Baxter operator and Baxter equation for q-Toda and Toda$_2$ chains.

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This work is dedicated to the memory of L.D. Faddeev.

Abstract

We construct the Baxter operator $\mathtt{Q}(\lambda)$ for the q-Toda chain and the Toda$_2$ chain (the Toda chain in the second Hamiltonian structure). Our construction builds on the relation between the Baxter operator and Bäcklund transformations that were unravelled in [13]. We construct a number of quantum intertwiners ensuring the commutativity of $\mathtt{Q}(\lambda)$ with the transfer matrix of the models and the one of $\mathtt{Q}$'s between each other. Most importantly, $\mathtt{Q}(\lambda)$ is modular invariant in the sense of Faddeev. We derive the Baxter equation for the eigenvalues $q(\lambda)$ of $\mathtt{Q}(\lambda)$ and show that these are entire functions of λ. This last property will ultimately lead to the quantisation of the spectrum for the considered Toda chains, in a subsequent publication [1].

1 Introduction.

The Toda chain was introduced by Toda [30] on the classical level and constituted, for over 50 years, a toy model for developing various exact solvability techniques be it on the classical or quantum level. In particular it is this model that saw the birth of the classical separation of variables in the approach using analytical properties on the spectral curve [6, 11, 25] and of the quantum separation of variables [29] following the earlier ideas of separability developed in the series of papers by Gutzwiller [14, 15]. Due to its natural relation with the representation theory of $\mathfrak{gl}_n$ [21, 27], the model can also be solved by harmonic analysis on the associated Lie group, see [33] for a review. In the modern language, the quantum integrability of the Toda chain is related with the Yang-Baxter equation associated with the representation theory of the Yangian $\mathcal{Y}(\widehat{\mathfrak{sl}}_2)$, as first established in Gaudin's book [12]. This aroused the natural question of constructing the Baxter operator for the Toda chain. In the process of answering this question, a relation with Bäcklund transformation was unveiled in [13]. This triggered a series of deep works on the subject by Sklyanin and Kuznetsov, see *e.g.* [23].

It is natural to study the q-deformations of the Toda chain, which are related to the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$. These include the q-Toda chain, first introduced in [28], and the much less known Toda$_2$ model

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which corresponds to the quantisation of the Toda chain in the *second* Hamiltonian structure [2]. One of the many interesting features of these models consists in their modular invariance as advocated by L.D. Faddeev [8, 9]. This was already used in [20] to construct the eigenfunctions of the open q-Toda chain, and it is also crucial in our construction Baxter operator. The Toda$_2$ chain is interesting by its direct relation to a discretisation of the Virasoro algebra [2]. The q-Toda chain has recently received a lot of attention in various domains see e.g. [16, 17, 19, 24, 26]. In particular, the work [26] proposed a construction of a Q operator for the q-Toda. This construction was given in terms of a formal series in the spectral parameter with operator valued combinatorial coefficients. When compared with our construction, this result seems much more implicit; in particular, the modular invariance of that result is not explicitly manifest.

The paper is organised as follows. Section 2 introduces the models of interest and lists several of their properties of relevance to our study. Section 3 focuses on the construction of various intertwining operators that play an important role in the construction of the Baxter Q-operator. Section 4 is devoted to the construction of the Q-operator as well as to the characterisation of its main properties. In particular, we construct there the operator valued Baxter equation and use it to obtain the scalar $t - q$ equation for the eigenvalues $q(\lambda)$ of the operator $Q(\lambda)$. We then use the explicit form of the operator $Q(\lambda)$ to prove that $q(\lambda)$ are *entire* functions of λ. This last property will ultimately yield [1] the quantisation conditions for the spectrum of the considered Toda chains along the lines developed in [13, 22].

Various technical aspects of the analysis developed in the paper are gathered into several appendices. Appendix A presents the Bäcklund transformation. Appendix B provides a general scheme for constructing intertwiners on the quantum space for the class of Lax matrices of interest to the study. Appendix C summarises the main properties and definitions of the special functions of interest to the present study.

2 The models of interest

2.1 The Lax and transfer matrices

The q-Toda and the Toda$_2$ chains are most conveniently constructed by means of the quantum inverse scattering method. The local Lax matrix which encompasses both models takes the form

$$
\mathsf{L}_{0V_{x_n}}(\lambda) = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} - \mathrm{e}^{-\omega_1 \mathsf{X}_n} & q^2 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}[d_2 + d_1 q \mathrm{e}^{-\omega_1 \mathsf{X}_n}]\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathsf{x}_n} \\ -q^{-2}\mathrm{e}^{\frac{2\pi}{\omega_2}\mathsf{x}_n} & -d_2 \end{pmatrix} . \tag{2.1}
$$

The Lax operator is realised as a 2×2 matrix on the auxiliary space $V_0 \simeq \mathbb{C}^2$ while its entries act on the local quantum space $V_{x_n} \simeq L^2(\mathbb{R})$. Above, X_n, x_n are canonically conjugated operators

$$
[\mathsf{X}_n, \mathsf{x}_n] = -\mathrm{i}
$$

so that the operator part of the entries of the above Lax matrix form a Weyl pair:

$$
\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathsf{x}_n}\mathrm{e}^{-\omega_1 \mathsf{X}_n} = q^2 \mathrm{e}^{-\omega_1 \mathsf{X}_n}\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathsf{x}_n}, \quad \text{with} \quad q = \mathrm{e}^{\mathrm{i}\pi\frac{\omega_1}{\omega_2}} . \tag{2.2}
$$

We shall henceforth realise x_n as a multiplication operator on V_{x_n}.

The remaining parameters d_1 and d_2 are free and their specialisations provide one with the Lax matrices of

- q-Toda ($d_2 = 0$)

$$\mathrm{L}^{q-\mathrm{Toda}}_{0V_{x_n}}(\lambda) = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} - \mathrm{e}^{-\omega_1 \mathrm{x}_n} & q^3 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} d_1 \mathrm{e}^{-\omega_1 \mathrm{x}_n} \mathrm{e}^{-\frac{2\pi}{\omega_2}\mathrm{x}_n} \\ -q^{-2} \mathrm{e}^{\frac{2\pi}{\omega_2}\mathrm{x}_n} & 0 \end{pmatrix} , \tag{2.3}$$

- Toda$_2$ ($d_1 = 0$)

$$\mathrm{L}^{\mathrm{Toda}_2}_{0V_{x_n}}(\lambda) = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} - \mathrm{e}^{-\omega_1 \mathrm{x}_n} & \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} q^2 d_2 \mathrm{e}^{-\frac{2\pi}{\omega_2}\mathrm{x}_n} \\ -q^{-2} \mathrm{e}^{\frac{2\pi}{\omega_2}\mathrm{x}_n} & -d_2 \end{pmatrix} . \tag{2.4}$$

The Lax matrix $\mathrm{L}_{0V_{x_n}}(\lambda)$ satisfies the usual Yang-Baxter equation

$$R_{12}(\lambda_1 - \lambda_2)\mathrm{L}_{1V_{x_n}}(\lambda_1)\mathrm{L}_{2V_{x_n}}(\lambda_2) = \mathrm{L}_{2V_{x_n}}(\lambda_2)\mathrm{L}_{1V_{x_n}}(\lambda_1)R_{12}(\lambda_1 - \lambda_2) \tag{2.5}$$

with the 4×4 quantum R-matrix acting on the tensor product of two auxiliary spaces 1 and 2:

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q^{-1}\sinh\frac{\pi\lambda}{\omega_2}}{\sinh\frac{\pi}{\omega_2}(\lambda + i\omega_1)} & \frac{(q-q^{-1})\mathrm{e}^{-\frac{\pi\lambda}{\omega_2}}}{2\sinh\frac{\pi}{\omega_2}(\lambda + i\omega_1)} & 0 \\ 0 & \frac{(q-q^{-1})\mathrm{e}^{\frac{\pi\lambda}{\omega_2}}}{2\sinh\frac{\pi}{\omega_2}(\lambda + i\omega_1)} & \frac{q\sinh\frac{\pi\lambda}{\omega_2}}{\sinh\frac{\pi}{\omega_2}(\lambda + i\omega_1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.6}$$

This R-matrix is twisted in respect to the "usual" 6vertex model R-matrix.

A first set of integrals of motion for the model is obtained by the standard recipe of the quantum inverse scattering method. The transfer matrix which generates the commuting quantities is defined as the trace of a monodromy matrix

$$\mathrm{t}(\lambda) = \mathrm{Tr}_0 \left[\mathrm{L}_{0V_{x_N}}(\lambda) \cdots \mathrm{L}_{0V_{x_1}}(\lambda) \right] .$$

t is an operator valued polynomial of degree N in $\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}$.

$$\mathrm{t}(\lambda) = \sum_{j=0}^{N} (-1)^j \mathrm{H}_j \mathrm{e}^{-\frac{2\pi}{\omega_2}(N-j)\lambda} .$$

Explicitly, one has

$$\mathrm{H}_0 = \mathrm{id} ,$$

$$\mathrm{H}_1 = \sum_{n=1}^{N} \left\{ \left(1 + q^{-1}d_1 e^{-\frac{2\pi}{\omega_2}(\mathrm{x}_n - \mathrm{x}_{n-1})}\right) e^{-\omega_1 \mathrm{x}_n} + d_2 e^{-\frac{2\pi}{\omega_2}(\mathrm{x}_n - \mathrm{x}_{n-1})} \right\} ,$$

$$\vdots$$

$$\mathrm{H}_N = \prod_{n=1}^{N} e^{-\omega_1 \mathrm{x}_n} + d_2^N . \tag{2.7}$$

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One can exhibit a slightly stronger conserved quantity than H_N, namely $\mathsf{P}_{\mathrm{tot}} = \sum_{a=1}^{N} \mathsf{X}_a$. Indeed, one can readily check that

$$e^{i\gamma\mathsf{X}_a}\mathsf{L}_{0V_{x_n}}(\lambda)e^{-i\gamma\mathsf{X}_a} \;=\; e^{-\frac{\pi\gamma}{\omega_2}\sigma_3}\mathsf{L}_{0V_{x_n}}(\lambda)e^{\frac{\pi\gamma}{\omega_2}\sigma_3} \;, \tag{2.8}$$

what ensures that, for any γ,

$$e^{i\gamma\mathsf{P}_{\mathrm{tot}}}\mathsf{t}(\lambda)e^{-i\gamma\mathsf{P}_{\mathrm{tot}}} \;=\; \mathsf{t}(\lambda) \;.$$

Then $\big[\mathsf{P}_{\mathrm{tot}}, \mathsf{t}(\lambda)\big] = 0$ follows upon taking the γ-derivative at $\gamma = 0$ of the previous relation.

2.2 The dual models

As was already pointed out by many authors and, in particular, by Faddeev [8, 9], since the entries of the Lax matrix $\mathsf{L}_{0V_{x_n}}(\lambda)$ only involve explicitly the Weyl pair $\left(e^{-\frac{2\pi}{\omega_2}\mathsf{X}_n}, e^{-\omega_1\mathsf{X}_n}\right)$, the representation induced on V_{x_n} by the algebra generated by the entries of $\mathsf{L}_{0n}(\lambda)$ is reducible. Indeed, any operator belonging to the algebra generated by the dual Weyl pair

$$e^{-\frac{2\pi}{\omega_1}\mathsf{X}_n}e^{-\omega_2\mathsf{X}_n} = \widetilde{q}^{\,2}e^{-\omega_2\mathsf{X}_n}e^{-\frac{2\pi}{\omega_1}\mathsf{X}_n} \;, \qquad \widetilde{q} = e^{i\pi\frac{\omega_2}{\omega_1}}$$

commutes with the former one. This reducibility makes ambiguous several steps occurring in the resolution of the joint spectral problem associated with $\{\mathsf{H}_j\}_{j=0}^{N}$. A way of avoiding such ambiguities is to directly consider the representation on V_{x_n} of the modular double, namely by simultaneously considering the Lax matrix $\mathsf{L}_{0V_{x_n}}(\lambda)$ and its dual $\widetilde{\mathsf{L}}_{0V_{x_n}}(\lambda)$ which is obtained from $\mathsf{L}_{0V_{x_n}}(\lambda)$ by exchanging ω_1 and ω_2 and upon replacing the coupling constants by tilded quantities $(d_1, d_2) \to (\widetilde{d}_1, \widetilde{d}_2)$. It is straightforward to see that

$$\left[\big[\mathsf{L}_{0V_{x_n}}(\lambda)\big]_{ab}, \big[\widetilde{\mathsf{L}}_{0V_{x_n}}(\lambda)\big]_{cd}\right] \;=\; 0 \;.$$

While the two dual algebras generated by $\mathsf{L}_{0V_{x_n}}$ and $\widetilde{\mathsf{L}}_{0V_{x_n}}$ do not see each other, they allow for an unambiguous fixing of various constants which, in case one would solely consider one of the two sub-algebras, would be only fixed to be quasi-constants. In particular, the fact of constructing the joint spectrum of the transfer matrix t and its dual transfer matrix $\widetilde{\mathsf{t}}$ imposes that all objects used to built up this spectrum have to be modular invariant in the terminology of Faddeev, *i.e.* invariant under the transformations $\omega_1 \leftrightarrow \omega_2$ and $(d_1, d_2) \to (\widetilde{d}_1, \widetilde{d}_2)$. In particular, the Baxter operator will exhibit such a modular invariance.

The dual transfer matrix provides one with the second set of conserved quantities

$$\widetilde{\mathsf{t}}(\lambda) = \sum_{j=0}^{N}(-1)^j\widetilde{\mathsf{H}}_j e^{-\frac{2\pi}{\omega_1}(N-j)\lambda}$$

where now

$$\widetilde{\mathsf{H}}_0 = \mathrm{id} \;,$$

$$\widetilde{\mathsf{H}}_1 = \sum_{n=1}^{N}\left\{\left(1 + \widetilde{q}^{-1}\widetilde{d}_1 e^{-\frac{2\pi}{\omega_1}(\mathsf{x}_n - \mathsf{x}_{n-1})}\right)e^{-\omega_2\mathsf{X}_n} + \widetilde{d}_2 e^{-\frac{2\pi}{\omega_1}(\mathsf{x}_n - \mathsf{x}_{n-1})}\right\} \;,$$

$$\vdots$$

$$\widetilde{\mathsf{H}}_N = \prod_{n=1}^{N}e^{-\omega_2\mathsf{X}_n} \;+\; \widetilde{d}_2^{\,N} \;. \tag{2.9}$$

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Obviously, the Hamiltonians $\mathtt{H}_j$ commute with the Hamiltonians $\widetilde{\mathtt{H}}_k$. It appears that one can now impose two conditions on the parameters at play so as to have interesting reality relations between the Hamiltonians.

i) ω_1 and ω_2 are real numbers. Then we can impose that the coupling constants verify

$$d_1 = \overline{d_1}, \quad d_2 = \overline{d_2}, \quad \widetilde{d}_1 = \overline{\widetilde{d}_1}, \quad \widetilde{d}_2 = \overline{\widetilde{d}_2} \; .$$

In such a case, one gets that

$$\mathtt{H}_j^\dagger = \mathtt{H}_j, \quad \widetilde{\mathtt{H}}_j = \widetilde{\mathtt{H}}_j^\dagger \; .$$

ii) ω_1 and ω_2 are complex numbers such that $\omega_2 = \overline{\omega}_1$. Upon imposing the coupling constants to satisfy

$$\widetilde{d}_1 = \overline{d_1}, \quad \widetilde{d}_2 = \overline{d_2}$$

one gets that

$$\mathtt{H}_j^\dagger = \widetilde{\mathtt{H}}_j \; .$$

In both cases we are lead to parameterise

$$d_1 = \mathrm{e}^{-\frac{2\pi}{\omega_2}\kappa_1}, \quad d_2 = \mathrm{e}^{-\frac{2\pi}{\omega_2}\kappa_2}, \quad \widetilde{d}_1 = \mathrm{e}^{-\frac{2\pi}{\omega_1}\kappa_1}, \quad \widetilde{d}_2 = \mathrm{e}^{-\frac{2\pi}{\omega_1}\kappa_2}$$

with κ_1 and κ_2 real and modular invariant. This choice of parametrisation will be made from now on. The q-Toda and Toda$_2$ Hamiltonians are obtained from $\mathtt{H}_1$ upon enforcing the respective specialisations $d_2 = 0$ for q-Toda and $d_1 = 0$ for Toda$_2$ (see [2] for a discussion of this model):

- q-Toda $(d_2 = 0)$

$$\mathtt{H}_1^{\mathrm{q-Toda}} = \sum_{n=1}^{N} \left[1 + q^{-1} \mathrm{e}^{-\frac{2\pi}{\omega_2}\kappa_1} \mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathtt{x}_n - \mathtt{x}_{n-1})} \right] \mathrm{e}^{-\omega_1 \mathtt{x}_n} \; ;$$

- Toda$_2$ $(d_1 = 0)$

$$\mathtt{H}_1^{\mathrm{Toda2}} = \sum_{n=1}^{N} \left\{ \mathrm{e}^{-\omega_1 \mathtt{x}_n} + \mathrm{e}^{-\frac{2\pi}{\omega_2}\kappa_2} \mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathtt{x}_n - \mathtt{x}_{n-1})} \right\} \; .$$

3 The various intertwiners of interest

3.1 $\mathbb{L}\mathtt{L}\mathtt{M} = \mathtt{M}\mathtt{L}\mathbb{L}$ - Operator.

The first step in constructing the $\mathtt{Q}$-operator is to build an appropriate, modular invariant, intertwiner for a product of two $\mathtt{L}$ operators, both sharing the same auxiliary space but acting on different quantum spaces. The prototypes of such intertwiners go back to [10, 31]. More precisely, one introduces two representation spaces V_x and V_u endowed with the respective action of Weyl-pairs $(\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}}, \mathrm{e}^{-\omega_1 \mathtt{X}})$, $(\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{u}}, \mathrm{e}^{-\omega_1 \mathtt{U}})$ and their duals. Then, as motivated in Appendix A on Bäcklund transformations, one focuses on building an

intertwiner for the two Lax operators $\mathtt{L}_{0V_x}(\lambda)$ and $\mathtt{M}_{0V_u}(\lambda; t)$, and their duals, where $\mathtt{M}_{0V_u}(\lambda; t)$ is obtained from $\mathtt{L}_{0V_u}(\lambda)$ by the substitution

$$\mathtt{U} \hookrightarrow \mathtt{U} + \frac{2\pi}{\omega_1\omega_2} t \qquad d_1 \hookrightarrow q^{-2} e^{\frac{2\pi}{\omega_2} t} \qquad d_2 \hookrightarrow -q^{-1} \ , \tag{3.1}$$

so that it takes the explicit form

$$\mathtt{M}_{0V_u}(\lambda; t) \ = \ \begin{pmatrix} e^{-\frac{2\pi}{\omega_2}\lambda} - e^{-\frac{2\pi}{\omega_2}t} e^{-\omega_1 \mathtt{U}} & -q e^{-\frac{2\pi}{\omega_2}\lambda}(1 - e^{-\omega_1 \mathtt{U}}) e^{-\frac{2\pi}{\omega_2}\mathtt{u}} \\ -q^{-2} e^{\frac{2\pi}{\omega_2}\mathtt{u}} & q^{-1} \end{pmatrix} . \tag{3.2}$$

This Lax matrix corresponds to a slight deformation of the classical formula eq.(A.5). The extra factors q are dictated by the requirement that $\mathtt{M}_{0V_u}(\lambda; t)$ satisfies the Baxter equation eq.(3.31) and that the Baxter $\mathtt{Q}$ operator we will construct is modular invariant, and thus also intertwines the dual Lax operators.

The construction of the intertwiner

$$\mathbb{L}_{V_x V_u}(t) \, \mathtt{L}_{0V_x}(\lambda) \, \mathtt{M}_{0V_u}(\lambda; t) \ = \ \mathtt{M}_{0V_u}(\lambda; t) \, \mathtt{L}_{0V_x}(\lambda) \, \mathbb{L}_{V_x V_u}(t) \tag{3.3}$$

is obtained by the general recipe discussed in Appendix B and more precisely in Subsection B.1. It takes the form

$$\mathbb{L}_{V_x V_u}(t) \ = \ C_{\mathbb{L}}(t) \cdot \mathtt{P}_{xu} \cdot \phi_{14}(\mathtt{x} - \mathtt{u}) \cdot \psi_{24}(\mathtt{U}) \cdot \phi_{23}(\mathtt{x} - \mathtt{u}) \tag{3.4}$$

where the building blocks of $\mathbb{L}_{V_x V_u}(t)$ satisfy to first order finite difference equations. These can be explicitly solved in terms of the double sine function $\mathcal{S}$ whose main properties are recalled in Appendix C.2:

$$\phi_{14}(x) = \mathcal{S}^{-1}\left(x - t + \mathrm{i}\frac{\Omega}{2}\right) \cdot e^{-\frac{2\mathrm{i}\pi t}{\omega_1\omega_2}x} \tag{3.5}$$

$$\phi_{23}(x) = \mathcal{S}\left(x + \kappa_1 - \mathrm{i}\frac{\Omega}{2}\right) \tag{3.6}$$

$$\psi_{24}(p) = e^{\mathrm{i}p\left(\kappa_2 - \mathrm{i}\frac{\Omega}{2}\right)} \cdot \mathcal{S}\left(\frac{\omega_1\omega_2}{2\pi}p + t + \kappa_1 - \kappa_2 - \mathrm{i}\frac{\Omega}{2}\right) \cdot \mathcal{S}^{-1}\left(\frac{\omega_1\omega_2}{2\pi}p\right) \ . \tag{3.7}$$

Here, we introduced the useful shorthand notation

$$\Omega = \omega_1 + \omega_2 \ . \tag{3.8}$$

Finally, the constant prefactor is expressed in terms of the quantum dilogarithm, *c.f.* C.3, takes the form

$$C_{\mathbb{L}}(t) \ = \ \sqrt{\omega_1\omega_2} \frac{e^{-\frac{2\mathrm{i}\pi\alpha_0}{\omega_1\omega_2}\left(\alpha_0 + \frac{\mathrm{i}\Omega}{2}\right)}}{\varpi\left(2\alpha_0 - \frac{\mathrm{i}\Omega}{2}\right)} \qquad \text{with} \qquad \alpha_0 = \frac{t + \kappa_1 - \kappa_2}{2} - \frac{\mathrm{i}\Omega}{4} \ . \tag{3.9}$$

Clearly, the value of the constant $C_{\mathbb{L}}(t)$ does not impact the intertwining property of $\mathbb{L}_{V_x V_u}(t)$, but fixing the constant factor to the above value will appear convenient in the following.

The operator product in (3.4) can be recast in a more compact form:

$$\mathbb{L}_{V_x V_u}(t) \ = \ C'_{\mathbb{L}}(t) \, \mathtt{P}_{xu} e^{\mathrm{i}\alpha\mathtt{U}} e^{\frac{2\mathrm{i}\pi\beta}{\omega_1\omega_2}(\mathtt{u} - \mathtt{x})} \frac{S\left(-aq e^{-\frac{2\pi}{\omega_2}t}(1 - e^{-\omega_1 \mathtt{U}}) e^{\frac{2\pi}{\omega_2}(\mathtt{x} - \mathtt{u})}\right)}{S\left(b(1 - e^{-\omega_1 \mathtt{U}}) e^{\frac{2\pi}{\omega_2}(\mathtt{x} - \mathtt{u})}\right)} \tag{3.10}$$

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where $C'_{\mathbb{L}}(t)$ is some constant whose explicit expression is irrelevant for our purposes,

$$a = e^{\frac{2\pi}{\omega_2}(\kappa_2 + \frac{i}{2}\Omega)}, \quad b = e^{\frac{2\pi}{\omega_2}(\kappa_1 + \frac{i}{2}\Omega)}, \quad \alpha = \kappa_2 - \frac{i}{2}\Omega, \quad \beta = \kappa_2 - \kappa_1 + \frac{i}{2}\Omega. \tag{3.11}$$

The main building block of the formula is the function S defined in (C.12), which is closely related to the double sine function, see Appendix C.2 for more details.

We refer to Subsection B.2 of Appendix B and, in particular eq.(B.18), for a proof of this representation.

3.2 Integral kernel of $\mathbb{L}_{V_x V_u}(t)$

In this subsection we realise $\mathbb{L}_{V_x V_u}(t)$ as an integral operator acting on the space of functions on the spectrum of the operators $\mathtt{x} \otimes \mathtt{u}$. More precisely, we show that

$$\mathbb{L}_{V_x V_u}(t) = \mathbb{E}_{V_x V_u} \mathbb{L}^{(c)}_{V_x V_u}(t) \mathbb{E}^{-1}_{V_x V_u} \quad \text{with} \quad \mathbb{E}_{V_x V_u} = e^{-\frac{3}{4}\Omega(\mathtt{x}+\mathtt{u})}. \tag{3.12}$$

The operator $\mathbb{L}^{(c)}_{V_x V_u}(t)$ is realised as an integral operator on $L^2(\mathbb{R})$:

$$\left(\mathbb{L}^{(c)}_{V_x V_u}(t) \cdot f\right)(x, u) = \int_{\mathbb{R}} dy \int_{\mathbb{R}} \mathbb{L}_t(x, u; y, v) f(y, v) dv \tag{3.13}$$

in which the integral kernel takes the manifestly modular invariant form:

$$\mathbb{L}_t(x, u; y, v) = \delta(y - u) e^{\frac{2i\pi}{\omega_1\omega_2}t(x-y)} \frac{S\left(x - v + \kappa_2 - \frac{3i}{2}\Omega\right) S\left(y - v + \kappa_1 - \frac{i}{2}\Omega\right)}{S\left(y - x - t + \frac{i}{2}\Omega\right) S\left(x - v + \kappa_1 + t - i\Omega\right)}. \tag{3.14}$$

Let us specialise this result to the q–Toda and Toda$_2$ cases

- q–Toda ($d_2 = 0$)

$$\mathbb{L}_t(x, u; y, v) = \delta(y - u) \frac{e^{\frac{2i\pi}{\omega_1\omega_2}t(x-y)} S\left(y - v + \kappa_1 - \frac{i}{2}\Omega\right)}{S\left(y - x - t + \frac{i}{2}\Omega\right) S\left(x - v + \kappa_1 + t - i\Omega\right)}; \tag{3.15}$$

- Toda$_2$ ($d_1 = 0$)

$$\mathbb{L}_t(x, u; y, v) = \delta(y - u) e^{\frac{2i\pi}{\omega_1\omega_2}t(x-y)} \frac{S\left(x - v + \kappa_2 - \frac{3i}{2}\Omega\right)}{S\left(y - x - t + \frac{i}{2}\Omega\right)}. \tag{3.16}$$

The operator form of $\mathbb{L}_{V_x V_u}(t)$ given in (3.4) can be recast as

$$\mathbb{L}_{V_x V_u}(t) = \mathsf{P}_{xu} \cdot \frac{e^{-\frac{2i\pi}{\omega_1\omega_2}t(\mathtt{x}-\mathtt{u})}}{S\left(\mathtt{x} - \mathtt{u} - t + i\frac{\Omega}{2}\right)} \cdot e^{i\mathtt{u}\left(\kappa_2 - i\frac{\Omega}{2}\right)} \cdot$$

$$\frac{S\left(\frac{\omega_1\omega_2}{2\pi}\mathtt{u} + t + \kappa_1 - \kappa_2 - i\frac{\Omega}{2}\right)}{S\left(\frac{\omega_1\omega_2}{2\pi}\mathtt{u}\right)} \cdot S\left(\mathtt{x} - \mathtt{u} + \kappa_1 - i\frac{\Omega}{2}\right). \tag{3.17}$$

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Starting from the Fourier transform of the function $D_\alpha(x)$ which is recalled in (C.18) and observing that

$$\frac{\mathcal{S}\left(p + t + \kappa_1 - \kappa_2 - i\frac{\Omega}{2}\right)}{\mathcal{S}(p)} = D_{\alpha_0}(p + \beta_0) e^{\frac{2i\pi}{\omega_1\omega_2}\alpha_0(p+\beta_0)} \tag{3.18}$$

with $2\alpha_0 = t + \kappa_1 - \kappa_2 - i\frac{\Omega}{2}$ and $2\beta_0 = t + \kappa_1 - \kappa_2 + i\frac{\Omega}{2}$, one has the integral representation

$$\frac{\mathcal{S}\left(p + t + \kappa_1 - \kappa_2 - i\frac{\Omega}{2}\right)}{\mathcal{S}(p)} = \frac{\mathcal{A}(\alpha_0)}{\sqrt{\omega_1\omega_2}} e^{\frac{2i\pi}{\omega_1\omega_2}\alpha_0\beta_0} \int_{\mathbb{R}} dv \frac{\mathcal{S}(v - \alpha_0 - i\Omega)}{\mathcal{S}(v + \alpha_0)} e^{-iv\frac{2\pi}{\omega_1\omega_2}p} . \tag{3.19}$$

Therefore, given any sufficiently regular function, one has the realisation as an integral representation

$$\left(e^{iX\left(\kappa_2 - i\frac{\Omega}{2}\right)} \frac{\mathcal{S}\left(\frac{\omega_1\omega_2}{2\pi}X + t + \kappa_1 - \kappa_2\right)}{\mathcal{S}\left(\frac{\omega_1\omega_2}{2\pi}X + i\frac{\Omega}{2}\right)} \cdot f \right)(x)$$

$$= \frac{\mathcal{A}(\alpha_0)}{\sqrt{\omega_1\omega_2}} e^{\frac{2i\pi}{\omega_1\omega_2}\alpha_0\beta_0} \int_{\mathbb{R}} dv \frac{\mathcal{S}\left(x - v + \kappa_2 - \frac{3i}{4}\Omega\right)}{\mathcal{S}\left(x - v + t + \kappa_1 - \frac{i}{4}\Omega\right)} f\left(v - \frac{3i}{4}\Omega\right) \tag{3.20}$$

where, after acting with the translation operator on f, we made a linear change of variables:

$$v \mapsto x - v + \frac{1}{2}(t + \kappa_1 + \kappa_2) . \tag{3.21}$$

Thus, upon acting with the multiplication operators appearing to the right and left of (3.17), one gets for any sufficiently regular function of two variables

$$\left(\mathbb{L}_{V_x V_u}(t) \cdot f \right)(x, u) = \int_{\mathbb{R}} dv \, \mathcal{L}_t\left(x, u; v - \frac{3i}{4}\Omega\right) f\left(u, v - \frac{3i}{4}\Omega\right) . \tag{3.22}$$

where

$$\mathcal{L}_t(x, u; v) = e^{\frac{2i\pi}{\omega_1\omega_2}t(x-u)} \frac{\mathcal{S}\left(x - v + \kappa_2 - \frac{3i}{2}\Omega\right)\mathcal{S}\left(u - v + \kappa_1 - i\frac{\Omega}{2}\right)}{\mathcal{S}\left(x - v + t + \kappa_1 - i\Omega\right)\mathcal{S}\left(u - x - t + i\frac{\Omega}{2}\right)} \tag{3.23}$$

It remains to observe that

$$\mathcal{L}_t\left(x, u; v - \frac{3i}{4}\Omega\right) = \mathcal{L}_t\left(x + \frac{3i}{4}\Omega, u + \frac{3i}{4}\Omega; v\right)$$

what allows one to obtain the below integral representation

$$\left(\mathbb{E}_{V_x V_u}^{-1} \mathbb{L}_{V_x V_u}(t) \, \mathbb{E}_{V_x V_u} f \right)(x, u) = \int_{\mathbb{R}} dv \, \mathcal{L}_t(x, u; v) f(u, v) . \tag{3.24}$$

Thus, representing the action on the u variable as an integral versus a Dirac mass, the claimed form of the kernel follows.

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3.3 The M − M intertwiner

In this subsection, we construct the intertwiner $\mathbb{R}_{V_u V_v}(t, t')$ satisfying

$$\mathbb{R}_{V_u V_v}(t, t') \, \mathbb{M}_{0V_u}(\lambda; t) \mathbb{M}_{0V_v}(\lambda; t') \; = \; \mathbb{M}_{0V_v}(\lambda; t') \mathbb{M}_{0V_u}(\lambda; t) \, \mathbb{R}_{V_u V_v}(t, t') \, . \tag{3.25}$$

Here λ is the spectral parameter, t, t' are two different Bäcklund parameters. In fact, $\mathbb{R}_{V_u V_v}(t, t')$ can be directly inferred from the intertwiner $\mathbb{L}_{V_u V_v}(t)$ arising in (3.3) owing to the transformation (3.1) that turns $\mathbb{L}_{0V_u}(\lambda)$ into $\mathbb{M}_{0V_u}(\lambda; t)$. One finds

$$\mathbb{R}_{V_u V_v}(t, t') = \mathbb{P}_{uv} \frac{S\left(-q^3 e^{-\frac{2\pi}{\omega_2}t'}(1 - e^{-\omega_1 \mathbb{V}}) e^{-\frac{2\pi}{\omega_2}(\mathbb{v}-\mathbb{u})}\right)}{S\left(-q^3 e^{-\frac{2\pi}{\omega_2}t}(1 - e^{-\omega_1 \mathbb{V}}) e^{-\frac{2\pi}{\omega_2}(\mathbb{v}-\mathbb{u})}\right)} \, . \tag{3.26}$$

Here, for convenience, we have set the constant prefactor present in $\mathbb{L}$ to 1 as the latter does not play a role on the intertwining property of $\mathbb{R}_{V_u V_v}(t, t')$. Also, it is manifest from the above formula that $\mathbb{R}_{V_u V_v}(t, t')$ is an invertible operator for almost all values of t, t'.

The proof goes as follows. Starting from (3.3), one makes the substitutions

$$t \hookrightarrow t' \quad \mathbb{U} \hookrightarrow \mathbb{U} \qquad d_1 \hookrightarrow q^{-2} e^{\frac{2\pi}{\omega_2}t} \qquad d_2 \hookrightarrow -q^{-1} \, , \tag{3.27}$$

what recasts (3.3) in the form

$$e^{\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} \mathbb{R}_{V_u V_v}(t, t') e^{-\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} \, \tilde{\mathbb{M}}_{0V_u}(\lambda; t) \mathbb{M}_{0V_v}(\lambda; t')$$
$$= \mathbb{M}_{0V_v}(\lambda; t') \tilde{\mathbb{M}}_{0V_u}(\lambda; t) \, e^{\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} \mathbb{R}_{V_u V_v}(t, t') e^{-\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} \tag{3.28}$$

where

$$\tilde{\mathbb{M}}_{0V_u}(\lambda; t) = \begin{pmatrix} e^{-\frac{2\pi}{\omega_2}\lambda} - e^{-\omega_1 \mathbb{U}} & -q e^{-\frac{2\pi}{\omega_2}\lambda}(1 - e^{\frac{2\pi}{\omega_2}t} e^{-\omega_1 \mathbb{U}}) e^{-\frac{2\pi}{\omega_2}\mathbb{u}} \\ -q^{-2} e^{\frac{2\pi}{\omega_2}\mathbb{u}} & q^{-1} \end{pmatrix} . \tag{3.29}$$

One can then move $e^{-\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}}$ occurring in the *rhs* of (3.28) to the left and $e^{\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}}$ occurring in the *lhs* of this equation to the right. Since

$$e^{-\omega_1 \mathbb{U}} e^{\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} \; = \; e^{-\frac{2\pi}{\omega_2}t} e^{\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} e^{-\omega_1 \mathbb{U}} \quad \text{and} \quad e^{-\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} e^{-\omega_1 \mathbb{U}} \; = \; e^{-\frac{2\pi}{\omega_2}t} e^{-\omega_1 \mathbb{U}} e^{-\frac{2i\pi t}{\omega_1 \omega_2}\mathbb{u}} \tag{3.30}$$

one then recovers, upon simplifying the position operator dependent exponents, eq. (3.25).

3.4 The $\mathbb{L} − \mathbb{L}$ intertwiner

Using the commutation relations eqs.(3.3), (3.25) we can write the transformation

$$\mathbb{L}_{0V_x}(\lambda) \mathbb{M}_{0V_u}(\lambda; t) \mathbb{M}_{0V_v}(\lambda; t') \to \mathbb{M}_{0V_v}(\lambda; t') \mathbb{M}_{0V_u}(\lambda; t) \mathbb{L}_{0V_x}(\lambda)$$

in two different ways. As usual we expect the compatibility relation

$$\mathbb{R}_{V_u V_v}(t, t') \mathbb{L}_{V_x V_v}(t') \mathbb{L}_{V_x V_u}(t) = \mathbb{L}_{V_x V_u}(t) \mathbb{L}_{V_x V_v}(t') \mathbb{R}_{V_u V_v}(t, t') \tag{3.31}$$

We now establish this relation directly. Writing

$$\mathbb{R}_{V_u V_v}(t,t') = \mathsf{P}_{uv}\check{\mathbb{R}}_{V_u V_v}(t,t'), \quad \mathbb{L}_{V_x V_u}(t) = \mathsf{P}_{xu}\check{\mathbb{L}}_{V_x V_u}(t) ,$$

eq. (3.31) becomes equivalent to

$$\check{\mathbb{R}}_{V_x V_u}(t,t')\check{\mathbb{L}}_{V_u V_v}(t')\check{\mathbb{L}}_{V_x V_u}(t) = \check{\mathbb{L}}_{V_u V_v}(t)\check{\mathbb{L}}_{V_x V_u}(t')\check{\mathbb{R}}_{V_u V_v}(t,t') \tag{3.32}$$

where $\check{\mathbb{L}}_{V_u V_v}(t)$ can be read from eq.(3.10) and $\check{\mathbb{R}}_{V_u V_v}(t,t')$ from eq.(3.26). In the following, it will appear convenient to introduce the shorthand notations

$$\mathcal{U} = (1 - e^{-\omega_1 \mathtt{U}})e^{-\frac{2\pi}{\omega_2}(\mathtt{u}-\mathtt{x})} ,$$
$$\mathcal{V} = (1 - e^{-\omega_1 \mathtt{V}})e^{-\frac{2\pi}{\omega_2}(\mathtt{v}-\mathtt{x})} ,$$
$$\mathcal{W} = (1 - e^{-\omega_1 \mathtt{V}})e^{-\frac{2\pi}{\omega_2}(\mathtt{v}-\mathtt{u})} .$$

Given α, β as in (3.11), one uses

$$\mathcal{W}e^{i\alpha\mathtt{U}} = -qd_2 e^{i\alpha\mathtt{U}}\mathcal{W} , \quad \text{and} \quad e^{-\omega_1\mathtt{U}}e^{\frac{2i\pi}{\omega_1\omega_2}\beta(\mathtt{v}-\mathtt{u})} = -qd_1 d_2^{-1} e^{\frac{2i\pi}{\omega_1\omega_2}\beta(\mathtt{v}-\mathtt{u})}e^{-\omega_1\mathtt{U}} ,$$

etc., to push these factors to the left of each side of the above equation, leading to

$$\check{\mathbb{R}}_{V_x V_u}(t,t')\check{\mathbb{L}}_{V_u V_v}(t')\check{\mathbb{L}}_{V_x V_u}(t) = \mathtt{E} \cdot S^{-1}\Big(q^2 ba^{-1}\mathcal{W}\Big) \cdot \mathsf{C}_L \cdot S\Big(-q^3 e^{-\frac{2\pi}{\omega_2}t'}\mathcal{W}\Big) \tag{3.33}$$

and

$$\check{\mathbb{L}}_{V_u V_v}(t)\check{\mathbb{L}}_{V_x V_u}(t')\check{\mathbb{R}}_{V_u V_v}(t,t') = \mathtt{E} \cdot S^{-1}\Big(q^2 ba^{-1}\mathcal{W}\Big) \cdot \mathsf{C}_R \cdot S\Big(-q^3 e^{-\frac{2\pi}{\omega_2}t'}\mathcal{W}\Big) . \tag{3.34}$$

Here, a, b are as defined in (3.11), and we agree upon

$$\mathtt{E} = C_{\mathbb{L}}(t)C_{\mathbb{L}}(t')e^{i(\mathtt{U}+\mathtt{V})\left(\kappa_2 - i\frac{\Omega}{2}\right)}e^{\frac{2i\pi}{\omega_1\omega_2}(\mathtt{x}-\mathtt{v})\left(\kappa_1 - \kappa_2 - i\frac{\Omega}{2}\right)}e^{-\frac{2i\pi}{\omega_1\omega_2}\left(\kappa_2 - i\frac{\Omega}{2}\right)\left(\kappa_1 - \kappa_2 - i\frac{\Omega}{2}\right)} . \tag{3.35}$$

Finally,

$$\mathsf{C}_L = S(q^2 ba^{-1}\mathcal{W}) \cdot \frac{S\Big(-aq e^{-\frac{2\pi}{\omega_2}t'}\mathcal{U}\Big)}{S\Big(-aq e^{-\frac{2\pi}{\omega_2}t}\mathcal{U}\Big)} \cdot S^{-1}(q^2 ba^{-1}\mathcal{W})$$
$$\times S\Big(-q^3 e^{-\frac{2\pi}{\omega_2}t'}\mathcal{W}\Big) \cdot \frac{S\Big(-aq e^{-\frac{2\pi}{\omega_2}t}\mathcal{U}\Big)}{S(b\,\mathcal{U})} \cdot S^{-1}\Big(-q^3 e^{-\frac{2\pi}{\omega_2}t'}\mathcal{W}\Big) ,$$

and

$$\mathsf{C}_R = S\Big(-q^3 e^{-\frac{2\pi}{\omega_2}t}\mathcal{W}\Big)\frac{S\Big(-aq e^{-\frac{2\pi}{\omega_2}t'}\mathcal{U}\Big)}{S(b\,\mathcal{U})} \cdot S^{-1}\Big(-q^3 e^{-\frac{2\pi}{\omega_2}t}\mathcal{W}\Big) .$$

C_L and C_R can be simplified further by means of the below rewriting of the adjoint action which is valid for any constant c:

$$S(c\mathcal{W})\,\mathcal{U}\,S^{-1}(c\mathcal{W}) = [1 - S(c\mathcal{W})\,e^{-\omega_1\mathtt{U}}\,S^{-1}(c\mathcal{W})]e^{-\frac{2\pi}{\omega_2}(\mathtt{u}-\mathtt{x})}$$
$$= [1 - S(c\mathcal{W})S^{-1}(q^2 c\mathcal{W})\,e^{-\omega_1\mathtt{U}}\,]e^{-\frac{2\pi}{\omega_2}(\mathtt{u}-\mathtt{x})}$$
$$= [1 - (1 - c\mathcal{W})e^{-\omega_1\mathtt{U}}]e^{-\frac{2\pi}{\omega_2}(\mathtt{u}-\mathtt{x})}$$
$$= \mathcal{U} + cq^{-2}\mathcal{V}\,e^{-\omega_1\mathtt{U}} .$$

Note that, in the intermediate equations, we have used the finite difference equation (C.14). Thence, upon setting $\mathbf{y} = \mathcal{U}$ and $\mathbf{Y} = \mathcal{V} \, \mathrm{e}^{-\omega_1 U}$, one gets

$$\mathtt{C}_L = \frac{S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\,(\mathbf{y}+ba^{-1}\mathbf{Y})\Big)}{S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t}(\mathbf{y}+ba^{-1}\mathbf{Y})\Big)} \cdot \frac{S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t}(\mathbf{y}-q\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\mathbf{Y})\Big)}{S\Big(b(\mathbf{y}-q\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\mathbf{Y})\Big)} \tag{3.36}$$

and

$$\mathtt{C}_R = \frac{S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}(\mathbf{y}-q\mathrm{e}^{-\frac{2\pi}{\omega_2}t}\mathbf{Y})\Big)}{S\Big(b(\mathbf{y}-q\mathrm{e}^{-\frac{2\pi}{\omega_2}t}\mathbf{Y})\Big)} \; . \tag{3.37}$$

Notice that $\mathcal{U}$ and $\mathcal{V}$ commute so that $(\mathbf{y}, \mathbf{Y})$ form again a Weyl pair. It remains to use the Schützenberger relation (C.15)

$$S(c_1\mathbf{y} + c_2\mathbf{Y}) = S(c_1\mathbf{y})S(c_2\mathbf{Y})$$

to recast these operators as

$$\mathtt{C}_L = S^{-1}\Big(-bq\mathrm{e}^{-\frac{2\pi}{\omega_2}t}\mathbf{Y}\Big)S^{-1}\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t}\mathbf{y}\Big)S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\mathbf{y}\Big)S\Big(-bq\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\mathbf{Y}\Big)$$
$$\times S^{-1}\Big(-bq\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\mathbf{Y}\Big)S^{-1}(b\mathbf{y})S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t}\mathbf{y}\Big)S\Big(aq^2\mathrm{e}^{-\frac{2\pi}{\omega_2}(t+t')}\mathbf{Y}\Big)$$
$$\mathtt{C}_R = S^{-1}\Big(-bq\mathrm{e}^{-\frac{2\pi}{\omega_2}t}\mathbf{Y}\Big)S^{-1}(b\mathbf{y})S\Big(-aq\mathrm{e}^{-\frac{2\pi}{\omega_2}t'}\mathbf{y}\Big)S\Big(aq^2\mathrm{e}^{-\frac{2\pi}{\omega_2}(t+t')}\mathbf{Y}\Big) \; .$$

At this stage, it becomes evident that $\mathtt{C}_R = \mathtt{C}_L$. Exactly the same techniques applies so as to show that $\mathbb{R}_{V_u V_v}(t, t')$ satisfies to the Yang–Baxter equation. An alternative proof of a similar identity using only the pentagonal identity can be found in [18].

4 The Q operator and the Baxter equation

4.1 Definition and basic properties of the Q operator

In exact parallel to the transfer matrix

$$\mathtt{t}(\lambda) \; = \; \mathrm{Tr}_0 \, \Big[\mathtt{L}_{0V_{x_N}}(\lambda) \cdots \mathtt{L}_{0V_{x_1}}(\lambda) \Big] \; ,$$

the Baxter Q-operator is defined, *modulo* a gauge transformation, as a trace of $\mathbb{L}$ matrices

$$\mathtt{Q}(\lambda) \; = \; \mathbb{E}_{\otimes V_{x_a}}^{-1} \cdot \mathrm{Tr}_{V_u} \, \Big[\mathbb{L}_{V_{x_N}, V_u}(\lambda) \cdots \mathbb{L}_{V_{x_1}, V_u}(\lambda) \Big] \cdot \mathbb{E}_{\otimes V_{x_a}} \quad \text{with} \quad \mathbb{E}_{\otimes V_{x_a}} \; = \; \prod_{a=1}^{N} \mathrm{e}^{-\frac{3}{4}\Omega \mathsf{X}_a} \; .$$

One has the properties:

   i) $[\mathtt{t}(\lambda), \mathtt{t}(\lambda')] = 0$,

  ii) $[\mathtt{t}(\lambda), \mathtt{Q}(\lambda')] = 0$,

 iii) $[\mathtt{Q}(\lambda), \mathtt{Q}(\lambda')] = 0$.

Property i) follows from

$$R_{00'}(\lambda, \lambda')\mathtt{L}_{0V_u}(\lambda)\mathtt{L}_{0'V_u}(\lambda') \;=\; \mathtt{L}_{0'V_u}(\lambda')\mathtt{L}_{0V_u}(\lambda)R_{00'}(\lambda, \lambda') \,,$$

property ii) from the fact that $\big[\mathtt{t}(\lambda), \sum_{a=1}^{N} \mathtt{X}_a\big] = 0$ and the intertwining relation

$$\mathtt{M}_{0V_u}(\lambda; \lambda')\mathtt{L}_{0V_x}(\lambda)\mathbb{L}_{V_x V_u}(\lambda') \;=\; \mathbb{L}_{V_x V_u}(\lambda')\mathtt{L}_{0V_x}(\lambda)\mathtt{M}_{0V_u}(\lambda; \lambda') \,,$$

while property iii) follows from

$$\mathbb{R}_{V_u V_v}(\lambda, \lambda')\mathbb{L}_{V_x V_v}(\lambda')\mathbb{L}_{V_x V_u}(\lambda) = \mathbb{L}_{V_x V_u}(\lambda)\mathbb{L}_{V_x V_v}(\lambda')\mathbb{R}_{V_u V_v}(\lambda, \lambda') \,.$$

We now establish that the $\mathtt{Q}(\lambda)$ operator is realised concretely as an integral operator on $L^2(\mathbb{R}^N)$:

$$\big(\mathtt{Q}(\lambda) \cdot f\big)(\boldsymbol{x}) \;=\; \int_{\mathbb{R}^N} \mathcal{Q}_\lambda(\boldsymbol{x}, \boldsymbol{y})f(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \tag{4.1}$$

with $\boldsymbol{x} = (x_1, \ldots, x_N)$ and $\boldsymbol{y} = (y_1, \ldots, y_N)$.

Indeed, by doing a similarity transformation under the trace, one can recast $\mathtt{Q}(\lambda)$ as

$$\mathtt{Q}(\lambda) \;=\; \mathrm{Tr}_{V_u} \big[\mathbb{L}^{(c)}_{V_{x_N}, V_u}(\lambda) \cdots \mathbb{L}^{(c)}_{V_{x_1}, V_u}(\lambda)\big]$$

in which the operator $\mathbb{L}^{(c)}_{V_{x_N}, V_u}(\lambda)$ has been introduced in (3.13). Then, for $f \in L^2(\mathbb{R}^N \times \mathbb{R})$, by using the kernel representation eq.(3.16) one has

$$\Big(\mathbb{L}^{(c)}_{V_{x_N}, V_u}(\lambda) \cdots \mathbb{L}^{(c)}_{V_{x_1}, V_u}(\lambda) \cdot f\Big)(\boldsymbol{x}, u)$$
$$= \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{y} \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{v} \prod_{a=1}^{N} \mathbb{L}_\lambda\big(x_a, v_{a+1}; y_a, v_a\big) \, f(\boldsymbol{y}, v_1) \,. \tag{4.2}$$

Here, we agree upon $v_{N+1} = u$. The latter allows one to identify the integral kernel of the above operator and, upon taking the partial trace and agreeing upon periodic boundary conditions $v_{N+1} \equiv v_1$, one gets that

$$Q_\lambda(\boldsymbol{x}, \boldsymbol{y}) = \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{v} \prod_{a=1}^{N} \mathbb{L}_\lambda\big(x_a, v_{a+1}; y_a, v_a\big) = \prod_{a=1}^{N} \mathcal{L}_\lambda\big(x_a, y_a; y_{a-1}\big) \,. \tag{4.3}$$

The integrals have been done thanks to the delta functions and the kernel $\mathcal{L}_\lambda$ has been introduced in (3.23). Here, we agree upon the periodic boundary conditions in the variables y: $y_0 = y_N$. Note that $\mathcal{L}_\lambda(x_a, y_a; y_{a-1})$ captures all the dependence of the kernel on the variable x_j attached to the jth site. For further convenience, it is useful to introduce the functions w_a given by

$$w_a(\boldsymbol{x}, \boldsymbol{y}) \;=\; \mathcal{L}_\lambda(x_a, y_a; y_{a-1}) \,. \tag{4.4}$$

Here $\boldsymbol{x}$, $\boldsymbol{y}$ are N-dimensional vectors as given below of (4.1).

4.2 The Baxter equation

We now derive the operator form of the Baxter equation:

$$
\mathsf{t}(\lambda)\mathbb{Q}(\lambda) =
$$
$$
\frac{\mathrm{e}^{-\omega_1 \mathsf{P}_{\mathrm{tot}}} \cdot \mathbb{Q}(\lambda - \mathrm{i}\omega_1)}{\left(-q^3 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)^N} + \left(-q\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)^N \left(d_2 + q^{-1} d_1 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)^N \mathbb{Q}(\lambda + \mathrm{i}\omega_1)\,, \tag{4.5}
$$

where we remind that $\mathsf{P}_{\mathrm{tot}} = \sum_{a=1}^{N} \mathsf{X}_a$. The dual transfer matrix solves the dual equation

$$
\widetilde{\mathsf{t}}(\lambda)\mathbb{Q}(\lambda) =
$$
$$
\frac{\mathrm{e}^{-\omega_2 \mathsf{P}_{\mathrm{tot}}} \cdot \mathbb{Q}(\lambda - \mathrm{i}\omega_2)}{\left(-\widetilde{q}^3 \mathrm{e}^{-\frac{2\pi}{\omega_1}\lambda}\right)^N} + \left(-\widetilde{q}\mathrm{e}^{-\frac{2\pi}{\omega_1}\lambda}\right)^N \left(\widetilde{d}_2 + \widetilde{q}^{-1} \widetilde{d}_1 \mathrm{e}^{-\frac{2\pi}{\omega_1}\lambda}\right)^N \mathbb{Q}(\lambda + \mathrm{i}\omega_2)\,, \tag{4.6}
$$

We only discuss the proof of eq. (4.5) as the dual case follows, for instance, upon making the duality transformation on the level of eq. (4.5) and using that $\mathbb{Q}(\lambda)$ is modular invariant.

One has

$$
\Big(\mathsf{t}(\lambda) \cdot Q_\lambda\Big)(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Tr}_0 \left[\Big(\mathsf{L}_{0 V_{x_N}}(\lambda) \cdot w_N\Big) \cdots \Big(\mathsf{L}_{0 V_{x_1}}(\lambda) \cdot w_1\Big)\right](\boldsymbol{x}, \boldsymbol{y})
$$

where $\big(\mathsf{L}_{0j}(\lambda) \cdot w_j\big)(\boldsymbol{x}, \boldsymbol{y})$ is the below matrix function

$$
\big(\mathsf{L}_{0j}(\lambda) \cdot w_j\big)(\boldsymbol{x}, \boldsymbol{y}) =
$$
$$
\begin{pmatrix} (\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} - \mathrm{e}^{-\omega_1 \mathsf{X}_j})\mathcal{L}_\lambda(x_j, y_j; y_{j-1}) & q^2 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}(d_2 + q d_1 \mathrm{e}^{-\omega_1 \mathsf{X}_j})\mathrm{e}^{-\frac{2\pi}{\omega_2}x_j}\mathcal{L}_\lambda(x_j, y_j; y_{j-1}) \\ -q^{-2}\mathrm{e}^{\frac{2\pi}{\omega_2}x_j}\mathcal{L}_\lambda(x_j, y_j; y_{j-1}) & -d_2 \mathcal{L}_\lambda(x_j, y_j; y_{j-1}) \end{pmatrix}.
$$

In the 11 matrix element, we use

$$
\Big(\mathrm{e}^{-\omega_1 \mathsf{X}_j}\mathcal{L}_\lambda\Big)(x_j, y_j; y_{j-1}) = \mathcal{L}_\lambda(x_j + \mathrm{i}\omega_1, y_j; y_{j-1})
$$
$$
= q^{-1}\frac{(q\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} + \mathrm{e}^{-\frac{2\pi}{\omega_2}(y_j - x_j)})(1 + q d_2 \mathrm{e}^{-\frac{2\pi}{\omega_2}(x_j - y_{j-1})})}{1 - d_1 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\mathrm{e}^{-\frac{2\pi}{\omega_2}(x_j - y_{j-1})}}\mathcal{L}_\lambda(x_j, y_j; y_{j-1}) \tag{4.7}
$$

and in the 12 matrix element we use

$$
\mathrm{e}^{-\omega_1 \mathsf{X}_j}\mathrm{e}^{-\frac{2\pi}{\omega_2}x_j}\mathcal{L}_\lambda(x_j, y_j; y_{j-1}) = q^{-2}\mathrm{e}^{-\frac{2\pi}{\omega_2}x_j}\mathcal{L}_\lambda(x_j + \mathrm{i}\omega_1, y_j; y_{j-1})\,.
$$

The q-factors adapt themselves to ensure the triangularisation property which is at the core of the Baxter-Bäcklund approach

$$
\big(\mathsf{L}_{0j}(\lambda) \cdot w_j\big)(\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} 1 & 0 \\ q^{-1}\mathrm{e}^{\frac{2\pi}{\omega_2}y_j} & 1 \end{pmatrix} \begin{pmatrix} A_j & B_j \\ 0 & D_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q^{-1}\mathrm{e}^{\frac{2\pi}{\omega_2}y_{j-1}} & 1 \end{pmatrix}
$$

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where (compare with eq.(A.7))

$$A_j = -q^{-1}\mathrm{e}^{-\frac{2\pi}{\omega_2}(y_j-x_j)}\left(1+qd_2\mathrm{e}^{-\frac{2\pi}{\omega_2}(x_j-y_{j-1})}\right)\mathcal{L}_\lambda(x_j,y_j;y_{j-1})\,,$$

$$D_j = -(d_2+q^{-1}d_1\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda})\mathrm{e}^{-\frac{2\pi}{\omega_2}(x_j-y_j)}\frac{(q\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}+\mathrm{e}^{-\frac{2\pi}{\omega_2}(y_j-x_j)})}{1-d_1\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\mathrm{e}^{-\frac{2\pi}{\omega_2}(x_j-y_{j-1})}}\mathcal{L}_\lambda(x_j,y_j;y_{j-1})\,.$$

Hence,

$$\left(\mathtt{t}(\lambda)\cdot Q_\lambda\right)(\boldsymbol{x},\boldsymbol{y})=\prod_{j=1}^N A_j+\prod_{j=1}^N D_j\,. \tag{4.8}$$

It is easy to see using the explicit formula, eq.(3.23), that

$$A_j = -q^{-3}\mathrm{e}^{\frac{2\pi}{\omega_2}\lambda}\,\mathcal{L}_{\lambda-\mathrm{i}\omega_1}(x_j+\mathrm{i}\omega_1,y_j;y_{j-1})\,,$$

$$D_j = -q\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\left(d_2+q^{-1}d_1\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)\mathcal{L}_{\lambda+\mathrm{i}\omega_1}(x_j,y_j;y_{j-1})\,.$$

Inserting this back into eq.(4.8), upon using $\boldsymbol{e}=(1,\dots,1)\in\mathbb{R}^N$, we obtain that

$$\left(\mathtt{t}(\lambda)\cdot Q_\lambda\right)(\boldsymbol{x},\boldsymbol{y})\;=\;\left(-q^3\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)^{-N}Q_{\lambda-\mathrm{i}\omega_1}(\boldsymbol{x}+\mathrm{i}\omega_1\boldsymbol{e},\boldsymbol{y})$$
$$+\left(-q\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)^N\left(d_2+q^{-1}d_1\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}\right)^N Q_{\lambda+\mathrm{i}\omega_1}(\boldsymbol{x},\boldsymbol{y})\,. \tag{4.9}$$

This yields the operator $\mathtt{t}-\mathtt{Q}$ equation given in (4.5). This form of the equation is somewhat impractial in that it invloves the operator $\mathsf{P}_{\mathrm{tot}}$ which has a purely continuous spectrum. Due to translation invariance, the same holds for $\mathtt{t}(\lambda)$ and $\mathtt{Q}(\lambda)$. In order to factor out the continuous part of the spectrum and work with a spectral problem associated with the pointwise part of the spectrum, one considers the unitary map $\mathcal{F}:L^2(\mathbb{R}^{N-1}\times\mathbb{R})\mapsto L^2(\mathbb{R}^N)$ such that

$$\mathcal{F}[\varphi](\boldsymbol{x})\;=\;\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}px_N}\varphi(x_{1N},\dots,x_{N-1N};p) \tag{4.10}$$

for functions belonging to a suitable dense subset in $L^2(\mathbb{R}^{N-1}\times\mathbb{R})$ and where we agree upon $x_{ab}=x_a-x_b$. Then, on an appropriate dense subspace, one has

$$\mathrm{e}^{-\omega_1\mathsf{P}_{\mathrm{tot}}}\cdot\mathcal{F}[\varphi](\boldsymbol{x})\;=\;\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}px_N}\mathrm{e}^{-\omega_1 p}\,\varphi(x_{1N},\dots,x_{N-1N};p)\,. \tag{4.11}$$

Likewise, it holds

$$\mathtt{Q}(\lambda)\cdot\mathcal{F}[\varphi](\boldsymbol{x})\;=\;\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\int_{\mathbb{R}^N}\mathrm{d}\boldsymbol{y}Q_\lambda(\boldsymbol{x},\boldsymbol{y})\mathrm{e}^{\mathrm{i}py_N}\varphi(y_{1N},\dots,y_{N-1N};p)\,. \tag{4.12}$$

Changing the variables to $\boldsymbol{y}\mapsto\boldsymbol{z}+(\boldsymbol{0},y)$ where $\boldsymbol{z}=(y_{1N},\dots,y_{N-1N},0)$ $y=y_N$, and setting $\boldsymbol{x}=\boldsymbol{u}+x_N\boldsymbol{e}$ where $\boldsymbol{u}=(x_{1N},\dots,x_{N-1N},0)$ and $\boldsymbol{e}=(1,\dots,1)\in\mathbb{R}^N$, one gets

$$\mathtt{Q}(\lambda)\cdot\mathcal{F}[\varphi](\boldsymbol{x})\;=\;\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\int_{\mathbb{R}^{N-1}}\mathrm{d}\boldsymbol{z}\int_{\mathbb{R}}\mathrm{d}y\,Q_\lambda(\boldsymbol{u}+x_N\boldsymbol{e},\boldsymbol{z}+y\boldsymbol{e})\,\mathrm{e}^{\mathrm{i}py}\,\varphi(z_1,\dots,z_{N-1};p)\,. \tag{4.13}$$

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Upon using the translation invariance of $\mathbb{Q}$'s kernel $Q_\lambda(\boldsymbol{x} + r\boldsymbol{e}, \boldsymbol{y} + r\boldsymbol{e}) = Q_\lambda(\boldsymbol{x}, \boldsymbol{y})$ and changing variables, one gets

$$\mathbb{Q}(\lambda) \cdot \mathcal{F}[\varphi](\boldsymbol{x}) \;=\; \int\limits_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px_N} \int\limits_{\mathbb{R}^{N-1}} \mathrm{d}\boldsymbol{z} \, \overline{Q}_\lambda(\boldsymbol{u}, \boldsymbol{z}; p) \, \varphi(z_1, \ldots, z_{N-1}; p) \tag{4.14}$$

where

$$\overline{Q}_\lambda(\boldsymbol{u}, \boldsymbol{z}; p) \;=\; \int\limits_{\mathbb{R}} \mathrm{d}y \, Q_\lambda(\boldsymbol{u}, \boldsymbol{z} + y\boldsymbol{e}) \, \mathrm{e}^{\mathrm{i}py} \; . \tag{4.15}$$

Likewise, due to translation invariance, $\mathtt{t}$ passes through the action of $\mathcal{F}$, namely

$$\mathtt{t}(\lambda) \cdot \mathcal{F}[\varphi](\boldsymbol{x}) \;=\; \int\limits_{\mathbb{R}} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px_N} \Big(\overline{\mathtt{t}}(\lambda; p) \cdot \varphi \Big)(\boldsymbol{u}; p) \; . \tag{4.16}$$

with $\boldsymbol{u}$ defined as above. Above, $\overline{\mathtt{t}}(\lambda; p)$ is an operator on functions depending on the reduced set of variables and is a multiplication operator in p. Thence, by projecting out, one gets a reduced $\mathtt{t} - \mathbb{Q}$ equation on a sector with a fixed continuous eigenvalue p_0 of $\mathsf{P}_{\mathrm{tot}}$:

$$\overline{\mathtt{t}}(\lambda; p_0) \overline{\mathbb{Q}}(\lambda; p_0) = \frac{\mathrm{e}^{-\omega_1 p_0} \cdot \overline{\mathbb{Q}}(\lambda - \mathrm{i}\omega_1; p_0)}{\left(-q^3 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} \right)^N}$$
$$+ \left(-q \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} \right)^N \left(d_2 + q^{-1} d_1 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} \right)^N \overline{\mathbb{Q}}(\lambda + \mathrm{i}\omega_1; p_0) \; . \tag{4.17}$$

Clearly, the dual reduced equation holds as well. It seems already reasonable to assume that the reduced operators $\overline{\mathtt{t}}(\lambda; p_0)$ and $\overline{\mathbb{Q}}(\lambda; p_0)$ have, for fixed λ, a point spectrum. We will make this assumption in the following and leave its proof to some subsequent work.

4.3   Analytic properties of the solution of Baxter equation.

We now argue that for both limiting cases, the q-Toda and the Toda_2 chains, any Eigenvalue $\mathfrak{q}(\lambda)$ of the reduced operator $\overline{\mathbb{Q}}(\lambda; p)$ is an entire function of the spectral parameter. However, for generic values of the coupling constants d_1 and d_2, q is a meromorphic function of the spectral parameter and has N^{th} order poles on the lattice

$$\kappa_2 - \kappa_1 + \mathrm{i}\frac{\Omega}{2} + \mathrm{i}\mathbb{N}\omega_1 + \mathrm{i}\mathbb{N}\omega_2 \; .$$

Thus, the two limiting cases seem to be very special in respect to their analytic structure. The pattern of poles can be argued by following the reasoning developed by Bytsko-Teschner [4]. Let $\boldsymbol{z} \mapsto \Psi_{\mathfrak{q}}(\boldsymbol{z}; p)$ be an Eigenfunction of $\overline{\mathbb{Q}}(\lambda; p)$ associated with the Eigenvalue $\mathfrak{q}(\lambda)$ and let φ be a test function. The pointwisness of the spectrum on the reduced space ensures that $\Psi_{\mathfrak{q}}(\boldsymbol{z}; p)$ decays fast enough at infinity. Although a precise estimate of this decay would demand additional work, for the purpose of the handlings below, we simply assume that it is fast enough for our needs and leave the study of this question for later investigation. Then, it holds,

$$\mathfrak{q}(\lambda) \cdot \big(\varphi, \Psi_{\mathfrak{q}} \big) = \int\limits_{\mathbb{R}^{N-1}} \mathrm{d}\boldsymbol{u} \int\limits_{\mathbb{R}^{N-1}} \mathrm{d}\boldsymbol{z} \int\limits_{\mathbb{R}} \mathrm{d}v \mathrm{e}^{\mathrm{i}pv} Q_\lambda(\boldsymbol{u}_N, \boldsymbol{z}_N + v\boldsymbol{e}) \varphi^*(\boldsymbol{u}) \Psi_{\mathfrak{q}}(\boldsymbol{z}; p) \; . \tag{4.18}$$

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Here $(\cdot, \cdot)$ stands for the canonical $L^2(\mathbb{R}^{N-1})$ scalar product, $\boldsymbol{z}_N$ and $\boldsymbol{u}_N$ stand for the canonical embeddings of of $\boldsymbol{z}, \boldsymbol{u} \in \mathbb{R}^{N-1}$ into $\mathbb{R}^N$: $\boldsymbol{z}_N = (\boldsymbol{z}, 0)$, $\boldsymbol{u}_N = (\boldsymbol{u}, 0)$. Also, we remind that $\boldsymbol{e} = (1, \ldots, 1) \in \mathbb{R}^N$. Upon a change of variables, we get

$$\mathfrak{q}(\lambda) \cdot \big(\varphi, \Psi_\mathfrak{q}\big) = \int\limits_{\mathbb{R}^{N-1}} \mathrm{d}\boldsymbol{u} \int\limits_{\mathbb{R}^N} \mathrm{d}\boldsymbol{w} \mathrm{e}^{\mathrm{i} p w_N} Q_\lambda(\boldsymbol{u}_N, \boldsymbol{w}) \varphi^*(\boldsymbol{u}_{w_N}) \Psi_\mathfrak{q}(\boldsymbol{z}; p) \tag{4.19}$$

where $\boldsymbol{u}_{w_N} = \big(u_1 + w_N, \ldots, u_{N-1} + w_N\big)$.

Since $\Psi_\mathfrak{q}(\boldsymbol{z}; p)$ is independent of λ and owing to the good decay properties at ∞ of the test function φ and the Eigenfunction $\Psi_\mathfrak{q}$, the sole mechanism that can give rise to a pole of $\mathfrak{q}(\lambda)$ is when the integration contour gets pinched between two poles of the integral kernel of the $\mathtt{Q}$-operator, one coming from the upper and the other from the lower half plane. Agreeing below on the convention $u_N = 0$, the integral kernel $Q_\lambda(\boldsymbol{u}_N, \boldsymbol{w})$, as follows from inspection of eqns. (4.3), (3.23), has poles at

$$u_a - w_{a-1} + \kappa_2 - \frac{3\mathrm{i}\Omega}{2} = -\mathrm{i} m \omega_1 - \mathrm{i} n \omega_2, \quad m, n \geq 1 \,,$$

$$w_a - w_{a-1} + \kappa_1 - \frac{\mathrm{i}\Omega}{2} = -\mathrm{i} m \omega_1 - \mathrm{i} n \omega_2, \quad m, n \geq 1 \,,$$

$$u_a - w_{a-1} + \kappa_1 + \lambda - \mathrm{i}\Omega = \mathrm{i} m' \omega_1 + \mathrm{i} n' \omega_2, \quad m', n' \geq 0 \,,$$

$$w_a - u_a - \lambda + \frac{\mathrm{i}\Omega}{2} = \mathrm{i} m' \omega_1 + \mathrm{i} n' \omega_2, \quad m', n' \geq 0 \,.$$

The first two sets of poles are located solely in the upper half–plane, while the third one is only located in the lower one, just as most of the poles belonging to the last set. As can be inferred from doing contour deformation or through computing explicitly the boundary values, the pinching in λ which generates effective poles will only occur at those values of λ which are independent of $\boldsymbol{u}, \boldsymbol{w}$. Thus, this can happen only when the closest to the real axis pole of the first set pinches with some pole present in the third set set of pole, namely when

$$\lambda = \kappa_2 - \kappa_1 + \frac{\mathrm{i}}{2}\Omega + \mathrm{i} m'' \omega_1 + \mathrm{i} n'' \omega_2, \quad m'', n'' \geq 0 \,.$$

Since there are N factors in the kernel giving rise to such poles, all–in–all, this gives rise to an Nth order pole.

Note that, in the q–Toda or Toda$_2$ cases one of the two building blocks of the kernel which are both responsible for the generation of these poles is absent (either $\mathcal{S}(x_j - y_{j-1} + \kappa_2 - \frac{3\mathrm{i}}{2}\Omega)$ or $\mathcal{S}(x_j - y_{j-1} + \kappa_1 + \lambda - \mathrm{i}\Omega)$) so that no pinching can arise. This ensures that $\mathfrak{q}(\lambda)$ is an *entire* function of λ in these two cases.

4.4 Quantisation of the spectrum by means of a scalar $\mathtt{t} - \mathtt{Q}$ equation

The hypothesis of a pointwise spectrum for the reduced operators $\overline{\mathtt{t}}(\lambda; p_0)$ and $\overline{\mathtt{Q}}(\lambda; p_0)$ and their commutativity allows one to project the reduced $\mathtt{t} - \mathtt{Q}$ equation (4.17) onto a given joint Eigenvector of $\overline{\mathtt{t}}(\lambda; p_0)$ and $\overline{\mathtt{Q}}(\lambda; p_0)$ associated with the respective eigenvalues $t(\lambda)$ and $\mathfrak{q}(\lambda)$. Here, for the sake of compactness of notations, we drop the p_0 dependence of these eigenvalues since p_0 appears explicitly in the equation. This scalar $t - q$ equation reads:

$$\begin{aligned} t(\lambda)\mathfrak{q}(\lambda) =&(-q^3)^{-N} \mathrm{e}^{\frac{2\pi}{\omega_2} N \lambda} \mathrm{e}^{-\omega_1 p_0} \mathfrak{q}(\lambda - \mathrm{i}\omega_1) \\ &+ (-q)^N \mathrm{e}^{-\frac{2\pi}{\omega_2} N \lambda} (d_2 + q^{-1} d_1 \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda})^N \mathfrak{q}(\lambda + \mathrm{i}\omega_1) \,. \end{aligned} \tag{4.20}$$

Note that the *same* $\mathfrak{q}$ satisfies as well the dual $t-q$ equation. We now specialise this equation to the q–Toda and Toda$_2$ cases. In both cases, we recast the equation in a canonical form what will allow us to simplify the analysis to come. Finally, note that the dual results holds upon making a modular transformation.

- The q–Toda case $(d_2 = 0)$

  The Baxter equation then takes the form

  $$t(\lambda)\mathfrak{q}(\lambda) = (-1)^N q^{-3N} e^{\frac{2\pi}{\omega_2}N\lambda} e^{-\omega_1 p_0} \mathfrak{q}(\lambda - i\omega_1) + (-1)^N d_1^N e^{-\frac{4\pi}{\omega_2}N\lambda} \mathfrak{q}(\lambda + i\omega_1) .$$

  Recall that in the case of interest, $t(\lambda)$ is of the form

  $$t(\lambda) = \left[e^{-\frac{2\pi}{\omega_2}N\lambda} + \cdots + (-1)^N e^{-\omega_1 p_0} \right] .$$

  Upon multiplying by $e^{\frac{\omega_1}{2}p_0} e^{\frac{N\pi}{\omega_2}\lambda}$, one gets a more convenient factorisation

  $$e^{\frac{\omega_1}{2}p_0} e^{\frac{N\pi}{\omega_2}\lambda} t(\lambda) = \left[e^{-\frac{N\pi}{\omega_2}\lambda} e^{\frac{\omega_1}{2}p_0} + \cdots + (-1)^N e^{\frac{N\pi}{\omega_2}\lambda} e^{-\frac{\omega_1}{2}p_0} \right]$$

  $$= (-1)^N \prod_{k=1}^{N} \left\{ 2\sinh\frac{\pi}{\omega_2}(\lambda - \tau_k) \right\}$$

  where the τ_k are subjected to the constraint

  $$\prod_{k=1}^{N} e^{-\frac{\pi}{\omega_2}\tau_k} = e^{-\frac{1}{2}\omega_1 p_0} , \tag{4.21}$$

  as can be inferred from (2.7). This handling transforms the Baxter equation into

  $$\prod_{k=1}^{N} \left\{ 2\sinh\frac{\pi}{\omega_2}(\lambda - \tau_k) \right\} \cdot \mathfrak{q}(\lambda) = (q^3)^{-N} e^{\frac{3\pi}{\omega_2}N\lambda} e^{-\frac{\omega_1}{2}p_0} \mathfrak{q}(\lambda - i\omega_1)$$

  $$+ d_1^N e^{-\frac{3\pi}{\omega_2}N\lambda} e^{\frac{\omega_1}{2}p_0} \mathfrak{q}(\lambda + i\omega_1) . \tag{4.22}$$

  In order to put this equation in a canonical form, it is convenient to make a change of unknown function

  $$\mathfrak{q}(\lambda) = e^{-\frac{3i\pi N}{2\omega_1 \omega_2}\lambda^2 + \left(-\frac{3\pi N}{2\omega_1 \omega_2}\Omega + \frac{i}{2}p_0 - \frac{i\pi N}{\omega_1 \omega_2}\kappa_1 \right)\lambda} q(\lambda) .$$

  This recasts the Baxter equation eq.(4.22) as

  $$\prod_{k=1}^{N} \left\{ 2\sinh\frac{\pi}{\omega_2}(\lambda - \tau_k) \right\} q(\lambda) =$$

  $$e^{-\frac{\pi\kappa_1}{\omega_2}N} \left((-i)^N q(\lambda - i\omega_1) + i^N q(\lambda + i\omega_1) \right) . \tag{4.23}$$

- The Toda$_2$ case $(d_1 = 0)$

  The scalar Baxter equation takes the form

  $$t(\lambda)\mathfrak{q}(\lambda) =$$

  $$(-q^3)^{-N} e^{\frac{2N\pi}{\omega_2}\lambda} e^{-\omega_1 p_0} \mathfrak{q}(\lambda - i\omega_1) + (-q)^N e^{-\frac{2N\pi}{\omega_2}\lambda} d_2^N \mathfrak{q}(\lambda + i\omega_1) . \tag{4.24}$$

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Now $t(\lambda)$ is of the form

$$t(\lambda) = \left[e^{-\frac{2\pi N}{\omega_2}\lambda} + \cdots + (-1)^N (e^{-\omega_1 p_0} + d_2^N) \right] = \prod_{k=1}^{N} \left\{ e^{-\frac{2\pi}{\omega_2}\lambda} - e^{-\frac{2\pi}{\omega_2}\tau_k} \right\} ,$$

and the roots τ_k are now subjected to the constraint

$$\prod_{k=1}^{N} e^{-\frac{2\pi}{\omega_2}\tau_k} = e^{-\omega_1 p_0} + d_2^N . \tag{4.25}$$

as can be inferred from (2.7). It is again convenient to make a change of unknown function

$$\mathfrak{q}(\lambda) = e^{-\frac{i\pi N}{\omega_1 \omega_2}\lambda^2 + \left(-\frac{2\pi N}{\omega_1 \omega_2}\Omega - \frac{i\pi N}{\omega_1 \omega_2}\kappa_2 \right)\lambda} \, q(\lambda) . \tag{4.26}$$

This puts the Baxter equation eq.(4.24) in the form

$$\prod_{k=1}^{N} \left\{ e^{-\frac{2\pi}{\omega_2}\lambda} - e^{-\frac{2\pi}{\omega_2}\tau_k} \right\} q(\lambda) =$$

$$(-1)^N e^{-\frac{\pi}{\omega_2}N\kappa_2} e^{-\frac{\omega_1}{2}p_0} \left(e^{-\omega_1 p_0} q(\lambda - i\omega_1) + q(\lambda + i\omega_1) \right) . \tag{4.27}$$

The Baxter equations for both models can thus be put in the canonical form

$$t_{\boldsymbol{\tau}}(\lambda) q(\lambda) = g^{N\omega_1} \left(\sigma \, \varkappa^{\omega_1} q(\lambda - i\omega_1) + \sigma^{-1} q(\lambda + i\omega_1) \right) , \tag{4.28}$$

where

- q-Toda

$$t_{\boldsymbol{\tau}}(\lambda) = \prod_{k=1}^{N} \left\{ 2 \sinh \frac{\pi}{\omega_2}(\lambda - \tau_k) \right\}, \quad \sigma = (-i)^N, \quad g = e^{-\frac{\pi\kappa_1}{\omega_1 \omega_2}}, \quad \varkappa = 1 ; \tag{4.29}$$

- Toda$_2$

$$t_{\boldsymbol{\tau}}(\lambda) = \prod_{k=1}^{N} \left\{ e^{-\frac{2\pi}{\omega_2}\lambda} - e^{-\frac{2\pi}{\omega_2}\tau_k} \right\}, \quad \sigma = (-1)^N, \quad g = e^{-\frac{\pi\kappa_2}{\omega_1 \omega_2}}, \quad \varkappa = e^{-p_0} . \tag{4.30}$$

The main difference between the q-Toda and Toda$_2$ chains is that, in the former model, $\varkappa$ depends explicitly on the zero mode p_0 and the transfer matrix eigenvalue polynomial only grows in one direction $\Re(\lambda/\omega_2) \to -\infty$. For further applications, it will be important to study the regularity properties in p_0 of the solution q to the $t-q$ equations governing the spectrum of the Toda$_2$ chain. We leave this to a subsequent publication.

5 Conclusion

In this paper we have constructed the Baxter operator $\mathbb{Q}(\lambda)$ for the q-Toda and Toda$_2$ chains, two q-deformations of the Toda chain. We used, as a starting point, the relation, in the classical limit, between

Baxter operator and Bäcklund transformations. However, in the quantum case, we used Faddeev's modular invariance as a guideline to properly define the fully quantum operator $\mathtt{Q}(\lambda)$. We then derived Baxter $t-q$ equation and showed, using our operator $\mathtt{Q}(\lambda)$, that its solutions should be requested to be *entire* functions of λ. This last property paves the way to the quantisation conditions yielding the spectrum of the q-Toda and Toda$_2$ chains. This matter will be investigated in a forthcoming publication [1].

Acknowledgment

A   Bäcklund transformation

By taking the $q \to 1$ limit of eq.(2.1), one gets the classical Lax matrix

$$L(\lambda \mid \hat{x}_n, \hat{X}_n) \;=\; \begin{pmatrix} \lambda - \hat{X}_n & \lambda[d_2 + d_1\hat{X}_n]\hat{x}_n \\ -\hat{x}_n^{-1} & -d_2 \end{pmatrix} \tag{A.1}$$

which is expressed in terms of exponents

$$\hat{X}_n = \mathrm{e}^{X_n}, \quad \hat{x}_n = \mathrm{e}^{x_n}$$

of Darboux canonical coordinates $\{x_n, X_m\} = \delta_{nm}$, in which $\{*, *\}$ is a Poisson bracket. Note that the notation is consistent since all the dependence on n of the classical Lax matrix is contained in the Darboux coordinates x_n, X_n or, rather, its exponentiated counterparts $\hat{x}_n, \hat{X}_n$. To construct Baxter's $\mathtt{Q}$ operator, we will use its relation to Bäcklund transformations. The main observation of [13] is that Bäcklund transformations are related to the triangulation of the matrix $L(\lambda \mid \hat{x}_n, \hat{X}_n)$ by a gauge transformation, just as Baxter constructed his $\mathtt{Q}$ operator.

Bäcklund transformations are canonical transformations $(x_n, X_n) \to (y_n, Y_n)$, $n = 1, \ldots, N$, that preserve the form of the Hamiltonians. This last property is achieved if the transformation acts on the Lax matrix by a gauge transformation *i.e.* there exist matrices $M(\lambda; t \mid \hat{u}_n, \hat{U}_n)$, depending on the dynamical variables $\hat{u}_n, \hat{U}_n$ and on a parameter t, such that

$$L(\lambda \mid \hat{x}_n, \hat{X}_n)M(\lambda; t \mid \hat{u}_n, \hat{U}_n) = M(\lambda; t \mid \hat{u}_{n+1}, \hat{U}_{n+1})L(\lambda \mid \hat{y}_n, \hat{Y}_n) \tag{A.2}$$

A nice way to derive the Bäcklund transformation and the matrix $M(\lambda; t \mid \hat{u}_n, \hat{U}_n)$ was devised by Kuznetsov and Sklyanin [23]. It is explained below, but let us first state the result in the form

$$\hat{X}_n = \frac{(1 + d_1\hat{x}_{n+1}\hat{x}_n^{-1})}{(1 + d_1\hat{x}_n\hat{x}_{n-1}^{-1})} \frac{(\hat{t} + \hat{x}_n\hat{y}_n^{-1})(1 + d_2\hat{y}_{n+1}\hat{x}_n^{-1})}{(1 - td_1\hat{y}_{n+1}\hat{x}_n^{-1})}, \tag{A.3}$$

$$\hat{Y}_n = \frac{(t + \hat{x}_n\hat{y}_n^{-1})(1 + d_2\hat{y}_n\hat{x}_{n-1}^{-1})}{(1 - td_1\hat{y}_n\hat{x}_{n-1}^{-1})} \;. \tag{A.4}$$

Here $\hat{x}_n = \mathrm{e}^{x_n}$, $\hat{X}_n = \mathrm{e}^{X_n}$, *etc.*, and t is the parameter of the Bäcklund tranformation. This is a slight generalisation of the results of [3] or a limit of the results of [23].

The matrix $M(\lambda; t \mid \hat{u}_n, \hat{U}_n)$ reads

$$M(\lambda; t \mid \hat{u}_n, \hat{U}_n) = \begin{pmatrix} \lambda - t\hat{U}_n & -\lambda(1 - \hat{U}_n)\hat{u}_n \\ -\hat{u}_n^{-1} & 1 \end{pmatrix} , \tag{A.5}$$

where the dynamical variables are expressed as

$$\hat{u}_{n+1} = \hat{x}_n, \qquad \hat{U}_n = \frac{(1 + d_1 \hat{x}_n \hat{x}_{n-1}^{-1})(1 + d_2\ \hat{y}_n \hat{x}_{n-1}^{-1})}{1 - d_1 t\ \hat{y}_n \hat{x}_{n-1}^{-1}} .$$

One can check by direct calculation that indeed equation (A.2) holds.

The relation of (A.2) with the triangulation of the Lax matrix is as follows. Since

$$\det\left[M(\lambda; t \mid \hat{u}_n, \hat{U}_n) \right] = (\lambda - t)\hat{U}_n,$$

the matrix $M(t; t \mid \hat{u}_n, \hat{U}_n)$ is of rank one. Then, the kernel is

$$M(t; t \mid \hat{u}_n, \hat{U}_n) \begin{pmatrix} 1 \\ \hat{x}_{n-1}^{-1} \end{pmatrix} = 0 .$$

Since the kernel is one dimensional, eq.(A.2) implies that

$$L(t \mid \hat{y}_n, \hat{Y}_n) \begin{pmatrix} 1 \\ \hat{x}_{n-1}^{-1} \end{pmatrix} \propto \begin{pmatrix} 1 \\ \hat{x}_n^{-1} \end{pmatrix} .$$

As a consequence, we have the triangulation property by a gauge transformation

$$\begin{pmatrix} 1 & 0 \\ -\hat{x}_n^{-1} & 1 \end{pmatrix} L_n(t \mid \hat{y}_n, \hat{Y}_n) \begin{pmatrix} 1 & 0 \\ \hat{x}_{n-1}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ 0 & D_n \end{pmatrix} .$$

Straightforward algebra then yields

$$A_n = -\hat{x}_n \hat{y}_n^{-1}(1 + d_2 \hat{y}_n \hat{x}_{n-1}^{-1}) , \tag{A.6}$$

$$D_n = -(d_2 + d_1 t)\hat{y}_n \hat{x}_n^{-1} \frac{t + \hat{x}_n \hat{y}_n^{-1}}{1 - d_1 t \hat{y}_n \hat{x}_{n-1}^{-1}} . \tag{A.7}$$

We now recall the Kuztnetsov-Sklyanin construction leading to the non-intuitive formulae eqs.(A.19, A.20) for the Bäcklund transformation of our model.

One starts with the matrices $L(\lambda \mid \hat{x}_n, \hat{X}_n)$, $M(\lambda, t \mid \hat{u}_n, \hat{U}_n)$, $L(\lambda \mid \hat{y}_n, \hat{Y}_n)$ and $M(\lambda, t \mid \hat{v}_n, \hat{V}_n)$ all satisfying the Sklyanin bracket (symplectic orbits of dimension 2):

$$\{N_1(\lambda_1), N_2(\lambda_2)\} = [r_{12}(\lambda_1, \lambda_2), N_1(\lambda_1)N_2(\lambda_2)] . \tag{A.8}$$

$r_{12}(\lambda_1, \lambda_2)$ appearing above is the classical limit of the quantum R-matrix eq.(2.6):

$$r_{12}(\lambda_1, \lambda_2) = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{A.9}$$

Then the matrices $L(\lambda \mid \hat{x}_n, \hat{X}_n)M(\lambda, t \mid \hat{u}_n, \hat{U}_n)$ and $M(\lambda, t \mid \hat{v}_n, \hat{V}_n)L(\lambda \mid \hat{y}_n, \hat{Y}_n)$ also satisfy the bracket eq.(A.8). The equation

$$L(\lambda \mid \hat{x}_n, \hat{X}_n)M(\lambda; \hat{u}_n, \hat{U}_n) = M(\lambda; \hat{v}_n, \hat{V}_n)L(\lambda \mid \hat{y}_n, \hat{Y}_n) \tag{A.10}$$

defines a symplectic transformation $(\hat{x}_n, \hat{X}_n, \hat{u}_n, \hat{U}_n) \to (\hat{y}_n, \hat{Y}_n, \hat{v}_n, \hat{V}_n)$ on a symplectic leaf of Sklyanin bracket. Imposing the constraints

$$\hat{v}_n = \hat{u}_{n+1}, \quad \hat{V}_n = \hat{U}_{n+1}$$

yields a solution of eq.(A.2). The success of this approach relies on the proper choice of the elementary solutions of eq.(A.8) which we now describe.

The most elementary L-matrices solutions of eq.(A.8) depending on a single Weyl pair $(viz.\{\hat{x}, \hat{y}\} = \hat{y}\hat{x})$ are

$$L(\lambda, P) = \begin{pmatrix} \alpha & \beta\hat{x} \\ \gamma\lambda^{-1}\hat{x}^{-1}\hat{y} & \delta\hat{y} \end{pmatrix}, \quad P = \begin{pmatrix} \alpha & \beta \\ \gamma\lambda^{-1} & \delta \end{pmatrix} . \tag{A.11}$$

Here λ is the spectral parameter and the coefficients of P are otherwise arbitrary constants. From this elementary L-matrix, we can construct a more general family of L-matrices still depending on a single Weyl pair. Start with two independent Weyl pairs

$$\{\hat{x}_i, \hat{y}_j\} = \hat{y}_j\hat{x}_i\delta_{ij}, \quad i = 1, 2 \tag{A.12}$$

one has that the product of two elementary L-matrices multiplied by λ

$$L(\lambda, P_1, P_2) = \lambda \begin{pmatrix} \alpha_1 & \beta_1\hat{x}_1 \\ \gamma_1\lambda^{-1}\hat{x}_1^{-1}\hat{y}_1 & \delta_1\hat{y}_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \gamma_2\hat{x}_2 \\ \beta_2\lambda^{-1}\hat{x}_2^{-1}\hat{y}_2 & \delta_2\hat{y}_2 \end{pmatrix} \tag{A.13}$$

also satisfies the bracket eq.(A.8). Now, this L-matrix is invariant under the group action

$$\hat{x}_1 \to s\hat{x}_1, \quad \hat{y}_1 \to s\hat{y}_1, \quad \hat{y}_2 \to s^{-1}\hat{y}_2, \quad \hat{x}_2 \to \hat{x}_2$$

which preserves the quadratic bracket eq.(A.12). The generator of the group action is

$$h = \log \hat{h} \quad \text{with} \quad \hat{h} = \hat{x}_1\hat{y}_1^{-1}\hat{x}_2^{-1} .$$

Invariant functions are generated by $\hat{h}$ itself (which is set to 1), $\hat{x} = \hat{x}_2$ and $\hat{X} = \hat{y}_1\hat{y}_2$. The L-matrix being invariant, it can be expressed in terms of the three invariant functions $\hat{h}, \hat{x}, \hat{X}$ and hence lives on the reduced phase space where it still satisfies eq.(A.8). Hence the Poisson bracket of the L-matrix depends only on the Poisson brackets of these three quantities. $\hat{h}$ being the generator of the group, it Poisson commutes with the invariant functions $\hat{x}$ and $\hat{X}$, and of course with itself. So it plays no role and can be set to a constant. Here, we choose $\hat{h} = 1$, meaning that $\hat{y}_1\hat{x}_2 = \hat{x}_1$.

Only the Poisson bracket of $\hat{x}$ and $\hat{X}$ matters and turns out to be

$$\{\hat{x}, \hat{X}\} = \hat{x}\hat{X} .$$

After the reduction, we can finally write

$$L(\lambda, P_1, P_2) \;=\; \begin{pmatrix} 1 & \\ & \hat{x}^{-1} \end{pmatrix} P_1 \begin{pmatrix} 1 & \\ & \hat{X} \end{pmatrix} P_2^t \begin{pmatrix} 1 & \\ & \hat{x} \end{pmatrix}, \quad P_1 = \begin{pmatrix} \alpha_1\lambda & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2\lambda \end{pmatrix}$$

The transposition on P_2 was introduced for later convenience. The Lax matrix eq.(A.1) can indeed be written in this factorised form:

$$L(\lambda \mid \hat{x}, \hat{X}) \;=\; \begin{pmatrix} 1 & \\ & \hat{x}^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & \hat{X} \end{pmatrix} \begin{pmatrix} 1 & d_2 \\ -1 & d_1\lambda \end{pmatrix} \begin{pmatrix} 1 & \\ & \hat{x} \end{pmatrix},$$

We look for $M(\lambda, t \mid \hat{u}, \hat{U})$ in this general set of factorised matrices

$$M(\lambda, t \mid \hat{u}, \hat{U}) = \begin{pmatrix} 1 & 0 \\ 0 & \hat{u}^{-1} \end{pmatrix} \begin{pmatrix} \alpha_1\lambda & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{U} \end{pmatrix} \begin{pmatrix} \alpha_2 & \gamma_2 \\ \beta_2 & \delta_2\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{u} \end{pmatrix}$$

and just replacing $(\hat{u}, \hat{U}) \rightarrow (\hat{v}, \hat{V})$ for $M(\lambda, t \mid \hat{v}, \hat{V})$. Then, eq.(A.10) can be solved for $(\hat{y}_n, \hat{Y}_n, \hat{v}_n, \hat{V}_n)$ as functions of $(\hat{x}_n, \hat{X}_n, \hat{u}_n, \hat{U}_n)$. These provide $4N$ equations relating the $8N$ variables

$$(\hat{x}_n, \hat{X}_n, \hat{u}_n, \hat{U}_n, \hat{y}_n, \hat{Y}_n, \hat{v}_n, \hat{V}_n) \,. \tag{A.14}$$

In particular, we find

$$\hat{v}_n = -\hat{x}_n \frac{\beta_1\gamma_1\hat{u}_n + \gamma_1 d_2\delta_1\hat{x}_n - \alpha_1\delta_1\hat{u}_n\hat{X}_n + \gamma_1 d_1\delta_1\hat{x}_n\hat{X}_n}{\alpha_1(d_2\delta_1\hat{x}_n + \beta_1\hat{u}_n)} \,. \tag{A.15}$$

Next, we have to impose the constraints

$$\hat{v}_n = \hat{u}_{n+1}, \quad \hat{V}_n = \hat{U}_{n+1} \,. \tag{A.16}$$

This adds $2N$ equations to our $4N$ equations. Hence we can solve everything in terms of the $2N$ variables, say $(\hat{x}_n, \hat{y}_n)$. The first condition in eq.(A.16) combined with eq.(A.15) in general leads to highly non local and untractable formulae. However, we remark that, if

$$\delta_1 = 0 \,, \tag{A.17}$$

then, the equation simplifies drastically

$$\hat{u}_{n+1} = -\frac{\gamma_1}{\alpha_1}\hat{x}_n \,.$$

The latter entails that the rest of the equations also simplifies and we get

$$\hat{U}_n = -\frac{\alpha_1\gamma_2}{\beta_1\delta_2} \frac{(1 + d_1\hat{x}_n\hat{x}_{n-1}^{-1})(1 + d_2 s\, \hat{y}_n\hat{x}_{n-1}^{-1})}{(1 - d_1 t s\, \hat{y}_n\hat{x}_{n-1}^{-1})} \,, \tag{A.18}$$

$$\hat{X}_n = \frac{(1 + d_1\hat{x}_{n+1}\hat{x}_n^{-1})}{(1 + d_1\hat{x}_n\hat{x}_{n-1}^{-1})} \frac{(t + s^{-1}\hat{x}_n\hat{y}_n^{-1})(1 + d_2 s\, \hat{y}_{n+1}\hat{x}_n^{-1})}{(1 - t d_1 s\, \hat{y}_{n+1}\hat{x}_n^{-1})} \,, \tag{A.19}$$

$$\hat{Y}_n = \frac{(t + s^{-1}\hat{x}_n\hat{y}_n^{-1})(1 + d_2 s\, \hat{y}_n\hat{x}_{n-1}^{-1})}{(1 - t d_1 s\, \hat{y}_n\hat{x}_{n-1}^{-1})} \,, \tag{A.20}$$

where

$$t = \frac{\gamma_2\beta_2}{\alpha_2\delta_2}, \quad s = \frac{\alpha_1\alpha_2}{\gamma_1\gamma_2}$$

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The parameter s can be eliminated by a rescaling of the variables $\hat{y}_n$ which is indeed a symmetry of the theory, so we can set it equal to one. The Bäcklund transformation obtained in this way is derived from the generating function

$$\log \hat{X}_n = \hat{x}_n \frac{\partial}{\partial \hat{x}_n} F, \quad \log \hat{Y}_n = -\hat{y}_n \frac{\partial}{\partial \hat{y}_n} F$$

where F reads

$$F = \sum_n \int^{\frac{\hat{x}_n}{\hat{y}_n}} \frac{dx}{x} \log(t+x) - \int^{\frac{\hat{y}_{n+1}}{\hat{x}_n}} dx \log(1+d_2 x)$$

$$+ \int^{\frac{\hat{y}_{n+1}}{\hat{x}_n}} \frac{dx}{x} \log(1 - d_1 t x) - \int^{\frac{\hat{x}_{n+1}}{\hat{x}_n}} \frac{dx}{x} \log(1 + d_1 x) .$$

The matrices M in (A.2) can be taken as

$$M(\lambda, t \mid \hat{u}_n, \hat{U}_n) = \begin{pmatrix} 1 & 0 \\ 0 & \hat{u}_n^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{U}_n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -t & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{u}_n \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - t\hat{U}_n & -\lambda(1 - \hat{U}_n)\hat{u}_n \\ -\hat{u}_n^{-1} & 1 \end{pmatrix} . \tag{A.21}$$

At the quantum level, the Bäcklund canonical transformation is replaced by a similarity transformation

$$(y, Y, v, V) = \mathbb{L}_{V_x V_u}^{-1}(t)(x, X, u, U)\mathbb{L}_{V_x V_u}(t) ,$$

where now $(\mathrm{e}^{-\frac{2\pi}{\omega_2} x}, \mathrm{e}^{-\omega_1 X})$ and $(\mathrm{e}^{-\frac{2\pi}{\omega_2} u}, \mathrm{e}^{-\omega_1 U})$ are Weyl pairs:

$$\mathrm{e}^{-\frac{2\pi}{\omega_2} x} \mathrm{e}^{-\omega_1 X} = q^2 \mathrm{e}^{-\omega_1 X} \mathrm{e}^{-\frac{2\pi}{\omega_2} x}, \quad \mathrm{e}^{-\frac{2\pi}{\omega_2} u} \mathrm{e}^{-\omega_1 U} = q^2 \mathrm{e}^{-\omega_1 U} \mathrm{e}^{-\frac{2\pi}{\omega_2} u} .$$

Hence eq.(A.2) becomes

$$\mathbb{L}_{V_x V_u}(t) \, \mathsf{L}_{0V_u}(\lambda) \, \mathsf{M}_{0V_u}(\lambda; t) \; = \; \mathsf{M}_{0V_u}(\lambda; t) \, \mathsf{L}_{0V_x}(\lambda) \, \mathbb{L}_{V_x V_u}(t) \tag{A.22}$$

where the operator $\mathbb{L}_{V_x V_u}(t)$ is independent of λ but depends on the Bäcklund parameter t, and $\mathsf{L}_{0V_x}(\lambda)$, resp. $\mathsf{M}_{0V_u}(\lambda; t)$, is the quantum deformation of the classical object $L(\lambda \mid \hat{x}, \hat{X})$, resp. $M(\lambda, t \mid \hat{u}, \hat{U})$.

B   Quantum intertwiners

The purpose of this appendix is to construct the intertwiner $\mathbb{L}_{V_x V_u}(t)$ arising in (3.3). Starting from the factorisation insight issuing from the implementation of the Bäcklund transformations, we construct $\mathbb{L}_{V_x V_u}(t)$ by means of the representation theory of the symmetric group $\mathfrak{S}_4$ following the strategy devised by Derkachov in [5] where he considered the cases of the $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$ quantum XXX magnet.

B.1   The double sine function representation

Consider an auxiliary Lax matrix depending on two matrices

$$P_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

and given by

$$\mathsf{L}(P_1, P_2) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\frac{2\pi}{\omega_2}\mathtt{x}} \end{pmatrix} P_1 \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\omega_1\mathtt{X}} \end{pmatrix} P_2^t \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}} \end{pmatrix} .$$

Here $(\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}}, \mathrm{e}^{-\omega_1\mathtt{X}})$ is a Weyl pair

$$\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}}\mathrm{e}^{-\omega_1\mathtt{X}} = q^2\mathrm{e}^{-\omega_1\mathtt{X}}\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}} .$$

The matrices $\mathsf{L}_{0,V_x}(\lambda)$ and $\mathsf{M}_{0,V_u}(\lambda; t)$ appearing in Sub-section 3.1 are of this form :

$$\mathsf{L}_{0,V_x}(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\frac{2\pi}{\omega_2}\mathtt{x}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} & 1 \\ -q^{-2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\omega_1\mathtt{X}} \end{pmatrix} \begin{pmatrix} 1 & q^2 d_2 \\ -1 & q^3\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}} \end{pmatrix}$$

$$\mathsf{M}_{0,V_u}(\lambda; t) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\frac{2\pi}{\omega_2}\mathtt{u}} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} & 1 \\ -q^{-2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\omega_1\mathtt{U}} \end{pmatrix} \begin{pmatrix} 1 & -q \\ -\mathrm{e}^{-\frac{2\pi}{\omega_2}t} & q\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{u}} \end{pmatrix}$$

In order to build $\mathbb{L}_{V_xV_u}(t)$, it is thus enough to look for an intertwiner $\mathbb{L}(P_1, P_2, P_3, P_4)$ such that

$$\mathbb{L}(P_1, P_2, P_3, P_4)\mathsf{L}(P_1, P_2)\mathsf{L}'(P_3, P_4) = \mathsf{L}'(P_3, P_4)\mathsf{L}(P_1, P_2)\mathbb{L}(P_1, P_2, P_3, P_4) . \tag{B.1}$$

Above, the $'$ indicates that the corresponding Lax matrix is defined in terms of a Weyl pair built up from independent operators $\mathtt{x}'$ and $\mathtt{X}'$:

$$(\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}'}, \mathrm{e}^{-\omega_1\mathtt{X}'}) \qquad \text{so that} \qquad \mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}'}\mathrm{e}^{-\omega_1\mathtt{X}'} = q^2\mathrm{e}^{-\omega_1\mathtt{X}'}\mathrm{e}^{-\frac{2\pi}{\omega_2}\mathtt{x}'} .$$

It is convenient to look for a solution to (B.1) in the form

$$\mathbb{L}(P_1, P_2, P_3, P_4) = \mathsf{P}_{xx'} \cdot \check{\mathbb{L}}(P_1, P_2, P_3, P_4) \tag{B.2}$$

where $\mathsf{P}_{xx'}$ is the permutation operator

$$\mathsf{P}_{xx'} \mathtt{x} = \mathtt{x}'\mathsf{P}_{xx'}, \quad \mathsf{P}_{xx'} \mathtt{X} = \mathtt{X}'\mathsf{P}_{xx'}, \quad \mathsf{P}_{xx'}^2 = \mathrm{id} .$$

This operator intertwines between the two irreducible representations of the Weyl algebra and its dual.

Then, $\check{\mathbb{L}}(P_1, P_2, P_3, P_4)$ solves the equation

$$\check{\mathbb{L}}(P_1, P_2, P_3, P_4)\mathsf{L}(P_1, P_2)\mathsf{L}'(P_3, P_4) = \mathsf{L}(P_3, P_4)\mathsf{L}'(P_1, P_2)\check{\mathbb{L}}(P_1, P_2, P_3, P_4) . \tag{B.3}$$

The main idea for constructing $\check{\mathbb{L}}(P_1, P_2, P_3, P_4)$ is to first find quantum intertwiners of more elementary objects, namely the solutions $\Psi_{P_1P_2}$ to

$$\Psi_{P_1P_2}\mathsf{L}(P_1, P_2) = \mathsf{L}(P_2, P_1)\Psi_{P_1P_2} \tag{B.4}$$

and $\Phi_{P_2P_3}$ to

$$\Phi_{P_2P_3}\mathsf{L}(P_1, P_2)\mathsf{L}'(P_3, P_4) = \mathsf{L}(P_1, P_3)\mathsf{L}'(P_2, P_4)\Phi_{P_2P_3} \tag{B.5}$$

It seems natural to look for the solution $\Psi_{P_1 P_2}$ to (B.4) as a sole function of $\mathrm{e}^{-\omega_1 \mathtt{X}}$, *viz.* $\Psi_{P_1 P_2} = \check{\psi}_{P_1 P_2}(\mathtt{X})$. Then, (B.4) reduces to

$$
\begin{pmatrix} \check{\psi}_{P_1 P_2}(\mathtt{X}) & 0 \\ 0 & \check{\psi}_{P_1 P_2}\left(\mathtt{X} - \frac{2\mathrm{i}\pi}{\omega_2}\right) \end{pmatrix} P_1 \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\omega_1 \mathtt{X}} \end{pmatrix} P_2^t
$$

$$
= P_2 \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\omega_1 \mathtt{X}} \end{pmatrix} P_1^t \begin{pmatrix} \check{\psi}_{P_1 P_2}(\mathtt{X}) & 0 \\ 0 & \check{\psi}_{P_1 P_2}\left(\mathtt{X} - \frac{2\mathrm{i}\pi}{\omega_2}\right) \end{pmatrix} \quad \text{(B.6)}
$$

i.e. the above matrix should be symmetric. Writing this condition yields the finite difference equation

$$
\frac{\check{\psi}_{P_1 P_2}\left(\mathtt{X} - \frac{2\mathrm{i}\pi}{\omega_2}\right)}{\check{\psi}_{P_1 P_2}(\mathtt{X})} = \frac{\alpha_1 \gamma_2 + \beta_1 \delta_2 \mathrm{e}^{-\omega_1 \mathtt{X}}}{\gamma_1 \alpha_2 + \delta_1 \beta_2 \mathrm{e}^{-\omega_1 \mathtt{X}}} \quad \text{(B.7)}
$$

along with it modular dual. The finite difference equation can be readily solved in terms of the double sine function $\mathcal{S}$, or equivalently, upon a slight change of parametrisation, in terms of the functions S which are both discussed in Appendix C.2. This will be done later on.

Similarly, when solving the equation (B.5) it seems natural to assume that $\Phi_{P_2 P_3} = \check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}')$. The equation becomes

$$
\begin{pmatrix} \check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}') & \\ & \check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}' - \mathrm{i}\omega_1) \end{pmatrix} P_2^t \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathtt{x} - \mathtt{x}')} \end{pmatrix} P_3
$$

$$
= P_3^t \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathtt{x} - \mathtt{x}')} \end{pmatrix} P_2 \begin{pmatrix} \check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}') & \\ & \check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}' - \mathrm{i}\omega_1) \end{pmatrix}
$$

saying that the matrix is symmetric. Writing this condition explicitly yields

$$
\frac{\check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}' - \mathrm{i}\omega_1)}{\check{\phi}_{P_2 P_3}(\mathtt{x} - \mathtt{x}')} = \frac{\alpha_2 \beta_3 + \gamma_2 \delta_3 \mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathtt{x} - \mathtt{x}')}}{\beta_2 \alpha_3 + \delta_2 \gamma_3 \mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathtt{x} - \mathtt{x}')}} \; . \quad \text{(B.8)}
$$

The two elementary operators $\check{\psi}$ and $\check{\phi}$ allow one to build the more complicated intertwiner $\check{\mathbb{L}}(P_1, P_2, P_3, P_4)$ as

$$
\check{\mathbb{L}}(P_1, P_2, P_3, P_4) = \check{C} \Phi_{14} \Psi'_{24} \Psi_{13} \Phi_{23} \; . \quad \text{(B.9)}
$$

Here $\check{C}$ is a constant yet to be fixed and we used the shorthand notation

$$
\Phi_{ij} = \check{\phi}_{P_i P_j}(\mathtt{x} - \mathtt{x}') \; , \qquad \Psi_{ij} = \check{\psi}_{P_i P_j}(\mathtt{X}) \quad \text{and} \quad \Psi'_{ij} = \check{\psi}_{P_i P_j}(\mathtt{X}') \; .
$$

The proof of this identity goes through a successive exchange of the matrices P_i:

$$
\begin{aligned}
\Phi_{23} \mathrm{L}(P_1, P_2) \mathrm{L}'(P_3, P_4) &= \mathrm{L}(P_1, P_3) \mathrm{L}'(P_2, P_4) \Phi_{23} \\
\Psi_{13} \Phi_{23} \mathrm{L}(P_1, P_2) \mathrm{L}'(P_3, P_4) &= \mathrm{L}(P_3, P_1) \mathrm{L}'(P_2, P_4) \Psi_{13} \Phi_{23} \\
\Psi'_{24} \Psi_{13} \Phi_{23} \mathrm{L}(P_1, P_2) \mathrm{L}'(P_3, P_4) &= \mathrm{L}(P_3, P_1) \mathrm{L}'(P_4, P_2) \Psi'_{24} \Psi_{13} \Phi_{23} \\
\Phi_{14} \Psi'_{24} \Psi_{13} \Phi_{23} \mathrm{L}(P_1, P_2) \mathrm{L}'(P_3, P_4) &= \mathrm{L}(P_3, P_4) \mathrm{L}'(P_1, P_2) \Phi_{14} \Psi'_{24} \Psi_{13} \Phi_{23}
\end{aligned}
$$

Having in mind the application of this analysis to the construction of the intertwiner $\mathbb{L}_{V_x V_u}(t)$ in (3.3), we can simplify the expressions of the building blocks of $\check{\mathbb{L}}(P_1, P_2, P_3, P_4)$ by specialising the parameters as

$$
\alpha_1 = \alpha_3 = \mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda}, \quad \beta_1 = \beta_3 = 1,
$$

$$
\alpha_2 = \alpha_4 = 1, \quad \beta_2 = -1, \quad \beta_4 = -\mathrm{e}^{-\frac{2\pi}{\omega_2}t},
$$

as well as

$$\gamma_1 = \gamma_3 = -q^{-2}, \quad \gamma_2 = q^2 d_2, \quad \gamma_4 = -q,$$

$$\delta_1 = \delta_3 = 0, \quad \delta_2 = q^3 d_1 e^{-\frac{2\pi}{\omega_2}\lambda}, \quad \delta_4 = q e^{-\frac{2\pi}{\omega_2}\lambda} \, .$$

This puts the finite difference equations in the form

$$\frac{\check{\phi}_{P_1 P_4}(x - i\omega_1)}{\check{\phi}_{P_1 P_4}(x)} = e^{-\frac{2\pi}{\omega_2}(\lambda + t + i\frac{\omega_2}{2})}\left(1 - e^{\frac{2\pi}{\omega_2}(t - x - i\frac{\Omega}{2})}\right) \tag{B.10}$$

$$\frac{\check{\phi}_{P_2 P_3}(x - i\omega_1)}{\check{\phi}_{P_2 P_3}(x)} = \frac{e^{\frac{2\pi}{\omega_2}(\lambda + i\frac{\omega_2}{2})}}{1 - e^{-\frac{2\pi}{\omega_2}(x + \kappa_1 - i\frac{\Omega}{2})}} \tag{B.11}$$

$$\frac{\check{\psi}_{P_1 P_3}\left(X - \frac{2i\pi}{\omega_2}\right)}{\check{\psi}_{P_1 P_3}(X)} = 1 \tag{B.12}$$

$$\frac{\check{\psi}_{P_2 P_4}\left(X - \frac{2i\pi}{\omega_2}\right)}{\check{\psi}_{P_2 P_4}(X)} = e^{\frac{2\pi}{\omega_2}(\kappa_2 - i\frac{\Omega}{2})}\frac{1 - e^{-\frac{2\pi}{\omega_2}\left(\frac{\omega_1\omega_2}{2\pi}X + \lambda + i\frac{\omega_2}{2}\right)}}{1 - e^{-\frac{2\pi}{\omega_2}\left(\frac{\omega_1\omega_2}{2\pi}X + \lambda + t + \kappa_1 - \kappa_2 - i\frac{\omega_1}{2}\right)}} \tag{B.13}$$

along with the dual equations. Since the construction is independent of constants, one can always choose $\check{\psi}_{13}(z) = 1$. The other functions are uniquely fixed, up to a constant, to be

$$\check{\psi}_{P_2 P_4}(X) = e^{iX(\kappa_2 - i\frac{\Omega}{2})}\frac{\mathcal{S}\left(\frac{\omega_1\omega_2}{2\pi}X + t + \kappa_1 - \kappa_2 + \lambda - i\frac{\omega_1}{2}\right)}{\mathcal{S}\left(\frac{\omega_1\omega_2}{2\pi}X + \lambda + i\frac{\omega_2}{2}\right)} \, ,$$

$$\check{\phi}_{P_1 P_4}(x) = \frac{e^{-\frac{2i\pi x}{\omega_1\omega_2}(\lambda + t + i\frac{\omega_2}{2})}}{\mathcal{S}\left(x - t + i\frac{\Omega}{2}\right)} \, ,$$

$$\check{\phi}_{P_2 P_3}(x) = e^{\frac{2i\pi x}{\omega_1\omega_2}(\lambda + i\frac{\omega_2}{2})}\mathcal{S}\left(x + \kappa_1 - i\frac{\Omega}{2}\right) \, .$$

Starting from these representations, taking explicitly the operator products and moving the λ and operator dependent parts so as to cancel them out, one eventually obtains that

$$\tilde{\mathbb{L}}_{V_x V_u}(t) = C(\lambda) \cdot \mathsf{P}_{xu} \cdot \phi_{14}(\mathsf{x} - \mathsf{u}) \cdot \psi_{24}(\mathsf{U}) \cdot \phi_{23}(\mathsf{x} - \mathsf{u}) \tag{B.14}$$

in which the building blocks are as defined in eqs. (3.5-3.7) intertwines LM. Here, $C(\lambda)$ is some λ dependent constant that can be made explicit but is irrelevant to the intertwining property. Hence, upon changing this constant prefactor to the desired value, one indeed gets that (3.4) does enjoy the sought intertwiner property.

B.2 A compact representation

It is however useful to provide a second representation for the intertwiner, this time in terms of the S-function introduced in (C.12):

$$\mathsf{P}_{xu}\mathbb{L}_{V_x V_u}(t) =$$

$$C_L(t)e^{-\frac{2i\pi t}{\omega_1\omega_2}(\mathsf{x} - \mathsf{u})}e^{i\mathsf{U}(\kappa_2 - i\frac{\Omega}{2})}S^{-1}\left(q^{-2}\mathfrak{a}e^{-\frac{2\pi}{\omega_2}(\mathsf{x} - \mathsf{u})}\right)\frac{S\left(\mathfrak{b}\mathfrak{a}^{-1}e^{-\omega_1\mathsf{U}}\right)}{S\left(e^{-\omega_1\mathsf{U}}\right)}S\left(q^{-2}\mathfrak{b}e^{-\frac{2\pi}{\omega_2}(\mathsf{x} - \mathsf{u})}\right) \, .$$

Here, we have set

$$\mathfrak{a} = d_2 q^2 \mathrm{e}^{\frac{2\pi}{\omega_2}t} \quad \text{and} \quad \mathfrak{b} = -q^3 d_1 . \tag{B.15}$$

We can simplify the above formula by using the Schützenberger relation given in (C.15). In order to do so, we have to use the relation

$$S(z) = G(z)S^{-1}(q^2 z^{-1})$$

where $G(z)$ is such that

$$\frac{G(q^2 z)}{G(z)} = -z^{-1} . \tag{B.16}$$

Upon using

$$G^{-1}\Big(q^{-2}\mathfrak{b}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}'-\mathbf{x})}\Big)\mathrm{e}^{-\omega_1\mathbf{X}'}G\Big(q^{-2}\mathfrak{b}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}'-\mathbf{x})}\Big) = -q^4\mathfrak{b}^{-1}\mathrm{e}^{-\omega_1\mathbf{X}'}\mathrm{e}^{-\frac{2\pi}{\omega_2}(\mathbf{x}'-\mathbf{x})}$$

we obtain

$$\mathbb{P}_{xu}\mathbb{L}_{V_x V_u}(t) = C_{\mathbb{L}}(t)\mathrm{e}^{-\frac{2\mathrm{i}\pi t}{\omega_1\omega_2}(\mathbf{x}-\mathbf{u})}\mathrm{e}^{\mathrm{i}\mathbf{U}(\kappa_2-\mathrm{i}\frac{\Omega}{2})}G^{-1}(q^{-2}\mathfrak{a}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{u}-\mathbf{x})})G(q^{-2}\mathfrak{b}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{u}-\mathbf{x})})$$
$$\times S\Big(q^4\mathfrak{a}^{-1}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}-\mathbf{u})}\Big)S\Big(-q^4\mathfrak{a}^{-1}\mathrm{e}^{-\omega_1\mathbf{U}}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}-\mathbf{u})}\Big)$$
$$\times \Big[S\Big(q^4\mathfrak{b}^{-1}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}-\mathbf{u})}\Big)S\Big(-q^4\mathfrak{b}^{-1}\mathrm{e}^{-\omega_1\mathbf{U}}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}-\mathbf{u})}\Big)\Big]^{-1} . \tag{B.17}$$

We can now use Schützenberger relation and the composition

$$G^{-1}\Big(q^{-2}\mathfrak{a}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{u}-\mathbf{x})}\Big)G\Big(q^{-2}\mathfrak{b}\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{u}-\mathbf{x})}\Big) = C'\mathrm{e}^{-\frac{2\mathrm{i}\pi}{\omega_1\omega_2}(\mathbf{u}-\mathbf{x})(t+\kappa_1-\kappa_2-\mathrm{i}\frac{\Omega}{2})}$$

for some constant C' to obtain

$$\mathbb{P}_{xu}\mathbb{L}_{V_x V_u}(t) =$$
$$C_{\mathbb{L}}(t)C'\mathrm{e}^{-\frac{2\mathrm{i}\pi}{\omega_1\omega_2}(\mathbf{x}-\mathbf{u})(\kappa_2-\kappa_1+\mathrm{i}\frac{\Omega}{2})}\mathrm{e}^{\mathrm{i}\mathbf{U}(\kappa_2-\mathrm{i}\frac{\Omega}{2})}\frac{S\Big(q^4\mathfrak{a}^{-1}(1-\mathrm{e}^{-\omega_1\mathbf{U}})\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}-\mathbf{u})}\Big)}{S\Big(q^4\mathfrak{b}^{-1}(1-\mathrm{e}^{-\omega_1\mathbf{U}})\mathrm{e}^{\frac{2\pi}{\omega_2}(\mathbf{x}-\mathbf{u})}\Big)} . \tag{B.18}$$

C Special functions

C.1 q products

Given $|p| < 1$ one denotes

$$(z;p) = \prod_{k\geq 0}(1-zp^k) . \tag{C.1}$$

This allows one to define the θ function and its dual as

$$\theta(\lambda) = \big(\mathrm{e}^{-\frac{2\pi}{\omega_2}\lambda};q^2\big)\cdot\big(q^2\mathrm{e}^{\frac{2\pi}{\omega_2}\lambda};q^2\big) \quad \text{and} \quad \tilde{\theta}(\lambda) = \big(\mathrm{e}^{\frac{2\pi}{\omega_1}\lambda};\tilde{q}^{-2}\big)\cdot\big(\tilde{q}^{-2}\mathrm{e}^{-\frac{2\pi}{\omega_1}\lambda};\tilde{q}^{-2}\big) . \tag{C.2}$$

Note that, up to a constant and an exponential prefactor, $\theta(\lambda)$ coincides with the usual theta function $\theta_1(\lambda \mid \tau)$. The modular transformation formula for $\theta_3(\lambda \mid \tau)$ translates into

$$\theta(\lambda) = \tilde{\theta}(\lambda)e^{iB(\lambda)} \tag{C.3}$$

in which

$$B(z) = \frac{\pi}{\omega_1\omega_2}z^2 + i\frac{\pi\Omega}{\omega_1\omega_2}z - \frac{\pi}{6\omega_1\omega_2}\left(\omega_1^2 + 3\omega_1\omega_2 + \omega_2^2\right) \tag{C.4}$$

and where we introduced the useful quantity

$$\Omega = \omega_1 + \omega_2 .$$

C.2   The double sine function

The double sine function $\mathcal{S}$ is defined by the integral representation

$$\ln \mathcal{S}(z) = \int_{\mathbb{R}+i0^+} \frac{dt}{t} \frac{e^{izt}}{\left(e^{\omega_1 t} - 1\right)\left(e^{\omega_2 t} - 1\right)} . \tag{C.5}$$

$\mathcal{S}$ can be represented as a convergent infinite product in the case where $\Im\left(\frac{\omega_1}{\omega_2}\right) > 0$, *i.e.* $|q| < 1$ and $|\tilde{q}| > 1$:

$$\mathcal{S}(\lambda) = \frac{\left(e^{-\frac{2\pi}{\omega_2}\lambda}; q^2\right)}{\left(\tilde{q}^{-2}e^{-\frac{2\pi}{\omega_1}\lambda}; \tilde{q}^{-2}\right)} = e^{iB(\lambda)} \cdot \frac{\left(e^{\frac{2\pi}{\omega_1}\lambda}; \tilde{q}^{-2}\right)}{\left(q^2 e^{\frac{2\pi}{\omega_2}\lambda}; q^2\right)} . \tag{C.6}$$

The equivalence of these two representation is a consequence of the modular transformation relation for theta functions (C.3).

The double sine function satisfies the quasi-periodicity relations

$$\frac{\mathcal{S}(z - i\omega_1)}{\mathcal{S}(z)} = \frac{1}{1 - e^{-\frac{2\pi}{\omega_2}z}} \quad , \quad \frac{\mathcal{S}(z - i\omega_2)}{\mathcal{S}(z)} = \frac{1}{1 - e^{-\frac{2\pi}{\omega_1}z}} \tag{C.7}$$

and enjoys a reflection property

$$\mathcal{S}(\lambda)\,\mathcal{S}(-\lambda - i\Omega) = e^{iB(\lambda)} . \tag{C.8}$$

The zeroes and poles of $\mathcal{S}(z)$ are all simple and located on the lattices

$$im\omega_1 + in\omega_2, \quad m, n, \geq 0 \qquad\qquad \text{zeroes} \tag{C.9}$$
$$im\omega_1 + in\omega_2, \quad m, n, \leq -1 \qquad\qquad \text{poles} . \tag{C.10}$$

Finally, $\mathcal{S}$ has the $\lambda \to \infty$ asymptotics

$$\mathcal{S}(\lambda) \sim \begin{cases} 1 & \arg(\omega_1) - \frac{\pi}{2} < \arg(\lambda) < \arg(\omega_2) + \frac{\pi}{2} \\[4pt] e^{iB(\lambda)} & \arg(\omega_1) - 3\frac{\pi}{2} < \arg(\lambda) < \arg(\omega_2) - \frac{\pi}{2} \\[4pt] e^{iB(\lambda)}\left(q^2 e^{\frac{2\pi}{\omega_2}\lambda}; q^2\right)^{-1} & \arg(\omega_2) - \frac{\pi}{2} < \arg(\lambda) < \arg(\omega_1) - \frac{\pi}{2} \\[4pt] \left(q^2 e^{-\frac{2\pi}{\omega_2}\lambda}; q^2\right) & \arg(\omega_2) + \frac{\pi}{2} < \arg(\lambda) < \arg(\omega_1) + \frac{\pi}{2} \end{cases} . \tag{C.11}$$

It is sometimes more convenient to work with a closely related function which is denoted by $S(x)$ and is defined as

$$S(\mathrm{e}^{-\frac{2\pi}{\omega_2}z}) = \mathcal{S}(z) \tag{C.12}$$

so that, for $|q| < 1$,

$$S(x) = \frac{\left(x; q^2\right)}{\left(\tilde{q}^{-2}x^{\frac{\omega_2}{\omega_1}}; \tilde{q}^{-2}\right)} . \tag{C.13}$$

Since $q^{\frac{2\omega_2}{\omega_1}} = 1$, S satisfies the functional equation

$$\frac{S(q^2x)}{S(x)} = \frac{1}{1-x} . \tag{C.14}$$

Volkov [32] has argued that it satisfies Schützenberger relation

$$S(\mathtt{x} + \mathtt{X}) = S(\mathtt{x})S(\mathtt{X}), \quad \text{provided that} \quad \mathtt{x}\mathtt{X} = q^2\mathtt{X}\mathtt{x} . \tag{C.15}$$

This identity has been rigorously established by Woronowicz in [34]. Then, one should understand $\mathtt{x} + \mathtt{X}$ appearing in the *lhs* and $\mathtt{x}$, $\mathtt{X}$ appearing in the *rhs* as the self-adjoint extension of the respective operators.

C.3   The quantum dilogarithm

The quantum dilogarithm [7] ϖ is a meromorphic function that is directly related to the double sine function $\mathcal{S}$:

$$\varpi\left(\lambda + \mathrm{i}\frac{\Omega}{2}\right) = \mathrm{e}^{-\mathrm{i}\frac{B(\lambda)}{2}}\mathcal{S}(\lambda) . \tag{C.16}$$

The below ratio of quantum dilogarithms enjoys nice Fourier transformation properties

$$D_\alpha(x) = \frac{\varpi(x+\alpha)}{\varpi(x-\alpha)} , \tag{C.17}$$

namely, it holds

$$D_\alpha(p) = \frac{\mathcal{A}(\alpha)}{\sqrt{\omega_1\omega_2}} \int_{\mathbb{R}} D_{\alpha^\star}(v)\mathrm{e}^{-\frac{2\mathrm{i}\pi}{\omega_1\omega_2}vp} \cdot \mathrm{d}v \quad \text{with} \quad \begin{cases} \alpha^\star = -\alpha - \mathrm{i}\dfrac{\Omega}{2} \\ \mathcal{A}(\alpha) = \varpi(\alpha - \alpha^\star) \end{cases} . \tag{C.18}$$

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