

Numerical computation of H-bases

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Received: date / Accepted: date

Abstract This paper gives a numerically stable method to compute H-basis which is based on the computing a minimal basis for the module of syzygies using singular value decomposition. We illustrate the performance of this method by means of various examples.

Keywords H-basis · Syzygy · SVD

Mathematics Subject Classification (2010) 13P10 · 65F30

1 Introduction

The concept of Gröbner bases plays an important, if not the fundamental, role in the development of modern computational Algebraic Geometry and Computer Algebra systems. Indeed, it provides an important tool to study and solve numerous problems in many different areas, ranging from optimization, coding theory, cryptography, to signal and image processing, robotics, statistics and even more, cf. [5].

One of the main drawbacks of Gröbner bases, however, is the fact that their structure depends on a monomial ordering or *term order* which breaks the symmetry among the variables and consequently also loses symmetries present in the underlying problem. Moreover, Gröbner bases are **not** numerically stable and even small perturbations in the coefficients of the polynomials generating an ideal may change the result dramatically, cf. [18,24]. Thus, the development of other methods with more numerical stability which also preserve symmetry is needed.

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H-bases, or *Macaulay bases* as they are sometimes called nowadays, are even older than Gröbner bases and were introduced by Macaulay in 1914, cf. [16]. Macaulay computed an H-basis only for a specific example by determining *syzygies* among the maximal degree homogeneous parts of the polynomials, the so-called *leading forms*, but he did not give a general symbolic algorithm in this regard. A symbolic algorithm to construct H-bases without relying on monomial orderings was introduced in [21]. This algorithm is a quite direct generalization of Buchberger’s algorithm and relies on a *reduction algorithm* which is a generalization of euclidean division with remainder to the multivariate case. The crucial point in this reduction algorithm consists of *orthogonalizing* the leading forms instead of attempting the impossible task of cancelling them. This generalized reduction leads to a characterization of H-bases which is based on reducing a generating set of the *syzygy module* of leading forms. Therefore, determining a basis for the module of syzygies of finitely many homogeneous forms becomes a crucial part of the construction of the H-basis. Unfortunately, finding a basis for the syzygies between forms is far more intricate than finding a basis for the syzygies between terms.

According to our knowledge, there are essentially two general ways to construct a basis for the syzygy module of an ideal. The first one, described in [5], is based on computing a Gröbner basis for the underlying ideal, while the numerically more suitable approach is based on a Linear Algebra and addressed, for example in [12]; it is referred to as *degree by degree approach* there. The key component in this approach is to generate a homogeneous matrix of coefficient vectors of leading forms of polynomials.

A more general type of such matrices were introduced as *Macaulay matrices* and analysed in [1,2,3]. In [3], the degree of regularity of a polynomial system is described and a formula for the dimension of the null space of Macaulay matrices is derived. A recursive orthogonalization scheme for two subspaces of these matrices, the range, i.e., the row spaces, and their null spaces, are examined in [2]. A decomposition for the vector space of monomials for a given degree k , and numerical Gröbner and border bases are finally examined in [1].

In this article we describe how one can use a submatrix of Macaulay matrices to efficiently determine a basis for the syzygy module of leading forms by techniques from Numerical Linear Algebra and apply this to develop a numerical algorithm for determining an H-basis, relying again on well-understood techniques from Numerical Linear Algebra which are available as efficient and stable implementations for example in Matlab. Our motivation comes also from the fact that in practical problems the polynomial systems are often only given as results of preceding computations as in [22] and therefore “empirically”, i.e., with inaccurate coefficients, in the sense of [24]. In this case, using a symbolic algorithm to compute an H-basis is often not meaningful and may even lead to misleading results. Moreover the numerical methods are faster than the purely symbolic ones by orders of magnitude and also have much smaller memory requirements. We will, however, see in the examples later that there also exists ill-conditioned ideals where small roundoff errors contaminate the computations so heavily that the eventual result can even be wrong.

The paper is organized as follows. After introducing the necessary concepts and terminology, we will describe the algorithms for syzygy determination and reduction

and prove their validity; moreover, we will discuss a stopping criterion for the H-basis process. Finally, we apply the method to various examples of ideals that are known to be notoriously difficult and serve as benchmarks for ideal basis computations. The examples will show that there are ideals that are well-conditioned and that there are numerically ill-conditioned ideals, the latter due to the fact that some of the normalized reductions can result in very small remainders that are very hard to distinguish from zero remainders numerically.

2 Notations and Definitions

For a field \mathbb{K} , usually of characteristic zero, we consider the polynomial ring $\mathbb{P} = \mathbb{K}[x_1, \dots, x_n]$. Its subset and linear subspace of homogeneous polynomials of degree k is defined by

$$\mathbb{P}_k := \text{span}_{\mathbb{K}}\{x^\alpha : |\alpha| = k\},$$

where the *length* of a multiindex $\alpha \in \mathbb{N}_0^n$ is defined as $|\alpha| := \alpha_1 + \dots + \alpha_n$. Moreover,

$$\mathbb{P}_{\leq k} := \bigoplus_{i=0}^k \mathbb{P}_i$$

is the linear space of all polynomials of degree $\leq k$. A monomial $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ has the *multidegree* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and the *total degree* $|\alpha|$. When we speak of the *degree* of a polynomial, we always mean the total degree, i.e.

$$\deg(f) := \max\{|\alpha| : f_\alpha \neq 0\}, \quad f(x) = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha x^\alpha.$$

Let \mathbb{T} denote the set of all these monomials, as well as $\mathbb{T}_k := \mathbb{T} \cap \mathbb{P}_k = \{x^\alpha : |\alpha| = k\}$. Any polynomial f can be written as a finite linear combination

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha x^\alpha, \quad f_\alpha \in \mathbb{K},$$

where the coefficients f_α are indexed by using the standard multi-index notation. In practical implementations, however, we have to map the multiindices to \mathbb{N} by equipping \mathbb{T} with a fixed monomial ordering, cf. [4]. Since our ordering has to be compatible with the total degree, we conveniently equip \mathbb{T} with the *graded lexicographical* ordering and any polynomial can be identified with its coefficient vector. Note that our results do not depend on the choice of this monomial ordering, it only may affect the computational efficiency of the implementation.

For $f = f^0 + \dots + f^k$, $k = \deg f$, $f^i \in \mathbb{P}_i$, with $f^k \neq 0$, we call $\text{lf}(f) := f^k$ the *leading form* of f . For every ideal $\mathcal{I} = \langle p_1, \dots, p_s \rangle$, generated by p_1, \dots, p_s , the *homogeneous ideal*

$$\text{lf}(\mathcal{I}) := \{\text{lf}(p) : p \in \mathcal{I}\}$$

can contain polynomials p such that $\mathbf{lf}(p) \notin \langle \mathbf{lf}(p_1), \dots, \mathbf{lf}(p_s) \rangle$. The absence of this unwanted situation leads to the definition of an H-basis. To formulate it, we recall that a syzygy of polynomials (p_1, \dots, p_s) is a tuple $(q_1, \dots, q_s) \in \mathbb{P}^s$ such that

$$\sum_{j=1}^s q_j p_j = 0.$$

By $\text{Syz}(p_1, \dots, p_s)$ we denote the set of all such syzygies, the *syzygy module* of p_1, \dots, p_s .

Definition 2.1 The (finite) set $\{p_1, \dots, p_s\} \subset \mathbb{P}$ is called an H-basis for the ideal $\mathcal{I} := \langle p_1, \dots, p_s \rangle$, if one of the following equivalent conditions holds:

1. If $p \in \mathcal{I}$ then $\mathbf{lf}(p) \in \langle \mathbf{lf}(p_1), \dots, \mathbf{lf}(p_s) \rangle$.
2. $p \in \mathcal{I}$ is equivalent to

$$p = \sum_{i=1}^s h_i p_i, \quad h_i \in \mathbb{P}_{\leq \deg(p) - \deg(p_i)}, \quad i = 1, \dots, s. \quad (2.1)$$

3. If $(h_1, \dots, h_s) \in \text{Syz}(\mathbf{lf}(p_1), \dots, \mathbf{lf}(p_s))$ then there exist $g_1, \dots, g_s \in \mathbb{P}$ such that

$$\sum_{i=1}^s h_i p_i = \sum_{i=1}^s g_i p_i, \quad \deg(g_i p_i) \leq \deg\left(\sum_{i=1}^s h_i p_i\right), \quad i = 1, \dots, s.$$

Remark 2.1 The restriction that the H-basis is finite is no restriction at all. By Hilbert's Basissatz there always exists a finite basis for any polynomial ideal \mathcal{I} which can be transformed into an H-basis by a variant of Buchberger's algorithm. Moreover, condition (3) means that any syzygy of *homogeneous* leading forms can be reduced to zero. A proof of the equivalence of the above properties can be found, for example, in [17].

The condition (3) allows for a direct extension of Buchberger's algorithm to compute H-bases, cf. [21], but finding a basis for $\text{Syz}(\mathbf{lf}(p_1), \dots, \mathbf{lf}(p_s))$ is crucial in this extension. Having at hand a Gröbner basis for the ideal $\langle \mathbf{lf}(p_1), \dots, \mathbf{lf}(p_s) \rangle$, it is indeed possible to compute such a basis, but this is unsatisfactory since then one could use the Gröbner basis directly. A more direct way is the degree by degree approach presented in [12], which determines an H-basis degree by degree without having to rely on Gröbner bases at all.

To apply the degree by degree approach in our context here, we need to recall the following definition.

Definition 2.2 Given a set $F = \{f_1, \dots, f_s\} \subseteq \mathbb{P}$ and $k \in \mathbb{N}_0$, we define the subspace of degree k as

$$\mathcal{M}_k(F) := \left\{ \sum_{i=1}^s h_i \mathbf{lf}(f_i) : h_i \in \mathbb{P}_{k - \deg(f_i)}, 1 \leq i \leq s \right\} \subseteq \mathbb{P}_k, \quad (2.2)$$

and the space of homogeneous syzygies of degree k for the leading forms as

$$\begin{aligned} \text{Syz}_k(\mathbf{1f}(F)) &:= \text{Syz}_k(\mathbf{1f}(f_1), \dots, \mathbf{1f}(f_s)) \\ &:= \left\{ (h_1, \dots, h_s) : \sum_{i=1}^s h_i \mathbf{1f}(f_i) = 0, h_i \in \mathbb{P}_{k-\deg(f_i)} \right\}. \end{aligned} \quad (2.3)$$

In all these definitions we use the convention that $\mathbb{P}_k = \{0\}$ whenever $k < 0$.

A generating system for the space $\mathcal{M}_k(F)$ is given by the following matrices.

Definition 2.3 For $f \in \mathbb{P}$ we define the matrices

$$C_k(f) := [x^\alpha \mathbf{1f}(f)(x) : |\alpha| = k - \deg(f)] \in \mathbb{P}_k^{1 \times d_{k-\deg(f)}}, \quad d_k := \binom{k+n}{n-1},$$

and their concatenation

$$C_k(F) := [C_k(f_1), \dots, C_k(f_s)] \in \mathbb{P}_k^{1 \times (d_{k-\deg(f_1)} + \dots + d_{k-\deg(f_s)})}.$$

Identifying the polynomials with their coefficients with respect to the homogeneous monomial basis of degree k , we can also assume that

$$C_k(F) \in \mathbb{K}^{d_k \times d_{k-\deg(f_1)} + \dots + d_{k-\deg(f_s)}}, \quad (2.4)$$

which is exactly the way how this matrix will be represented on the computer with the multiindices in rows and columns ordered in the graded lexicographical way.

It can be easily seen that

$$\mathcal{M}_k(F) = \text{span} \{C_k(f) : f \in F, \deg(f) \leq k\} = \text{span} C_k(F). \quad (2.5)$$

To simplify (2.4), we introduce the abbreviation

$$d_{k-\deg(F)} := \sum_{i=1}^s d_{k-\deg(f_i)},$$

denote by $\mathcal{R}(C_k(F)) \subseteq \mathbb{P}$ the range of $C_k(F)$ and use

$$\mathcal{N}(C_k(F)) = \{v \in \mathbb{K}^{d_{k-\deg(F)}} : C_k(F)v = 0\} \subseteq \mathbb{K}^{d_{k-\deg(F)}}$$

for the *null space* or *kernel* of $C_k(F)$.

The space $\mathcal{M}_k(F)$ now allows us to establish a connection between $\mathcal{N}(C_k(F))$ and $\text{Syz}_k(\mathbf{1f}(F))$. Indeed, if $v \in \mathcal{N}(C_k(F))$, we see from the affine column space interpretation that the expression $C_k(F)v = 0$ is equivalent to

$$0 = \sum_{i=1}^s \sum_{|\alpha|=k-\deg(f_i)} v_{i,\alpha} x^\alpha \mathbf{1f}(f_i)(x) =: \sum_{i=1}^s h_i \mathbf{1f}(f_i) = 0. \quad (2.6)$$

Here, the vector v contains the coefficients of polynomials h_i . This fact along with the following concept helps us to compute H-basis degree by degree.

Definition 2.4 A set $\{p_1, \dots, p_s\} \subset \mathbb{P}$ of polynomials is called an *H-basis up to (degree) K* , $K \in \mathbb{N}_0$, if for every $k \leq K$ and any $(h_1, \dots, h_s) \in \text{Syz}_k(\text{lf}(p_1), \dots, \text{lf}(p_s))$ there exist $g_1, \dots, g_s \in \mathbb{P}$ such that

$$\sum_{i=1}^s h_i p_i = \sum_{i=1}^s g_i p_i, \quad g_i p_i \in \mathbb{P}_{\leq k-1}, \quad i = 1, \dots, s.$$

Note that $\{p_1, \dots, p_s\}$ is an H-basis if and only if it is an H-basis up to K for every positive integer K .

This argument shows that finding a basis for $\mathcal{N}(C_k(F))$ is crucial in this approach. On the other hand, one of the most robust and numerically stable ways to find the orthogonal basis for the null space is the singular value decomposition (SVD). The following classical theorem recalls how the basis vectors for $\mathcal{R}(C_k(F))$ and $\mathcal{N}(C_k(F))$ can be read from SVD of matrix $C_k(F)$, cf. [10].

Theorem 2.1 Let $A \in \mathbb{C}^{m \times n}$ with $\text{rank } A = r$ and let $A = U\Sigma V^H$ be SVD of A , then the vectors u_1, \dots, u_r form a basis for $\mathcal{R}(A)$ and the vectors v_{r+1}, \dots, v_n form a basis for $\mathcal{N}(A)$.

Indeed, every element of $\mathcal{N}(C_k)$ is a linear dependence relation between the columns of C_k and expresses a syzygy of the form $\sum_{i=1}^s h_i \text{lf}(p_i) = 0$. On the other hand each column of C_k corresponds to a certain monomial multiple $x^\alpha \text{lf}(p_i)$. Then, $x_j \sum_{i=1}^s h_i \text{lf}(p_i) = 0$, which means that all columns corresponding with $x_j x^\alpha \text{lf}(p_i) = x^{\alpha + \varepsilon_j} \text{lf}(p_i)$ in C_{k+1} will also be linear dependent. We will refer to $h = (h_1, \dots, h_s)$ in this case as a *basic syzygy* and its monomial multiple $x_j h$ as an *extended syzygy*.

In the next section we show that how SVD helps us to obtain a basis for the pure syzygies as well as a basis for the extended syzygies.

3 Numerical syzygy computation

For polynomials $F = \{f_1, \dots, f_s\} \subseteq \mathbb{P} \setminus \{0\}$ let $d_i := \deg(f_i) \leq k$, $i = 1, \dots, s$. We now present an approach to obtain an orthonormal basis for $\mathcal{N}(C_{k+1}(F))$ exploiting the structure of $C_k(F)$ and earlier computations of an orthogonal basis for $\mathcal{N}(C_k(F))$. Since F is the same all over this computation, we will simply use C_k in the description of the method and suppose that C_k is a $t \times q$ matrix whereas C_{k+1} is a $t' \times q'$ matrix. An orthogonal basis for the null space of C_k will be denoted by $N_k \in \mathbb{C}^{q \times d}$ matrix, where $d := \dim \mathcal{N}(C_k)$. By Definition 2.3, the matrix C_k can be partitioned as illustrated below.

$$C_k = t \begin{bmatrix} q_1 & & q_s \\ |C_k(f_1)| & \cdots & |C_k(f_s)| \end{bmatrix},$$

where $q_j := \#\mathbb{T}_{k-d_j} = d_{k-\deg(f_j)}$, $j = 1, \dots, s$, and $t = d_k$. Consequently N_k can be partitioned as

$$N_k = \begin{bmatrix} B_1 \\ \vdots \\ B_s \end{bmatrix}, \quad B_j \in \mathbb{K}^{q_j \times d}.$$

Now let $L_{ij} \in \mathbb{K}^{d_{k-d_i} \times d_{k-d_i+1}}$ be the *shift matrix* that represents the multiplication with x_j , $j = 1, \dots, n$, $i = 1, \dots, s$. Then the block diagonal matrix

$$L_j := \begin{bmatrix} L_{1j} & & \\ & \ddots & \\ & & L_{sj} \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{K}^{d' \times d}$$

has the property that

$$v \in \mathcal{N}(C_k) \quad \Rightarrow \quad w := L_j v \in \mathcal{N}(C_{k+1}), \quad j = 1, \dots, n, \quad (3.1)$$

Setting

$$A := [L_1 \cdots L_n] N_k$$

and $r := \text{rank } A$, the SVD of the matrix A will be

$$A = QSW^H, \quad Q \in \mathbb{C}^{d' \times d'}, W \in \mathbb{C}^{d \times d}.$$

Now Q can be partitioned as

$$Q = [Q_1 | Q_2], \quad Q_1 \in \mathbb{C}^{d' \times r}, Q_2 \in \mathbb{C}^{d' \times d-r}, \quad (3.2)$$

and we can compute yet another SVD

$$B := C_{k+1} Q_2 = U \Sigma V^H.$$

Let $r' := \text{rank } B$, partition V as

$$V = [V_1 | V_2], \quad V_1 \in \mathbb{C}^{d' \times r'}, V_2 \in \mathbb{C}^{d' \times d-r'}, \quad (3.3)$$

and define

$$N_{k+1} := [Q_1 | Q_2 V_2], \quad (3.4)$$

which finally allows us to draw the following conclusion.

Theorem 3.1 N_{k+1} is an orthogonal basis for $\mathcal{N}(C_{k+1})$ such that Q_1 is an orthogonal basis for the extended syzygies and $Q_2 V_2$ is an orthogonal basis for pure syzygies.

Proof We start with the observation that

$$C_{k+1} [Q_1 \ Q_2 V_2] = 0$$

since V_2 is a basis for $\mathcal{N}(C_{k+1} Q_2)$ and Q_1 is a basis for $\mathcal{B}(A) \subseteq \mathcal{N}(C_{k+1})$. This yields that

$$\text{span} [Q_1 \ Q_2 V_2] \subseteq \mathcal{N}(C_{k+1}).$$

On the other hand,

$$0 = \text{rank } C_{k+1} Q_2 V_2 \leq \text{rank } C_{k+1} + \text{rank } Q_2 V_2 - q',$$

hence

$$\dim \mathcal{N}(C_{k+1}) = q' - \text{rank } C_{k+1} \leq \text{rank } Q_2 V_2 \leq \text{rank} [Q_1 \ Q_2 V_2],$$

yields that $\text{rank} [Q_1 \ Q_2 V_2] = \dim \mathcal{N}(C_{k+1})$, and this completes the proof.

Theorem 3.1 can be immediately translated into Algorithm 1 to extract an orthogonal basis for $\mathcal{N}(C_{k+1})$ and consequently pure syzygies. τ and τ' which are used to decide the numerical ranks are introduced in Section 5.

Algorithm 1 SYZGY UPDATE

Input: N_k

Output: orthogonal basis N_{k+1}

construct the block diagonal matrices L_j for $j = 1, \dots, n$

$A \leftarrow [L_1 \cdots L_n] N_k$

$QSW^H \leftarrow \text{SVD}(A)$, $r \leftarrow \max\{r : s_r > \tau\}$

$[Q_1 | Q_2] \leftarrow Q$, $Q_1 \in \mathbb{C}^{d \times r}$, $Q_2 \in \mathbb{C}^{d \times d-r}$

$B \leftarrow C_{k+1} Q_2$

$U\Sigma V^H \leftarrow \text{SVD}(B)$, $r' \leftarrow \max\{r' : \sigma_{r'} > \tau'\}$

$[V_1 | V_2] \leftarrow V$, $V_1 \in \mathbb{C}^{d' \times r'}$, $V_2 \in \mathbb{C}^{d' \times d'-r'}$

$N_{k+1} \leftarrow [Q_1 | Q_2 V_2]$

It is now possible to formulate the degree by degree approach to compute H-bases using the iterative orthogonalization scheme of Theorem 3.1. In the next section first we remind a numerical description of reduction algorithm presented in [21], then we give an updated version of H-bases algorithm using Theorem 3.1.

4 Reduction and H-basis

To describe the reduction algorithm, we fix, according to [21], an inner product $(\cdot, \cdot) : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{K}$. Keeping the terminology of the previous section, the orthogonal complement of $\mathcal{M}_k(F)$ with respect to (\cdot, \cdot) is defined as

$$\mathcal{W}_k(F) := \mathbb{P}_k \ominus \mathcal{M}_k(F) = \{f \in \mathbb{P}_k : (f, \mathcal{M}_k(F)) = 0\},$$

or, equivalently,

$$\mathbb{P}_k = \mathcal{M}_k(F) \oplus \mathcal{W}_k(F).$$

These orthogonal vector spaces enable us to decompose every homogeneous polynomial $g \in \mathbb{P}_k$ in two parts, a part in $\mathcal{M}_k(F)$ and another part in $\mathcal{W}_k(F)$ orthogonal to it, in which makes a homogeneous remainder r^k :

$$g = \underbrace{\sum_{j=1}^p c_j v_j}_{\in \mathcal{M}_k(F)} + \underbrace{\sum_{j=1}^q c_{p+j} w_j}_{\in \mathcal{W}_k(F)} := \sum_{j=1}^p c_j v_j + r^k. \quad (4.1)$$

This is the main idea of the reduction algorithm which gives an orthogonal decomposition of every polynomial $p \in \mathbb{P}$ as

$$p = \sum_{f \in F} q_f f + r, \quad \deg(q_f f) \leq \deg(p), \quad r \in \bigoplus_{k=0}^{\deg(p)} \mathcal{W}_k(F). \quad (4.2)$$

Taken together, Definition 2.3 and Theorem 2.1 help us to find a basis for $\mathcal{M}_k(F)$ as well as $\mathcal{W}_k(F)$ and eventually the representation (4.1). Indeed if $C_k(F) = U\Sigma V^H$ is an SVD and $p = \text{rank } C_k(F)$, then

$$\mathcal{M}_k(F) = \text{span}\{u_1, \dots, u_p\}, \quad \mathcal{W}_k(F) = \text{span}\{u_{p+1}, \dots, u_q\}.$$

Thus, the coefficient vector $c = (c_j : j = 1, \dots, p)$ in (4.1) is obtained as a solution of the linear system

$$[u_1, \dots, u_p, u_{p+1}, \dots, u_q]c = g.$$

On the other hand, the thin SVD from [10] yields

$$u_j = \frac{1}{\sigma_j} C_k(F) v_j, \quad j = 1, \dots, p \quad (4.3)$$

and replacing (4.3) in (4.1) results in

$$g = \sum_{j=1}^p c_j u_j + r^k = C_k(F) \underbrace{\left(\sum_{j=1}^p \frac{c_j}{\sigma_j} v_j \right)}_{=: \tilde{c}} + r^k. \quad (4.4)$$

Partitioning $\tilde{c} = (\tilde{c}_f : f \in F)$ according to the blocks of $C_k(F)$ leads to

$$g = \sum_{f \in F} C_k(F) \tilde{c}_f + r^k, \quad \tilde{c}_f \in \mathbb{K}^{\dim \mathbb{P}_{k-\deg(f)}};$$

by adding proper zeros we finally transition each \tilde{c}_f to

$$g_{f,k} := \begin{bmatrix} 0 \\ \tilde{c}_f \end{bmatrix} \in \mathbb{K}^{\dim \mathbb{P}_{\leq k-\deg(f)}},$$

which results in the representation

$$g = \sum_{f \in F} g_{f,k} \mathbf{1}_f + r^k.$$

Repeating this process for each homogeneous part of p gives us the representation (4.2) where

$$q_f = \sum_{k=0}^{\deg(p)} \sum_{f \in F} g_{f,k}, \quad r = \sum_{k=0}^{\deg(p)} r^k.$$

This leads to the following definition

Definition 4.1 For a given sequence of polynomials F , a polynomial $p \in \mathbb{P}$ is called *reducible* module F and denoted by $p \rightarrow_F r$ if there exists the representation (4.2) for p . In addition, we call r the *reduced form* of p w.r.t. F .

Remark 4.1 It should be mentioned that this definition implies that the *reduced form* of a polynomial is only dependent on the order of polynomials and inner product. Moreover it is illustrated in [21] that if F is replaced by an H-basis then the *reduced form* is unique for a fixed inner product. This process can be summarized in the following algorithm

Algorithm 2 REDUCTION

Input: polynomial system $F = \{f_1, \dots, f_s\} \subseteq \mathbb{P}$, $p \in \mathbb{P}$
Output: $q_f \in \mathbb{P}$, $r \in \bigoplus_{k=0}^{\deg(p)} \mathcal{W}_k(F)$ with $p = \sum_{f \in F} q_f f + r$
 $r \leftarrow 0$, $q_f \leftarrow 0$, $f \in F$
while $p \neq 0$ **do**
 $g \leftarrow \mathbf{lf}(p)$, $k \leftarrow \deg(p)$
 construct $C_k(F)$
 if $C_k(F) \neq 0$ **then**
 decompose $C_k(F) = U\Sigma V^H$ using SVD
 compute homogeneous decomposition $g = \sum_{f \in F} g_f \mathbf{lf}(f) + r^k$
 $r \leftarrow r + r^k$
 $p \leftarrow p - \sum_{f \in F} g_f f - r^k$
 $q_f \leftarrow q_f + g_f$
 else
 $r = r + g$
 end if
end while

Now the pseudo-code for the update version of H-basis algorithm is shown in Algorithm 3. The algorithm starts for the initial degree $k = \min(d_i : 1 \leq i \leq s)$. An orthogonal basis for N_k in the early step of each iteration process is computed from the SVD of $C_k(F)$. The subsequent steps of the algorithm are then computing a basis for pure syzygies and reducing the corresponding polynomial using Algorithm 2. The updating of the *bound* is explained in the following.

Algorithm 3 NUMERICAL H-BASIS

Input: polynomial system $F = \{f_1, \dots, f_s\} \subseteq \mathbb{P}$ of degree d_1, \dots, d_s
Output: H-basis for the ideal $\langle F \rangle$
 $k \leftarrow \min(d_i : 1 \leq i \leq s)$, $bound \leftarrow 2 \max(d_i : 1 \leq i \leq s)$
 $N_j \leftarrow \emptyset$, $0 \leq j \leq k-1$
while $k \leq bound$ **do**
 construct $C_k(F)$
 if $N_{k-1} = \emptyset$ **then**
 $\widetilde{N}_k := N_k \leftarrow$ an orthogonal basis for $\mathcal{N}(C_k(F))$
 else
 construct $[Q_1 | Q_2 V_2]$ using Algorithm 1
 $N_k \leftarrow [Q_1 | Q_2 V_2]$, $\widetilde{N}_k \leftarrow Q_2 V_2$
 end if
 while $\widetilde{N}_k \neq \emptyset$ **do**
 $p \leftarrow \sum_{i=1}^s \sum_{|\alpha|=k-d_i} v_{i,\alpha} x^\alpha f_i$, $v \in \widetilde{N}_k$
 if $p \rightarrow_F f_{s+1} \neq 0$ **then**
 $F \leftarrow F \cup \{f_{s+1}\}$, $s \leftarrow s+1$
 $k \leftarrow \deg(f_{s+1}) - 1$
 end if
 end while
 update *bound*
 $k \leftarrow k+1$
end while

In view of the above algorithm, we need to choose an appropriate degree *bound* to ensure that a generating set for $\text{Syz}(\text{lf}(F))$ has been constructed. Since we generate syzygies degree by degree, reaching such a bound tells us that all syzygies reduce to zero and thus the polynomial system is indeed an H-basis. Finding this *stopping criterion* has already been discussed in [1, 12] in detail. However, for the sake of completeness, we give a short analysis here.

A key component in finding the termination degree relies on Schreyer's theorem. Schreyer showed in his diploma thesis [23] that a generating set, more precisely, even a Gröbner basis, for the syzygy module of a Gröbner basis is obtained by reducing every S-polynomial of each pair of polynomials of underlying Gröbner basis to zero, cf. [23]. This implies that if G is a Gröbner basis for $\langle G \rangle$, then $k = \max\{\deg \tau_{ij} : \tau_{ij} = \text{lcm}(\text{lm}(f_i), \text{lm}(f_j)), f_i, f_j \in G\}$ is a maximal degree on $h_i g_i$, where $(h_1, \dots, h_m) \in \text{Syz}(G)$ and $\text{lm}(f)$ stands for a leading monomial of a polynomial f under a given monomial ordering.

On the other hand, Buchberger's criterion provides an algorithm to construct a Gröbner basis by computing the remainder of each S-polynomial and adding the non-zero remainders to the candidate set [6]. Lazard showed that computing such a remainder is equivalent to bring a resultant matrix into triangular form, [14]. By this argument, the pivots of a row reduced echelon form of $C_k(F)$ correspond to the leading monomials of the reduced Gröbner basis for $\text{lf}(F)$ for sufficiently large k , provided that all columns of the transpose $C_k(F)$ are reversed. We refer to such a matrix by $R_k(F)$. This tells us that if we bring $C_k(F)$ into a triangular form $R_k(F)$, then the maximum degree of least common multiple of each pair of leading monomials provides an upper bound to find all of syzygies of $\text{lf}(F)$. For this k the reduced Gröbner basis of $\text{lf}(F)$ can be retrieved from $R_k(F)$ which is discussed in detail in [1].

This argument gives an approach to find an appropriate degree bound to terminate the above algorithm. In doing so, we compare the pivot elements of $R_k(F)$ with the pivots of $R_{k-1}(F)$ which have been transitioned to the monomials in \mathbb{T}_k by multiplying with the shift matrices from Section 3. If there exists any new leading monomial, we update the bound by computing the maximum degree of least common multiple of each pair of leading monomials. Otherwise, if the algorithm reaches the updated bound and no new leading monomial appears up to this degree, it follows that a Gröbner basis of $\mathcal{M}_{\text{bound}}(F)$ and consequently a basis of $\text{Syz}(\text{lf}(F))$ is found. In this case, the algorithm can be terminated if there is no non-zero remainder of reduction. Since the bound will become stable after finitely steps due to the ascending chain condition property of \mathbb{P} , cf. [6], the algorithm will always terminate.

Remark 4.2 It should be mentioned that we use the theory of Gröbner basis here only to find the upper bound for the algorithm by simply computing the row reduced echelon form of a matrix without computing any Gröbner basis or any S-polynomials. The algorithm itself is still free of computing Gröbner bases.

Remark 4.3 As mentioned in Remark 4.1, the reduced form of a polynomial w.r.t. an H-basis is unique. This helps us to solve the ideal membership problem easily by finding an H-basis first and then applying reduction to a given polynomial. Finding

the common zeros of a set of polynomials by means of eigenvectors of a generalization of Frobenius companion matrices is another problem which is addressed in [19] and discussed using H-basis technique in [17].

5 Numerical results

Any numerical implementation of the above algorithms has to rely on a *tolerance* τ to decide the numerical rank and a *threshold* ε that determines whether a float number is numerically zero or not. The latter one is usually chosen as the *machine accuracy* which depends on the mantissa length of the underlying floating point arithmetic and is roughly the value 2.22×10^{-16} for double precision floating point numbers, the most frequently used arithmetic on contemporary processors. A standard choice for τ in [10] is then given by $\tau = \max\{n, m\} \max_j \sigma_j \varepsilon$, where ε the σ_j are the singular values of the underlying matrix with m columns and n rows. This defines the meaning of the term *rank* in Numerical Linear Algebra, often referred to as *numerical rank*, in contrast to its meaning in Linear Algebra. In other words, if A is an $m \times n$ matrix with singular values $\sigma_1 \geq \dots \geq \sigma_n$, then the numerical rank r is chosen such that

$$\sigma_1 \geq \dots \geq \sigma_r \geq \tau \geq \sigma_{r+1} \geq \dots \geq \sigma_n \quad (5.1)$$

cf. [2]. For an exact definition and detailed discussion of numerical rank, we refer to [15]. (5.1) shows that the correct determination of the numerical rank strongly depends on a good choice of τ . On the other hand, the determination of numerical rank is a crucial step in Algorithm 1 to distinguish new syzygies from extended ones and consequently finding a correct H-basis. So, the good choice of τ to guarantee the correct result is a must.

Our numerical tests illustrate that a standard choice of τ works truly for most experiments though it fails for some polynomial systems like Caprasse4. We will observe that in this case there is a large gap between singular values in index k but the default value of τ is smaller than σ_k , so the numerical rank detection fails and the syzygies are *ill-conditioned*. The ratio σ_r/σ_{r+1} , the so called approx-rank gap influences the accuracy of rank-revealing computation. This is discussed in details in [8,9]. Indeed a well-defined numerical rank leads to a choice of r which maximises this approx-rank gap.

Apart from rank revelation, also the choice of the threshold ε clearly affects the outcome of the algorithm. An inappropriate value for ε results in apparently very small remainders in the reduction, which forces the algorithm to end up with the basis 1, falsely claiming that the polynomials have no common zeros. To avoid this problem, we have to increase the value ε to the suitable value which then yields the correct result. We will discuss this effect in details for a polynomial system named after R. Sendra.

5.1 Experiment setup

Here we compare the efficiency and the stability of the numerical H-basis algorithm with the symbolic algorithm presented in [12]. All numerical experiments are carried

out on a 2.5 GHz seven-core personal computer with 8 GB RAM using 64-bit Matlab and usually the machine precision $\varepsilon \approx 2.22 \times 10^{-16}$. Our numerical H-basis algorithm is implemented as a Matlab module H-BASIS that is electronically available from the authors upon request.

In the first group of experiments we will show how much a numerical implementation can speed up the degree-by-degree approach to compute H-bases. The capability of the algorithm is evaluated for a benchmark set of examples with different Krull-dimensions which confirms that our approach is not restricted to zero dimensional ideals. In the second part, however, we will discuss the problems arising from the floating numbers in the numerical implementation. In each experiment we use the Hilbert polynomial of the ideal to check the correctness of the obtained numerical H-basis. Indeed the Hilbert polynomial of an ideal equals to the Hilbert polynomial of the leading form ideal, cf. [13].

In what follows, the run time is measured in seconds. The H-basis column shows that how many polynomials are detected in corresponding H-basis. d_{max} is the maximum degree of polynomials in the H-basis and *bound* shows the degree that algorithm is terminated. The names of the benchmark ideals are due to

<http://homepages.math.uic.edu/~jan/Demo/>.

5.2 Correct experiments

We begin by listing some examples where the numerical method performed correctly and obtained the proper H-basis, at least when the thresholding parameter was chosen properly.

Weispfenning94. In the first numerical experiment, the capability of the algorithm is tested for a 0-dimensional ideal which consists of 2 polynomials of total degree 5 and a polynomial of degree 4 in 3 variables. The numerical approach speeds up the calculation of the H-bases by a factor of 96.

$$\begin{cases} f_1 : & xy^2z + y^4 + x^2 - 2xy + y^2 + z^2 \\ f_2 : & -x^3y^2 + xyz^3 + xy^2z + y^4 - 2xy \\ f_3 : & xy^4 + yz^4 - 2x^2y - 3 \end{cases}$$

Weispfenning94	ε	time	H-basis	d_{max}	<i>bound</i>
SYMBOLIC	-	2507	6	6	18
NUMERICAL	10^{-12}	26.3	11		
	10^{-10}	26.3	11	6	18
	10^{-8}	26.7	11		

Liu. For the second numerical experiment, we consider a one dimensional polynomial system consists of 4 polynomials in 5 variables of total degree 2.

$$\begin{cases} f_1 : & yz - zw - x + u \\ f_2 : & -xy + yw - z + u \\ f_3 : & xw - zw - y + u \\ f_4 : & -xy + xz - w + u \end{cases}$$

Liu	ε	time	H-basis	d_{max}	bound
SYMBOLIC	-	526	5	2	8
NUMERICAL	10^{-12}	13.44	5	2	8
	10^{-10}	13.6	5		
	10^{-8}	13.26	5		

Gerdt2. In the third numerical experiment, we consider a three dimensional polynomial system that consists of 2 polynomials in 5 variables of degree 4.

$$\begin{cases} f_1 & 5xy^3 - 140y^3z - 3x^2y + 45xyz + 210y^2w - 420yz^2 - 25xw + 126yu + 70zw \\ f_2 & 35y^4 - 30xy^2 - 210y^2z + 3x^2 + 30xz + 140yw - 105z^2 - 21u \end{cases}$$

Gerdt2	ε	time	H-basis	d_{max}	bound
SYMBOLIC	-	3840	6	6	12
NUMERICAL	10^{-10}	-	constant	6	12
	10^{-9}	165	6		
	10^{-6}	165	6		

The following tables show further results of successful runs. The polynomial system Schwartz can be found in [7] and Lazard-Mora is defined in [14]. All the other polynomial systems in this section and the next section are available at the aforementioned website.

Lorentz	ε	time	H-basis
SYMBOLIC	-	6s	5
NUMERICAL	10^{-10}	3.05	5

Conform1	ε	time	H-basis
SYMBOLIC	-	19	10
NUMERICAL	10^{-10}	1.82	10

Redeco5	ε	time	H-basis
SYMBOLIC	-	252	5
NUMERICAL	10^{-10}	34.7	5

Noon	ε	time	H-basis
SYMBOLIC	-	7660	7
NUMERICAL	10^{-10}	172.1	9

Schwartz	ε	time	H-basis
SYMBOLIC	-	18672	6
NUMERICAL	10^{-10}	183.3	6

Lazard-Mora	ε	time	H-basis
SYMBOLIC	-	> 2 hours	3
NUMERICAL	10^{-10}	338.2	3

5.3 Ill-conditioned ideals

Next, we show some examples where numerical ill-conditioning occurred and discuss the reasons for failure in some more detail.

Sendra. In this example we illustrate the failure of the numerical algorithm to find an H-basis because of small remainders. The polynomial system consists of 2 polynomials in two variables of total degree 7 with Krull-dimension 0.

Sendra	ε	time	H-basis	d_{max}	$bound$
SYMBOLIC	-	6	4	12	22
NUMERICAL	10^{-10}	-	constant	-	-
	10^{-4}	1.15	4	12	22

The algorithm starts with $k = 7$ but first syzygies appear in $k = 13$. $C_{13}(F)$ is a 14×14 matrix with density 28% and singular values

$$\sigma_{13} = 0.364 > \tau \approx 10^{-11} > 1.7 \times 10^{-11} = \sigma_{14}.$$

Thus, $\text{rank } C_{13}(F) = 13$ and $\dim \mathcal{N}(C_{13}(F)) = 1$. The corresponding syzygy is reduced to the non-zero remainder of degree 11 and 2-norm 4.039×10^3 . The second non-zero remainder is detected in $k = 12$, where $C_{12}(F)$ is a 13×14 matrix with density 38%, the tolerance and singular values

$$\sigma_{12} = 0.029 > \tau \approx 10^{-11} > 1.14 \times 10^{-13} = \sigma_{13}.$$

The numerical rank is determined by the default tolerance as 12. Therefore, the null space satisfies $\dim \mathcal{N}(C_{12}(F)) = 2$ and the corresponding polynomial to this new syzygy gives a non-zero remainder of total degree 11 and 2-norm 9.37×10^4 . The third step of the algorithm is started with $k = 11$, and the first new syzygies are appeared in $k = 12$. $C_{12}(F)$ is a 13×16 matrix with $\dim \mathcal{N}(C_{12}(F)) = 3$. The exact computations in Maple show that their corresponding polynomials are reduced to zero. However in Matlab we have a non-zero remainder of degree 10 and 2-norm 1.744×10^{-6} . Continuing this process in the next steps results a very small non-zero remainder in each step but still greater than threshold ε , so that the algorithm ends up with a constant non-zero remainder -1.36×10^{-10} . To get ride of this obstacle we increased the ε from 10^{-10} up to 10^{-4} and observed that the algorithm terminates in two steps with a correct H-basis.

Remark 5.1 It is worth mentioning that if we divide each polynomial in the initial polynomial system as well as the obtained remainders in each step by 1-norm, 2-norm and ∞ -norm, then all of small remainders will be vanished and we will obtain a correct H-basis in two steps for $\varepsilon = 10^{-9}$. It means that in case that the algorithm fails to get a correct H-basis due to the small remainders, normalizing the polynomials can counteract the bad affect of small reminders by minifying the coefficients of underlying polynomials in the reduction process and give the correct result.

Caprasse4. The polynomial system so-called Caprasse4 demonstrates the failure of rank-revealing for a default numerical tolerance. Our observations show that the failure of rank-revealing during one of the iterations destroys the result of all of the

consequent iterations in a way that the algorithm ends up with very small non-zero constant which is not an acceptable H-basis. This polynomial system consists of 4 polynomials of degrees 3,3,4,4 on 4 variables with Krull-dimension 0.

For the first observation we suppose that $\varepsilon = 10^{-10}$ and observe that the first failure of rank-revealing occurs in step = 4 (each step is whenever a new non-zero remainder is added to the candidate set) at $k = 5$. Inspecting $\sigma_{37} = 0.0017$ and $\sigma_{38} = 1.56 \times 10^{-12}$ shows that the numerical rank should be 37 instead of 40 although the default tolerance is 8.53×10^{-14} . It means that $\dim \mathcal{N}(C_5(F)) = 0$ and hence $\dim \text{Syz}_5(1f(F)) = 0$. The symbolic implementation however shows that the rank is 37 and $\dim \mathcal{N}(C_5(F)) = 3$. In spite of this numerical rank, the polynomials corresponding to these three detected syzygy in Maple algorithm is reduced to zero. At $k = 6$ the numerical rank is estimated to be 79. While the approx-rank gap is maximised at 74 with $\sigma_{74}/\sigma_{75} = 1.71 \times 10^8$. It implies $\dim \text{Syz}_6(1f(F)) = 11$ with a non-zero remainder of 2-norm 3.05×10^{-10} . If we let the execution of the algorithm continues, very small non-zero remainders detected in the next steps (of 2-norm almost 3.8×10^{-9}) make the algorithm to end up with a non-zero but very small constant.

To release the affect of small non-zero remainders and exploring the impact of wrong rank-revealing we increased the threshold to $\varepsilon = 10^{-7}$. As we expect we will not have any non-zero remainder in step = 4 at $k = 6$ and algorithm proceeds with $k = 7$. Having syzygies at $k = 6$, a 40×44 matrix A of extended syzygies is generated. It's singular values show that the rank-revealing determines the numerical rank correctly to be 44. The approx-rank gap $\sigma_{110}/\sigma_{111} = 2.85 \times 10^9$ of matrix B however shows that another wrong rank-revealing has occurred since the numerical rank is estimated to be 112. Regardless of fault of rank-revealing both symbolic and numerical algorithms find a non-zero remainder of total degree 5. The non-zero remainder computed in Maple has 2-norm 17.06 while 2-norm of non-zero remainder of Matlab implementation is 0.34. So, step 5 starts from $k = 5$ with the same values for singular values and tolerance as reported for $\varepsilon = 10^{-10}$. For $k = 6$ approx-rank gap at 78 is $\sigma_{78}/\sigma_{79} = 2.96 \times 10^8$ which illustrates the numerical rank should be 78 but it is estimated to be 79 instead. Here all of new detected syzygies are reduced to zero. In symbolic implementation though a non-zero remainder of degree 4 is detected. Tracing the algorithm in the next steps show that the next non-zero remainder is of total degree 5 which is detected at $k = 7$. In the next steps of the algorithm the small remainders are appeared such that again the algorithm is ended up with a non-zero small remainder.

The above observations show that even a minor error in the rank-revealing in one step is caused the magnitude faults in the following steps and eventually the wrong result. This comes back to the sensitivity of the floating numbers as well as the rank-revealing. In the case that there is a large gap in singular values at index k but the tolerance is larger than σ_k the other rank-revealing methods such as L-curve analysis are the better choice to determine the numerical rank cf. [11].

6 Conclusion

We have presented a numerical stable algorithm to compute H-bases. The approach benefits the SVD to compute a minimal generating set for the syzygy of a given degree k of the underlying ideal which speeds up computing H-bases dramatically. This considerable achievement in run time comes at the cost of losing an accurate H-basis in some examples because of the rank-revealing fault or very small remainders due to the floating numbers property. But these kind of ideals do not appear frequently in practice.

It is worthwhile to be mentioned that as the proposed approach computes a minimal generating set for syzygies in each degree, it can be applied to compute the second syzygies, third syzygies and etc. It means that for a given ideal we can make the finite chain of syzygies (with the length at most equals to the number of variables), so-called free resolutions which are very important tools in commutative and computational algebra and releases important invariants such as Betti numbers, Hilbert regularity and Krull-dimension, cf. [20]. To deal with this issue needs very precise discussion which is left to the future for further discussion.

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