

ON THE DIMENSION DATUM OF A SUBGROUP. II.

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ABSTRACT. This paper studies three aspects around dimension datum: (1), a generalization of the dimension datum, which we call the τ -dimension datum; (2), dimension data of disconnected subgroups; (3), compactness of isospectral sets of normal homogeneous spaces. In the first aspect, we find interesting examples of pairs (H_i, τ_i) ($i = 1, 2$) of subgroups and irreducible representations with the same τ -dimension datum, and give an application in constructing isospectral hermitian vector bundles. We show that τ -dimension data for all linear characters together determine the image of the homomorphism from a connected subgroup up to isomorphism. In the second aspect, we express dimension datum in terms of *characters* supported on *maximal commutative connected subsets*, and give formulas for these characters using data associated to *affine root systems*. In the last aspect, we show that any compact semisimple Lie group has only finitely many possible normal homogeneous quotients up to diffeomorphism. We also list some open questions about dimension datum and τ -dimension datum.

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1. INTRODUCTION

Dimension datum is first seriously studied by Larsen and Pink ([11]), motivated by its possible application in arithmetic geometry ([7], [12]). A striking result they showed is that any semisimple closed subgroup is determined by its dimension datum up to isomorphism. Recently dimension datum catches more attention due to its critical role in Langlands' program of beyond endoscopy (cf. [9] and [2]). Motivated by this, we started a systematic study of dimension data of connected closed subgroups which are not necessarily semisimple in [1]. A particular important finding in [1] is a family of non-isomorphic connected closed subgroups which have the same dimension datum. This is a new feature for dimension data outside the semisimple subgroups case, and it has an interesting application in constructing isospectral manifolds. Further in [17], we classified connected closed subgroups with the same dimension datum, and we also classified connected closed subgroups with linearly dimension data in some sense. This could be viewed as a nearly complete understanding for dimension data of connected closed subgroups in a compact Lie group. In another aspect, topological properties of dimension datum (finiteness, rigidity, Coxeter number) are studied in [1] and [10], and stronger results with different proofs are shown in [18].

The goal of this paper is to extend the study of dimension datum in three aspects. The first is to study a more general notion of τ -dimension datum. We show interesting examples of different pairs (H_i, τ_i) with the same τ -dimension datum, and use them to construct isospectral hermitian vector bundles. We also show that τ -dimension data for all linear characters suffices to determine the image of the homomorphism from a connected compact Lie group up to isomorphism. The second is to study dimension data of disconnected subgroups. For this we set up a strategy by studying characters associated to affine root systems. The third is to further study compactness of isospectral sets of normal homogeneous spaces. In every aspect the study is not complete yet, major remaining questions are asked in Questions 2.2, 3.1, 4.1. Moreover, in the last section, we propose some other open questions about dimension datum and τ -dimension datum.

Notation and conventions. For a compact Lie group G , write G^0 for the neutral subgroup containing $e \in G$; write \mathfrak{g}_0 for the Lie algebra of G ; write $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ for the complexified Lie algebra of G .

For a connected compact Lie group G , write G^s (or G_{der}) for the derived subgroup, i.e.,

$$G^s = G_{der} = [G, G].$$

For a compact abelian group S , write

$$X^*(S) = \text{Hom}(S, \text{U}(1))$$

for the character group of S . In case S is connected, i.e., it is a torus, write

$$\Lambda_S = X^*(S).$$

It is also called the *weight lattice* of S . Write

$$X_*(S) = \text{Hom}(U(1), S)$$

for the cocharacter group of a torus S .

2. τ -DIMENSION DATUM

Let G be a compact Lie group. For a closed subgroup H of G and an irreducible finite-dimensional complex linear representation τ of H , let $\mathcal{D}_{H,\tau}$ denote the map from \widehat{G} to \mathbb{Z} ,

$$\mathcal{D}_{H,\tau} : \rho \mapsto \dim \text{Hom}_H(\tau, \rho|_H),$$

where \widehat{G} is the set of equivalence classes of irreducible finite-dimensional complex linear representations of G and $\text{Hom}_H(\tau, \rho|_H)$ is the space of H -equivariant linear maps from τ to $\rho|_H$. We call $\mathcal{D}_{H,\tau}$ the τ -*dimension datum* of H . When $\tau = 1$, $\mathcal{D}_{H,\tau} = \mathcal{D}_H$ is the *dimension datum* of H ([11], [1], [17]). In [1] and [17], we have constructed non-isomorphic connected closed subgroups with the same dimension datum, e.g. two closed subgroups H_1, H_2 in $G = \text{SU}(4n+1)$ with $H_1 \cong U(2n+1)$ and $H_2 \cong \text{Sp}(n) \times \text{SO}(2n+2)$. In [17], we made a classification of connected closed subgroups with the same dimension datum or with linearly-dependent dimension data. The dimension datum problem ([17]) asks: to what extent is H (up to G -conjugacy) determined by its dimension datum \mathcal{D}_H ? Analogously, we could ask the following question.

Question 2.1. *To what extent is a pair (H, τ) (up to G -conjugacy) determined by the τ -dimension datum $\mathcal{D}_{H,\tau}$?*

It is also interesting to construct non-isomorphic connected closed subgroups H_1, H_2 in a compact Lie group G and non-trivial irreducible representations $\tau_1 \in \widehat{H}_1$ and $\tau_2 \in \widehat{H}_2$ such that $\mathcal{D}_{H_1,\tau_1} = \mathcal{D}_{H_2,\tau_2}$, and to study linear relations among τ -dimension data.

In this section we present a family of tuples $(G, H_1, \tau_1, H_2, \tau_2)$ ($\tau_1 \neq 1$, $\tau_2 \neq 1$) such that $\mathcal{D}_{H_1,\tau_1} = \mathcal{D}_{H_2,\tau_2}$, and we give a short remark for what we know about Question 2.1. This example has an application in constructing isospectral hermitian vector bundles. In the last subsection, we show that the τ -dimension data for all linear together determine the image up to isomorphism for a homomorphism from a given connected compact Lie group to a connected compact Lie group.

2.1. An interesting example. In $G = \text{SU}(4n+2)$, set

$$H_1 = \{A, \overline{A} : A \in U(2n+1)\},$$

$$H_2 = \{(A, B) : A \in \text{Sp}(2n), B \in \text{SO}(2n+2)\}.$$

Then, $H_1 \cong U(2n+1)$, $H_2 \cong \text{Sp}(n) \times \text{SO}(2n+2)$. For a sequence of integers $a_1 \geq a_2 \geq \cdots \geq a_{2n+1}$ with $a_i + a_{2n+2-i} = 0$ for any i , $1 \leq i \leq n$, write $\lambda = (a_1, a_2, \dots, a_{2n+1})$ for a weight of $H_1 \cong U(2n+1)$. Write $\lambda_1 = (a_1, \dots, a_n)$ for a weight of $\text{Sp}(2n)$, $\lambda_2 = (a_1, \dots, a_{n+1})$ for a weight of $\text{SO}(2n+2)$, and

$\lambda' = (\lambda_1, \lambda_2)$ for a weight of H_2 . Write τ_λ ($\tau_{\lambda'}$) for an irreducible representation of H_1 (of H_2) with highest weight λ (λ').

Theorem 2.1. *In the above setting, $\mathcal{D}_{H_1, \tau_\lambda} = \mathcal{D}_{H_2, \tau_{\lambda'}}$.*

Note that, when $a_1 = a_2 = \cdots = a_{2n+1} = 0$, Theorem 2.1 is just [1, Theorem 1.5(1)].

The proof of Theorem 2.1 consists of three steps.

Step 1, root systems, characters, and dimension datum.

Let T be a closed torus in G . Write Λ_T for the weight lattice of T . Write

$$\Gamma^\circ = N_G(T)/Z_G(T).$$

Choose a biinvariant Riemannian metric on G . Restricting to T it gives a positive definite inner product on the Lie algebra of T , hence also induces a positive definite inner product on Λ_T , both are Γ° invariant. Let (\cdot, \cdot) denote the induced inner product on Λ_T . As in [17, Definition 2.2], a root system in Λ_T is a finite subset Φ of Λ_T satisfying the following conditions,

- (1) For any two roots $\alpha \in \Phi$ and $\beta \in \Phi$, the element $\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$.
- (2) (**Strong integrality**) For any root α and any vector $\lambda \in \Lambda_T$, the number $\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$ is an integer.

Let Ψ'_T be the root system in Λ_T generated by root systems $\Phi(H, T)$ where H run over connected closed subgroups of G with T a maximal torus of H . It is clear that Ψ'_T is Γ° stable. By [17, Corollary 3.4], Ψ'_T is the union of root systems $\Phi(H, T)$ where H run over connected closed subgroups of G with T a maximal torus of H . Hence, $W_{\Psi'_T} \subset \Gamma^\circ$. Choose (and fix) a system of positive roots Ψ'^+_T .

Analogous to [17, Definition 3.5], we introduce some characters. For a root system Φ in Λ_T , set

$$\delta_\Phi = \frac{1}{2} \sum_{\alpha \in \Phi \cap \Psi'^+_T} \alpha.$$

For a root system Φ in Λ_T and a weight $\lambda \in \Lambda_T$ which is dominant and integral for Φ , set

$$A_{\Phi, \lambda} = \sum_{w \in W_\Phi} \text{sgn}(w) [\lambda + \delta_\Phi - w\delta_\Phi] \in \mathbb{Q}[\Lambda_T].$$

For a finite group W between W_Φ and Γ° , set

$$F_{\Phi, \lambda, W} = \frac{1}{|W|} \sum_{\gamma \in W} \gamma(A_{\Phi, \lambda}) \in \mathbb{Q}[\Lambda_T].$$

For a weight $\lambda \in \Lambda_T$ and a finite subgroup W of Γ° , set

$$\chi_{\lambda, W}^* = \frac{1}{|W|} \sum_{\gamma \in W} [\gamma\lambda] \in \mathbb{Q}[\Lambda_T].$$

Proposition 2.2. *If $\mathcal{D}_{H_1, \tau_1} = \mathcal{D}_{H_2, \tau_2}$ for two connected closed subgroups H_1, H_2 and irreducible representations $\tau_1 \in \widehat{H_1}$ and $\tau_2 \in \widehat{H_2}$, then H_1 and H_2 have conjugate maximal tori.*

Assume that T is a maximal torus of both H_1 and H_2 , write $\Phi_i \subset \Lambda_T$ for the root system of H_i ($i = 1, 2$). Then, $\mathcal{D}_{H_1, \tau_1} = \mathcal{D}_{H_2, \tau_2}$ if and only if

$$F_{\Phi_1, \lambda_1, \Gamma^\circ} = F_{\Phi_2, \lambda_2, \Gamma^\circ},$$

where $\lambda_i \in \Lambda_T$ is highest weight of τ_i ($i = 1, 2$).

Proof. This is analogous to Propositions 3.7 and 3.8 in [17]. Based on the proof given there, the only new adding is calculation for $F_\Phi(t)\chi_\lambda(t)$, where H is a connected closed subgroup of G with T a maximal torus of H , $\Phi \subset \Lambda_T$ is root system of H , F_Φ is Weyl product of H , and χ_λ is the character of an irreducible representation of H with highest weight λ . The calculation goes as follows,

$$\begin{aligned} |W_\Phi| F_\Phi(t)\chi_\lambda(t) &= \chi_\lambda \prod_{\alpha \in \Phi} (1 - [\alpha]) \\ &= \prod_{\alpha \in \Phi^+} \left(\left[\frac{-\alpha}{2} \right] - \left[\frac{\alpha}{2} \right] \right) (\chi_\lambda \prod_{\alpha \in \Phi^+} \left(\left[\frac{\alpha}{2} \right] - \left[\frac{-\alpha}{2} \right] \right)) \\ &= \left(\sum_{w \in W_\Phi} \text{sgn}(w) [-w\delta] \right) \left(\sum_{\tau \in W_\Phi} \text{sgn}(\tau) [\tau(\lambda + \delta)] \right) \\ &= \sum_{w, \tau \in W} \text{sgn}(w) \text{sgn}(\tau) [-w\delta + \tau(\lambda + \delta)] \\ &=_{w \rightarrow \tau w} \sum_{\tau \in W_\Phi} \tau \left(\sum_{w \in W_\Phi} \text{sgn}(w) [\lambda + \delta - w\delta] \right) \\ &= |W_\Phi| F_{\Phi, \lambda, W_\Phi}. \end{aligned}$$

In the above $\delta = \delta_\Phi$. Due to $W_\Phi \subset \Gamma^0$, we also note that

$$\frac{1}{|\Gamma^0|} \sum_{\gamma \in \Gamma^0} \gamma \cdot F_{\Phi, \lambda, W_\Phi} = F_{\Phi, \lambda, \Gamma^\circ}.$$

□

In the situation of Theorem 2.1, write

$$T = \{\text{diag}\{a_1, \dots, a_{2n+1}, a_1^{-1}, \dots, a_{2n+1}^{-1}\} : |a_1| = \dots = |a_{2n+1}| = 1\}.$$

Then, T is conjugate to maximal tori of H_1 and H_2 . By calculation we see that

$$\Psi'_T = \text{BC}_{2n+1}$$

and

$$\Gamma^\circ = W_{\text{BC}_{2n+1}} = \{\pm 1\}^{2n+1} \rtimes S_{2n+1}.$$

Substituting H_2 by a conjugate subgroup H'_2 , we may assume that T is a maximal torus of both H_1 and H'_2 , and identify the root system of H_1 (of H'_2)

with the sub-root system A_{2n} ($C_n \sqcup D_{n+1}$) of $\Psi'_T = BC_{2n+1}$. By Proposition 2.2, the conclusion of Theorem 2.1 is equivalent to

$$(1) \quad F_{A_{2n}, \lambda, W_{BC_{2n+1}}} = F_{C_n \sqcup D_{n+1}, \lambda', W_{BC_{2n+1}}}.$$

Step 2, sub-root systems of $\Psi = BC_n$, polynomials $a_n(\lambda)$, $b_n(\lambda)$, $c_n(\lambda)$, $d_n(\lambda)$ and their multiplicative relations.

There is a good idea in [11] which transfers characters $F_{\Phi, 0, W_{BC_n}}$ into polynomials. In [1] and [17], we further find matrix expression for the resulting polynomials. Here, we extend these to the characters $F_{\Phi, \lambda, W_{BC_n}}$.

Following [17, Section 7], we briefly recall the idea of [11] which identifies the direct limit of character groups with polynomial ring. Set

$$\begin{aligned} \mathbb{Z}^n &:= \mathbb{Z} BC_n = \Lambda_{BC_n} = \text{span}_{\mathbb{Z}}\{e_1, e_2, \dots, e_n\}, \\ W_n &:= \text{Aut}(BC_n) = W_{BC_n} = \{\pm 1\}^n \rtimes S_n, \\ \mathbb{Z}_n &:= \mathbb{Q}[\mathbb{Z}^n], \\ Y_n &:= \mathbb{Z}_n^{W_n}. \end{aligned}$$

For $m \leq n$, the injection

$$\mathbb{Z}^m \hookrightarrow \mathbb{Z}^n : (a_1, \dots, a_m) \mapsto (a_1, \dots, a_m, 0, \dots, 0)$$

extends to an injection $i_{m,n} : \mathbb{Z}_m \hookrightarrow \mathbb{Z}_n$. Define $\phi_{m,n} : \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ by

$$\phi_{m,n}(z) = \frac{1}{|W_n|} \sum_{w \in W_n} w(i_{m,n}(z)).$$

Thus $\phi_{m,n} \phi_{k,m} = \phi_{k,n}$ for any $k \leq m \leq n$ and the image of $\phi_{m,n}$ lies in Y_n . Hence $\{Y_m : \phi_{m,n}\}$ forms a direct system and we define

$$Y = \lim_{\rightarrow n} Y_n.$$

Define the map $j_n : \mathbb{Z}_n \rightarrow Y$ by composing $\phi_{n,p}$ with the injection $Y_p \hookrightarrow Y$. The isomorphism $\mathbb{Z}^m \oplus \mathbb{Z}^n \rightarrow \mathbb{Z}^{m+n}$ gives a canonical isomorphism $M : \mathbb{Z}_m \otimes_{\mathbb{Q}} \mathbb{Z}_n \rightarrow \mathbb{Z}_{m+n}$. Given two elements of Y represented by $y \in Y_m$ and $y' \in Y_n$ we define

$$yy' = j_{m+n}(M(y \otimes y')).$$

This product is independent of the choice of m and n and makes Y a commutative associative algebra.

The monomials $[e_1]^{k_1} \cdots [e_n]^{k_n}$ ($k_1, k_2, \dots, k_n \in \mathbb{Z}$) form a \mathbb{Q} basis of \mathbb{Z}_n , where $[e_i]^{k_i} = [k_i e_i] \in \mathbb{Z}_1$ is a linear character. Hence Y has a \mathbb{Q} basis

$$e(k_1, k_2, \dots, k_n) = j_n([e_1]^{k_1} \cdots [e_n]^{k_n})$$

indexed by $n \geq 0$ and $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$. Mapping $e(k_1, k_2, \dots, k_n)$ to $x_{k_1} x_{k_2} \cdots x_{k_n}$, we get a \mathbb{Q} linear map

$$E : Y \rightarrow \mathbb{Q}[x_0, x_1, \dots, x_n, \dots].$$

This map E is an algebra isomorphism. Here $x_0 = 1$ and write x_0 for notational convenience. For any $k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$ and $\lambda = k_1 e_1 + k_2 e_2 + \cdots + k_n e_n$,

$$\chi_{\lambda, W_n}^* = e(k_1, k_2, \dots, k_n) \in Y$$

and

$$E(\chi_{\lambda, W_n}^*) = x_{k_1} x_{k_2} \cdots x_{k_n}.$$

Given $f \in \mathbb{Q}[x_0, x_1, \dots]$, set

$$\sigma(f)(x_0, x_1, \dots, x_{2n}, x_{2n+1}, \dots) = f(x_0, -x_1, \dots, x_{2n}, -x_{2n+1}, \dots).$$

Then, σ is an involutive automorphism of $\mathbb{Q}[x_0, x_1, \dots]$.

Write $a_n(\lambda)$, $b_n(\lambda)$, $c_n(\lambda)$, $d_n(\lambda)$ for the image of $j_n(F_{\Phi, \lambda, W_n})$ under E for $\Phi = A_{n-1}$, B_n , C_n or D_n , and a weight $\lambda \in \mathbb{Z}^n$. Observe that $a_n(\lambda)$, $b_n(\lambda)$, $c_n(\lambda)$, $d_n(\lambda)$ are homogeneous polynomials of degree n with integer coefficients. Write $b'_n(\lambda) = \sigma(b_n(\lambda))$.

For $n \geq 1$ and a weight $\lambda = a_1 e_1 + \cdots + a_n e_n$, define the matrices

$$\begin{aligned} A_n(\lambda) &= (x_{|a_j+i-j|})_{n \times n}, \\ B_n(\lambda) &= (x_{|a_j+i-j|} - x_{|a_j+2n+1-i-j|})_{n \times n}, \quad B'_n(\lambda) = (x_{|a_j+i-j|} + x_{|a_j+2n+1-i-j|})_{n \times n}, \\ C_n(\lambda) &= (x_{|a_j+i-j|} - x_{|a_j+2n+2-i-j|})_{n \times n}, \quad D_n(\lambda) = (x_{|a_j+i-j|} + x_{|a_j+2n-i-j|})_{n \times n}, \\ D'_n(\lambda) &= (y_{i,j})_{n \times n}, \end{aligned}$$

where $y_{i,j} = x_{|a_j+i-j|} + x_{|a_j+2n-i-j|}$ if $i, j \leq n-1$, $y_{n,j} = \sqrt{2}x_{|a_j+n-j|}$, $y_{i,n} = \frac{\sqrt{2}}{2}(x_{|a_n+i-n|} + x_{|a_n+n-i|})$ and $y_{n,n} = x_{|a_n|}$.

Lemma 2.3. *We have*

$$\begin{aligned} \det A_n(\lambda) &= a_n(\lambda), \quad \det B_n(\lambda) = b_n(\lambda), \quad \det B'_n(\lambda) = b'_n(\lambda), \\ \det C_n(\lambda) &= c_n(\lambda), \quad \frac{1}{2} \det D_n(\lambda) = \det D'_n(\lambda) = d_n(\lambda). \end{aligned}$$

Proof. For $\Phi = A_{n-1}$, B_n , C_n , these follow from comparing $E(j_n([\lambda + \delta - w\delta]))$ ($w \in W_\Phi$) with terms in the expansion of $\det A_n(\lambda)$, $\det B_n(\lambda)$, $\det C_n(\lambda)$. Applying the involutive automorphism σ , we get $\det B'_n(\lambda) = b'_n(\lambda)$. For $\Phi = D_n$, we define a new character $\epsilon' : W_n \rightarrow \{1\}$ by $\epsilon'|_{W_{D_n}} = \epsilon|_{W_{D_n}}$, and $\epsilon'(s_{e_1}) = 1$ (rather than $\epsilon(s_{e_1}) = -1$). Due to $s_{e_n}(\delta_{D_n}) = \delta_{D_n}$, we have

$$F_{D_n, \lambda, W_{D_n}} = \frac{1}{2} \sum_{w \in W_n} \epsilon'(w)[\lambda + \delta - w\delta].$$

From this, we get

$$\frac{1}{2} \det D_n(\lambda) = d_n(\lambda).$$

Apparently, $\frac{1}{2} \det D_n(\lambda) = \det D'_n(\lambda)$. □

Proposition 2.4. *Assume $a_1 \geq a_2 \cdots \geq a_n \geq 0$, then each of $b_n(\lambda)$, $b'_n(\lambda)$, $c_n(\lambda)$, is an irreducible polynomial. Assume $a_1 \geq a_2 \cdots \geq |a_n| \geq 0$, then $d_n(\lambda)$ is an irreducible polynomial.*

Proof. For $b_n(\lambda)$, the indeterminate appearing in it with largest index is x_{a_1+2n-1} , and

$$b_n(\lambda) = -x_{a_1+2n-1} b_{n-1}(\lambda') + \cdots,$$

where $\lambda' = (a_2, \dots, a_n)$. By induction, $b_{n-1}(\lambda')$ is irreducible. Apparently, $b_{n-1}(\lambda')$ does not divide $b_n(\lambda)$, hence $b_n(\lambda)$ is irreducible. Similarly, $b'_n(\lambda)$, $c_n(\lambda)$, $d_n(\lambda)$ are irreducible. \square

Proposition 2.5. *If $n = 2m + 1$ is odd, $a_1 \geq a_2 \cdots \geq a_n$, and $a_{n+1-i} + a_i = 0$, $\forall i$, $1 \leq i \leq m$, then*

$$a_{2m+1}(\lambda) = c_m(\lambda_1)d_{m+1}(\lambda_2),$$

where $\lambda_1 = (a_1, \dots, a_m)$, $\lambda_2 = (a_1, \dots, a_{m+1})$.

If $n = 2m$ is even, $a_1 \geq a_2 \cdots \geq a_n \geq 0$, and $a_{n+1-i} + a_i = 0$, $\forall i$, $1 \leq i \leq m$, then

$$a_{2m}(\lambda) = b_m(\lambda_1)b'_m(\lambda_2),$$

where $\lambda_1 = \lambda_2 = (a_1, \dots, a_m)$.

For $n \geq 1$, assume $a_1 \geq a_2 \cdots \geq a_n$, then $a_n(\lambda)$ is reducible only if $a_i + a_{n+1-i} = 0$ for any i , $1 \leq i \leq n/2$.

Proof. Define L_m inductively by $L_1 = 1$ and

$$L_m = \begin{pmatrix} & & 1 \\ & L_{m-2} & \\ 1 & & \end{pmatrix}$$

for any $m \geq 2$. Then, $L_m^2 = I$. The matrix $A_{2m}(\lambda)$ is of the form

$$\begin{pmatrix} X & Y \\ L_m Y L_m & L_m X L_m \end{pmatrix},$$

where X, Y are two $m \times m$ matrices. By calculation we have

$$\begin{aligned} & \frac{1}{2} \begin{pmatrix} I & L_m \\ -L_m & I \end{pmatrix} \begin{pmatrix} X & Y \\ L_m Y L_m & L_m X L_m \end{pmatrix} \begin{pmatrix} I & -L_m \\ L_m & I \end{pmatrix} \\ &= \begin{pmatrix} X + Y L_m & 0 \\ 0 & L_m X L_m - L_m Y \end{pmatrix}. \end{aligned}$$

One can check that $X + Y L_m$ ($X - Y L_m$) is just the matrix $B'_m(\lambda_2)$ ($B_n(\lambda_1)$), thus

$$a_{2m}(\lambda) = b_m(\lambda_1)b'_m(\lambda_2).$$

The matrix $A_{2m+1}(\lambda)$ is of the form

$$\begin{pmatrix} X & \beta^t & Y \\ \alpha & z & \alpha L_m \\ L_m Y L_m & \gamma^t & L_m X L_m \end{pmatrix},$$

where X, Y are two $m \times m$ matrices, α, β, γ are $1 \times m$ vectors. By calculation we have

$$\begin{aligned} & \frac{1}{2} \begin{pmatrix} I & & L_m \\ & \sqrt{2} & \\ -L_m & & I \end{pmatrix} \begin{pmatrix} X & \beta^t & Y \\ \alpha & z & \alpha L_m \\ L_m Y L_m & \gamma^t & L_m X L_m \end{pmatrix} \begin{pmatrix} I & & -L_m \\ & \sqrt{2} & \\ L_m & & I \end{pmatrix} \\ &= \begin{pmatrix} X + Y L_m & \frac{\sqrt{2}}{2}(\beta^t + L_m \gamma^t) & 0 \\ \sqrt{2}\alpha & z & 0 \\ 0 & \frac{\sqrt{2}}{2}(-L_m \beta^t + \gamma^t) & L_m X L_m - L_m Y \end{pmatrix}. \end{aligned}$$

The matrix

$$\begin{pmatrix} X + YL_m & \frac{\sqrt{2}}{2}(\beta^t + L_m\gamma^t) \\ \sqrt{2}\alpha & z \end{pmatrix}$$

is just $D'_{m+1}(\lambda_2)$, and the matrix $X - YL_m$ is just $C_m(\lambda_1)$. Thus,

$$a_{2m+1}(\lambda) = c_m(\lambda_1)d_m(\lambda_2).$$

We prove the last statement by induction on n . When $n = 1$, it is trivial. When $n = 2$, it is easy to check. Now we assume $n \geq 3$ and the statement is true for $1, 2, \dots, n-1$, and show it in the n case. First we show $a_1 + a_n = 0$. Suppose $a_1 + a_n \neq 0$ and $a_n(\lambda)$ is reducible. By symmetry we may assume that $a_1 + a_n \geq 1$. Among the indeterminates appearing in $a_n(\lambda)$, the one of largest index is x_{a_1+n-1} , and

$$a_n(\lambda) = x_{a_1+n-1}a_{n-1}(\lambda') + \dots,$$

where $\lambda' = (a_2 - 1, \dots, a_n - 1)$. Write $a_n(\lambda) = (x_{a_1+n-1}u_1 + v_1)u_2$ with $\deg u_2 \geq 1$. There are two cases according to $\deg u_1 > 0$ or $\deg u_1 = 0$. In the case of $\deg u_1 > 0$, we have $a_{n-1}(\lambda') = u_1u_2$. By induction, $a_i + a_{n+2-i} = 2, \forall i, 2 \leq i \leq n$. In this case, $a_1 + n - 2 \geq a_2 + n - 2 > a_2 + n - 3 = -(a_n + 1 - n)$. The sum of terms in $a_n(\lambda)$ dividing $x_{a_1+n-2}x_{a_2+n-2}$ is $x_{a_1+n-2}x_{a_2+n-2}a_{n-2}(\lambda'')$, where $\lambda'' = (a_2 - 2, \dots, a_n - 2)$. The indeterminates x_{a_1+n-2} and x_{a_2+n-2} can only appear in v_1 . Hence, u_2 is a factor of $a_{n-2}(\lambda'')$. If $n \geq 4$, then $a_i + a_{n+3-i} = 4, \forall i, 3 \leq i \leq n$. This leads to $a_3 > a_2$, hence a contradiction. If $n = 3$, then $u_2 = \pm a_1(\lambda'')$ and $a_1(\lambda'')|a_2(\lambda)$, which does not hold, hence a contradiction. In the case of $\deg u_1 = 0$, we may assume that $u_1 = 1$ and $u_2 = a_{n-1}(\lambda')$. Due to $a_1 + n - 2 \geq \max |a_j + i - j| : 1 \leq i \leq n-1, 2 \leq j \leq n$, then v_1 has a term $\pm x_{a_1+n-2}$. Then, the determinant of the remaining matrix of $A_n(\lambda)$ by removing the last row and the first column (which is equal to $a_{n-1}(\lambda')$) is equal to the determinant of the remaining matrix of $A_n(\lambda)$ by removing the $n-1$ -th row and the first column. This is impossible as their leading terms are different. Now assume $a_1 + a_n = 0$, then

$$a_n(\lambda) = x_{a_1+n-1}x_{-(a_1+1-n)}a_{n-2}(\lambda''') + \dots,$$

where $\lambda''' = (a_2 - 1, \dots, a_{n-1} - 1)$. One can show that $a_{n-2}(\lambda''')$ does not divide $a_n(\lambda)$ (in case $n \geq 4$). Thus, $a_{n-2}(\lambda''')$ is reducible. By induction, $a_i + a_{n+1-i} = 0$ for any $i, 2 \leq i \leq n/2$. \square

Step 3, final conclusion.

By Proposition 2.5,

$$a_{2n+1}(\lambda) = c_n(\lambda_1)d_{n+1}(\lambda_2).$$

This shows Equation 1, hence the conclusion of Theorem 2.1.

2.2. Equalities and linear relations among τ -dimension data. Like in [17, Section 4], Proposition 2.2 leads us to study equalities among characters

$$\{F_{\Phi, \lambda, \text{Aut}(\Psi)} : \Phi \subset \Psi, \lambda \in \Lambda_{\Phi}(\mathbb{Q}\Psi)\}$$

for a given root system Ψ , where the lattices $\mathbb{Q}\Psi$ and $\Lambda_{\Phi}(\mathbb{Q}\Psi)$ are as introduced in [17, Pages 2687-2688]). Similarly, if we want to study linear relations among τ -dimension data for connected subgroups, then we need to study linear relations among the characters $\{F_{\Phi, \lambda, \text{Aut}(\Psi)} : \Phi \subset \Psi, \lambda \in \Lambda_{\Phi}(\mathbb{Q}\Psi)\}$. As in [17, Section 7], both studies may reduce to the case of Ψ is an irreducible root system.

When $\Psi = \text{BC}_n$ and $\lambda \in \mathbb{Z}\Psi$, the equalities/linear relations among $F_{\Phi, \lambda, \Psi}$ correspond to multiplicative/algebraic relations among the polynomials

$$\{a_n(\lambda), b_n(\lambda), c_n(\lambda), d_n(\lambda) : n \geq 1\}.$$

Propositions 2.4 and 2.5 give all such multiplicative relations.

Question 2.2. *Is there a generalization of Theorem [1, Theorem 1.5(2)] to τ -dimension data of more general weights?*

Moreover, can we find all generating relations of algebraic relations among the polynomials $\{a_n(\lambda), b_n(\lambda), c_n(\lambda), d_n(\lambda) : n \geq 1\}$?

When $\Psi = \text{B}_n, \text{C}_n$ or D_n and $\lambda \in \mathbb{Z}_n = \mathbb{Z}\text{BC}_n$, it reduces to the $\Psi = \text{BC}_n$ case, so is for $\Psi = \text{A}_{n-1}$. However, for these root systems, the weight λ is not necessarily $\lambda \in \mathbb{Z}_n$. To have a complete study we need to consider polynomials associated to non-integral weights. We don't know if this arises new complication, or gives new interesting phenomenon.

When Ψ is an exceptional irreducible root system, from

$$A_{\Phi, \lambda} = \sum_{w \in W_{\Phi}} \text{sgn}(w)[\lambda + \delta_{\Phi} - w\delta_{\Phi}]$$

and

$$F_{\Phi, \lambda, \text{Aut}(\Psi)} = \frac{1}{|\text{Aut}(\Psi)|} \sum_{\gamma \in \text{Aut}(\Psi)} \gamma(A_{\Phi, \lambda}),$$

the orbits $\text{Aut}(\Psi)\lambda$ and $\text{Aut}(\Psi)(\lambda + 2\delta_{\Phi})$ are determined, as well as the polynomial

$$\sum_{w \in W_{\Phi}} \text{sgn}(w)t^{|\lambda + \delta_{\Phi} - w\delta_{\Phi}|^2} = t^{|\lambda|^2} \prod_{\alpha \in \Phi^+} (1 - t^{(\lambda + \delta_{\Phi}, 2\alpha)})$$

(cf. [17, Proposition 5.2]). These invariants all involve the weight λ . The only invariant without involving λ which could be extracted from them easily is $\prod_{\alpha \in \Phi^+} (1 - t^{(\lambda + \delta_{\Phi}, 2\alpha)})$, but this seems too weak. We do not know how to use the three invariants in an effective way, so as to find all equalities among $\{F_{\Phi, \lambda, \text{Aut}(\Psi)} : \Phi \subset \Psi, \lambda \in \Lambda_{\Phi}(\mathbb{Q}\Psi)\}$. We have no idea yet for how to study linear relations among the characters $\{F_{\Phi, \lambda, \text{Aut}(\Psi)} : \Phi \subset \Psi, \lambda \in \Lambda_{\Phi}(\mathbb{Q}\Psi)\}$.

2.3. Isospectral hermitian vector bundles. Let H be a closed subgroup of a connected compact Lie group G , and (V_τ, τ) (V_τ is the representation space of $\tau \in \widehat{H}$) be a finite-dimensional irreducible complex linear representation of H . Write $E_\tau = G \times_\tau V_\tau$ for a G -equivariant vector bundle on $X = G/H$ induced from V_τ . As a set, E_τ is the set of equivalence classes in $G \times V_\tau$,

$$(g, v) \sim (g', v') \Leftrightarrow \exists x \in H \text{ s.t. } g' = gx, v' = x^{-1} \cdot v.$$

Write $C^\infty(G/H, E_\tau)$ for the space of smooth sections of E_τ . Then,

$$C^\infty(G/H, E_\tau) = (C^\infty(G, V_\tau))^H,$$

where $C^\infty(G, V_\tau)$ the space of smooth functions $f : G \rightarrow V_\tau$ and H acts on it through

$$(xf)(g) = x \cdot f(gx).$$

The group G acts on $C^\infty(G/H, E_\tau)$ through

$$(g'f)(g) = f(g'^{-1}g).$$

By differentiation, we get an action of $\mathfrak{g}_0 = \text{Lie } G$ on $C^\infty(G/H, E_\tau)$, and so an action of the universal enveloping algebra $U(\mathfrak{g}_0)$ on $C^\infty(G/H, E_\tau)$. Let Δ_τ denote the resulting differential operator on $C^\infty(G/H, E_\tau)$ from the Casimir element $C \in Z(\mathfrak{g}_0) = U(\mathfrak{g}_0)^G$. The action of Δ_τ on $C^\infty(G/H, E_\tau)$ commutes with the action by G , and it is a second order elliptic differential operator.

Choose an H -invariant unitary form $\langle \cdot, \cdot \rangle$ on V_τ (which is unique up to a scalar). It induces a hermitian metric on E_τ and makes it a hermitian vector bundle¹. Define a hermitian pairing (\cdot, \cdot) on $C^\infty(G/H, E_\tau)$ by

$$(f_1, f_2) = \int_{G/H} \langle f_1(g), f_2(g) \rangle d(gH),$$

where $d(gH)$ is a G -equivariant measure on G/H of volume 1. As Δ_τ is an elliptic differential operator, any eigenform of it in $L^2(G/H, E_\tau)$ is a smooth section. By the Peter-Weyl theorem,

$$L^2(G/H, E_\tau) = \bigoplus_{\rho \in \widehat{G}} L^2(G/H, E_\tau)_\rho$$

and the ρ -isotropic subspace, $L^2(G/H, E_\tau)_\rho = \text{Hom}(\tau, \rho|_H) \otimes_{\mathbb{C}} \rho$. Also, we know that Δ_τ acts on the ρ -isotropic component $L^2(G/H, E_\tau)_\rho$ by a scalar determined by ρ . By this, we have the following fact: if $\mathcal{D}_{H_1, \tau_1} = \mathcal{D}_{H_2, \tau_2}$, then the Hermitian vector bundles $E_{\tau_1} = G \times_{\tau_1} V_{\tau_1}$ (on G/H_1) and $E_{\tau_2} = G \times_{\tau_2} V_{\tau_2}$ (on G/H_2) are isospectral. Here, isospectral means the multiplicities of eigenspaces of Δ_{τ_1} and Δ_{τ_2} are the same for any eigenvalue.

Corollary 2.6. *For $G = \text{SU}(4n + 2)$, subgroups H_1, H_2 and representations τ_λ and $\tau_{\lambda'}$ as in Theorem 2.1, the hermitian vector bundles $E_{\tau_\lambda} = G \times_{\tau_\lambda} V_{\tau_\lambda}$ (on G/H_1) and $E_{\tau_{\lambda'}} = G \times_{\tau_{\lambda'}} V_{\tau_{\lambda'}}$ (on G/H_2) are isospectral.*

¹We could define a Laplace-Beltrami operator on $C^\infty(G/H, E_\tau)$ from the hermitian metric on E_τ such that it coincides with Δ_τ up to a scalar.

2.4. Generalization of a theorem of Larsen-Pink. Langlands associated a (conjectural) group H^π to an automorphic form π ([9]). If the Langlands group L_F exists (which is a much bigger conjecture), then H^π should be the image of the Langlands L -homomorphism for π . Besides dimension datum, it is interesting to see if more invariant theory data could determine H^π . The following proposition² concerns this. In case H is semisimple, it is [11, Theorem 1].

Proposition 2.7. *let G, H be connected compact Lie groups, and $f_1, f_2 : H \rightarrow G$ be two homomorphisms. If*

$$\dim((\rho \circ f_1) \otimes \chi)^H = \dim((\rho \circ f_2) \otimes \chi)^H$$

for any $\rho \in \widehat{G}$ and any $\chi \in \mathcal{X}(H) = \text{Hom}(H, \text{U}(1))$, then $f_1(H) \cong f_2(H)$.

Proof of Proposition 2.7. The tors case. To motivate the proof in the general case, we first show Proposition 2.7 in the case of H is a torus. First we show $\ker f_1 = \ker f_2$. Suppose no. Without loss of generality we assume that $\ker f_1 \not\subset \ker f_2$. Then, there exists $\chi \in \mathcal{X}(H)$ such that $\chi|_{\ker f_1} \neq 1$ and $\chi|_{\ker f_2} = 1$. For any $\rho \in \widehat{G}$, $\rho \circ f_1|_{\ker f_1} = 1$, hence $\dim((\rho \circ f_1) \otimes \chi)^H = 0$. As $\chi|_{\ker f_2} = 1$, χ descends to a linear character χ' of $f_2(H) \subset G$. Choose some $\rho \in \widehat{G}$ such that $\rho \subset \text{Ind}_{f_2(H)}^G(\chi'^*)$. Then, $\dim((\rho \circ f_2) \otimes \chi)^H > 0$. This is in contradiction with $\dim((\rho \circ f_1) \otimes \chi)^H = \dim((\rho \circ f_2) \otimes \chi)^H$ for any $\rho \in \widehat{G}$. Thus, we have showed $\ker f_1 = \ker f_2$.

By considering $H/\ker f_1$ instead, we may assume that both f_1 and f_2 are injections. By considering the support of the Sato-Tate measures of $f_1(H)$ and $f_2(H)$, we know that $f_1(H)$ and $f_2(H)$ are conjugate in G ([17, Proposition 3.7]). We may assume that $f_1(H) = f_2(H)$, and denote it by T . Write $\Gamma^\circ = N_G(T)/Z_G(T)$.

We identify H with T through f_1 , and regard f_2 as an automorphism of T , denoted by ϕ . Then, the condition is equivalent to:

$$F_{\emptyset, \chi, \Gamma^\circ} = F_{\emptyset, \phi^*(\chi), \Gamma^\circ}.$$

Which is equivalent to: $\phi^*(\chi) \in \Gamma^\circ \cdot \chi$. We show that $\phi = \gamma|_T$ for some $\gamma \in \Gamma^\circ$. Suppose it is not the case. For any $\gamma \in \Gamma^\circ$, due to $\phi \neq \gamma|_T$,

$$X_\gamma = \{\chi \in \mathcal{X}(H) : \phi^*(\chi) = \gamma \cdot \chi\}$$

is a sublattice of $\mathcal{X}(H)$ with positive corank. Hence,

$$\bigcup_{\gamma \in \Gamma^\circ} X_\gamma \neq \mathcal{X}(H).$$

This is in contradiction with $\phi^*(\chi) \in \Gamma^\circ \cdot \chi$ for any $\chi \in \mathcal{X}(H)$.

The general case. First we show $H_{\text{der}} \ker f_1 = H_{\text{der}} \ker f_2$. Here $H_{\text{der}} = [H, H]$ is the derived subgroup of H . Suppose no. Without loss of generality we assume that $H_{\text{der}} \ker f_1 \not\subset H_{\text{der}} \ker f_2$. Then, there exists $\chi \in \mathcal{X}(H)$ such that

²Proposition 2.7 is proposed by Professor Richard Taylor. The author would like to thank him for discussion about dimension datum and related subjects.

$\chi|_{H_{der} \ker f_1} \neq 1$ and $\chi|_{H_{der} \ker f_2} = 1$. For any $\rho \in \widehat{G}$, $\rho \circ f_1|_{\ker f_1} = 1$, hence $\dim((\rho \circ f_1) \otimes \chi)^H = 0$. As $\chi|_{H_{der} \ker f_2} = 1$, χ descends to a linear character χ' of $f_2(H) \subset G$. Choose some $\rho \in \widehat{G}$ such that $\rho \subset \text{Ind}_{f_2(H)}^G(\chi')$. Then, $\dim((\rho \circ f_2) \otimes \chi)^H > 0$. This is in contradiction with $\dim((\rho \circ f_1) \otimes \chi)^H = \dim((\rho \circ f_2) \otimes \chi)^H$ for any $\rho \in \widehat{G}$. Thus, we have showed

$$H_{der} \ker f_1 = H_{der} \ker f_2.$$

Write $H_i = f_i(H)$. Due to $H/H_{der} \ker f_i \cong H_i/(H_i)_{der}$, we have

$$H_1/(H_1)_{der} \cong H_2/(H_2)_{der}.$$

Choose a maximal torus T_i of H_i ($i = 1$ or 2). Write $(T_i)_s = T_i \cap (H_i)_{der}$. Then,

$$T_i = Z(H_i)^0 \cdot (T_i)_s.$$

Due to $T_i/(T_i)_s \cong H_i/(H_i)_{der}$, we have

$$T_1/(T_1)_s \cong T_2/(T_2)_s.$$

By considering the support of the Sato-Tate measures of H_1 and H_2 , we know that T_1 and T_2 are conjugate in G ([17, Proposition 3.7]). We may assume that $T_1 = T_2$, and denote it by T . Write $\Gamma^\circ = N_G(T)/Z_G(T)$.

Choose a biinvariant Riemannian metric on G , which induces a Γ° invariant inner product on the Lie algebra of T , and also a Γ° invariant inner product on the weight lattice Λ_T . Write $\Phi_1 \subset \Lambda_T$ ($\Phi_2 \subset \Lambda_T$) for root system of H_1 (of H_2). Write

$$X_i = \mathcal{X}(T_i/(T_i)_s) \subset \Lambda_T.$$

Due to $T_1/(T_1)_s \cong T_2/(T_2)_s$, we have an isomorphism $\phi : X_1 \rightarrow X_2$. For any $\chi_1 \in X_1$, write $\chi_2 = \phi(\chi_1)$. Due to $W_{\Phi_i} \subset \Gamma^\circ$ ($i = 1, 2$) and the Γ° invariance the inner product on the Lie algebra of T , the Lie algebra of $Z(H_i)^0$ is orthogonal to the Lie algebra of $(T_i)_s$. By the condition in the question and Proposition 2.2,

$$F_{\Phi_1, \chi_1, \Gamma^\circ} = F_{\Phi_2, \chi_2, \Gamma^\circ}.$$

Due to χ_i is orthogonal to $\delta_{\Phi_i} - w\delta_{\Phi_i}$ for any $w \in W_{\Phi}$, $\chi_{\chi_i, \Gamma^\circ}^*$ is the shortest term in the expansion of $F_{\Phi_i, \chi_i, \Gamma^\circ}$. Thus, $\chi_2 = \gamma \cdot \chi_1$ for some $\gamma \in \Gamma^\circ$. Arguing similarly as in the torus case, one can show that $\phi = \gamma|_{X_1}$ for some $\gamma \in \Gamma^\circ$. By this, we may assume that $\phi = \text{id}$. That is to say, $X_1 = X_2$ and $(T_1)_s = (T_2)_s$. As the Lie algebra of $Z(H_i)^0$ is orthogonal to the Lie algebra of $(T_i)_s$, we also have $Z(H_1)^0 = Z(H_2)^0$. Write $Z = Z(H_i)^0$, $T_s = (T_i)_s$, and $X = X_i$. Write $G' = Z_G(Z)$ and

$$\Gamma' = N_{G'}(T_s)(T_s)/Z_{G'}(T_s).$$

There is an identification³

$$\Gamma' = \{\gamma \in \Gamma^\circ : \gamma|_Z = \text{id}\} = \{\gamma \in \Gamma^\circ : \gamma|_X = \text{id}\}.$$

For any $\gamma \in \Gamma^\circ - \Gamma'$,

$$X_\gamma = \{\chi \in X : \gamma \cdot \chi = \chi\}$$

³Here we use the fact of the Lie algebra of Z is orthogonal to the Lie algebra of T_s .

is a sublattice of positive corank. Thus,

$$\bigcup_{\gamma \in \Gamma^\circ} X_\gamma \neq X.$$

Choose $\chi_0 \in X - \bigcup_{\gamma \in \Gamma^\circ} X_\gamma$. Write

$$c = \min\{|\gamma \cdot \chi_0 - \chi_0| : \gamma \in \Gamma^\circ - \Gamma'\} > 0$$

and

$$c' = \max\{|\delta_{\Phi_2} - w_2 \delta_{\Phi_2} - \gamma(\delta_{\Phi_1} - w_1 \delta_{\Phi_1})| : w_1 \in W_{\Phi_1}, w_2 \in W_{\Phi_2}, \gamma \in \Gamma^\circ\} \geq 0.$$

Take $m \geq 1$ such that $mc > c'$. Put $\chi = m\chi_0$. Then, for $\gamma \in \Gamma^\circ$, $w_1 \in W_{\Phi_1}$, $w_2 \in W_{\Phi_2}$,

$$\gamma(\chi + \delta_{\Phi_1} - w_1 \delta_{\Phi_1}) = \chi + \delta_{\Phi_2} - w_2 \delta_{\Phi_2}$$

if and only if $\gamma \in \Gamma'$ and

$$\gamma(\delta_{\Phi_1} - w_1 \delta_{\Phi_1}) = \delta_{\Phi_2} - w_2 \delta_{\Phi_2}.$$

Then, the equation

$$F_{\Phi_1, \chi, \Gamma^\circ} = F_{\Phi_2, \chi, \Gamma^\circ}$$

implies

$$F_{\Phi_1, 0, \Gamma'} = F_{\Phi_2, 0, \Gamma'}.$$

By [17, Proposition 3.8], this means the subgroups $(H_1)_{der}$ and $(H_2)_{der}$ of G' have the same dimension datum. By [11, Theorem 1], $(H_1)_{der}$ is isomorphic to $(H_2)_{der}$. A more detailed argument using the method in [17] shows that this isomorphism could extend to an isomorphism from H_1 and H_2^4 , hence finish the proof. \square

⁴Shortly to say, the argument goes in this way: define Ψ_{T_s} as in [17, Definition 3.1]. Then, $\Gamma' \subset \text{Aut}(\Psi_{T_s})$. Thus,

$$F_{\Phi_1, 0, \text{Aut}(\Psi_{T_s})} = F_{\Phi_2, 0, \text{Aut}(\Psi_{T_s})}.$$

By this, results in [17, Section 7] imply that $\Phi_2 = \gamma \cdot \Phi_1$ for some $\gamma \in \text{Aut}(\Psi_{T_s})$. This leads to an isomorphism $\eta : (H_1)_{der} \rightarrow (H_2)_{der}$ which stabilizes T_s with $\eta|_{T_s} = \gamma$. We have

$$Z \cap (H_1)_{der} \subset T_s \cap Z(G').$$

Decompose Ψ_{T_s} into an orthogonal union of irreducible root systems, which corresponds to a decomposition of T_s . Due to BC_n is its own dual lattice, $T_s \cap Z(G')$ is contained in the product of those factors of T_s corresponding to reduced irreducible factors of Ψ_{T_s} . The results in [17, Section 7] imply that there exists $\gamma' \in \Gamma'$ such that $\gamma'^{-1}\gamma$ acts trivially on reduced irreducible factors of Ψ_{T_s} . Hence,

$$\eta|_{T_s \cap Z(G')} = \gamma|_{T_s \cap Z(G')} = \gamma'|_{T_s \cap Z(G')} = \text{id}.$$

Defining $\eta|_Z = \text{id}$, then η extends to an isomorphism $\eta : H_1 \rightarrow H_2$.

3. DIMENSION DATUM OF A DISCONNECTED SUBGROUP

After the papers [11], [1], [17], we have a pretty well understanding of dimension data of connected closed subgroups of a compact Lie group. In this section we study dimension data of disconnected closed subgroups. Extending the strategy in [17], we transfer the study of dimension data to the study of certain characters, supported on *maximal commutative connected subsets*. We introduce a kind of *affine root system*. The character is associated to affine root system, and we use data from affine root system to get clear expression for the character. The final formula is the same as in the torus case.

3.1. Generalized Cartan subgroup and Weyl integration formula. Let G be a compact Lie group. A closed abelian subgroup S of G is called a (*generalized*) *Cartan subgroup* if S contains a dense cyclic subgroup and

$$S^0 = (Z_G(S))^0$$

([3, Definition 4.1]). We call a closed commutative connected subset S' of G a *maximal commutative connected subset* if

$$s^{-1}S' = (Z_G(S'))^0$$

for any $s \in S'$. There is a close relationship between generalized Cartan subgroups and maximal commutative connected subsets: if S' is a maximal commutative connected subset of G , then $S = \langle S' \rangle$ is a generalized Cartan subgroup; if S is a generalized Cartan subgroup of G and $s \in S$ generates S/S^0 , then $S' = sS^0$ is a maximal commutative connected subset. For any element $g \in G$, choose a maximal torus T^g of $Z_G(g)^0$. Set

$$S = \langle T^g, g \rangle$$

and

$$S' = gT^g.$$

Then, S is a generalized Cartan subgroup of G , and S' is a maximal commutative connected subset of G . Moreover, all generalized Cartan subgroups (and maximal commutative connected subsets) of G arise in this way.

For a generalized Cartan subgroup S in G ,

$$S^0 \subset S \cap G^0 \subset Z_{G^0}(S).$$

We would like to remind that in general $S^0 \neq S \cap G^0$ (i.e, $S \cap G^0$ is not necessarily connected) and $S \cap G^0 \neq Z_{G^0}(S)$.

Example 3.1. Set $G = (\mathrm{SU}(4)/\langle -I \rangle) \rtimes \langle \sigma \rangle$, where $\sigma^2 = [iI]$, and

$$\sigma[X]\sigma^{-1} = [L\bar{X}L^{-1}]$$

for $L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. Write

$$T = \{\mathrm{diag}\{\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}\} : |\lambda_1| = |\lambda_2| = 1\},$$

and $S = \langle T, \sigma \rangle$. Then, S is a generalized Cartan subgroup of G . In this case,

$$\sigma^2 = [iI] \in S \cap G^0 - S^0.$$

Thus, $S^0 \neq S \cap G^0$.

Example 3.2. Set $G = (\mathrm{SU}(4)/\langle -I \rangle) \rtimes \langle \sigma \rangle$, where $\sigma^2 = 1$, and

$$\sigma[X]\sigma^{-1} = [L\bar{X}L^{-1}]$$

$$\text{for } L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \text{ Write}$$

$$T = \{\mathrm{diag}\{\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1}\} : |\lambda_1| = |\lambda_2| = 1\},$$

and $S = \langle T, \sigma \rangle$. Then, S is a generalized Cartan subgroup of G . In this case,

$$[iI] \in Z_{G^0}(S) - S \cap G^0.$$

Thus, $S \cap G^0 \neq Z_{G^0}(S)$.

Given a maximal commutative connected subset S' of G , set

$$W(G, S') = N_{G^0}(S')/s^{-1}S'$$

($s \in S'$) and call it the *Weyl group* of S' . Given a generalized Cartan subgroup S of G , set

$$W(G, S) = N_{G^0}(S)/S^0$$

and call it the *Weyl group* of S . Both $W(G, S')$ and $W(G, S)$ are finite groups. If S' is a maximal commutative connected subset and $S = \langle S' \rangle$, then S is a generalized Cartan subgroup, and $N_{G^0}(S') = N_{G^0}(S)$. Thus, $W(G, S') = W(G, S)$. The group $W(G, S')$ ($= W(G, S)$) acts on S' (and on S) through conjugation.

Proposition 3.1. ([3, Propositions 4.2, 4.3, 4.7]) *Given a compact Lie group G and a connected component gG^0 of G , any two maximal commutative connected subsets in gG^0 are conjugate.*

If S' is a maximal commutative connected subset in gG^0 , then every G^0 conjugacy class in gG^0 contains an element in S' . Two elements in S' are in a G^0 conjugacy class if and only if they are in the same $W(G, S')$ orbit.

By Proposition 3.1, G^0 conjugacy classes in a connected component gG^0 are parameterized by $S'/W(G, S')$.

Let S' be a maximal commutative connected subset of G . Write $S = \langle S' \rangle$. Choose $s_0 \in S'$.

Lemma 3.2. *The map*

$$q : G^0/S^0 \times S' \rightarrow s_0G^0, \quad (gS^0, s) \mapsto gsg^{-1}$$

is surjective, and the degree of the map q is equal to $|W(G, S')|$

Proof. Surjectivity of the map q is shown in [3, Lemma 4.5]. It is shown in [15, Lemma 2.2]) that the degree of the map q is equal to $|W(G, S')|$. \square

Write dg for a left G^0 invariant measure of volume one on s_0G^0 , $d\bar{g}$ for a G^0 invariant measure on G^0/S^0 of volume one, and ds for a left S^0 invariant measure on S' of volume one. Then, $q^*dg = \det(q)d\bar{g} \wedge ds$ for some positive-valued function $\det(q)$. Write \mathfrak{g} for the complexified Lie algebra of G and \mathfrak{s} for the complexified Lie algebra of S^0 . Then, we have the following formula for $\det(q)$.

Lemma 3.3. *For any $g \in G^0$ and $s \in S'$,*

$$\det(q)(gS^0, s) = \det(\text{Ad}(s) - 1)|_{\mathfrak{g}/\mathfrak{s}}.$$

Proof. This is [15, Lemma 2.1]. □

Write

$$(2) \quad D(s) = \frac{1}{|W(G, S')|} \det(\text{Ad}(s) - 1)|_{\mathfrak{g}/\mathfrak{s}}, \quad s \in S',$$

and call it the *density function* on S' .

Proposition 3.4. *For any G^0 conjugation invariant continuous function f on s_0G^0 ,*

$$\int_{s_0G^0} f(g)dg = \int_{S'} f(s)D(s)ds.$$

Proof. This is [15, Proposition 2.3]. □

Choose a maximal commutative connected subset in each connected component of G . Write S'_1, S'_2, \dots, S'_m for these chosen subsets. For each i , write $D_i(s)$ for the corresponding density function as in (2). Write dg for a normalized Haar measure on G . For each i , we denote by ds a left $s_i^{-1}S'_i$ ($s_i \in S'_i$) invariant measure of volume one on S'_i . The following is Weyl integration formula on G . It follows directly from Proposition 3.4 directly.

Proposition 3.5. *For any G^0 conjugation invariant continuous function f on G ,*

$$\int_G f(g)dg = \frac{1}{|G/G^0|} \sum_{1 \leq i \leq m} \int_{S'_i} f(s)D(s)ds.$$

3.2. Affine root datum. Now we endow G with a biinvariant Riemannian metric, which induces a positive definite inner product on the character group of S^0 , denoted by (\cdot, \cdot) . For two characters λ and μ of S , define

$$(\lambda, \mu) = (\lambda|_{S^0}, \mu|_{S^0}).$$

We introduce an affine root system $(R(G, S), R^\vee)$ from the conjugation action of S on \mathfrak{g} .

The conjugation action of S on \mathfrak{g} gives a decomposition

$$\mathfrak{g} = \sum_{\lambda \in X^*(S)} \mathfrak{g}_\lambda,$$

where

$$\mathfrak{g}_\lambda = \{Y \in \mathfrak{g} : \text{Ad}(s)Y = \lambda(s)Y, \forall s \in S\}.$$

Since $(Z_G(S))^0 = S^0$, the zero-weight space is equal to the complexified Lie algebra of S . Set

$$R = R(G, S) = \{\alpha \in X^*(S) - \{0\} : \mathfrak{g}_\alpha \neq 0\}.$$

An element $\alpha \in R(G, S)$ is called a *root* and \mathfrak{g}_α is called the *root space* for a root α . A root α is called an *infinite root* if $\alpha|_{S^0} \neq 0$; it is called a *finite root* if $\alpha|_{S^0} = 0$.

Lemma 3.6. *Let α be an infinite root. Then $\dim \mathfrak{g}_\alpha = 1$ and 2α is not a root.*

Proof. Set

$$G^{[\alpha]} = Z_G(\ker \alpha)$$

and

$$G_{[\alpha]} = [(G^{[\alpha]})^0, (G^{[\alpha]})^0].$$

Write $T_{[\alpha]} = G_{[\alpha]} \cap S$. Then, $T_{[\alpha]}$ is a maximal torus of $G_{[\alpha]}$, and $\dim T_{[\alpha]} = 1$. Thus, the complexified Lie algebra of $G_{[\alpha]}$ is semisimple and is of rank one. Hence, it is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. By this, $\dim \mathfrak{g}_\alpha = 1$ and 2α is not a root. \square

There is a unique cocharacter $\check{\alpha} \in X_*(S)$ whose image lies in $T_{[\alpha]}$ and such that $\langle \alpha, \check{\alpha} \rangle = 2$. Define

$$(3) \quad s_\alpha(x) = x\check{\alpha}(\alpha(x)^{-1}), \quad \forall x \in S.$$

The following proposition follows from the classical \mathfrak{sl}_2 theory (cf. [8] and [3]).

Lemma 3.7. *There exists*

$$n_\alpha \in N_{G_{[\alpha]}}(T_{[\alpha]}) = G_{[\alpha]} \cap N_G(A)$$

such that $\text{Ad}(n_\alpha)|_A = s_\alpha$.

We have $s_\alpha|_{\ker \alpha} = \text{id}$, $s_\alpha|_{\text{Im } \check{\alpha}} = -1$ and $s_\alpha^2 = 1$.

Given a finite root α of order n , set

$$G^{[\alpha]} = Z_G(\ker(\alpha)).$$

Its complexified Lie algebra is

$$\mathfrak{g}^{[\alpha]} = \mathfrak{s} \oplus \left(\bigoplus_{m \in \mathbb{Z}^*} \mathfrak{g}_{m\alpha} \right).$$

Lemma 3.8. *The subalgebra $\mathfrak{g}^{[\alpha]}$ is abelian.*

Proof. Write $\mathfrak{h} = Z_{\mathfrak{g}}(S^0)$, and $\mathfrak{h} = z(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}]$ for the Levi decomposition of \mathfrak{h} . Then, $\mathfrak{s} \subset z(\mathfrak{h})$. Since α is a finite root, $S^0 \subset \ker \alpha$. By this,

$$\mathfrak{g}^{[\alpha]} \subset \mathfrak{h} = z(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}].$$

As the conjugation action of S on \mathfrak{h} stabilizes both $z(\mathfrak{h})$ and $[\mathfrak{h}, \mathfrak{h}]$, we have

$$\mathfrak{g}^{[\alpha]} = (\mathfrak{g}^{[\alpha]} \cap z(\mathfrak{h})) \oplus (\mathfrak{g}^{[\alpha]} \cap [\mathfrak{h}, \mathfrak{h}]).$$

Choose an element $y \in S$ generates $S/\ker \alpha$. Then,

$$\mathfrak{s} = Z_{\mathfrak{g}}(S) = (\mathfrak{g}^{[\alpha]})^y.$$

Apparently, $\mathfrak{s} \subset \mathfrak{g}^{[\alpha]} \cap z(\mathfrak{h})$. Hence, $(\mathfrak{g}^{[\alpha]} \cap [\mathfrak{h}, \mathfrak{h}])^y = 0$. By a theorem of Borel, this indicates that $\mathfrak{g}^{[\alpha]} \cap [\mathfrak{h}, \mathfrak{h}]$ is abelian. Therefore, $\mathfrak{g}^{[\alpha]} \subset z(\mathfrak{h}) \oplus (\mathfrak{g}^{[\alpha]} \cap [\mathfrak{h}, \mathfrak{h}])$ is also abelian. \square

Let

$$\mathfrak{g}_{[\alpha]} = \bigoplus_{m \in \mathbb{Z}^*} \mathfrak{g}_{m\alpha},$$

and $G_{[\alpha]}$ be the image of $\mathfrak{g}_{[\alpha]}$ under the exponential map.

Lemma 3.9. *$G_{[\alpha]}$ is a closed abelian subgroup of G and $G_{[\alpha]} \cap S$ is a finite group.*

Proof. Choose $y \in S$ generates $S/\ker \alpha$. The conjugation action of y on the torus $(G^{[\alpha]})^0$ is given by some $y_* \in \text{Aut}((G^{[\alpha]})^0)$. Then, $(\text{Fix } y_*)^0 = S^0$. Write n for the order of y_* , and $z_* = I + y_* + \cdots + y_*^{n-1}$. Then, z_* is an endomorphism of $(G^{[\alpha]})^0$ and $G_{[\alpha]}$ is equal to the neutral subgroup of

$$\{x \in (G^{[\alpha]})^0 : z_*(x) = 1\}.$$

Thus, $G_{[\alpha]}$ is a closed subgroup of G . As $\mathfrak{g}_{[\alpha]}$ is abelian, so is $G_{[\alpha]}$. Since $\mathfrak{g}_{[\alpha]} \cap \mathfrak{s} = 0$, we know that $G_{[\alpha]} \cap S$ is a finite group. \square

For any $\xi \in \text{Hom}(S/\ker \alpha, G_{[\alpha]} \cap S)$, define $s_{\alpha, \xi} : S \rightarrow S$ by

$$s_{\alpha, \xi}(x) = x\xi(x), \quad \forall x \in S.$$

Defined in this way, $s_{\alpha, \xi}$ is a self-map of S .

Lemma 3.10. *For any $\xi \in \text{Hom}(S/\ker \alpha, G_{[\alpha]} \cap S)$, there exists*

$$n \in G_{[\alpha]} \cap N_G(S)$$

such that

$$\text{Ad}(n)|_S = s_{\alpha, \xi}.$$

Particularly, $s_{\alpha, \xi}$ is an automorphism of S .

Proof. As $\text{Ad}(y^{-1}) - I$ is an endomorphism of $\mathfrak{g}_{[\alpha]}$ without eigenvalue 0 due to $Z_{\mathfrak{g}_{[\alpha]}}(y) = 0$, we know that

$$\text{Ad}(y^{-1}) - I : G_{[\alpha]} \rightarrow G_{[\alpha]}$$

is surjective. Hence, there exists $n \in G_{[\alpha]}$ such that $y^{-1}ny^{-1} = \xi(y)$. Thus, $ny^{-1}n^{-1} = y\xi(y)$. Since $\ker \alpha$ commutes with $G_{[\alpha]}$ as $\ker \alpha$ acts trivially on $\mathfrak{g}_{[\alpha]}$, we have $nxn^{-1} = x$ for any $x \in \ker \alpha$. Thus,

$$n \in G_{[\alpha]} \cap N_G(S)$$

and

$$\text{Ad}(n)|_S = s_{\alpha, \xi}.$$

\square

Write

$$R^\vee(\alpha) = \text{Hom}(S/\ker \alpha, G_{[\alpha]} \cap S),$$

and call it the root transvection group. We call $s_{\alpha, \xi}$ ($\xi \in R^\vee(\alpha)$) a transvection.

Definition 3.1. Let $W_{small}(G, S)$ denote the subgroup of $N_{G^0}(S)/Z_{G^0}(S)$ generated by s_α (for infinite roots α) and transvections $s_{\alpha, \xi}$ (for finite roots α and $\xi \in R^\vee(\alpha)$). We call $W_{small}(G, S)$ the small Weyl group of S in G .

By Lemma 3.10, $W_{small}(G, S)$ is a subgroup of $N_{G^0}(S)/Z_{G^0}(S)$. We also note that $N_{G^0}(S)/Z_{G^0}(S)$ is a quotient group of the Weyl group $N_{G^0}(S)/S^0$. The action of $W_{small}(G, S)$ on S induces an action on the character group $X^*(S)$.

Proposition 3.11. The set $R(G, S)$ is stable under the action of $W_{small}(G, S)$, the map $\alpha \mapsto \check{\alpha}$ and the root transvection groups $R^\vee(\alpha)$ are permuted under the action of $W_{small}(G, S)$.

Proof. This follows from Lemmas 3.7 and 3.10. □

Definition 3.2. We call $R(G, S)$ together with the coroots $\alpha \mapsto \check{\alpha}$ for infinite roots and the root transvection groups $R^\vee(\alpha)$ for finite roots the affine root datum for G with respect to S .

By Proposition 3.11, one can show that $(R(G, S), W_{small}(G, S))$ inherits many structures and properties like the classical root system. A related structure of affine root datum is studied in [14]. Different with that in [14], we employ ideas in [5] to consider all finite roots and their root transvection groups. A more general theory of *twisted root datum* and a more complete investigation of the structures are given in [19].

Write

$$R_0 = R_0(G, S)$$

for the set of finite roots. Write

$$R' = R'(G, S) = \{\alpha|_{S^0} : \alpha \in R(G, S) - R_0(G, S)\}.$$

Proposition 3.12. $R'(G, S)$ in a root system in $X^*(S^0)$ in the sense of [17, Definition 2.2].

Proof. This follows from Proposition 3.11. □

For the application in the study of dimension datum, we see in Subsection 3.3 that only infinite roots and their corresponding reflections matter. We call infinite roots

$$R(G, S) - R_0(G, S)$$

together with the coroots $\alpha \mapsto \check{\alpha}$ ($\alpha \in R(G, S) - R_0(G, S)$) the *affine root system* for G with respect to S .

3.3. Weyl product. Write

$$T = Z_{G^0}(S^0).$$

The following fact is well known.

Lemma 3.13. T is a maximal torus of G^0 .

Proof. Choose $y \in S$ generates S/S^0 . Then, $\mathfrak{t}^y = \mathfrak{s}$. Apparently, $\mathfrak{s} \subset z(\mathfrak{t})$. Thus, $\mathfrak{t}^y \subset z(\mathfrak{t})$. Write $\mathfrak{t} = z(\mathfrak{t}) \oplus \mathfrak{t}^s$ with $\mathfrak{t}^s = [\mathfrak{t}, \mathfrak{t}]$ the derived subalgebra. Then, $\mathfrak{t}^y \subset z(\mathfrak{t})$ implies $(\mathfrak{t}^s)^y = 0$. By a theorem of Borel, this indicates that \mathfrak{t}^s is abelian. Thus, \mathfrak{t} is abelian. We know T is connected and it contains any maximal torus which contains S^0 . Therefore, T is a maximal torus. \square

Lemma 3.14. *There is an exact sequence*

$$1 \rightarrow N_T(S)/S^0 \rightarrow W(G, S) \rightarrow W_{R'} \rightarrow 1.$$

Proof. Restricting the action of elements in $W(G, S)$ to S^0 , it induces an action of $W(G, S)$ on R' . Thus, we have a natural homomorphism

$$\phi : W(G, S') \rightarrow \text{Aut}(R').$$

Apparently the image is equal to $W_{R'}$. For any $gS^0 \in W(G, S')$ ($g \in G^0$), $gS^0 \in \ker \phi$ if and only $\alpha(gxg^{-1}) = \alpha(x)$ for any $x \in S^0$ and any infinite root α . Since $\alpha(gxg^{-1}) = \alpha(x) = 1$ for any $x \in S^0$ and any finite root α . Then, $gS^0 \in \ker \phi$ if and only $\alpha(gxg^{-1}) = \alpha(x)$ for all roots α . Which is equivalent to $gxg^{-1}x^{-1} \in Z(G^0)$ for any $x \in S^0$. Then,

$$gxg^{-1}x^{-1} \in Z(G^0) \cap [G^0, G^0] = Z((G^0)_{der})$$

for any $x \in S^0$. As $Z((G^0)_{der})$ is a finite group, we get $gxg^{-1}x^{-1} = 1$ for any $x \in S$ by continuity. Hence, $g \in T = Z_{G^0}(S^0)$. This proves the conclusion. \square

Lemma 3.15. *For any $s \in S'$,*

$$\prod_{\alpha \in R_0} (1 - \alpha(s)) = |N_T(S)/S^0|.$$

Proof. The group $N_T(S)$ consists of elements t in T such that $tst^{-1}s^{-1} \in S^0$. Considering the quotient group T/S^0 and the induced action by s , then

$$N_T(S)/S^0 \cong \{tS^0 \in T/S^0 : s(tS^0)s^{-1}(tS^0)^{-1} = S^0\}.$$

The latter is just the fixed point group of s in T/S^0 . The action of s on T/S^0 is given by a matrix $X \in \text{GL}(k, \mathbb{Z})$, where $k = \dim(T/S^0)$. Write $f(u)$ for the characteristic polynomial of X . Then,

$$(T/S^0)^s \cong (I - X)^{-1}(\mathbb{Z}^k)/\mathbb{Z}^k$$

and

$$f(u) = \prod_{\alpha \in R_0} (u - \alpha(s)).$$

Thus,

$$|(T/S^0)^s| = |\det(I - X)| = |f(1)| = \prod_{\alpha \in R_0} (1 - \alpha(s)).$$

Together with the identification $N_T(S)/S^0 \cong (T/S^0)^s$ shown in the above, we get

$$\prod_{\alpha \in R_0} (1 - \alpha(s)) = |N_T(S)/S^0|.$$

\square

For each $\alpha' \in R'$ with $\frac{1}{2}\alpha' \notin R'$, set

$$R_{1,\alpha'} = \{\beta \in R(G, A) : \beta|_{S^0} = \alpha'\}, \quad R_{2,\alpha'} = \{\beta \in R(G, A) : \beta|_{S^0} = 2\alpha'\}$$

and

$$R_{\alpha'} = R_{1,\alpha'} \cup R_{2,\alpha'}.$$

Write

$$m_{1,\alpha'} = |R_{1,\alpha'}|, \quad m_{2,\alpha'} = |R_{2,\alpha'}|$$

and

$$m_{\alpha'} = m_{1,\alpha'} + 2m_{2,\alpha'}.$$

Lemma 3.16. *For any $\alpha \in R_{1,\alpha'}$,*

$$(4) \quad \prod_{\beta \in R_{\alpha'}} (1 - \beta(s)) = 1 - \alpha(s)^{m_{\alpha'}}.$$

Proof. Write

$$S_{\alpha'} = \ker(\alpha') = \ker \alpha \cap S^0.$$

Set

$$H_{\alpha'} = Z_G(S_{\alpha'}).$$

Then, all root spaces \mathfrak{g}_β ($\beta \in R_{\alpha'}$) are contained in $\mathfrak{h}_{\alpha'}$. We may assume that $G = H_{\alpha'}$, i.e., all infinite roots are in the set $R_{\alpha'}$. Write $\mathfrak{h}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$. Let

$$\phi : G \rightarrow \text{Aut}(\mathfrak{h}_0)$$

be the natural homomorphism from the adjoint action of G on \mathfrak{h}_0 . Write $H = \text{Aut}(\mathfrak{h}_0)$. We have $\ker \phi = Z_G([G^0, G^0])$. As $\mathfrak{g}_\beta \subset [\mathfrak{g}_0, \mathfrak{g}_0]$ for any $\beta \in R_{\alpha'}$, we have $\beta|_{S \cap \ker \phi} = 1$ for any $\beta \in R_{\alpha'}$. Hence, it is equivalent to consider $\phi(S) \subset H$, which is a generalized Cartan subgroup of dimension one.

Up to isomorphism, the pair $(\phi(S), \mathfrak{h}_0)$ has only two possibilities: (1), $\mathfrak{u}_0 = \underbrace{\mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_1}_m$, $\phi(S) = \langle \Delta(S_1), \theta \rangle$, where $\mathfrak{u}_1 \cong \mathfrak{su}(2)$, S_1 is a maximal torus of $\text{Int}(\mathfrak{u}_1)$,

$$\Delta : \text{Int}(\mathfrak{u}_1) \rightarrow \prod_{1 \leq i \leq m} \text{Int}(\mathfrak{u}_1)$$

is the diagonal map, and

$$\theta(X_1, \dots, X_m) = (X_2, \dots, X_m, X_1).$$

(2), $\mathfrak{u}_0 = \underbrace{\mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_1}_m$, $\phi(S) = \langle \Delta(S_1), \theta \rangle$, where $\mathfrak{u}_1 \cong \mathfrak{su}(3)$, $\sigma \in \text{Aut}(\mathfrak{u}_1)$

such that $\mathfrak{u}_1^\sigma \cong \mathfrak{so}(3)$ and hence $\text{Int}(\mathfrak{u}_1)^\sigma \cong \text{SO}(3)$, S_1 is a maximal torus of $\text{Int}(\mathfrak{u}_1)^\sigma$,

$$\Delta : \text{Int}(\mathfrak{u}_1) \rightarrow \prod_{1 \leq i \leq m} \text{Int}(\mathfrak{u}_1)$$

is the diagonal map, and

$$\theta(X_1, \dots, X_m) = (X_2, \dots, X_m, \sigma(X_1)).$$

In case (1), define $\beta_0 \in \text{Hom}(S, \text{U}(1))$ by $\beta_0|_{S^0} = 1$ and $\beta_0(\theta) = e^{\frac{2\pi i}{m}}$. Then,

$$R_{\alpha'} = \{\alpha + k\beta_0 : 0 \leq k \leq m-1\}.$$

Thus, $m_{1,\alpha'} = m$, $m_{2,\alpha'} = 0$, and $m_{\alpha'} = m$.

In case (2), define $\beta_0 \in \text{Hom}(S, \text{U}(1))$ by $\beta_0|_{S^0} = 1$ and $\beta_0(\theta) = e^{\frac{\pi i}{m}}$. Then,

$$R_{\alpha'} = \{\alpha + k\beta_0 : 0 \leq k \leq 2m-1\} \cup \{2\alpha + (2k+1)\beta_0 : 0 \leq k \leq m-1\}.$$

Thus, $m_{1,\alpha'} = 2m$, $m_{2,\alpha'} = m$, and $m_{\alpha'} = 4m$. In either case, the conclusion follows from the elementary formula

$$\prod_{0 \leq k \leq l-1} (1 - e^{\frac{2k\pi i}{l}} t) = 1 - t^l$$

for any $l \geq 1$. □

Proposition 3.17. *Let $\alpha' \in R$ with $\frac{1}{2}\alpha' \notin R$. For any two roots $\alpha, \tilde{\alpha} \in R_{1,\alpha'}$, $[m_{1,\alpha'}\alpha] = [m_{1,\alpha'}\tilde{\alpha}]$, and $m_{\alpha'} = m_{1,\alpha'}$ or $2m_{1,\alpha'}$.*

Proof. In the proof of Lemma 3.16, we have described the structure of the set $R_{\alpha'}$. From it we get the conclusion of this proposition. □

Choose a simple system $\{\alpha'_1, \dots, \alpha'_t\}$ of the root system R' by choosing a simple system for each irreducible factor of R' . In case some simple factor of R' is not reduced, i.e, it is of type BC_l for some $l \geq 1$, a simple system means simple system for the corresponding root system B_l . For each i , choose $\alpha_i \in R(G, S)$ such that

$$\alpha_i|_{S^0} = \alpha'_i.$$

Since the characters $\alpha'_1, \dots, \alpha'_t$ are linearly independent, there exists $s_0 \in S'$ such that

$$\alpha_1(s_0) = \dots = \alpha_t(s_0) = 1.$$

Let R_{s_0} be the sub-root system of $R(G, S)$ generated by $\alpha_1, \dots, \alpha_t$. Then, the map

$$R_{s_0} \rightarrow R', \quad \alpha \mapsto \alpha|_{S^0}$$

is an isomorphism of root systems. For each $\alpha \in R_{s_0}$, we write $\alpha' = \alpha|_{S^0}$ to denote this bijection.

Choose a positive system R'^+ in R' . Set⁵

$$\delta = \delta_R = \frac{1}{2} \sum_{\alpha' \in R'^+, \text{reduced}} m_{\alpha'} \alpha.$$

In the above the summation takes over positive roots α' in R' with $\frac{1}{2}\alpha' \notin R'$; for any such α' , we choose a root $\alpha \in R_{1,\alpha'}$.

⁵Precisely to say, 2δ is a well-defined character of S , but δ may be not. While we use δ below, it appears in the form of $\delta - w\delta$ ($w \in W_{\text{small}}(G, S)$), which are all well-defined characters of S . Thus, the express of δ as above arises no ambiguity.

Definition 3.3. *Define*

$$A_R = \frac{1}{|W_{R'}|} \sum_{w \in W_{R_{s_0}}} \epsilon(w) [\delta_R - w\delta_R].$$

Lemma 3.18. *$2\delta_R$ and $\delta_R - w\delta_R$ ($W \in W_{R_{s_0}}$) are in $X^*(S)$ and they are independent of the choice of α for $\alpha' \in R'$.*

$A_R \in \mathbb{Q}[X^(S)]$ depends only on R , not on the choice of $\alpha_1, \dots, \alpha_t$ (or of s_0).*

Proof. By definition, $2\delta_R$ is an integral combination of $\{m_{\alpha'}\alpha : \alpha' \in R'\}$. Thus, it is in $X^*(S)$. From basic properties of root system and Weyl group, each term $\delta_R - w\delta_R$ ($W \in W_{R_{s_0}}$) is an integral combination of $\{m_{\alpha'}\alpha : \alpha' \in R'\}$. Thus, they are in $X^*(S)$. By Proposition 3.17, $[m_{\alpha'}\alpha] = [m_{\alpha'}\tilde{\alpha}]$ for any $\alpha' \in R'$ and any $\alpha, \tilde{\alpha} \in R_{1,\alpha'}$. Thus, $2\delta_R$ and $\delta_R - w\delta_R$ ($W \in W_{R_{s_0}}$) are independent of the choice of α for $\alpha' \in R'$.

By the first statement shown above and the definition of A_R , we have

$$A_R \in \mathbb{Q}[X^*(S)]$$

and it is independent of the choice of $\alpha_1, \dots, \alpha_t$ (or of s_0). \square

Proposition 3.19. *For any $s \in S'$,*

$$D(s) = \frac{1}{|W_{R'}|} \sum_{\tau \in W_{s_0}} \tau \cdot A_R(s).$$

Proof. When S is connected, this is shown in the proof of [17, Proposition 3.7]. In general, Lemmas 3.14, 3.15, 3.16 reduce it to the connected case. \square

Proposition 3.20. *For any character $F \in \mathbb{Q}[S]$, $F|_{S'} = 0$ if and only if $F = 0$.*

Proof. Write $m = |S/S^0|$ and $\omega_m = e^{\frac{2\pi i}{m}}$. Choose $s \in S$ generates S/S^0 . Define $\beta_0 \in X^*(S)$ by $\beta_0|_{S^0} = 1$ and $\beta_0(s) = \omega_m$. For any two linear characters $\lambda, \lambda' \in X^*(S)$, $\lambda|_{S^0} = \lambda'|_{S^0}$ if and only if $\lambda' = \lambda + k\beta_0$ for some k , $0 \leq k \leq m-1$. The conclusion follows from the linear independence of characters of S^0 and the \mathbb{Q} -independence of $1, \omega_m, \dots, \omega_m^{m-1}$. \square

By Proposition 3.20, any character $F \in \mathbb{Q}[S]$ is determined by its restriction (evaluation) on S' . Particularly, if $F|_{S'} = 0$, then $F|_{S^0} = 0$.

3.4. Irreducible affine root systems.

Definition 3.4. *We say two pairs (G_1, S_1) and (G_2, S_2) of generalized Cartan subgroups in compact Lie groups isogenous if there is another pair (G_3, S_3) and homomorphisms $f_i : G_i \rightarrow G_3$ ($i = 1, 2$) such that f_1, f_2 are isomorphisms on Lie algebras, and $f_1(S_1) = f_2(S_2) = S_3$.*

We call a pair (G, S) irreducible if there is no pairs (G_1, S_1) and (G_2, S_2) with $\dim G_1 > 0$ and $\dim G_2 > 0$, such that (G, S) is isogenous to $(G_1 \times G_2, S_1 \times S_2)$.

Now we describe all possible pairs (G, S) up to isogeny, and give the multiples $m_{1,\alpha'}, m_{\alpha'}$. For an irreducible pair (G, S) , either G^0 is a torus, or \mathfrak{g}_0 is the product of several copies of a simple compact Lie algebra and a generator of S acts on \mathfrak{g}_0 by permuting its simple factors transitively. In the first case, either $S^0 = G^0$ is a one-dimensional torus, or $\dim G^0 = \phi(n)$ ($\phi(n)$ means Euler function) and $S = \langle \theta \rangle$ with $\theta_* = \text{Ad}(\theta) \in \text{GL}(X^*(G^0))$ having degree n cyclotomic polynomial as its characteristic polynomial. There are no infinite roots in the first case.

In the latter case, write Φ for affine root system of G , and $R' = R'(G, S)$. Then, the pair (Φ, R') determines (G, S) up to isogeny. All possible pairs (Φ, R') are as follows. We also indicate the multiples $m_{1,\alpha'}$ and $m_{\alpha'}$.

- (1) $(\Phi, R') = (m\Phi_0, \Phi_0)$, where Φ_0 is an irreducible root system, $m \geq 1$; $m_{\alpha'} = m_{1,\alpha'} = m$ for any $\alpha' \in R'$.
- (2) $(\Phi, R') = (mA_{2n}, BC_n)$ ($n \geq 1$); $m_{1,\alpha'} = 2$ and $m_{\alpha'} = 4$ for any short root $\alpha' \in R'$; $m_{\alpha'} = m_{1,\alpha'} = 2$ for middle root $\alpha' \in R'$.
- (3) $(\Phi, R') = (mA_{2n-1}, C_n)$ ($n \geq 2$); $m_{\alpha'} = m_{1,\alpha'} = 2$ for any short root $\alpha' \in R'$; $m_{\alpha'} = m_{1,\alpha'} = 1$ for long root $\alpha' \in R'$.
- (4) $(\Phi, R') = (mD_n, B_{n-1})$ ($n \geq 4$); $m_{\alpha'} = m_{1,\alpha'} = 2$ for any short root $\alpha' \in R'$; $m_{\alpha'} = m_{1,\alpha'} = 1$ for long root $\alpha' \in R'$.
- (5) $(\Phi, R') = (mD_4, G_2)$; $m_{\alpha'} = m_{1,\alpha'} = 3$ for any short root $\alpha' \in R'$; $m_{\alpha'} = m_{1,\alpha'} = 1$ for long root $\alpha' \in R'$.
- (6) $(\Phi, R') = (mE_6, F_4)$; $m_{\alpha'} = m_{1,\alpha'} = 2$ for any short root $\alpha' \in R'$; $m_{\alpha'} = m_{1,\alpha'} = 1$ for long root $\alpha' \in R'$.

3.5. Further about the character A_R . For a given R' , the character A_R is determined by the multiples $\{m_{\alpha'} : \alpha' \in R'\}$ and a set of lifting $\{\alpha_1, \dots, \alpha_t\}$ of simple roots of R' . The latter is determined by the base point s_0 .

For a given base point $s_0 \in S'$, the map $s \in S^0 \mapsto ss_0$ identifies S' with S^0 . The description of (R, R') reduces to irreducible case. By the description in the irreducible case as given above, we see that $\{m_{\alpha'}\alpha' : \alpha' \in R'\}$ looks like a root system. Compared with [17, Definition 2.2], it satisfies the first condition (Weyl group permutation), but does not satisfy the second condition (strong integrality) in general.

Write $m = |S/S^0|$. Choose and fix a point $\tilde{s} \in S'$ such that $o(\tilde{s}) = m$. Then,

$$S = S^0 \times \langle \tilde{s} \rangle.$$

Define $\beta_0 \in X^*(S)$ by $\beta_0|_{S^0} = 1$ and $\beta_0(s) = e^{\frac{2\pi i}{m}}$. For each $\alpha' \in R'$, let $\alpha \in X^*(S)$ be defined by $\alpha|_{S^0} = \alpha'$ and $\alpha(\tilde{s}) = 1$. Then

$$\{\alpha + k\beta_0 : 0 \leq k \leq m-1\}$$

are all liftings of α' in $X^*(S)$.

Lemma 3.21. *For any root $\alpha' \in R'$ with $\frac{1}{2}\alpha' \notin R'$, we have $m_{1,\alpha'} | m$.*

Proof. Choose one $\alpha \in R_{1,\alpha'}$. From the description of $R_{1,\alpha'}$ given in the proof of Lemma 3.16, we see that there exists a degree $m_{1,\alpha'}$ character $\beta \in X^*(S/S^0)$

such that

$$R_{1,\alpha'} = \{\alpha + k\beta : 0 \leq k \leq m_{1,\alpha'}\}.$$

Thus,

$$m_{1,\alpha'} || |S/S^0| = m.$$

□

Write

$$R' = \bigsqcup_{1 \leq j \leq k} R'_j$$

for the decomposition of R' into irreducible root systems. For each i , write $m_i = m_{1,\alpha'}$ for a short root in R'_i . From the description of $(R - R_0, R')$ given in Subsection 3.4, we see that there are three possibilities.

- (1) R'_i is a reduced irreducible root system, all $m_{\alpha'}$ ($\alpha' \in R_i$) equal to m_i . In this case $m_i | m$ by Lemma 3.21.
- (2) R'_i is a reduced non-simply-laced irreducible root system, and $m_{\alpha'} = n_{\alpha'}^{-1} m_i$, where $n_{\alpha'}$ is the square of the ratio of the length of α' and of short roots in R'_i . In this case $m_i | m$ by Lemma 3.21.
- (3) R'_i is a non-reduced irreducible root system, for a short root $\alpha' \in R'_i$, $m_{1,\alpha'} = m_i$, $m_{1,\alpha'} = 2m_i$; for a middle length root $\alpha' \in R'_i$, $m_{1,\alpha'} = m_{\alpha'} = m_i$. In this case $m_i | m$ by Lemma 3.21. Moreover, m_i is even in this case from the description of (R, R') given in Subsection 3.4.

Remark 3.1. *In the above, we have described the possibilities of the multiples m_α and and of the liftings $\{\alpha_1, \dots, \alpha_t\}$. From that, we could describe the $R - R_0$ and calculate A_R from the root system R' , the multiples m_α and and of the liftings $\{\alpha_1, \dots, \alpha_t\}$.*

3.6. Dimension data of disconnected subgroups. We call a closed abelian subgroup S of G a *generalized torus* if S/S^0 is a cyclic group. Write

$$\Gamma^0 = N_{G^0}(S)/Z_{G^0}(S).$$

Fix a connected component S' of S such that S' generates S . Then, S' is a closed *commutative connected subset* in G .

Choose a biinvariant Riemannian metric on G . By restriction it gives a Γ^0 invariant positive definite inner product on the Lie algebra of S^0 , hence also a Γ^0 invariant positive definite inner product on $X^*(S^0)$, denoted by (\cdot, \cdot) . Set

$$\Psi_{S^0} = \{0 \neq \alpha \in X^*(S^0) : \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}, \forall \lambda \in X^*(S^0)\}.$$

It is a root system in $X^*(S^0)$ ([17, Definition 2.2]). Choose and fix a positive system $\Psi_{S^0}^+$ of Ψ_{S^0} .

Set

$$\Psi_S = \{\alpha \in X^*(S) : \alpha|_{S^0} \in \Psi_{S^0}^+\}.$$

For any $\alpha \in \Psi_S$, there exists a unique $\check{\alpha} \in X_*(S^0)$ such that

$$\lambda(\check{\alpha}) = \frac{2(\lambda, \alpha|_{S^0})}{(\alpha|_{S^0}, \alpha|_{S^0}), \forall \lambda \in X^*(S^0).$$

Define $s_\alpha : X^*(S) \rightarrow X^*(S)$ by

$$s_\alpha(\lambda) = \lambda - (\lambda|_{S^0})(\check{\alpha})\alpha, \quad \forall \lambda \in X^*(S).$$

Then, $s_\alpha(\alpha) = -\alpha$, $s_\alpha^2 = 1$, and

$$s_\alpha(\lambda) = \lambda \Leftrightarrow (\lambda|_{S^0})(\check{\alpha}) = 0.$$

Definition 3.5. We call a subset R of $X^*(S)$ an affine root system on S if it satisfies the following conditions⁶,

- (1) $R \subset \Psi_S$.
- (2) $s_\alpha(R) = R$ for any $\alpha \in R$.
- (3) for any $\alpha \in R$, $2\alpha \notin R$.
- (4) for any $\alpha \in R$ with $2\alpha|_{S^0} \notin \{\beta|_{S^0} : \beta \in R\}$, there exists $\delta \in X^*(S/S^0)$ ($\subset X^*(S)$) of order m ($\in \mathbb{Z}_{\geq 1}$) such that

$$\begin{aligned} & \{\beta \in R : \beta|_{S^0} = \alpha|_{S^0}\} \\ &= \{\alpha + k\delta : 0 \leq k \leq m-1\}. \end{aligned}$$

- (5) for any $\alpha \in R$ with $2\alpha|_{S^0} \in \{\beta|_{S^0} : \beta \in R\}$, there exists $\delta \in X^*(S/S^0)$ ($\subset X^*(S)$) of order $2m$ ($m \in \mathbb{Z}_{\geq 1}$) such that

$$\begin{aligned} & \{\beta \in R : \beta|_{S^0} \in \mathbb{Z} \cdot \alpha|_{S^0}\} \\ &= \{\alpha + k\delta : 0 \leq k \leq 2m-1\} \cup \{2\alpha + (2k+1)\delta : 0 \leq k \leq m-1\}. \end{aligned}$$

Lemma 3.22. Let H be a compact subgroup of G with S a generalized Cartan subgroup of H . Then the affine root system of H with respect to S satisfies the Definition 3.5.

Proof. This follows from Lemma 3.6, Lemma 3.7, and the proof of Lemma 3.16 (or from the description of the set $(R - R_0, R')$ given in Subsection 3.4). \square

Definition 3.6. For an affine root system R on S , write $W_R (\subset \text{Aut}(S))$ for the finite group generated by s_α ($\alpha \in R$), and call it the Weyl group of R .

Write $R' = \{\alpha|_{S^0} : \alpha \in R\}$.

By Definition 3.5, it is clear that R' is a root system in $X^*(S^0)$ in the sense of [17, Definition 2.2].

Choose a simple system $\{\alpha'_1, \dots, \alpha'_t\}$ of the root system R' . For each i , choose $\alpha_i \in R$ such that $\alpha_i|_{S^0} = \alpha'_i$. Take $s_0 \in S'$ such that

$$\alpha_1(s_0) = \dots = \alpha_t(s_0) = 1.$$

Let

$$R_{s_0} = \{\alpha \in R : \alpha(s_0) = 1\}.$$

Then, the map

$$R_{s_0} \rightarrow R', \quad \alpha \mapsto \alpha' = \alpha|_{S^0}$$

is an isomorphism of root systems.

⁶Note that S/S^0 is a finite cyclic group. Actually one can show that the axioms (1)-(3) imply the axioms (4) and (5) (cf. [19]).

For each $\alpha' \in R'$ with $\frac{1}{2}\alpha' \notin R'$, set

$$R_{1,\alpha'} = \{\beta \in R : \beta|_{S^0} = \alpha'\}, \quad R_{2,\alpha'} = \{\beta \in R(G, A) : \beta|_{S^0} = 2\alpha'\}$$

and

$$R_{\alpha'} = R_{1,\alpha'} \cup R_{2,\alpha'}.$$

Write

$$m_{1,\alpha'} = |R_{1,\alpha'}|, \quad m_{2,\alpha'} = |R_{2,\alpha'}|$$

and $m_{\alpha'} = m_{1,\alpha'} + 2m_{2,\alpha'}$.

Write $R' = R \cap \Psi_{S^0}^+$.

Definition 3.7. *Set*

$$\delta = \delta_R = \frac{1}{2} \sum_{\alpha' \in R'^+, \text{reduced}} m_{\alpha'} \alpha.$$

Define

$$A_R = \frac{1}{|W_{R'}|} \sum_{w \in W_{R_{s_0}}} \epsilon(w) [\delta_R - w\delta_R].$$

The following lemma could be shown the same as for Lemma 3.18.

Lemma 3.23. *$2\delta_R$ and $\delta_R - w\delta_R$ ($W \in W_{R_{s_0}}$) are in $X^*(S)$ and they are independent of the choice of α for $\alpha' \in R'$.*

$A_R \in \mathbb{Q}[X^(S)]$ depends only on R , not on the choice of $\alpha_1, \dots, \alpha_t$ (or of s_0).*

Lemma 3.24. *We have*

$$A_R = \frac{1}{|W_R|} \sum_{w \in W_R} \epsilon(w) [\delta_R - w\delta_R].$$

Proof. Write $R^+ = \{\beta \in R : \beta|_{S^0} \in R'^+\}$. Due to

$$\delta_R - w\delta_R = \sum_{\beta \in R^+ \cap -w \cdot R^+} \beta,$$

we see that $\delta_R - w\delta_R$ ($w \in W_R$) depends only on the image of projection of w in $W_{R'}$. Thus,

$$A_R = \frac{1}{|W_R|} \sum_{w \in W_R} \epsilon(w) [\delta_R - w\delta_R].$$

□

Definition 3.8. *For an affine root system R on S , and a finite group W between W_R and $\text{Aut}(S)$, set*

$$F_{R,W} = \frac{1}{|W|} \sum_{\gamma \in W} \gamma \cdot A_R.$$

The following proposition is a generalization of [17, Proposition 3.8].

Proposition 3.25. *Given a compact Lie group G , let $H_1, H_2, \dots, H_s \subset G$ ($s \geq 2$) be a collection of closed subgroups of G . For non-zero constants c_1, \dots, c_s ,*

$$\sum_{1 \leq i \leq s} c_i \mathcal{D}_{H_i} = 0$$

if and only if: for any closed commutative connected subset S' of G ,

$$\sum_{1 \leq j \leq t} \frac{c_{i_j}}{|G_{i_j}/G_{i_j}^0|} F_{\Phi_j, \Gamma^\circ}|_{S'} = 0.$$

Here

$$\Gamma^\circ = N_G(S')/Z_G(S'),$$

$\{H_{i_j} | i_1 \leq i_2 \leq \dots \leq i_t\}$ are all subgroups amongst $\{H_i | 1 \leq i \leq s\}$ with H_{i_j} contains a maximal commutative connected subset conjugate to S' and Φ_j is the affine root system of H_{i_j} with respect to $S = s^{-1}S'$ ($s \in S'$). If a subgroup H_i contains k maximal commutative connected subsets conjugate to S' in G , then H_i appears k times in the summation (however the corresponding affine root systems on S' might be different).

Proof. With Propositions 3.5 and 3.19 available, the proof of this proposition is similar to the proof of [17, Proposition 3.8]. \square

By Proposition 3.25, to study equalities and linear relations among dimension data of disconnected subgroups, it suffices to study linear relations among the characters F_{R, Γ° for a fixed generalized torus S and affine root systems R on S . Write $\Psi = \Psi_S$ and $\Psi' = \Psi'_S$. First one can reduce it to the case of Ψ' is an irreducible root system and⁷

$$\bigcap_{\alpha' \in \Psi'} \ker \alpha' = 1.$$

Question 3.1. *Let S be a generalized torus with an inner product on its Lie algebra, and with $\Psi' = \Psi_{S^0}$ an irreducible root system such that*

$$\bigcap_{\alpha' \in \Psi'} \ker \alpha' = 1.$$

How to find all equalities and linear relations among the characters F_{R, W_Ψ} for sub-affine root systems R of $\Psi = \Psi_S$?

To solve Question 3.1, we need to first classify affine sub-root systems $R \subset \Psi$. As remarked in Subsection 3.5, this consists of three parts: (1), classify sub-root systems R' of Ψ ; (2), specify the multiplicity m_i for an irreducible factor R'_i of R' and multiplicities $m_{1, \alpha'}$ and $m_{\alpha'}$ for roots $\alpha' \in R'$; (3), specify liftings in R of simple roots of R' .

⁷This is analogous to the property of root system for a connected compact semisimple Lie group of adjoint type.

In the case of $\Psi' = \text{BC}_n$, we should associate polynomials to each irreducible sub-affine root systems and study their properties and multiplicative and algebraic relations. These polynomials are variants of the polynomials a_n, b_n, c_n, d_n in the connected case (cf. [17, Page 2713]), subject to operations on indeterminates and coefficients. It looks to us [17, Proposition 7.2]) has a generalization to the affine case. The cases of $\Psi' = \text{B}_n, \text{C}_n, \text{D}_n, \text{A}_{n-1}$ reduce to the BC_n case.

In case Ψ' is an exceptional irreducible root system, like in [17] we can calculate the dominant terms $2\delta_R$ and the polynomials

$$f_R(t) = \frac{1}{|W_R|} \sum_{w \in W_R} \epsilon(w) t^{|\delta_R - w\delta_R|^2}.$$

However, these invariants seem too weak in the affine case. We haven't thought seriously yet concerning the case of Ψ' is an exceptional irreducible root system.

4. COMPACTNESS OF ISOSPECTRAL SET

A conjecture of Osgood-Philipps-Sarnak states that set of closed Riemannian manifolds with a given Laplace spectrum (called an *isospectral set* of Riemannian manifolds) should be compact ([13], [4]). By considering normal homogeneous spaces, we proved a result before.

Theorem 4.1. ([18, Theorem 3.6]) *Given a compact Lie group G equipped with a bi-invariant Riemannian metric m_0 , up to isometry, any collection of isospectral normal homogeneous spaces of the form $(G/H, m_0)$, where $H \subset G$ is a closed subgroup, must be finite. That is, for any closed subgroup H of G , up to conjugacy, there are finitely many closed subgroups H_1, \dots, H_k of G such that the normal homogeneous space $(G/H_j, m_0)$ is isospectral to $(G/H, m_0)$.*

Here we prove a generalization of the above theorem by allowing the metric on G to vary.

Theorem 4.2. *Given a spectrum and a compact semisimple Lie group G , there are only finitely many conjugacy classes of closed subgroups H of G such that there exists a bi-invariant Riemannian metric m on G which induces a normal homogeneous space $(G/H, m)$ with Laplace spectrum equals to the given spectrum.*

Proof. First we may assume that G is connected and simply connected. Write $G = G_1 \times \dots \times G_s$ for the decomposition of G into simple factors. For each i , choose a bi-invariant Riemannian metric m_i on G_i . By normalization we may assume that the Laplace operator and Casimir operator coincide on $C^\infty(G_i)$ ($1 \leq i \leq s$).

Let $\{(G/H_n, m_n) : n \geq 1\}$ be a sequence of normal homogeneous spaces such that the Laplace spectrum of each $(G/H_n, m_n)$ is equal to a given spectrum, and H_n ($n \geq 1$) are non-conjugate to each other. Write

$$m_n = \bigoplus_{1 \leq i \leq s} a_i^{(n)} m_i.$$

By [18, Theorem 1.1], there exists a closed subgroup H of G , a subsequence $\{H_{n_j} : j \geq 1\}$ and a sequence $\{g_j : j \geq 1, g_j \in G\}$ such that for all $j \in \mathbb{N}$,

$$[H^0, H^0] \subset g_j H_{n_j} g_j^{-1} \subset H,$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_{n_j}} = \mathcal{D}_H.$$

Substituting $\{(G/H_n, m_n) : n \geq 1\}$ by a subsequence if necessary we may assume that: for any $n \geq 1$,

$$[H^0, H^0] \subset H_n \subset H,$$

and

$$\lim_{j \rightarrow \infty} \mathcal{D}_{H_n} = \mathcal{D}_H.$$

Since H_n are assumed to be non-conjugate to each other, at most finitely many of them contain H^0 . By removing such exceptions. we may assume that $\dim H_n < \dim H$ for all n .

We may also assume that each sequence $\{a_i^{(n)} : n \geq 1\}$ converges. Write

$$a_i = \lim_{n \rightarrow \infty} a_i^{(n)} \in [0, \infty].$$

Without loss of generality we assume that

$$\begin{aligned} a_1 &= \cdots = a_u = 0, \\ 0 &< a_{u+1}, \dots, a_v < \infty, \\ a_{v+1} &= \cdots = a_s = \infty, \end{aligned}$$

where $0 \leq u \leq v \leq s$. Write

$$\begin{aligned} G' &= \prod_{u+1 \leq i \leq v} G_i, \\ H' &= G' \cap H \prod_{1 \leq i \leq u} G_i, \end{aligned}$$

and

$$m' = \bigoplus_{u+1 \leq i \leq v} a_i m_i.$$

Claim 4.1. *We have*

$$\prod_{v+1 \leq i \leq s} G_i \subset H \prod_{1 \leq i \leq v} G_i,$$

and the Laplace spectrum of $(G'/H', m')$ equals to the given spectrum.

Write

$$G'' = \prod_{1 \leq i \leq v} G_i, \quad G''' = \prod_{v+1 \leq i \leq s} G_i.$$

Since

$$G''' \subset H G''$$

by Claim 4.1, we get

$$\dim G/H_n > \dim G/H = \dim G''/G'' \cap H \geq \dim G'/H'.$$

By Claim 4.1, $(G'/H', m')$ and $(G/H_n, m_n)$ are isospectral. As Laplace spectrum determines dimension ([4]), we get a contradiction. \square

Proof of Claim 4.1. Write $\chi_j(\rho_j)$ ($1 \leq j \leq s$) for the eigenvalue of Laplace operator associated to (G_j, m_j) on matrix coefficients of $\rho_j \in \widehat{G_j}$. We know that: $\chi_j(\rho_j) \geq 0$; $\chi_j(\rho_j) = 0$ if and only if $\rho_j = 1$; for any positive real number c , there are only finitely many $\rho_j \in \widehat{G_j}$ such that $\chi_j(\rho_j) \leq c$.

Suppose

$$\prod_{v+1 \leq i \leq s} G_i \not\subset H \prod_{1 \leq i \leq v} G_i.$$

Then, there exists a nontrivial irreducible representation

$$\rho = \bigotimes_{v+1 \leq i \leq s} \rho_i$$

of $\prod_{v+1 \leq i \leq s} G_i$ such that $\dim V_\rho^{\prod_{v+1 \leq i \leq s} G_i \cap H \prod_{1 \leq i \leq v} G_i} > 0$. Take

$$0 \neq v \in V_\rho^{\prod_{v+1 \leq i \leq s} G_i \cap H \prod_{1 \leq i \leq v} G_i}$$

and $0 \neq \alpha \in V_\rho^*$. Set

$$f_{v,\alpha}(g_1, \dots, g_s) = \alpha((g_{v+1}, \dots, g_s)^{-1}v).$$

Then, $f_{v,\alpha} \in C^\infty(G/H) \subset C^\infty(G/H_n)$ for any $n \geq 1$. The Laplace eigenvalue for $f_{v,\alpha} \in C^\infty(G/H_n)$ is equal to

$$\sum_{v+1 \leq i \leq s} \frac{1}{a_i^{(n)}} \chi_i(\rho_i) > 0.$$

When $n \rightarrow \infty$, this value tends to 0, which is in contradiction with the fact that the Laplace spectrum of all G/H_n are equal to a given spectrum.

Now we assume

$$G''' \subset HG''.$$

Due to $[H^0, H^0] \subset H_n$ for any $n \geq 1$, we have $G''' \subset H_n G''$ for any $n \geq 1$. Each H_n is of the form

$$H_n = (H_n \cap G'') \times \{(\phi_n(x), x) : x \in G'''\}$$

for some homomorphism $\phi_n : G''' \rightarrow G''$. Since G''' is semisimple,

$$\text{Hom}(G''', G'') / \sim_{G''}$$

($\sim_{G''}$ means conjugation action of G'' on $\text{Hom}(G''', G'')$ by acting on the target of homomorphisms) is a finite set. Replacing H_n by a subsequence if necessary, we may assume that there exists $\phi \in \text{Hom}(G''', G'')$ such that $\phi_n = \phi$ for any $n \geq 1$. Moreover we may assume that

$$H = (H \cap G'') \times \{(\phi(x), x) : x \in G'''\}$$

and

$$\lim_{n \rightarrow \infty} \mathcal{D}_{H_n \cap G''} = \mathcal{D}_{H \cap G''}$$

(equivalent to say, as dimension data of subgroups of G'' , or as dimension data of subgroups of G). Due to $G''' \subset H_n G''$, we have $G/H_n = G''/G''' \cap H_n$.

Let c be a positive real number. Suppose matrix coefficients of

$$\rho = \bigotimes_{1 \leq i \leq s} \rho_i$$

contributes to Laplace spectrum of G/H_n in the eigenvalue scope $[0, c]$. Then,

$$\sum_{1 \leq i \leq s} \frac{1}{a_i^{(n)}} \chi_i(\rho_i) \leq c.$$

Due to $a_i^{(n)} \rightarrow a_i$ and $a_i = 0$ for $1 \leq i \leq u$, we have: ρ_i ($u+1 \leq i \leq s$) lies in a finite set when $n \gg 0$; $\rho_i = 1$ ($1 \leq i \leq u$) when $n \gg 0$. Due to H_n is of the form

$$H_n = (H_n \cap G'') \times \{(\phi(x), x) : x \in G'''\},$$

$\bigotimes_{v+1 \leq i \leq s} \rho_i$ is determined by $\bigotimes_{1 \leq i \leq v} \rho_i$ up to finitely many possibility. Thus, each ρ_i ($u+1 \leq i \leq s$) lies in a finite set when $n \gg 0$. Using $\lim_{n \rightarrow \infty} \mathcal{D}_{G'' \cap H_n} = \mathcal{D}_{G'' \cap H}$, we can show that the eigenvalues stabilize and the invariant dimensions $\dim V_{\rho_i}^{G' \cap H_n \prod_{1 \leq i \leq u} G_i}$ ($u+1 \leq i \leq s$) stabilize to $\dim V_{\rho_i}^{G' \cap H \prod_{1 \leq i \leq u} G_i}$. This indicates: the Laplace spectrum of $(G'/H', m')$ is larger than the given spectrum. Using $H_n \subset H$, we can show that the Laplace spectrum of $(G'/H', m')$ is smaller than the given spectrum. Therefore, the Laplace spectrum of $(G'/H', m')$ equals to the given spectrum. \square

Motivated by the compactness conjecture of isospectral sets, we think the following statement should hold.

Question 4.1. *Given a spectrum, can we show that there exist only finitely many normal homogeneous spaces $(G/H, m)$ up to isometry with Laplace spectrum equal to the given spectrum?*

In case G and m is given, this follows from Theorem 4.1.

Any normal homogeneous space is of the form $M = G/H$, where

$$G = T \prod_{1 \leq i \leq s} G_i$$

with T a torus and each G_i ($1 \leq i \leq s$) a connected and simply-connected compact simple Lie group, $H \cap T = 1$, and $G_i \not\subset H$ for any i . Let $M = G/H$ be of the this form.

In case G is semisimple, as $\dim G/H$ is determined from Laplace, we can show that there are only finitely many possible G by the condition $G_i \not\subset H$ for any i . For a fixed G , there are only finitely many possible G/H by Theorem 4.2. Then, Question 4.1 reduces the the following statement, which is an algebraic question and seems not hard to prove.

Question 4.2. *Given a spectrum, a connected compact Lie group G , and a closed subgroup H , can we show that there exist only finitely many normal*

homogeneous spaces $(G/H, m)$ up to isometry with Laplace spectrum equal to the given spectrum?

In case G is a torus, $H = 1$ by the above assumption. In this case the statement is a theorem of Kneser. A simple proof is given in [16], which is based on the Mahler compactness theorem for lattices.

In case G is neither semisimple nor a torus, again we have only finitely many possible G by dimension reason. We can't show yet finitely many possibility of G/H when G is fixed. The difficulty is due to there may exist infinitely many metrics on the T part which give isometric metrics on G/H . Probably a sophisticated use of Mahler compactness theorem could overcome this difficulty.

5. QUESTIONS

Besides some un-solved questions asked in the main body of this paper, Questions 2.2, 3.1, 4.1, we have some more questions concerning dimension datum and τ -dimension datum.

Question 5.1. *For subgroups G, H_1, H_2 as in Theorem 2.1, do we have explicit formula for $\dim \text{Hom}_{H_1}(\tau_\lambda, \rho|_{H_1})$ ($= \dim \text{Hom}_{H_2}(\tau_{\lambda'}, \rho|_{H_2})$)?*

Question 5.2. *It is hard to calculate the τ -dimension datum in general. But, do we have an asymptotic formula for $\dim \text{Hom}_H(\tau, \rho|_H)$?*

We tried to calculate dimension datum or its asymptotics from the character F_{Φ, Γ^0} , but failed. Another way one might think is whether there is a connection of asymptotics of dimension datum and the asymptotics studied in [6].

Question 5.3. *Let G be a connected compact Lie group, and H_1, H_2 be two closed subgroups. If*

$$\dim V_\rho^{H_1} = \dim V_\rho^{H_2}$$

for all but finitely many $\rho \in \widehat{G}$, is then $\mathcal{D}_{H_1} = \mathcal{D}_{H_2}$?

For this question, one may first study the asymptotics of $\dim V_\rho^{H_i}$ ($i = 1, 2$) and use the asymptotical coincidence to show equality of dimension data.

Question 5.4. *Let G be a connected compact Lie group, and H_1, H_2 be two connected closed subgroups. Suppose $(Z_G(H_i))^0 = Z(G)^0$. Does $\mathcal{D}_{H_1} = \mathcal{D}_{H_2}$ imply that $H_1 \sim H_2$?*

We call a connected closed subgroup H of a connected compact Lie group G an *elliptic subgroup* if $(Z_G(H))^0 = Z(G)^0$. Note that, if $\mathcal{D}_{H_1} = \mathcal{D}_{H_2}$ and H_1 is an elliptic subgroup, then so is H_2 . One proves this by just taking ρ the adjoint module of G .

Question 5.4 is related to Questions asked in Arthur's paper [2]. We have made some progress along this direction. It remains to be completed.

Besides τ -dimension datum and dimension data for disconnected subgroups studied in this paper, in the most generality we may study of equalities and linear relations of τ -dimension data for disconnected subgroups. In this case the dimension datum is equivalent to a kind of characters which have a mixed form of the characters used in studying τ -dimension datum or in studying dimension datum of disconnected subgroups. We may develop a theory for this situation in future.

Now we know that the dimension datum nearly determines the conjugacy class of a connected closed subgroup, but there are a few exceptions. Several mathematicians asked if there are extra information besides dimension datum extracted from L-function which together with dimension datum could determine the conjugacy class of the H^π defined by Langlands ([9]). This is an interesting question to think.

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