Existence and smoothness of the density for the stochastic continuity equation

David A.C. Mollinedo * Christian Olivera[†] Ciprian A. Tudor [‡]

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Abstract

We consider the stochastic continuity equation driven by Brownian motion. We use the techniques of the Malliavin calculus to show that the law of the solution has a density with respect to the Lebesgue measure. We also prove that the density is Hölder continuous and satisfies some Gaussian-type estimates.

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1 Introduction

In this paper we consider the stochastic continuity equation given by

$$\begin{cases} \partial_t u(t,x) + div \left((b(t,x) + \frac{dB_t}{dt}) \cdot u(t,x) \right) = 0, \quad t \in [0,T], x \in \mathbb{R}^d \\ u(0,x) = u_0(x) \text{ for } x \in \mathbb{R}^d. \end{cases}$$
(1)

where $(B_t)_{t \in [0,T]}$ is a Brownian motion (Bm) in \mathbb{R}^d and the stochastic integration is understood in the Stratonovich sense. This equation constitutes a well-known model for several physical phenomena arising, among others, in fluid dynamics and conservation of laws (see e.g. the monographs [10], [11] or [5]). The stochastic continuity equation with random perturbation given by a standard Brownian noise has been first studied in the celebrated works by Kunita [8], [9]. More recent works on these topics are, among others, [1], [6], [7] [12] and [13].

^{*}Universidade Tecnológica Federal do Parana, Brazil. E-mail: davida@utfpr.edu.br

[†]Departamento de Matemática, Universidade Estadual de Campinas, Brazil. E-mail: *colivera@ime.unicamp.br*.

[‡]Université de Lille 1. UFR Mathématiques. E-mail: *Ciprian.Tudor@math.univ-lille1.fr*

Our purpose is to analyze the density of the solution to the continuity equation in dimension d = 1 by using Malliavin calculus. It is widely recognized today that this theory constitutes a powerful tool in order to study the densities of random variables in general, and of solutions to stochastic (partial) differential equations in particular. By using several criteria in terms of the Malliavin derivative of the solution, we are able to show that the unique solution to (1) admits a density which is Hölder continuous and satisfies some Gaussian estimates. These criteria, proved in [2], [3], and [14] are recalled in the next section of our work.

In order to control the Malliavin derivative of the solution to (1), we will use the representation theorem of the solution. It is well-known from [8], [9] that one can associate to (1) a so-called equation of characteristics whose solution generates a C^{1} - flow of diffeomorphism. Then the solution to the continuity equation can be expressed as the initial condition applied to the inverse flow multiplied by the Jacobian of the inverse flow. Therefore, via a careful analysis of the Malliavin derivatives of the inverse flow, and by assuming suitable properties on the initial condition u_0 and on the drift b of the equation, we are able to control the Malliavin derivatives of the solution and to prove several properties of the density of the law of the solution.

We organize our paper as follows. In Section 2, we present some preliminaries on Malliavin calculus and the method of characteristics. In Section 3, we study the properties of the Malliavin derivative of the solution to (1) while in section 4, we state and prove our main results on regularity in law.

2 Preliminaries

In this preliminary section we present the basic elements from Malliavin calculus that we will need in this work. In the second part we present three criteria based on Malliavin calculus that we will use in order to prove the absolute continuity of the law of the solution with respect to the Lebesque measure and various properties of the density. The last paragraph contains some known facts on the continuity equation and on its strong solution. In particular, we give the representation of the solution in terms of the initial condition and of the inverse flow which constitutes a key result for our approach.

2.1 Notions of Malliavin Calculus

This subsection is devoted to present the basics tools from Malliavin calculus that will be used in the paper. We refer to [15] for a complete exposition. Consider \mathcal{H} a real separable Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, that is, a centred Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$.

We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}, i = 1, ..., n)$

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} (B(\varphi_1), \dots, B(\varphi_n))\varphi_i.$$

It can be checked that the operator D is closable from S (the space of smooth functionals as above) into $L^2(\Omega; \mathcal{H})$ and it can be extended to the space $\mathbb{D}^{1,p}$ which is the closure of S with respect to the norm

$$\|F\|_{1,p}^p = \mathbf{E}F^p + \mathbf{E}\|DF\|_{\mathcal{H}}^p.$$

We can analogously define the kth iterated Malliavin derivative. We will denote by $\mathbb{D}^{k,p}$ with $k \geq 2$ the completition of the set of smooth random variables with respect to the norm

$$||F||_{k,p}^{p} = \mathbf{E}F^{p} + \sum_{j=1}^{k} \mathbf{E}||DF||_{\mathcal{H}^{\otimes j}}^{p}$$

We denote by $\mathbb{D}^{k,\infty} := \bigcap_{p \ge 1} \mathbb{D}^{k,p}$ for every $k \ge 1$. In this paper, \mathcal{H} will be the standard Hilbert space $L^2([0,T])$.

We will use the chain rule for the Malliavin derivative (see Proposition 1.2.4 in [15]) which says that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a differentiable function with bounded derivative and $F \in \mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \varphi'(F)DF.$$
(2)

We also recall the rule to differentiate a product of random variables: if $F, G \in \mathbb{D}^{1,2}$ such that $FG \in \mathbb{D}^{1,2}$, then

$$D(FG) = FDG + GDF.$$
(3)

2.2 Criteria for densities of random variables

The Malliavin calculus is widely recognized theory as a powerful theory that allows to analyze the existence and smoothness of the densities of certain random variables. We list below a collection of criteria from Malliavin calculus that we will emply in our work. The first is the well-known Bouleau-Hirsch criterium that says that the strict positivity of the norm of the Malliavin derivative implies the absolute continuity of the law. The second criterium (from [2]) gives sufficient conditions for the Hölder continuity of the density while the third criterium (given in [14]) allows to prove upper and lower Gaussian estimates for the density of a random variable.

A notable result is the Bouleau-Hirsch theorem, which provides a criteria in terms of Malliavin derivatives for a random variable to have a density (see [3] or [15]).

Theorem 1 Let F be a random variable of the space $\mathbb{D}^{1,2}$ and suppose that $\|DF\|_{L^2([0,T])} > 0$ a.s. Then the law of F is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .

In [2], Proposition 23, the authors have presented sufficient conditions for a random variable to have a Hölder continuous density.

Proposition 1 Let $F \in \bigcap_{p \in \mathbb{N}} \mathbb{D}^{2,p}$ and assume that $\|DF\|_{\mathcal{H}}^{-2} \in \bigcap_{p \in \mathbb{N}} L^p(\Omega)$. Then the law of F is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} and its density ρ_F is Hölder continuous of any exponent 0 < q < 1. Moreover, there exists some universal constant C > 0 and $p_i > 0$, $i = 1, \ldots, 5$, such that

$$\rho_F(z) \le C \mathbf{E} \left(\|DF\|_{\mathcal{H}}^{-2p_1} \right)^{p_2} \|F\|_{2,p_4}^{p_3} \left(\mathbb{P}(|F-z| \le 2) \right)^{1/p_5}$$

In particular, $\lim_{z\to\infty} |z|^p \rho_F(z) = 0$ for every $p \in \mathbb{N}$.

The main tool in order to obtain the Gaussian estimates for the density of the solution to the continuity equation is the following result given in [14], Corollary 3.5.

Proposition 2 If $F \in \mathbb{D}^{1,2}$, let

$$g_F(F) = \int_0^\infty d\theta e^{-\theta} \mathbf{E} \left[\mathbf{E}' \left(\langle DF, \widetilde{DF} \rangle_{\mathcal{H}} | F \right) \right]$$

where for any random variable X, we denote

$$\tilde{X}(\omega,\omega') = X(e^{-\theta}w + \sqrt{1 - e^{-2\theta}}\omega').$$
(4)

Here \tilde{X} is defined on a product probability space $(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}, P \times P')$ and \mathbf{E}' denotes the expectation with respect to the probability measure P'. If there exist two constants $\gamma_{min} > 0$ and $\gamma_{max} > 0$ such that almost surely

$$0 \le \gamma_{min}^2 \le g_F(F) \le \gamma_{max}^2$$

then F admits a density ρ . Moreover, for every $z \in \mathbb{R}$,

$$\frac{\mathbf{E}|F - \mathbf{E}F|}{2\gamma_{max}^2} e^{-\frac{(z - \mathbf{E}F)^2}{2\gamma_{min}^2}} \le \rho(z) \le \frac{\mathbf{E}|F - \mathbf{E}F|}{2\gamma_{min}^2} e^{-\frac{(z - \mathbf{E}F)^2}{2\gamma_{max}^2}}.$$

2.3 Representation of the solution

We will start by recalling some known facts on the solution to the stochastic continuity equation driven by a standard Wiener process in \mathbb{R}^d .

The equation (1) is interpreted in the strong sense, as the solution to the following stochastic integral equation

$$u(t,x) = u_0(x) - \int_0^t div(b(s,x)u(s,x)) \, ds - \sum_{i=0}^d \int_0^t \partial_{x_i}u(s,x) \circ dB_s^i$$
(5)

for $t \in [0, T]$ and $x \in \mathbb{R}^d$.

The solution to (1) is related with the so-called equation of characteristics. That is, for $0 \leq s \leq t$ and $x \in \mathbb{R}^d$, consider the following stochastic differential equation in \mathbb{R}^d

$$X_{s,t}(x) = x + \int_{s}^{t} b(r, X_{s,r}(x)) \, dr + B_t - B_s, \tag{6}$$

and denote by $X_t(x) := X_{0,t}(x), t \in [0,T], x \in \mathbb{R}^d$.

For $m \in \mathbb{N}$ and $0 < \alpha < 1$, let us assume the following hypothesis on b:

$$b \in L^1((0,T); C_b^{m,\alpha}(\mathbb{R}^d)) \tag{7}$$

where $C^{m,\alpha}(\mathbb{R}^d)$ denotes the class of functions of class C^m on \mathbb{R}^d such that the last derivative is Hölder continuous of order α . By considering the condition (7) it is well known that $X_{s,t}(x)$ is a stochastic flow of C^m -diffeomorphism (see for example [4] and [8]). Moreover, the inverse flow

$$Y_{s,t}(x) := X_{s,t}^{-1}(x)$$

satisfies the following backward stochastic differential equation

$$Y_{s,t}(x) = x - \int_{s}^{t} b(r, Y_{r,t}(x)) \, dr - (B_t - B_s), \tag{8}$$

for every $0 \le s \le t \le T$. We will denote $Y_{0,t}(x) := Y_t(x)$ for every $t \in [0,T], x \in \mathbb{R}$.

We have the following representation of the solution to the stochastic continuity equation in terms of the initial data and of the inverse flow (8). We refer to e.g. [8] or [4], Section 3 for the proof. By J we denote the Jacobian.

Lemma 1 Assume (7) for $m \ge 3, \delta > 0$ and let $u_0 \in C^{m,\delta}(\mathbb{R}^d)$. Then the Cauchy problem (5) has a unique solution $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ which can be represented as

$$u(t,x) = u_0(Y_t(x))JY_t(x), \quad t \in [0,T], x \in \mathbb{R}^d.$$
(9)

Our approach to prove the regularity of the solution in the sense of Malliavin calculus is based on the formula (9) by assuming that the initial condition is smooth. A similar representation to (9) exists for the transport equation (see e.g. [8]) and it has been used in [16] to obtain to absolute continuity of the law of the solution to the transport equation.

3 Properties of the inverse flow and Malliavin differentiability of the solution

Taking into account the formula (9), in order to control the Malliavin derivative of u(t, x), one needs to analyse the inverse flow. We will need to control its Malliavin derivative, its

Jacobian and the Malliavin derivative of its Jacobian. We will restrict, from now on, to the case d = 1.

Consider $(Y_{s,t}(x))_{0 \le s \le t \le T, x \in \mathbb{R}}$ the inverse flow $Y_{s,t}(x) = X_{s,t}^{-1}(x), 0 \le s \le t \le T, x \in \mathbb{R}$ with $X_{s,t}(x)$ from (6). Recall that satisfies the following backward stochastic differential equation (8).

We will keep the multidimensional notation $JY_{s,t}(x) = \frac{d}{dx}Y_{s,t}(x)$ although we work now in dimension d = 1. We also denote by $b'(t,x) = \frac{d}{dx}b(t,x)$, $b''(t,x) = \frac{d^2}{dx^2}b(t,x)$ and $b^{(n)}(t,x) = \frac{d^n}{dx^n}b(t,x)$. By $||f||_{\infty}$ we denote the infinity norm of the function f.

Concerning the Malliavin derivative of the inverse flow, we have the following estimate.

Lemma 2 Assume $b \in L^{\infty}((0,T), C_b^1(\mathbb{R}))$.

If $(Y_{s,t}(x))_{0 \le s \le t \le T, x \in \mathbb{R}}$ denotes the inverse flow (8), then for every $0 \le s \le t \le T$ and $x \in \mathbb{R}$ the random variable $Y_{s,t}(x)$ is Malliavin differentiable and we have, for every $\alpha \in (0,T]$

$$D_{\alpha}Y_{s,t}(x) = -1_{[s,t]}(\alpha)e^{-\int_{s}^{\alpha}b'(r,Y_{r,t})dr}.$$
(10)

Moreover, there exists a constant $C_1 > 0$ (not depending on ω) such that

$$\sup_{\alpha \in (0,T), 0 \le s \le t \le T} \sup_{x \in \mathbb{R}} |D_{\alpha} Y_{s,t}(x)| \le C_1.$$
(11)

Proof: The Malliavin differentiability of $Y_{s,t}(x)$ and the formula (10) have been proven in Proposition 2 in [16]. We also have, for every α, s, t, x ,

$$|D_{\alpha}Y_{s,t}(x)| \le e^{-\int_{s}^{\alpha} b'(r,Y_{r,t})dr} \le e^{T||b'||_{\infty}} := C_{1}$$

by using the fact that b' is uniformly bounded. This implies (11).

Now, we regard the Jacobian of Y.

Lemma 3 Assume $b \in L^{\infty}((0,T), C_b^1(\mathbb{R}))$. Consider $(Y_{s,t}(x))_{0 \le s \le t \le T, x \in \mathbb{R}}$ given by (8). Then, for every $0 \le s \le t \le T$ and $x \in \mathbb{R}$,

$$JY_{s,t}(x) = e^{-\int_{s}^{t} b'(v, Y_{v,t}(x))dv}$$
(12)

and there exists $C_1 := e^{T ||b'||_{\infty}}$ such that

$$\sup_{0 \le s \le t \le T} \sup_{x \in \mathbb{R}} |JY_{s,t}(x)| \le C_1.$$
(13)

Proof: By differentiating with respect to x in (8), we get

$$JY_{s,t}(x) = 1 - \int_{s}^{t} b'(r, Y_{r,t}(x)) JY_{r,t}(x) dr$$

and this implies (12). Since $b \in L^{\infty}((0,T), C_b^1(\mathbb{R}))$, we have

$$|JY_{s,t}(x)| \le e^{T ||b'||_{\infty}}$$

for every s, t, x and (13) follows.

We have a similar result for the Malliavin derivative of the inverse flow. We need in addition to assume the existence and boundness of the second derivative of the drift b.

Lemma 4 Assume $b \in L^{\infty}((0,T), C_b^2(\mathbb{R}))$. Then $JY_{s,t}(x)$ belongs to $\mathbb{D}^{1,p}$ for every $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$ and there exists $C_2 > 0$ with

$$\sup_{\alpha \in (0,T), 0 \le s \le t \le T} \sup_{x \in \mathbb{R}} |D_{\alpha}JY_{s,t}(x)| \le C_2.$$
(14)

Proof: From (12), it is clear that $JY_{s,t}(x)$ belongs to $\mathbb{D}^{1,p}$ for every $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}$. By using the chain rule (2),

$$D_{\alpha}JY_{s,t}(x) = -e^{-\int_{s}^{t} b'(v,Y_{v,t}(x))dv} \int_{s}^{t} b''(v,Y_{v,t}(x))D_{\alpha}Y_{v,t}(x)dv$$
(15)

and by (11), we obtain

$$|D_{\alpha}JY_{s,t}(x)| \le C_1^2 T ||b''||_{\infty} := C_2.$$

From the three lemmas above and the representation (9), we immediately obtain the following result on the Malliavin differentiability of the solution to the continuity equation.

Proposition 3 Let $(u(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ be given by (5).

1. Assume $u_0 \in C_b^1(\mathbb{R})$ and $b \in L^{\infty}((0,T), C_b^2(\mathbb{R}))$. Then u(t,x) belong to $\mathbb{D}^{1,p}$ for every t, x and for every $p \ge 2$ and there exists $C_3 > 0$ such that

$$\sup_{\alpha \in (0,T), 0 \le t \le T} \sup_{x \in \mathbb{R}} |D_{\alpha}u(t,x)| \le C_3.$$
(16)

2. Assume $u_0 \in C_b^2(\mathbb{R})$ and $b \in L^{\infty}((0,T), C_b^3(\mathbb{R}))$. Then u(t,x) belong to $\mathbb{D}^{2,p}$ for every t, x and for every $p \geq 2$ and there exists $C_4 > 0$ such that

$$\sup_{\alpha,\beta\in(0,T)} \sup_{0\le t\le T} \sup_{x\in\mathbb{R}} |D_{\beta}D_{\alpha}u(t,x)| \le C_4.$$
(17)

Proof: 1. We start by proving the first point. From (9) and Lemmas 2, 3, 4 it is clear that the random variable u(t, x) is differentiable in the Malliavin sense for all t, x. Using the product rule (3) and the chain rule (2) for the Malliavin derivative (recall that $Y_{0,t} = Y_t$),

$$D_{\alpha}u(t,x) = D_{\alpha}u_{0}(Y_{t}(x))JY_{t}(x) + u_{0}(Y_{t}(x))D_{\alpha}JY_{t}(x) = u_{0}'(Y_{t}(x))D_{\alpha}Y_{t}(x)JY_{t}(x) + u_{0}(Y_{t}(x))D_{\alpha}JY_{t}(x).$$
(18)

From the estimates (11), (13) and (14) and the assumption on u_0 , we obtain

$$\sup_{\alpha \in (0,T), 0 \le t \le T} \sup_{x \in \mathbb{R}} |D_{\alpha}u(t,x)| \le ||u_0'||_{\infty} C_1^2 + ||u_0||_{\infty} C_2 := C_3$$

The above relation also implies that $u(t, x) \in \mathbb{D}^{1,p}$ for every $p \geq 2$.

2. Concerning the second point, we note that obviously $D_{\alpha}u(t,x)$ is also Malliavin differentiable and from (18),

$$D_{\beta}D_{\alpha}u(t,x) = u_{0}'(Y_{t}(x))D_{\alpha}Y_{t}(x)D_{\beta}JY_{t}(x) + u_{0}'(Y_{t}(x))D_{\beta}D_{\alpha}Y_{t}(x)JY_{t}(x) + u_{0}''(Y_{t}(x))D_{\beta}Y_{t}(x)D_{\alpha}Y_{t}(x)JY_{t}(x) + u_{0}(Y_{t}(x))D_{\beta}D_{\alpha}JY_{t}(x) + u_{0}'(Y_{t}(x))D_{\beta}Y_{t}(x)D_{\alpha}JY_{t}(x).$$

We need to bound the second Malliavin derivative of $Y_t(x)$ and $JY_t(x)$. We have from (10) (remember that $Y_{0,t} = Y_t$)

$$D_{\beta}D_{\alpha}Y_{t}(x) = 1_{[0,t]}(\alpha)e^{-\int_{0}^{t}b'(v,Y_{v,t}(x))dv}\int_{0}^{t}b''(v,Y_{v,t}(x))D_{\beta}Y_{v,t}(x)dv$$
(19)

 \mathbf{SO}

$$|D_{\beta}D_{\alpha}Y_t(x)| \le e^{T\|b'\|_{\infty}}T\|b''\|_{\infty}C_1 = C_1^2T\|b''\|_{\infty} = C_2.$$
(20)

Also, from (15), for $\alpha, \beta \in (0, T]$

$$\begin{split} D_{\beta}D_{\alpha}JY_{t}(x) \\ &= -e^{-\int_{0}^{t}b'(v,Y_{v,t}(x))dv}\int_{0}^{t} \left(b^{(3)}(v,Y_{v,t}(x))D_{\beta}Y_{v,t}(x)D_{\alpha}Y_{v,t}(x) + b''(v,Y_{v,t}(x))D_{\beta}D_{\alpha}Y_{v,t}(x)\right)dv \\ &+ e^{-\int_{0}^{t}b'(v,Y_{v,t}(x))dv} \left(\int_{0}^{t}b''(v,Y_{v,t}(x))D_{\beta}Y_{v,t}(x))dv\right) \left(\int_{0}^{t}b''(v,Y_{v,t}(x))D_{\alpha}Y_{v,t}(x))dv\right). \end{split}$$

Considering the estimates (20) and (11), and the assumption on b, we obtain

$$|D_{\beta}D_{\alpha}JY_{t}(x)| \leq e^{T\|b'\|_{\infty}} \Big[TC_{1}^{2}\|b^{(3)}\|_{\infty} + TC_{2}\|b''\|_{\infty} + T^{2}C_{1}^{2}\|b''\|_{\infty}^{2} \Big]$$

$$\leq TC_{1}^{3} \Big[\|b^{(3)}\|_{\infty} + 2T\|b''\|_{\infty}^{2} \Big].$$

Hence, from the last two estimates, together with (11), (13) and (14), imply the conclusion of point 2.

4 Density of the solution

In this section, we prove the existence and various properties of the density of the solution to (1) via the three criteria presented in Section 2.2. Under a relatively strong control of the initial condition and of the drift, we will show that these criteria can be applied to (5).

4.1 Existence of the density

The first result gives the absolute continuity of the law of the solution when the second derivative of the drift is negative and u_0, u'_0 are bounded and positive.

Proposition 4 We will assume that $b \in L^{\infty}((0,T), C_b^2(\mathbb{R}))$ and

$$b''(t,x) \le 0 \text{ for all } t \in [0,T], x \in \mathbb{R}.$$
(21)

Assume $u_0 \in C_b^1(\mathbb{R})$ satisfies

$$u_0(x) > 0 \text{ and } u'_0(x) > 0 \text{ for every } x \in \mathbb{R}.$$
 (22)

If $(u(t,x))_{t\in[0,T],x\in\mathbb{R}}$ is the unique solution to the stochastic continuity equation (1), then, for every $t \in (0,T]$ and for every $x \in \mathbb{R}$, the law of u(t,x) is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} .

Proof: We need to show that $||Du(t,x)||_{\mathcal{H}} > 0$ almost surely where $\mathcal{H} = L^2([0,T])$. From (18), we have

$$\begin{aligned} \|Du(t,x)\|_{\mathcal{H}}^2 &= \left(u_0'(Y_t(x))\right)^2 (JY_t(x))^2 \|DY_t(x)\|_{\mathcal{H}}^2 + (u_0(Y_t(x)))^2 \|DJY_t(x)\|_{\mathcal{H}}^2 \\ &+ 2u_0'(Y_t(x))JY_t(x)u_0(Y_t(x))\langle DY_t(x), DJY_t(x)\rangle_{\mathcal{H}} \\ &:= A_1 + A_2 + A_3. \end{aligned}$$

From (22), by noticing that $(JY_t(x))^2$ and $||DY_t(x)||^2_{\mathcal{H}}$ are strictly positive due to (10) and (12), we have $A_1 > 0$. Clearly $A_2 \ge 0$. Concerning the summand A_3 , by the condition (21), we have from (10) and (15), for $\alpha \in (0, T]$, that

$$D_{\alpha}JY_{s,t}(x) = -e^{-\int_{s}^{t}b'(v,Y_{v,t}(x))dv} \int_{s}^{t}b''(v,Y_{v,t}(x))D_{\alpha}Y_{v,t}(x)dv$$

$$= e^{-\int_{s}^{t}b'(v,Y_{v,t}(x))dv} \int_{s}^{t}b''(v,Y_{v,t}(x))1_{[v,t]}(\alpha)e^{-\int_{v}^{\alpha}b'(r,Y_{r,t})dr}dv \qquad (23)$$

$$\leq 0.$$

Since $D_{\alpha}Y_t(x) \leq 0$ for every $\alpha, t \in (0, T], x \in \mathbb{R}$ due to (10) we have $\langle DY_t(x), DJY_t(x) \rangle_{\mathcal{H}} \geq 0$. 0. By using the assumption (22) we get $A_3 \geq 0$. Therefore $\|Du(t, x)\|_{\mathcal{H}} > 0$ and the conclusion follows from Theorem 1.

Remark 1 For example, the function $u_0(x) = \frac{\pi}{2} + \arctan(x) + \delta$ with $\delta > 0$ satisfies the assumption in Proposition 4 (and also in Lemma 5 and Propositions 5, 6 below).

4.2 Hölder continuity of the density

We apply now Proposition 1 in order to show that the density of u(t, x) is Hölder continuous. To this end, we will also need to control the second derivative of the solution. We start by proving that $||Du(t, x)||_{\mathcal{H}}^2$ is bounded below by a positive constant which does not depend on ω .

Lemma 5 Assume $b \in L^{\infty}((0,T), C_b^2(\mathbb{R}))$ such that

$$b''(t,x) \le -C < 0 \text{ for all } t \in [0,T], x \in \mathbb{R}.$$
 (24)

Assume $u_0 \in C_b^1(\mathbb{R})$ satisfies

$$u_0(x) \ge C > 0 \text{ and } u'_0(x) > 0 \text{ for every } x \in \mathbb{R}.$$
 (25)

Then for $t \in (0,T]$ and $x \in \mathbb{R}$, there exists a constant $C_5 = C_5(t) > 0$ (non depending on ω and x) such that

$$||Du(t,x)||_{\mathcal{H}}^2 \ge C_5(t)$$

Proof: Notice that from (10), for $t, \alpha \in (0, T], x \in \mathbb{R}$, and by the assumption (24) we have

$$\begin{split} \|DJY_{t}(x)\|_{\mathcal{H}}^{2} &= e^{-2\int_{0}^{t}b'(v,Y_{v,t}(x))dv} \Big\langle \int_{0}^{t}b''(Y_{v,t}(x))DY_{v,t}(x)dv, \int_{0}^{t}b''(Y_{v,t}(x))DY_{v,t}(x)dv \Big\rangle_{\mathcal{H}} \\ &\geq e^{-2T\|b'\|_{\infty}} \int_{0}^{T} \Big(\int_{0}^{t}b''(v,Y_{v,t}(x))\mathbf{1}_{[v,t]}(\alpha) \ e^{-\int_{v}^{\alpha}b'(r,Y_{r,t}(x))dr} \ dv \Big)^{2} \ d\alpha \\ &\geq C^{2}e^{-4T\|b'\|_{\infty}} \int_{0}^{t}\int_{0}^{t}\int_{0}^{t}\mathbf{1}_{[v,t]}(\alpha)\mathbf{1}_{[v_{1},t]}(\alpha)dv_{1} \ dv \ d\alpha \\ &= C^{2}e^{-4T\|b'\|_{\infty}} \int_{0}^{t}\int_{0}^{t}(t-\max\{v_{1},v\})dv_{1} \ dv \\ &= C^{2}e^{-4T\|b'\|_{\infty}} \int_{0}^{t}\int_{0}^{t}(t-\frac{v_{1}+v}{2}-\frac{|v_{1}-v|}{2})dv_{1} \ dv \\ &= C^{2}e^{-4T\|b'\|_{\infty}} \left(\frac{t^{3}}{3}\right) \\ &> 0, \text{ for all} \quad t > 0, \end{split}$$

Thus, by the hypothesis (25), for every $t \in (0, T]$, we see that

$$A_2 := (u_0(Y_t(x)))^2 \|DJY_t(x)\|_{\mathcal{H}}^2 \ge C^4 e^{-4T\|b'\|_{\infty}} \left(\frac{t^3}{3}\right) := C_5(t) > 0.$$

Now, recalling that $||Du(t,x)||_{\mathcal{H}}^2 = A_1 + A_2 + A_3$ with $A_1 > 0$, $A_2 \ge 0$ and $A_3 \ge 0$ defined in the proof of Proposition 4, we have, by the calculus above, that

$$||Du(t,x)||_{\mathcal{H}}^2 \ge C_5(t) > 0.$$
(26)

Therefore, the conclusion is obtained.

Using the Proposition 1 (see [2, Proposition 23]), we can prove the Hölder continuity of the density.

Proposition 5 Assume $b \in L^{\infty}((0,T), C_b^3(\mathbb{R}))$ such that (24) holds. Assume $u_0 \in C_b^1(\mathbb{R})$ such that (25) holds.

Let u(t,x) be the solution to the stochastic continuity equation (1). Then, for every $t \in [0,T]$ and for every $x \in \mathbb{R}$, the density $\rho_{u(t,x)}$ of the solution u(t,x) is Hölder continuous for any exponent q < 1. Moreover, there exist two universal constants C, p > 0 such that

$$\rho_{u(t,x)}(z) \le C \left(P(|u(t,x) - z| \le 2) \right)^{\frac{1}{p}}$$

Proof: From Proposition 3 we have that u(t, x) belongs to $\mathbb{D}^{2,p}$ for every $p \geq 2$. By Lemma 5, we have that $\|Du(t, x)\|_{\mathcal{H}}^{-2}$ belongs to $L^p(\Omega)$ for every $t \in (0, T]$ and for every $x \in \mathbb{R}$ and for each $p \geq 1$. Then we can apply Proposition 1.

4.3 Gaussian estimates

We finally prove the Gaussian estimate for the solution to the continuity equation.

Proposition 6 Assume $b \in L^{\infty}((0,T), C_b^2(\mathbb{R}))$ such that (24) holds. Assume $u_0 \in C_b^1(\mathbb{R})$ such that (25) holds. Let $(u(t,x))_{t \in [0,T], x \in \mathbb{R}}$ be the solution to the continuity equation (1). Then, for every $t \in [t_0,T]$ with $0 < t_0 < T$ and for every $x \in \mathbb{R}$, the random u(t,x) admits a density $\rho_{u(t,x)}$ and there exist two positive constants c_1, c_2 such that

$$\frac{\mathbf{E}|u(t,x)-m|}{2c_1t}e^{-\frac{(y-m)^2}{2c_2t}} \le \rho_{u(t,x)} \le \frac{\mathbf{E}|u(t,x)-m|}{2c_2t}e^{-\frac{(y-m)^2}{2c_1t}}.$$
(27)

Proof: We will show that

$$ct \le \langle Du(t,x), \widetilde{Du(t,x)} \rangle_{\mathcal{H}} \le Ct$$
 (28)

with 0 < c < C where Du(t, x) is constructed from Du(t, x) via the formula (4). From (18),

$$\langle Du(t,x), Du(t,x) \rangle_{\mathcal{H}}$$

$$= u_0'(Y_t(x))u_0'(Y_t(x))JY_t(x)JY_t(x)\langle DY_t(x), DY_t(x) \rangle_{\mathcal{H}}$$

$$+ u_0(Y_t(x))u_0(Y_t(x))\langle DJY_t(x), DJY_t(x) \rangle_{\mathcal{H}}$$

$$+ u_0'(Y_t(x))JY_t(x)u_0(Y_t(x))\langle DY_t(x), DJY_t(x) \rangle_{\mathcal{H}}$$

$$+ u_0'(Y_t(x))JY_t(x)u_0(Y_t(x))\langle DJY_t(x), DY_t(x) \rangle_{\mathcal{H}}$$

$$:= I_1 + I_2 + I_3 + I_4.$$

$$(29)$$

Note that, since $DY_t(x)$, $\widetilde{DY_t(x)}$, $DJY_t(x)$ and $\widetilde{DJY_t(x)}$ are all negative, we can replace $\langle Du(t,x), \widetilde{Du(t,x)} \rangle_{\mathcal{H}}$ by $\langle |Du(t,x)|, |\widetilde{Du(t,x)}| \rangle_{\mathcal{H}}$ in (29) (and similarly for the other inner products in (29)). We also have, from (10), for $t, \alpha \in (0,T], x \in \mathbb{R}$,

$$|D_{\alpha}Y_{t}(x)| = |1_{[0,t]}(\alpha)e^{-\int_{0}^{\alpha}b'(r,Y_{r,t})dr}| \ge C1_{[0,t]}(\alpha)e^{-\|b'\|_{\infty}\alpha}$$
(30)

and by (12),

$$|JY_t(x)| \ge e^{-\|b'\|_{\infty}t} \tag{31}$$

while from (23),

$$|D_{\alpha}JY_{t}(x)| \ge Cte^{\|b'\|_{\infty}t} \mathbf{1}_{[0,t]}(\alpha)e^{-\|b'\|_{\infty}\alpha}.$$
(32)

Under the hypothesis $b \in L^{\infty}((0,T), C_b^2(\mathbb{R}))$ and (24), the lower inequalities (30), (31) and (32) hold also for $\widetilde{DY_t(x)}$, $\widetilde{JY_t(x)}$ and $\widetilde{DJY_t(x)}$. Consequently, since $t \in [t_0,T]$ with $0 < t_0 < T$ the summand

$$\langle DJY_t(x), DJY_t(x) \rangle_{\mathcal{H}}$$

is bigger than cT with c > 0 and so $I_2 \ge ct$, via (25). Since I_1, I_3, I_4 are positive we obtained the lower bound in (28). By Lemmas 2, 3 and 4, the upper bound in (28) clearly holds. Therefore the conclusion is obtained.

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