(-f)(X) = f(-X)$ and for any $r \in \mathbb{R}$, the constant function $\mathbf{r}$ is the member of $\mathcal{R}L_\tau$ given by

$$\mathbf{r}(X) = \begin{cases} \top & \text{if } r \in X, \\ \bot & \text{if } r \notin X. \end{cases}$$

Also for any $f, g \in \mathcal{R}L_\tau$, we have

(1) $(f \diamond g)(X) = \bigvee \{f(\{y\}) \wedge g(\{z\}) \mid y \diamond z \in X\}$, for every $X \in \mathcal{P}(\mathbb{R})$.
(2) $f = g$ if and only if $f(\{r\}) = g(\{r\})$ for every $r \in \mathbb{R}$.

An important link between a topoframe L_τ and its ring of τ-real-continuous function on L_τ given by the *zero map* $z : \mathcal{R}L_\tau \to L$ taking every $f \in \mathcal{R}L_\tau$ to $z(f) = f(\{0\})$. The zero map has several important properties (see [18]) that we emphasize some of them. For every $f, g \in \mathcal{R}L_\tau$, we have.

(1) For every $n \in \mathbb{N}$, $z(f) = z(-f) = z(|f|) = z(f^n)$,
(2) $z(fg) = z(f) \vee z(g)$,
(3) $z(f + g) \ge z(f) \wedge z(g)$,
(4) $z(f + g) = z(f) \wedge z(g)$, while $f, g \ge \mathbf{0}$,
(5) $z(f) = \top$ if and only if $f = \mathbf{0}$,
(6) $z(f) = \bot$ if and only if f is a unit element of $\mathcal{R}L_\tau$.

A *zero element* of L_τ is an element of the form $z(f)$ for some $f \in \mathcal{R}L_\tau$. The *zero part* of L_τ, denoted by $Z(\mathcal{R}L_\tau)$ or $Z(L_\tau)$, is the regular sub-co-σ-frame consisting of all the zero elements of L_τ. Also a *cozero-element* of L_τ is defined by $coz(f) := f(\mathbb{R} \setminus \{0\})$ for some $f \in \mathcal{R}L_\tau$. Obviously, $z(f) = (coz(f))'$. Note that $z(f)$ and $coz(f)$ are a closed element and an open element in L, respectively. General properties of zero elements and cozero-elements of topoframes can be found in [6]. Here we highlight the following.

(1) If $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}L_\tau$, then there is a $f \in \mathcal{R}L_\tau$ such that

$$\bigvee_{n\in\mathbb{N}} coz(f_n) = coz(f) \text{ and } \bigwedge_{n\in\mathbb{N}} z(f_n) = z(f).$$

(2) For every $f \in \mathcal{R}L_\tau$, $coz(f) = coz(f \mid_{\mathfrak{O}\mathbb{R}}) \in \tau$.

3. Kasch of $\mathcal{R}L_\tau$

An ideal I of a ring A, throughout, by the term ring we mean a commutative ring with identity, is called *essential* in A if $I \cap J \neq (0)$ holds for every non-zero ideal J of A.

Let α be a cardinal number. An ideal I in a ring A is called α-*generated* if it has got a generating set G such that $|G| \leq \alpha$. The least element in the set of cardinal numbers of all generating set of I is denoted by $gen(I)$. A ring A is said to be an α-*Kasch ring* if for any proper ideal I with $gen(I) < \alpha$, then I is a non-essential ideal (see [4]).

We need the following proposition which will play a central role in the development of this article, but we omit its proof for it is similar to the proof of Proposition 3.1 in [8].

Proposition 1. *Let L_τ be a topoframe and a be an element of τ such that $a' \in \tau$. Then $f_a : \mathcal{P}(\mathbb{R}) \to L$ defined by*

$$f_a(X) = \begin{cases} \top & \text{if } 0, 1 \in X \\ a' & \text{if } 0 \in X \text{ and } 1 \notin X \\ a & \text{if } 0 \notin X \text{ and } 1 \in X \\ \bot & \text{if } 0 \notin X \text{ and } 1 \notin X, \end{cases}$$

is a real-continuous functions on L_τ and the following statements hold.

(1) *$f_a^2 = f_a$, $z(f_a) = a'$ and $f_a + f_{a'} = \mathbf{1}$.*
(2) *For every $f \in \mathcal{R}L_\tau$ and $X \in \mathcal{P}(\mathbb{R})$,*

$$ff_a(X) = \begin{cases} a' \vee f(X) & \text{if } 0 \in X, \\ a \wedge f(X) & \text{if } 0 \notin X. \end{cases}$$

(3) *For each pair of complemented elements $a, b \in L$ such that $a, a', b, b' \in \tau$, $f_a f_b = f_{a \wedge b}$.*

Lemma 1. *For $\{f_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{R}L_\tau$, the following statements hold.*

(1) *$\left(\bigvee_{\lambda\in\Lambda} coz(f_\lambda)\right)' = \bigwedge_{\lambda\in\Lambda} z(f_\lambda)$.*
(2) *If $a = \bigvee_{\lambda\in\Lambda} coz(f_\lambda)$ and $a' \in \tau$, then $f_\lambda f_a^n = f_\lambda$ and $f_\lambda f_{a'}^n = \mathbf{0}$.*

Proof. (1) By the first De Morgan law, we have.

$$\left(\bigvee_{\lambda\in\Lambda} coz(f_\lambda)\right)' = \left(\bigvee_{\lambda\in\Lambda} coz(f_\lambda)\right)^* = \bigwedge_{\lambda\in\Lambda} (coz(f_\lambda)^* = \bigwedge_{\lambda\in\Lambda} (coz(f_\lambda))' = \bigwedge_{\lambda\in\Lambda} z(f_\lambda).$$

(2) Consider $\lambda \in \Lambda$ and $X \subseteq \mathbb{R}$. If $0 \notin X$, then $f_\lambda(X) \leq coz(f_\lambda) \leq a$, we can conclude that

$$f_\lambda f_a(X) = a \wedge f_\lambda(X) = f_\lambda(X)$$

and

$$f_\lambda f_{a'}(X) = a' \wedge f_\lambda(X) = \bot.$$

Also if $0 \in X$, then $f_\lambda(X) = f_\lambda(X) \vee a'$, we conclude that

$$f_\lambda f_a(X) = a' \vee f_\lambda(X) = f_\lambda(X)$$

and

$$f_\lambda f_{a'}(X) = a \vee f_\lambda(X) = a \vee (f_\lambda(X) \vee a') = \top.$$

Hence $f_\lambda f_a = f_\lambda$ and $f_\lambda f_{a'} = \mathbf{0}$. $\qquad\square$

Proposition 2. *The following statements hold for every $f \in \mathcal{R}L_\tau$.*

(1) *For every $n \in \mathbb{N}$, if $z(f) \in \tau$, then $f f_{coz(f)}^n = f$ and $f f_{z(f)}^n = \mathbf{0}$.*
(2) *If f is a nonunit element in $\mathcal{R}L_\tau$ and $z(f) \in \tau$, then f is a zerodivisor.*
(3) *If (f) is a proper ideal in $\mathcal{R}L_\tau$ and $z(f) \in \tau$, then (f) is a non-essential ideal.*

Proof. (1) By Lemma 1, it is trivial.

(2) Let $\mathbf{0} \neq f$ be a nonunit element in $\mathcal{R}L_\tau$. Then $coz(f) \neq \bot$ and $z(f) \neq \top$, we conclude from the statement (1) that $f f_{z(f)} = \mathbf{0}$, where $f_{z(f)} \neq \mathbf{0}$.

(3) It is trivial by the statement (2). $\qquad\square$

Corollary 1. *If $\theta : \mathcal{R}L_\tau \longrightarrow \mathcal{R}L_\tau$ is a $\mathcal{R}L_\tau$-monomorphism and $z(\theta(1)) \in \tau$, then θ is an isomorphism.*

Proof. Since $\theta(g) = g\theta(1)$ for all $g \in \mathcal{R}L_\tau$, we conclude that $\theta(1)$ is non-zero-divisor. It follows, by Proposition 2(2), that $\theta(1)$ is a unit, i.e., there is a $h \in \mathcal{R}L_\tau$ such that $\theta(h) = h\theta(1) = 1$. Thus $\theta(fh) = f$, for every $f \in \mathcal{R}L_\tau$. Consequently θ is also an epimorphism and the proof is complete. $\qquad\square$

Corollary 2. *Every countably generated ideal I in $\mathcal{R}L_\tau$ with*

$$\bot \neq \left(\bigvee_{f\in I} coz(f)\right)' \in \tau$$

is in a non-essential principal ideal.

Proof. Let $I = (f_1, f_2, \ldots)$ be a countably generated ideal in $\mathcal{R}L_\tau$ such that

$$\bigvee_{n\in\mathbb{N}} coz(f_n) = \bigvee_{f\in I} coz(f) \neq \top.$$

Then, there is an element $f \in \mathcal{R}L_\tau$ such that $\bigvee_{n\in\mathbb{N}} coz(f_n) = coz(f)$, which follows from Proposition 2 that $(f_{coz(f)})$ is a non-essential ideal. For every $n \in \mathbb{N}$, by Lemma 1, $f_n = f_n f_{coz(f)} \in (f_{coz(f)})$. Hence $I \subseteq (f_{coz(f)})$ and the proof is complete.

$\qquad\square$

Let A be a ring and $x \in A$. Then

(a) x is called a regular element in A if $xy = 0$ and $y \in A$ implies $x = 0$, and

(b) an ideal I of A is called regular if it contains a regular element in A.

Corollary 3. $\mathcal{R}L_\tau$ *has no proper non-zero regular ideal.*

Proof. It is trivial from Proposition 2.   □

Recall that a P- space (P-frame) is one in which every zero set (cozero element) is open (complemented). Now, we are going to extend these concepts on topoframes.

Definition 1. If $Z(L_\tau) \subseteq \tau$, then L_τ is called a P-topoframe.

Let A be a ring and $S \subseteq A$. We denote the annihilator of S by $Ann(S)$.

Theorem 1. *inj17 If L_τ is a P-topoframe, then the ring $\mathcal{R}L_\tau$ is a $\aleph_0$-Kasch ring.*

Proof. Let $I = (f_1, \ldots, f_n)$ be a proper finitely generated ideal in $\mathcal{R}L_\tau$. Since $\sum_{i=1}^n f_i^2 \in I$ is not unit, we conclude from Proposition 2 that there exists $\mathbf{0} \neq g \in \mathcal{R}L_\tau$ such that $g \sum_{i=1}^n f_i^2 = \mathbf{0}$, which follows that

$$\top = z(0) = z\left(g \sum_{i=1}^n f_i^2\right) = \bigwedge_{i=1}^n z(gf_i).$$

Then $gf_i = \mathbf{0}$, because $z(gf_i) = \top$, for every $1 \leq i \leq n$. Hence $g \in Ann(I) \neq (0)$ and we infer that I is a non-essential ideal in $\mathcal{R}L_\tau$. Therefore $\mathcal{R}L_\tau$ is a $\aleph_0$-Kasch ring.   □

4. REGULARITY OF $\mathcal{R}L_\tau$

A ring A is said to be regular (in the sense of Von Neumann) if for every $x \in A$ there is $y \in A$ with $x = x^2 y$. It is shown that for a completely regular frame L, $\mathcal{R}L$ is a regular ring if and only if L is a P-frame (see [3, Proposition 3.9]). The purpose of this section is to extend this result to topoframes.

Remark 1. Let e be an idempotent element of $\mathcal{R}L_\tau$. Then $e(\{x\}) = \bot$, for every $x \in \mathbb{R} \setminus \{0, 1\}$. Hence $z(e) = e(\{x \in \mathbb{R} : -1 < x < 1\}) \in \tau$ and $e = f_{coz(e)}$.

Remark 2. Let a be an element of a regular ring of A. Then there is an element $x \in A$ such that $a = xa^2$ and $a = ba^2$, where $b = ax^2$. If $u := 1 + b - ab$, then u is a unit element of A and au is an idempotent element of A.

Theorem 2. *For a topoframe L_τ, the following statements are equivalent.*

(1) L_τ *is a P-topoframe.*

(2) $\mathcal{R}L_\tau$ *is a regular ring.*

Proof. $(1) \Rightarrow (2)$. Consider $f \in \mathcal{R}L_\tau$. We define $g : \mathcal{P}(\mathbb{R}) \to L$ by

$$g(X) = \begin{cases} \bigvee_{0 \neq x \in X} f(\{\frac{1}{x}\}) \vee z(f) & \text{if } 0 \in X \\ \bigvee_{x \in X} f(\{\frac{1}{x}\}) & \text{if } 0 \notin X \end{cases}$$

for every $X \subseteq \mathbb{R}$. Then $g(\emptyset) = \bigvee_{x \in \emptyset} f(\{\frac{1}{x}\}) = \bot$ and $g(\mathbb{R}) = \bigvee_{0 \neq x \in \mathbb{R}} f(\{\frac{1}{x}\}) \vee z(f) = coz(f) \vee z(f) = \top$.

Consider $X, Y \subseteq \mathbb{R}$. If $0 \in X \cap Y$, then

$$
\begin{aligned}
g(X) \wedge g(Y) &= [\bigvee_{0 \neq x \in X} f(\{\tfrac{1}{x}\}) \vee z(f)] \wedge [\bigvee_{0 \neq y \in Y} f(\{\tfrac{1}{y}\}) \vee z(f)] \\
&= \bigvee_{0 \neq x \in X, 0 \neq y \in Y} f(\{\tfrac{1}{x}\} \cap \{\tfrac{1}{y}\}) \vee z(f) \\
&= \bigvee_{0 \neq x \in X \cap Y} f(\{\tfrac{1}{x}\}) \vee z(f) \\
&= g(X \cap Y).
\end{aligned}
$$

If $0 \in X$ and $0 \notin Y$, then

$$
\begin{aligned}
g(X) \wedge g(Y) &= [\bigvee_{0 \neq x \in X} f(\{\tfrac{1}{x}\}) \vee z(f)] \wedge \bigvee_{y \in Y} f(\{\tfrac{1}{y}\}) \\
&= \bigvee_{0 \neq x \in X, y \in Y} f(\{\tfrac{1}{x}\} \cap \{\tfrac{1}{y}\}) \vee [z(f) \wedge \bigvee_{y \in Y} f(\{\tfrac{1}{y}\})] \\
&= \bigvee_{x \in X \cap Y} f(\{\tfrac{1}{x}\}) \vee \bot \\
&= g(X \cap Y).
\end{aligned}
$$

If $0 \notin X$ and $0 \notin Y$, then

$$
\begin{aligned}
g(X) \wedge g(Y) &= \bigvee_{x \in X} f(\{\tfrac{1}{x}\}) \wedge \bigvee_{y \in Y} f(\{\tfrac{1}{y}\}) \\
&= \bigvee_{x \in X \cap Y} f(\{\tfrac{1}{x}\}) \\
&= g(X \cap Y).
\end{aligned}
$$

Hence g preserves all finite meets. Consider $\{X_i\}_{i \in I} \subseteq P(\mathbb{R})$. If $0 \notin \bigcup_{i \in I} X_i$, then

$$
\begin{aligned}
\bigvee_{i \in I} g(X_i) &= \bigvee_{i \in I} \bigvee_{x \in X_i} f(\{\tfrac{1}{x}\}) \\
&= \bigvee_{x \in \bigcup_{i \in I} X_i} f(\{\tfrac{1}{x}\}) \\
&= g(\bigcup_{i \in I} X_i).
\end{aligned}
$$

If $0 \in \bigcup_{i \in I} X_i$, then

$$
\begin{aligned}
\bigvee_{i \in I} g(X_i) &= \bigvee_{\substack{i \in I, \\ 0 \notin X_i}} \bigvee_{x \in X_i} f(\{\tfrac{1}{x}\}) \vee [z(f) \vee \bigvee_{\substack{i \in I, \\ 0 \in X_i}} \bigvee_{0 \neq x \in X_i} f(\{\tfrac{1}{x}\})] \\
&= z(f) \vee \bigvee_{0 \neq x \in \bigcup_{i \in I} X_i} f(\{\tfrac{1}{x}\}) \\
&= g(\bigcup_{i \in I} X_i).
\end{aligned}
$$

Hence g preserves arbitrary joins. Consider $p, q \in Q$ with $p < q$. If $p < 0 < q$, then $\{\frac{1}{x} : p < x < q, x \neq 0\} = \mathbb{R} \setminus \{r \in \mathbb{R} : \frac{1}{p} \leq r \leq \frac{1}{q}\}$ and since $f \in \mathcal{R}L_\tau$, we conclude from $z(f) \in \tau$ that $g(]\!]p, q[\![) \in \tau$. If $0 \notin]\!]p, q[\![$, then $g(]\!]p, q[\![) = f(]\!]\frac{1}{q}, \frac{1}{p}[\![) \in \tau$. Therefore, $g \in \mathcal{R}L_\tau$. Since $z(f) = z(g)$, we conclude that $z(f) = z(gf^2)$. We claim that $f = gf^2$. In order to prove our claim, we consider $0 \neq x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
gf^2(\{x\}) &= \bigvee\{g(\{y\}) \wedge f^2(\{\tfrac{x}{y}\}) : 0 \neq y \in \mathbb{R}\} \\
&= \bigvee\{f(\{\tfrac{1}{y}\}) \wedge f(\{z\}) \wedge f(\{\tfrac{x}{zy}\}) : y, z \in \mathbb{R} \setminus \{0\}\} \\
&= f(\{x\}).
\end{aligned}
$$

Therefore, $\mathcal{R}L_\tau$ is a regular ring.

(2) $\Rightarrow$ (1). Consider $f \in \mathcal{R}L_\tau$. By Remark 2, there is a unit element of g in $\mathcal{R}L_\tau$ such that fg is an idempotent element of $\mathcal{R}L_\tau$. By Remark 1, we have $z(f) = z(fg) \in \tau$. Therefore, L_τ is a P-topoframe. $\qquad\square$

5. Injectivity of $\mathcal{R}L_\tau$

A ring A is said to be *self injective* ($\aleph_0$-*selfinjective*) if every A-homomorphism from an ideal (a countably generated ideal) of A to A can be extended to an A-homomorphism from A to A. The principal purposes in this section are to find the properties of a topoframe L_τ which are equivalent to the ring-theoretic properties of $\mathcal{R}L_\tau$, in particular: $\aleph_0$-selfinjectivity and selfinjectivity. First, we investigate the $\aleph_0$-selfinjectivity of $\mathcal{R}L_\tau$. In order to survey this property we need some background.

A subset S of a ring A is said to be *orthogonal* provided $xy = 0$ for all $x, y \in S$ with $x \neq y$. If $S \cap T = \emptyset$ and $S \cup T$ is an orthogonal set in A, then $a \in A$ is said to *separate S from T* if $a \in Ann(T)$ and $s^2 a = s$, for every $s \in S$ (see [11]). In [12] it is shown that there exists an element in A which separates S from T if and only if there is an element b in A such that $b \in Ann(T)$ and $s^2 = sb$, for every $s \in S$. For the proof of the next lemma see [11, Theorem 2.2], and [12, Proposition 1.2].

Lemma 2. *Let A be a reduced ring, then the following statements are equivalent.*

(1) *The ring A is selfinjective ($\aleph_0$-selfinjective).*
(2) *The ring A is a regular ring and whenever $S \cup T$ is an orthogonal (countable) set with $S \cap T = \emptyset$, then there exists an element in A which separates S from T.*

Lemma 3. *Let $S \cup T \subseteq \mathcal{R}L_\tau$ be an orthogonal set with $S \cap T = \emptyset$. If $s = \bigvee_{f \in S} coz(f)$ and $t = \bigvee_{f \in T} coz(f)$, then*

(1) $s \wedge t' = s$.
(2) *For every $R \subseteq S \cup T$ and every $f \in (S \cup T) \setminus R$,*
$$z(f) \vee \bigvee_{g \in R} z(g) = z(f) \text{ and } coz(f) \wedge \bigwedge_{g \in R} z(g) = coz(f).$$

Proof. (1) Consider $(f, g) \in S \times T$. Since $fg = \mathbf{0}$, we infer that $coz(f) \wedge coz(g) = \perp$, which follows that $coz(f) \leq z(g)$. Hence $\bigwedge_{g \in T} z(g)$ is an upper bounded $\{coz(f) : f \in S\}$ and so $s = \bigvee_{f \in S} coz(f) \leq \bigwedge_{g \in T} z(g) = t'$.

(2) Similar to the proof of statement (1). $\qquad\square$

The following theorem is a counterpart of Theorem 1 in [5] and we characterize P-topoframes in the following theorem.

Theorem 3. *The topoframe L_τ is a P-topoframe if and only if the ring $\mathcal{R}L_\tau$ is $\aleph_0$-selfinjective.*

Proof. Necessity. Let $S \cup T \subseteq \mathcal{R}L_\tau$ be an orthogonal countable set with $S \cap T = \emptyset$ and $s = \bigvee_{f \in S} coz(f)$. We define $h : \mathcal{P}(\mathbb{R}) \to L$ by

$$h(X) = \begin{cases} \bigvee_{f \in S} f(X \setminus \{0\}) \vee s' & \text{if } 0 \in X \\ \bigvee_{f \in S} f(X) & \text{if } 0 \notin X \end{cases}$$

for every $X \subseteq \mathbb{R}$. First of all, we show that $h \in \mathcal{R}L_\tau$. It is clear that $h(\emptyset) = \bigvee_{f \in S} f(\emptyset) = \bot$ and $h(\mathbb{R}) = \bigvee_{f \in S} f(\mathbb{R} \setminus \{0\}) \vee s' = s \vee s' = \top$. Consider $X, Y \subseteq \mathbb{R}$. If $0 \in X \cap Y$, then

$$
\begin{aligned}
h(X) \wedge h(Y) &= [\textstyle\bigvee_{f \in S} f(X \setminus \{0\}) \vee s'] \wedge [\bigvee_{f \in S} f(Y \setminus \{0\}) \vee s'] \\
&= \textstyle\bigvee_{f \in S} f((X \cap Y) \setminus \{0\}) \vee s' \\
&= h(X \cap Y).
\end{aligned}
$$

If $0 \in X$ and $0 \notin Y$, then

$$
\begin{aligned}
h(X) \wedge h(Y) &= [\textstyle\bigvee_{f \in S} f(X \setminus \{0\}) \vee s'] \wedge \bigvee_{f \in S} f(Y) \\
&= \textstyle\bigvee_{f \in S} f(X \cap Y) \vee [s' \wedge \bigvee_{f \in S} f(Y)] \\
&= \textstyle\bigvee_{f \in S} f(X \cap Y) \vee \bot \\
&= h(X \cap Y).
\end{aligned}
$$

If $0 \notin X$ and $0 \notin Y$, then

$$
h(X) \wedge h(Y) = \bigvee_{f \in S} f(X) \wedge \bigvee_{f \in S} f(Y) = \bigvee_{f \in S} f(X \cap Y) = h(X \cap Y).
$$

Hence h preserves all finite meets. Consider $\{X_i\}_{i \in I} \subseteq P(\mathbb{R})$. If $0 \notin \bigcup_{i \in I} X_i$, then

$$
\bigvee_{i \in I} h(X_i) = \bigvee_{i \in I} \bigvee_{f \in S} f(X_i) = \bigvee_{f \in S} f(\bigcup_{i \in I} X_i) = h(\bigcup_{i \in I} X_i).
$$

If $0 \in \bigcup_{i \in I} X_i$, then

$$
\begin{aligned}
\bigvee_{i \in I} h(X_i) &= \bigvee_{\substack{i \in I, \\ 0 \notin X_i}} \bigvee_{f \in S} f(X_i) \vee [s' \vee \bigvee_{\substack{i \in I, \\ 0 \in X_i}} \bigvee_{f \in S} f(X_i \setminus \{0\})] \\
&= s' \vee \bigvee_{f \in S} f(\bigcup_{i \in I} X_i \setminus \{0\}) \\
&= h(\bigcup_{i \in I} X_i).
\end{aligned}
$$

Hence h preserves arbitrary joins. Therefore, $h \in \mathcal{R}L_\tau$. Now, we show that $h \in Ann(T)$. Consider $g \in T$ and $t = \bigvee_{f \in T} coz(f)$. Then, by Lemma 3, we have

$$
\begin{aligned}
z(hg) &= z(h) \vee z(g) \\
&= s' \vee z(g) \\
&\geq t \vee z(g) \\
&\geq coz(g) \vee z(g) \\
&= \top,
\end{aligned}
$$

which follows that $hg = \mathbf{0}$. Therefore $h \in Ann(T)$. Since $z(h) = s'$, we conclude from Lemma 3 that $z(fh) = z(f) \vee s' = z(f) = z(f^2)$, for every $f \in S$. Consider

$0 \neq x \in \mathbb{R}$. Then

$$
\begin{aligned}
hf(\{x\}) &= \bigvee\{h(\{y\}) \wedge f(\{\tfrac{x}{y}\}) : 0 \neq y \in \mathbb{R}\} \\
&= \bigvee\{f(\{y\}) \wedge f(\{\tfrac{x}{y}\}) : y \in \mathbb{R} \setminus \{0\}\} \\
&= f^2(\{x\}).
\end{aligned}
$$

Hence $hf = f^2$, which means that h separates S from T. Now, by Theorem 2 and Lemma 2, we are through.

   Sufficiency. Consider $f \in \mathcal{R}L_\tau$ and I is the generated ideal by f^2 in $\mathcal{R}L_\tau$. Since $h : I \to \mathcal{R}L_\tau$ given by $gf^2 \mapsto gf$ is a $\mathcal{R}L_\tau$-homomorphism, we conclude from statement (2) that there exists a $\mathcal{R}L_\tau$-homomorphism $\bar{h} : \mathcal{R}L_\tau \to \mathcal{R}L_\tau$ such that $\bar{h}|_I = h$. Hence

$$
f = h(f^2) = \bar{h}(\mathbf{1}f^2) = \bar{h}(\mathbf{1})f^2
$$

Then $\mathcal{R}L_\tau$ is a regular ring. Therefore, by Theorem 2, the topoframe L_τ is a P-topoframe.   □

   As an immediate consequence, by this theorem and Theorem 2, we have.

Corollary 4. *For a topoframe L_τ, the ring $\mathcal{R}L_\tau$ is regular if and only if it is $\aleph_0$-selfinjective*

   A frame L is called extremally disconnected if $a^{**} \vee a^* = \top$ for all $a \in L$.
   Also note that for every regular frame L, if $f : L \to M$ and $g : L \to M$ are frame maps with $f \leq g$, then $f = g$ (see [1]).

Proposition 3. *If L is an extremally disconnected frame and L_τ is a P-topoframe, then $\mathcal{R}L_\tau$ is a selfinjective ring. But the converse is not true*

Proof. Let $S \cup T \subseteq \mathcal{R}L_\tau$ be an orthogonal set with $S \cap T = \emptyset$ and $s = \bigvee_{f \in S} coz(f)$. We are to find an element in $\mathcal{R}L_\tau$ that separates S from T. We define $h : \mathcal{P}(\mathbb{R}) \to L$ by

$$
h(X) = \begin{cases} (\bigvee_{f \in S} f(X \setminus \{0\}))^{**} \vee s^* & \text{if } 0 \in X \\ (\bigvee_{f \in S} f(X))^{**} & \text{if } 0 \notin X \end{cases}
$$

for every $X \subseteq \mathbb{R}$. First of all, we show that $h \in \mathcal{R}L_\tau$. It is clear that

$$
h(\emptyset) = \bigvee_{f \in S} f(\emptyset) = \bot \text{ and } h(\mathbb{R}) = (\bigvee_{f \in S} f(\mathbb{R} \setminus \{0\}))^{**} \vee s^* = s^{**} \vee s^* = \top,
$$

since L is an extremally disconnected frame. Consider $X, Y \subseteq \mathbb{R}$. If $0 \in X \cap Y$, then

$$
\begin{aligned}
h(X) \wedge h(Y) &= [(\bigvee_{f \in S} f(X \setminus \{0\}))^{**} \vee s^*] \wedge [(\bigvee_{f \in S} f(Y \setminus \{0\}))^{**} \vee s^*] \\
&= (\bigvee_{f \in S} f((X \cap Y) \setminus \{0\}))^{**} \vee s^* \\
&= h(X \cap Y).
\end{aligned}
$$

If $0 \in X$ and $0 \notin Y$, then

$$
\begin{aligned}
h(X) \wedge h(Y) &= [(\bigvee_{f \in S} f(X \setminus \{0\}))^{**} \vee s^*] \wedge (\bigvee_{f \in S} f(Y))^{**} \\
&= (\bigvee_{f \in S} f(X \cap Y))^{**} \vee [s^* \wedge (\bigvee_{f \in S} f(Y))^{**}] \\
&= (\bigvee_{f \in S} f(X \cap Y))^{**} \vee \perp \\
&= h(X \cap Y).
\end{aligned}
$$

If $0 \notin X$ and $0 \notin Y$, then

$$
h(X) \wedge h(Y) = (\bigvee_{f \in S} f(X))^{**} \wedge (\bigvee_{f \in S} f(Y))^{**} = (\bigvee_{f \in S} f(X \cap Y))^{**} = h(X \cap Y).
$$

Hence h preserves all finite meets. Consider $\{X_i\}_{i \in I} \subseteq P(\mathbb{R})$. If $0 \notin \bigcup_{i \in I} X_i$, then

$$
\bigvee_{i \in I} h(X_i) = (\bigvee_{i \in I} \bigvee_{f \in S} f(X_i))^{**} = (\bigvee_{f \in S} f(\bigcup_{i \in I} X_i))^{**} = h(\bigcup_{i \in I} X_i).
$$

If $0 \in \bigcup_{i \in I} X_i$, then

$$
\begin{aligned}
\bigvee_{i \in I} h(X_i) &= \bigvee_{\substack{i \in I, \\ 0 \notin X_i}} (\bigvee_{f \in S} f(X_i))^{**} \vee [s^* \vee \bigvee_{\substack{i \in I, \\ 0 \in X_i}} (\bigvee_{f \in S} f(X_i \setminus \{0\}))^{**}] \\
&= s^* \vee (\bigvee_{f \in S} f(\bigcup_{i \in I} X_i \setminus \{0\}))^{**} \\
&= h(\bigcup_{i \in I} X_i).
\end{aligned}
$$

Hence h preserves arbitrary joins. Therefore, $h \in \mathcal{R}L_\tau$. Now, we show that $h \in Ann(T)$. Consider $g \in T$ and $t = \bigvee_{f \in S} coz(f)$. Then, by Lemma 3, we have

$$
\begin{aligned}
z(hg) &= z(h) \vee z(g) \\
&= s^* \vee z(g) \\
&\geq t \vee z(g) \\
&\geq coz(g) \vee z(g) \\
&= \top,
\end{aligned}
$$

which follows that $hg = \mathbf{0}$. Therefore, $h \in Ann(T)$. Since $z(h) = s^*$, we conclude from Lemma 3 that $z(fh) = z(f) \vee s^* = z(f) = z(f^2)$, for every $f \in S$. Consider $0 \neq x \in \mathbb{R}$. Then

$$
\begin{aligned}
hf(\{x\}) &= \bigvee\{h(\{y\}) \wedge f(\{\tfrac{x}{y}\}) : 0 \neq y \in \mathbb{R}\} \\
&= \bigvee\{(\bigvee_{f \in S} f(\{y\}))^{**} \wedge f(\{\tfrac{x}{y}\}) : 0 \neq y \in \mathbb{R}\} \\
&\geq \bigvee\{(\bigvee_{f \in S} f(\{y\})) \wedge f(\{\tfrac{x}{y}\}) : 0 \neq y \in \mathbb{R}\} \\
&= \bigvee\{f(\{y\}) \wedge f(\{\tfrac{x}{y}\}) : y \in \mathbb{R} \setminus \{0\}\} \\
&= f^2(\{x\}).
\end{aligned}
$$

Hence $hf(X) \geq f^2(X)$, for every $X \in P(\mathbb{R})$. Since $P(\mathbb{R})$ is a regular frame and also hf and f^2 are frame maps, we conclude that $hf = f^2$. Therefore, h separates S from T. Now, by Theorem 2 and Lemma 2, $\mathcal{R}L_\tau$ is a selfinjective ring.

Now, we show that the converse is not true. Let L be a connected frame, say $L = \mathfrak{O}\mathbb{R}$, then L has only two complemented elements, that is, $\bot$ and $\top$ are only two complemented elements of L. Consider $\mathbf{0} \neq f \in \mathcal{R}L_\tau$. Then $coz(f) = \top$ which follows that there exists a $0 \neq r \in \mathbb{R}$ such that $f(\{r\}) \neq \bot$. We claim that $f = \mathbf{r}$. In order to prove our claim, we consider $A \subseteq \mathbb{R}$ such that $r \in A$. From $f(A)$ which is the complement of $f(\mathbb{R} \setminus A)$ in L and $f(A) = \top$, we conclude that $f(\mathbb{R} \setminus A) = \bot$. Then $f = \mathbf{r}$. Therefore, $\mathcal{R}L_\tau \cong \mathbb{R}$. Since $\mathbb{R}$ is a field, we infer that $\mathcal{R}L_\tau$ is a selfinjective ring, but L is not an extremally disconnected frame. $\qquad\square$

Recall from [17] that the *semi-Heyting operation* $\to_\tau$ on a topoframe L_τ is defined by

$$a \to_\tau b = \bigvee \{x \in \tau \mid a \wedge x \leq b\},$$

where $a, b \in L$. We put $a^\perp = a \to_\tau \bot$, for every $a \in L$. Clearly if $a \in \tau$, then $a^\perp = a^*$, where the peudocomplement of a is formed in τ. It follows that for any topoframe L_τ, τ is a extremally disconnected frame if and only if $a^{\perp\perp} \vee a^\perp = \top$ for all $a \in \tau$. Also, $(\bigvee_{i \in I} a_i)^\perp = \bigwedge_{i \in I} a_i^\perp$ where $\{a_i\}_{i \in I} \subseteq \tau$.

The *closure* of $p \in L$ in a topoframe (L, τ) is the element

$$\overline{p} = Cl_{L_\tau}(p) = \bigwedge^{\tau'} \{x \in \tau' : p \leq x\}.$$

It is easy to see that for every $a \in \tau$, $(\overline{a})' = a^\perp = a^*$, where the peudocomplement of a is formed in τ. Thus if $\overline{a} \in \tau$, then $\overline{a} \leq a^{\perp\perp} = a^{**}$, for more details see [17].

Definition 2. A topoframe L_τ is called extremally disconnected if $\overline{a} \in \tau$ for all $a \in \tau$.

The *interior* of $p \in L$ in a topoframe (L, τ) is the element

$$p^o = Int_{L_\tau}(p) = \bigvee \{x \in \tau : x \leq p\}.$$

It is easy to see that $(\overline{a})^o = a^{\perp\perp} = a^{**}$ for every $a \in \tau$, where the peudocomplement of a is formed in τ and also $p \in \tau$ if and only if $p^o = p$, for more details see [17].

Proposition 4. *For any topoframe L_τ, τ is an extremally disconnected frame if and only if L_τ is an extremally disconnected topoframe.*

Proof. Necessity. We show that $(\overline{a})^o = \overline{a}$ for every $a \in \tau$. Given $a \in \tau$ we have $a^\perp \vee a^{\perp\perp} = \top$ and hence $(\overline{a})' \vee (\overline{a})^o = \top$. On the other hand $(\overline{a})' \wedge (\overline{a})^o \leq (\overline{a})' \wedge \overline{a} = \bot$, it follows that $(\overline{a})' \wedge (\overline{a})^o = \bot$. Consequently, $(\overline{a})^o = ((\overline{a})')' = \overline{a}$.

Sufficiency. Suppose $a \in \tau$ hence $\overline{a} \in \tau$. It follows that

$$\top = (\overline{a})' \vee \overline{a} = a^\perp \vee \overline{a} \leq a^\perp \vee a^{\perp\perp}.$$

Consequently, $a^\perp \vee a^{\perp\perp} = \top$. $\qquad\square$

Before the following proposition is proposed, we first recall some definitions. If I and J are ideals in a ring A we say I is essential in J if $I \subseteq J$ and every nonzero ideal inside J intersects I nontrivially, and recall that when we say I is essential, we mean it is essential in A. An ideal I in a ring A is called closed ideal (complement) if it is not essential in a larger ideal and a ring A is said to be CS-ring if every

closed ideal is a direct summand, see [16]. A ring A is called a Baer ring if for any subset S of A, we have $Ann_R(S) = eA$, where $e^2 = e$.

A topoframe L_τ is said to be *completely regular* if for every $a \in \tau$ there exists $\{f_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{R}L_\tau$ such that $a = \bigvee_{\lambda \in \Lambda} coz(f_\lambda)$. In [17] proved that for any topoframe L_τ, there exists a completely regular topoframe M_w such that $\mathcal{R}L_\tau$ is isomorphic to $\mathcal{R}M_w$.

Proposition 5. *The following statements are equivalent for any completely regular topoframe L_τ.*

(1) *L_τ is an extremally disconnected topoframe.*
(2) *$\mathcal{R}L_\tau$ is a Baer ring.*
(3) *Every nonzero ideal in $\mathcal{R}L_\tau$ is essential in a principal ideal generated by an idempotent.*
(4) *$\mathcal{R}L_\tau$ is a CS-ring.*

Proof. (1)$\Rightarrow$(2). Let $S \subseteq \mathcal{R}L_\tau$, we are to show that $AnnS = e\mathcal{R}L_\tau$, where $e^2 = e$. We put $s = \bigvee_{f \in S} coz(f)$. Since $s \in \tau$ and L_τ is extremally disconnected, we infer that $s^\perp \vee s^{\perp\perp} = s^* \vee s^{**} = \top$, where the peudocomplement of s is formed in τ, which follows that $s^\perp$ and $(s^\perp)'$ belong to τ. Consider $g \in Ann(S)$. Then $coz(f) \wedge coz(g) = coz(fg) = coz(\mathbf{0}) = \bot$, which follows that $coz(f) \leq coz(g)^\perp$, for every $f \in S$. Hence $s \leq coz(g)^\perp$ and so

$$coz(g) \leq coz(g)^{\perp\perp} \leq s^\perp.$$

Consider $X \subseteq \mathbb{R}$ and $f \in S$. If $0 \notin X$, then, by Proposition 1, $gf_{s^\perp}(X) = s^\perp \wedge g(X) = g(X)$, because $g(X) \leq coz(g) \leq s^\perp$. If $0 \in X$, then $gf_{s^\perp}(X) = (s^\perp)' \vee g(X) \geq g(X)$, by Proposition 1. Since g and $f_{s^\perp}$ are frame maps and $\mathcal{P}(\mathbb{R})$ is the regular frame, we conclude that $gf_{s^\perp} = g$ which means that $g \in f_{s^\perp}\mathcal{R}L_\tau$. Hence $Ann(S) \subseteq f_{s^\perp}\mathcal{R}L_\tau$. Now, suppose that $f \in S$ and hence $coz(f) \leq s$, consequently

$$coz(ff_{s^\perp}) = coz(f) \wedge coz(f_{s^\perp}) \leq coz(f)^{\perp\perp} \wedge s^\perp \leq s^{\perp\perp} \wedge s^\perp = \bot,$$

it follows that $ff_{s^\perp} = \mathbf{0}$. Thus $f_{s^\perp} \in Ann(S)$ and so $Ann(S) = f_{s^\perp}\mathcal{R}L_\tau$. Therefore $\mathcal{R}L_\tau$ is a Baer ring.

(2)$\Rightarrow$(3). Let I be a nonzero ideal in $\mathcal{R}L_\tau$, then there is an idempotent element e in $\mathcal{R}L_\tau$ such that $Ann(I) = e\mathcal{R}L_\tau = Ann((1 - e)\mathcal{R}L_\tau)$, which follows that $f = f(1 - e) \in (1 - e)\mathcal{R}L_\tau \cap I$, for every $f \in I$. Hence I is essential in $(1 - e)\mathcal{R}L_\tau$.

(3)$\Rightarrow$(4). Let I be a closed ideal in $\mathcal{R}L_\tau$, then there is an idempotent element e in $\mathcal{R}L_\tau$ such that I is essential in $e\mathcal{R}L_\tau$.

(4)$\Rightarrow$(2). Consider $S \subseteq \mathcal{R}L_\tau$ and $I = Ann(S)$. We claim that the ideal $Ann(S)$ is a closed ideal in $\mathcal{R}L_\tau$. Let $Ann(S)$ be essential in a larger ideal J, then $SJ \neq (\mathbf{0})$ implies that $SJ \cap Ann(S) \neq (\mathbf{0})$, but $(SJ \cap Ann(S))^2 = (\mathbf{0})$, which is impossible, since $\mathcal{R}L_\tau$ is a reduced ring. This shows that $Ann(S)$ is a closed ideal and by (4), I is generated by an idempotent.

(2)$\Rightarrow$(1). Consider $a \in \tau$, then there is $\{f_t\}_{t \in T} \subseteq \mathcal{R}L_\tau$ such that $a = \bigvee_{t \in T} coz(f_t)$. Since $\mathcal{R}L_\tau$ is a Baer ring, we conclude that there is an idempotent element $e \in \mathcal{R}L_\tau$ such that $Ann(\{f_t\}_{t \in T}) = e\mathcal{R}L_\tau$, which follows that for every $t \in T$

$$coz(e) \wedge coz(f_t) = coz(ef_t) = coz(\mathbf{0}) = \bot \Rightarrow coz(e) \leq coz(f_t)^\perp.$$

Then $coz(e) \leq \bigwedge_{t \in T} coz(f_t)^{\perp} = a^{\perp}$. Since $coz(e) \vee coz(1-e) = \top$ and $coz(e) \wedge coz(1-e) = \perp$, we conclude that $a^{\perp\perp} \leq coz(e)^{\perp} = coz(1-e)$. Suppose that $\{g_k\}_{k \in K} \subseteq \mathcal{R}L_{\tau}$ such that $a^{\perp} = \bigvee_{k \in K} coz(g_k)$. For every $(t,k) \in T \times K$, we have

$$coz(f_t g_k) = coz(f_t) \wedge coz(g_k) \leq coz(f_t)^{\perp\perp} \wedge a^{\perp} \leq a^{\perp\perp} \wedge a^{\perp} = \perp,$$

and so $f_t g_k = \mathbf{0}$. Then $g_k \in Ann(\{f_t\}_{t \in T}) = e\mathcal{R}L_{\tau}$, which follows that there is a $h_k \in \mathcal{R}L_{\tau}$ such that $g_k = eh_k$, for every $k \in K$. Therefore, $coz(g_k) = coz(eh_k) \leq coz(e)$ and so $a^{\perp} = \bigvee_{k \in K} coz(g_k) \leq coz(e)$. Consequently, $a^{\perp} = coz(e)$ and we immediate have $a^{\perp} \vee a^{\perp\perp} = coz(e) \vee coz(1-e) = \top$.   □

In order to state the following theorem we need some background. A lattice A is called upper continuous if A is complete and $a \wedge (\vee b_i) = \vee(a \wedge b_i)$ for all $a \in A$ and all linearly ordered subset $\{b_i\} \subseteq A$. A regular ring R is called continuous if the lattice of all principal ideals is upper continuous.

We recall from [10, Corollary 13.4] that a regular ring R is continuous if and only if every ideal of R is essential in a principal right ideal of R. Also, we recall from [10, Corollary 13.5] that every regular self injective ring is continuous. Also, every reduced self injective ring is regular ring which is Baer ring, see [14, Proposition 1.7].

Proposition 6. [2] *The following statements are equivalent.*

(1) *A is a Baer ring.*
(2) *A is a p.p. ring which is also the Boolean algebra B(A) of idempotents in A is complete.*
(3) *A is a p.p. ring and every set of orthogonal idempotents in A has a supremum.*

Theorem 4. *For any completely regular topoframe L_{τ}, the following statements are equivalent.*

(1) $\mathcal{R}L_{\tau}$ *is a Baer regular ring.*
(2) $\mathcal{R}L_{\tau}$ *is a continuous regular ring.*
(3) $\mathcal{R}L_{\tau}$ *is a complete regular ring.*
(4) L_{τ} *is an extremally disconnected P-topoframe.*
(5) $\mathcal{R}L_{\tau}$ *is a self injective ring.*

Proof. (1)⇒(2). It is clear by [10, Corollary 13.4] and Proposition 6.

(2)⇒(3). It is obvious.

(3)⇒(4). Since every regular ring is a p.p. ring, we conclude from Proposition 6 that $\mathcal{R}L_{\tau}$ is a Baer regular ring. Then, combining Theorem 2 and Proposition 5 imply that L_{τ} is an extremally disconnected P-topoframe.

(4)⇒(5). The proof is similar to Proposition 3.

(5)⇒(1). By [14, Proposition 1.7.], $\mathcal{R}L_{\tau}$ is a Baer regular ring.   □

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