MAXIMAL ANTIPODAL SETS IN IRREDUCIBLE COMPACT SYMMETRIC SPACES

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Abstract. We give an explicit classification of maximal antipodal sets in any irreducible compact symmetric space except for spin groups and half spin groups, and some quotient symmetric spaces associated to them.

Contents

1. Introduction

A closed Riemannian manifold M is said to be a compact symmetric space if for any point $x \in M$, there is a Riemannian isometry $s_x : M \to M$ such that: (i) $s_x = x$; (ii) the tangent map $(s_x)_* : T_x(M) \to T_x(M)$ is -1. For any compact symmetric space M, it is known that there exists a connected compact Lie group G and an involutive automorphism θ of it such that $M = G/G^\theta$ (cf. [8, Thm. 4.6, p. 185]). We call a nonempty subset X of M an *antipodal set* if

$$s_x(y) = y \ (\forall x, y \in X).$$

An antipodal set must be a finite set since it is a discrete set and M is compact. We call an antipodal set a *maximal antipodal set* if it is not properly contained in any other antipodal set. In [4], Chen and Nagano introduced and calculated the invariant 2-number $\#_2(M)$ of a compact symmetric space, which is the maximal cardinality of antipodal sets in a compact symmetric space M. After this paper, there are many studies on maximal antipodal sets. Particularly, Tanaka and Tasaki made the classification of maximal antipodal sets for some kinds of compact symmetric

2010 *Mathematics Subject Classification.* 22E46, 53C35.
Key words and phrases. Compact symmetric space, maximal antipodal set, Cartan quadratic morphism, elementary abelian 2-subgroup.

spaces ([12], [13], [14], [15]): symmetric R-spaces, some compact classical Lie groups, etc. The readers may consult [2] for an excellent survey on the study of 2-numbers and maximal antipodal sets. In this paper we deduce the classification of maximal antipodal sets from the classification of elementary abelian 2-subgroups in compact Lie groups ([17]).

Let G be a connected compact Lie group, and θ be an involutive automorphism of it. Put $M=G/G^{\theta}$. Set $\bar{G}=G\rtimes\langle\bar{\theta}\rangle$, where $\bar{\theta}^2=1$ and $\mathrm{Ad}(\bar{\theta})|_{G}=\theta$. Write $C_{\bar{\theta}}=\{g\bar{\theta}g^{-1}:g\in G\}$. The *Cartan quadratic morphism* (cf. [2]) is a map $\phi:G/G^{\theta}\to G$ defined by

$$\phi(gG^{\theta})=g\theta(g)^{-1},\ \forall g\in G.$$

Let X be a subset of M containing the origin $o=eG^{\theta}\in M$. Write

$$\phi(X)=\{\phi(x):x\in X\}\subset G,$$

$$F_1(X)=\langle\phi(X)\rangle\subset G$$

and

$$F_2(X)=\langle\phi(X),\bar{\theta}\rangle\subset\bar{G}.$$

Using the Cartan quadratic morphism, we show a correspondence between maximal antipodal sets in G/G^{θ} and certain elementary abelian 2-subgroups of $\bar{G}$.

Theorem 1.1. *Let X be a subset of $M=G/G^{\theta}$ containing the origin $o=eG^{\theta}\in M$. Then X is a maximal antipodal set if and only if $F_2(X)$ is a maximal element in the set of elementary abelian 2-subgroups of $\bar{G}$ which are generated by elements in $C_{\bar{\theta}}$, and*

$$X=\{x\in M:\phi(x)\in F_2(X)\cap C_{\bar{\theta}}\bar{\theta}^{-1}\}.$$

We call a compact symmetric space "irreducible" if it is not isogenous to the product of two positive-dimensional compact symmetric spaces. With an explicit list of irreducible compact symmetric spaces, we show a precise classification of maximal antipodal sets in most of them using Theorem 1.1. The only cases which haven't been treated are spin groups and half spin groups, and some quotient symmetric spaces of them.

The content of this paper is organized as follows. In Proposition 2.1, we give a criterion of antipodal sets using the Cartan quadratic morphism $\phi:G/G^{\theta}\to G$. With it, we show Theorem 1.1. In Subsection 2.2, we study Weyl groups of maximal antipodal sets. In Section 3, we give a precise list of irreducible compact symmetric spaces that are not of group form. In Section 4, we present an explicit classification of maximal antipodal sets in most irreducible compact symmetric spaces. The remaining ones which haven't been treated are listed in Subsection 5.3. In Subsection 5.1, we illustrate how to classify G^{θ} orbits in the fixed point set of s_0, which are related to polars defined by Chen and Nagano.

Notation and conventions. In this paper a compact Lie group G is said to be "simple" if its Lie algebra is a non-abelian simple Lie algebra. Let E_6^{sc} (or E_6) denote a connected and simply-connected compact simple Lie group of type E_6; let E_6^{ad} denote a connected adjoint type compact simple Lie group of type E_6. Similarly, we have the notations E_7^{sc}, E_7, E_7^{ad}, E_8, F_4, G_2. The last three are connected compact Lie groups which are both simply-connected and of adjoint type.

Write

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

In $\mathrm{Spin}(2n)$, write $c = c_n = e_1 \cdots e_{2n}$, where $\{e_1, e_2, \ldots, e_{2n}\}$ is a standard normal basis of the Euclidean space based on which $\mathrm{Spin}(2n)$ is defined. Involutive automorphisms σ_i of compact exceptional simple Lie algebras are as specified in [7, Table 1].

Write $\omega_m = e^{\frac{2\pi i}{m}}$, which is a primitive m-th root of unity.

Acknowledgements. A part of this work was done when the author visited MPI Bonn in the summer of 2016 and a draft was written when the author visited National University of Singapore in January 2018. The author would like to thank both institutions for their support and hospitality. I would like to thank the referees for providing useful references and giving very helpful comments. This research is partially supported by the NSFC Grant 11971036.

2. Characterization of antipodal sets

2.1. **Proof of Theorem 1.1.** Let G be a connected compact Lie group and θ be an involutive automorphism of it. Write $H = G^\theta$. Put $M = G/G^\theta$, which is a compact symmetric space. Let $o = eG^\theta$ denote the *origin*. There is a left G action on G/G^θ through

$$L_g(g'G^\theta) = g \cdot g'G^\theta = gg'G^\theta,$$

and there is a G-action on itself through

$$g * g' = gg'\theta(g)^{-1}.$$

The Cartan quadratic morphism map ϕ is G-equivariant with regard to these two actions, i.e.,

$$\phi(g \cdot x) = g * \phi(x), \ \forall g \in G, \forall x \in G/H.$$

It is clear that ϕ is an imbedding. Apparently, the translation by any element in G of an antipodal set in G/G^θ is still an antipodal set.

Proposition 2.1. *Let X be a subset of M containing the origin $o = eH \in M = G/H$. Then X is an antipodal set if and only if $\phi(x) \in H$ and $\phi(x)^2 = 1$ for any $x \in X$, and $\phi(x)$ commutes with $\phi(y)$ for any $x, y \in X$.*

Proof. We first show that: X is an antipodal set if and only if $\phi(g_2^{-1}g_1) \in H$ for any two points $x_1 = g_1H \in G/H$ and $x_2 = g_2H \in G/H$. Note that

$$s_o(gH) = \theta(g)H, \ \forall g \in G.$$

Since $L_{g_1}(o) = g_1H = x_1$, we have $s_{x_1} = L_{g_1}s_oL_{g_1}^{-1}$. Then,

$$s_{x_1}(x_2) = L_{g_1}s_oL_{g_1}^{-1}(g_2H) = L_{g_1}s_o(g_1^{-1}g_2H) = L_{g_1}(\theta(g_1^{-1}g_2)H) = g_1\theta(g_1^{-1}g_2)H.$$

Thus, $x_2 = s_{x_1}(x_2)$ if and only if $g_1\theta(g_1^{-1}g_2)H = g_2H$. This is equivalent to

$$\phi(g_2^{-1}g_1) = g_2^{-1}g_1\theta(g_1^{-1}g_2) \in H.$$

Necessarity. Suppose X is an antipodal set. Write $x = gH \in X$. Taking $x_1 = x$ and $x_2 = o$, we get $\phi(x) = g\theta(g)^{-1} \in H$. That is to say, $\theta(\phi(x)) = \phi(x)$. We also have

$$\theta(\phi(x)) = \theta(g\theta(g)^{-1}) = \theta(g)g^{-1} = \phi(x)^{-1}.$$

Thus, $\phi(x)^2 = 1$. Taking $x = g_1H \in X$ and $y = g_2H \in X$, we get $\phi(g_2^{-1}g_1) \in H$. By the argument above this leads to $\phi(g_2^{-1}g_1)^2 = 1$. Equivalently,

$$(g_2^{-1}g_1\theta(g_1^{-1})\theta(g_2))^2 = 1.$$

This is equivalent to $(\phi(x)\phi(y)^{-1})^2 = 1$. Since $\phi(x)^2 = \phi(y)^2 = 1$, it follows that: $\phi(x)$ commutes with $\phi(y)$.

Sufficiency. Suppose $\phi(x) \in H$ and $\phi(x)^2 = 1$ for any $x \in X$, and $\phi(x)$ commutes with $\phi(y)$ for any $x, y \in X$. For any two points $x, y \in M$, write $x = g_1H$ and $y = g_2H$. Reverse to the above argument, by the conditions of $\phi(x)^2 = \phi(y)^2 = 1$ and $\phi(x)$ commutes with $\phi(y)$, one gets $\phi(g_2^{-1}g_1)^2 = 1$. Again by the above argument, this is equivalent to $\phi(g_2^{-1}g_1) \in H$. Then, X is an antipodal set. □

Proof of Theorem 1.1. Assume that X is a maximal antipodal set. By Proposition 2.1, $F_2(X)$ is an elementary abelian 2-subgroup of $\bar{G}$ generated by elements in $C_{\bar{\theta}}$. Write

$$X' = \{x \in M : \phi(x) \in F_2(X) \cap C_{\bar{\theta}}\bar{\theta}^{-1}\}.$$

Then, $X \subset X'$ and $F_2(X') \subset F_2(X)$. By Proposition 2.1, X' is an antipodal set. By the maximality of X, we get $X = X'$. By a similar argument, one shows that $F_2(X)$ is a maximal element in the set of elementary abelian 2-subgroups of $\bar{G}$ which are generated by elements in $C_{\bar{\theta}}$. The converse is clear. □

Note that each elementary abelian 2-subgroup is contained in a maximal one. In practice, we first classify maximal elementary abelian 2-subgroups of $\bar{G}$ containing $\bar{\theta}$ up to conjugacy. Take such an F and let X be the set elements $x \in M$ such that $\phi(x) \in F$. Then, we remove such X which are not maximal and leave only the maximal ones. In this way, we get all maximal antipodal sets in G/G^θ up to conjugacy.

2.2. **Weyl group.** Define a map $\psi : G/H \to \bar{G}$ by

$$\psi(gH) = g\bar{\theta}g^{-1}.$$

Let X be a subset of $M = G/H$, not necessarily contain the origin. Put $\psi(X) = \{\psi(x) : x \in X\}$. By Proposition 2.1, one can show that X is an antipodal set in M if and only if $\psi(X)$ generates an elementary abelian 2-subgroup of G. Let it be still denote by $F_2(X)$. Set

$$N_G(X) = \{g \in G : g \cdot X = X\},$$
$$Z_G(X) = \{g \in G : g \cdot x = x,\ \forall x \in X\}$$

and $W_G(X) = N_G(X)/Z_G(X)$. Apparently, the conjugation action of any $g \in N_G(X)$ on G stabilizes $F_2(X)$, and the inducing action on $F_2(X)$ is trivial if and only if $g \in Z_G(X)$. Thus, we have an injective homomorphism $W_G(X) \hookrightarrow W_G(F_2(X))$. It is clear that

$$W_G(X) = \{w \in W_G(F_2(X)) : w \cdot \psi(X) = \psi(X)\}.$$

Proposition 2.2. *If X is a maximal antipodal set in M, then $W_G(X) = W_G(F_2(X))$.*

Proof. Write $X' = \{x \in M : \psi(x) \in F_2(X)\}$. Then, $X' \supset X$. Using Proposition 2.1 one can show that X' is an antipodal set. By the maximality of X, we get $X = X'$. Sine the conjugation action of each $w \in W_G(F_2(X))$ on $F_2(X)$ preserves conjugacy classes, it stabilizes $C_{\bar{\theta}} \cap F_2(X) = \psi(X') = \psi(X)$. Hence,

$$W_G(X) = \{w \in W_G(F_2(X)) : w \cdot \psi(X) = \psi(X)\} = W_G(F_2(X)).$$

□

2.3. **Irreducible compact symmetric spaces of adjoint type.** Now assume that G is a connected compact simple Lie group of *adjoint type*. Let $\mathfrak{u}_0$ be the Lie algebra of G, which is a compact simple Lie algebra. Then, $G \cong \mathrm{Int}(\mathfrak{u}_0)$. For simplicity we identify G with $\mathrm{Int}(\mathfrak{u}_0)$, and regard θ as an element of $\mathrm{Aut}(\mathfrak{u}_0)$ which acts on $G = \mathrm{Int}(\mathfrak{u}_0)$ by conjugation. Divide the discussion into two cases: (i)θ is an inner automorphism; (ii)θ is an outer automorphism. In the first case, $\theta \in \mathrm{Int}(\mathfrak{u}_0) = G$ and $\bar{\theta}\theta^{-1}$ is a central element of $\bar{G}$. Thus, $\bar{G} = G \times \langle \bar{\theta}\theta^{-1} \rangle$. Let $\pi : \bar{G} \to \mathrm{Int}(\mathfrak{u}_0) = G$ be the adjoint homomorphism. Then $\pi|_G = \mathrm{id}$ and $\ker \pi = \langle \bar{\theta}\theta^{-1} \rangle$. Write

$$C_\theta = \{g\theta g^{-1} : g \in \mathrm{Int}(\mathfrak{u}_0)\} \subset \mathrm{Int}(\mathfrak{u}_0).$$

Let $F(X) = p(F_2(X))$. Then $\theta \in F(X)$ and $F(X)$ is an elementary abelian 2-subgroup of $\mathrm{Int}(\mathfrak{u}_0)$ generated by elements in C_θ. In the second case, $\theta \in \mathrm{Aut}(\mathfrak{u}_0) - \mathrm{Int}(\mathfrak{u}_0)$. We could identify $\bar{\theta}$ with $\theta \in \mathrm{Aut}(\mathfrak{u}_0)$ and regard $\bar{G}$ as a subgroup of $\mathrm{Aut}(\mathfrak{u}_0)$. Let $F(X) = F_2(X)$. Then $\theta \in F(X)$ and $F(X)$ is generated by elements in

$$C_\theta = \{g\theta g^{-1} : g \in \mathrm{Int}(\mathfrak{u}_0)\} \subset \mathrm{Aut}(\mathfrak{u}_0).$$

The following theorem follows from Theorem 1.1 directly.

Theorem 2.3. *Let X be a subset of $M = \mathrm{Int}(\mathfrak{u}_0)/\mathrm{Int}(\mathfrak{u}_0)^\theta$ containing the origin. Then, X is a maximal antipodal set if and only if $F(X)$ is a maximal element in the set of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ generated by elements in C_θ and*

$$X = \{gH : g\theta g^{-1} \in F(X)\}.$$

Remark 2.4. *With Theorem 2.3, we can deduce the classification of maximal antipodal sets in $\mathrm{Int}(\mathfrak{u}_0)/\mathrm{Int}(\mathfrak{u}_0)^\theta$ from the classification of elementary abelian 2-subgroups of $\mathrm{Aut}(\mathfrak{u}_0)$ given in [17]. It is only a routine work, we omit the details here. Note that conjugacy classes of elements of each elementary abelian 2-subgroup of $\mathrm{Aut}(\mathfrak{u}_0)$ are described well in [17].*

3. A precise list of irreducible compact symmetric spaces

Analogous to Lie groups, we use coverings to define isogeny for symmetric spaces.

Definition 3.1. *Two compact symmetric spaces M_1, M_2 are said to be isogenous if they admit isomorphic universal coverings.*

We define irreducible symmetric spaces as follows.

Definition 3.2. *A compact symmetric space M is said to be irreducible if there exists no positive-dimensional compact symmetric spaces M_1, M_2 such that M is isogenous to $M_1 \times M_2$.*

The following theorem is from [8], which is pointed out to the author by an anonymous referee.

Theorem 3.3 ([8], p. 145, Theorem 4.6)**.** *Let M be a compact symmetric space. Then there is a compact Lie group G and an involutive automorphism θ of G such that $M \cong G/G^{\theta}$.*

One can show that (for example, use Theorem 3.3) any irreducible compact symmetric space M is isomorphic to one of the following: (i)S^1; (ii)a compact simple Lie group; (iii)G/G^{θ} with G a compact simple Lie group and θ an involutive automorphism of it. Compact symmetric spaces in cases (i)-(ii) are said to be of *group form.*

Definition 3.4. *Let M be a compact symmetric space. We call M semisimple if its fundamental group $\pi_1(M)$ is finite; we call M simply-connected if $\pi_1(M) = 1$; we call M of adjoint type if there is no proper Riemannian covering $M \to M'$ for M' another compact symmetric space.*

In this section we give an explicit list of irreducible compact symmetric spaces that are not of group form by using Theorem 3.3 and calculating symmetric subgroups G^{θ} (cf. [7, Table 2, p. 408]). Recall that simply-connected compact symmetric spaces are classified by Élie Cartan and can be found in the classical textbooks like [6], [9], [16]. The description and construction of non-simply connected compact symmetric spaces are given in some excellent monographs (cf. [1, Thm. 4.5, p. 103], [6, Thm. 9.1, p. 326], [9, Proposition 2.4, p. 68-69], [16, Thm. 8.3.11, p. 244]).

1, Grassmannians. Put $c = e_1 \dots e_n \in \mathrm{Spin}(n)$ and $L_{2n} = \frac{1+e_1e_{2n+1}}{\sqrt{2}} \cdots \frac{1+e_{2n}e_{4n}}{\sqrt{2}} \in \mathrm{Spin}(4n)$. Any irreducible compact symmetric space M which is isogenous to a (real, complex or quaternion) Grassmannian is isomorphic to G/G^{θ} for some (G, θ) as in the following list:

(i) adjoint type: $G = \mathrm{PSU}(p+q)$, $\mathrm{PSO}(p+q)$ or $\mathrm{PSp}(p+q)$ $(q \geq p \geq 1)$, $\theta = \mathrm{Ad}(I_{p,q})$.
(ii) $G = \mathrm{SU}(2p)$ $(p \geq 1)$, $\theta = \mathrm{Ad}(I_{p,p})$, and $G^{\theta} = S(U(p) \times \mathrm{U}(p))$.
(iii) $G = \mathrm{Sp}(2p)$ $(p \geq 1)$, $\theta = \mathrm{Ad}(I_{p,p})$, and $G^{\theta} \cong \mathrm{Sp}(p) \times \mathrm{Sp}(p)$.
(iv) $G = \mathrm{SO}(2p)$ $(p \geq 4)$, $\theta = \mathrm{Ad}(I_{p,p})$, and $G^{\theta} = S(\mathrm{O}(p) \times \mathrm{O}(p))$.

(v) $G = \mathrm{Spin}(p+q)$ $(q \geq p \geq 1)$, $\theta = \mathrm{Ad}(e_1 \dots e_p)$, $G^\theta = \mathrm{Spin}(p) \cdot \mathrm{Spin}(q))$.
(vi) $G = \mathrm{Spin}(4n)/\langle c \rangle$ $(n \geq 2)$, $\theta = \mathrm{Ad}(e_1 \dots e_{2n})$, and $G^\theta = ((\mathrm{Spin}(2n) \cdot \mathrm{Spin}(2n)) \rtimes \langle L_{2n} \rangle)/\langle c \rangle$.

2, Types AI and AII. Write $G_{n,m} = \mathrm{SU}(n)/\langle \omega_m I \rangle$ for any integer $m|n$. Put $J_k = \begin{pmatrix} 0_k & I_k \\ -I_k & 0_k \end{pmatrix}$. Let τ be the complex conjugation on $\mathrm{SU}(n)$ (and on $G_{n,m}$). When n is even, let $\tau' = \tau \circ \mathrm{Ad}(J_{n/2})$. Any irreducible compact symmetric space M which is of type AI or AII in Cartan's notation is isomorphic to $G_{n,m}/G_{n,m}^\tau$ $(m|n)$ or $G_{n,m}/G_{n,m}^{\tau'}$ ($m|n$ and n is even). The isomorphism types of the groups $G_{n,m}^\tau$, $G_{n,m}^{\tau'}$ are as follows:

(1) If m is odd, then $G_{n,m}^\tau \cong \mathrm{SO}(n)$ and $G_{n,m}^{\tau'} \cong \mathrm{Sp}(n/2)$ (in case n is even).
(2) If m and $\frac{n}{m}$ are both even, then $G_{n,m}^\tau \cong \mathrm{PSO}(n) \times \mathbb{Z}/2\mathbb{Z}$ and $G_{n,m}^{\tau'} \cong \mathrm{PSp}(n/2) \times \mathbb{Z}/2\mathbb{Z}$.
(3) If m is even and $\frac{n}{m}$ is odd, then $G_{n,m}^\tau \cong \mathrm{PO}(n)$ and $G_{n,m}^{\tau'} \cong \mathrm{PSp}(n/2)$.

3, Types CI and DIII. Any irreducible compact symmetric space which is of type CI or DIII in Cartan's notation is isomorphic to G/G^θ for some (G,θ) as in the following list:

(i) adjoint type: $G = \mathrm{PSp}(n)$ $(n \geq 1)$, $\theta = \mathrm{Ad}(\mathbf{i}I)$.
(ii) $G = \mathrm{Sp}(n)$ $(n \geq 1)$, $\theta = \mathrm{Ad}(\mathbf{i}I)$, and $G^\theta = \mathrm{U}(n)$.
(iii) adjoint type: $G = \mathrm{PSO}(2n)$ $(n \geq 3)$, $\theta = \mathrm{Ad}(J_n)$.
(iv) $G = \mathrm{SO}(2n)$ $(n \geq 3)$, $\theta = \mathrm{Ad}(J_n)$, and $G^\theta = \mathrm{U}(n)$.

4, Irreducible compact symmetric spaces of exceptional type. We call an irreducible compact symmetric space M of *exceptional type* if the neutral subgroup of its isometry group is a compact exceptional simple Lie group. Any irreducible compact symmetric space of exceptional type is isomorphic to G/G^θ for some (G,θ) as in the following list:

(i) adjoint type: when G is a connected compact simple Lie group of adjoint type, and θ is an involutive automorphism of G.
(ii) $G = \mathrm{E}_6^{sc}$, $\theta \sim \sigma_3$ or σ_4 as in [7, Table 1], $G^{\sigma_3} \cong \mathrm{F}_4$ and $G^{\sigma_4} \cong \mathrm{PSp}(4)$.
(iii) $G = \mathrm{E}_7^{sc}$, $\theta \sim \sigma_2$ or σ_3 as in [7, Table 1], $G^{\sigma_2} \cong (\mathrm{E}_6^{sc} \times \mathrm{U}(1))/\langle (c,1) \rangle$ (where c is a nontrivial central element of E_6^{sc}) and $G^{\sigma_3} \cong \mathrm{SU}(8)/\langle -I \rangle$.

4. Explicit classification of maximal antipodal sets

In this section we classify maximal antipodal sets in irreducible compact symmetric spaces.

4.1. **Irreducible compact symmetric spaces of group form.** Let $M = G$ be an irreducible compact symmetric space of group form. Then, either $G \cong \mathrm{U}(1)$, or G is a compact simple Lie group. In this case, the geodesic symmetry s_x acts by $s_x(y) = xy^{-1}x$ $(\forall x, y \in G)$. Let X be a subset of M containing the origin. It is clear that X is a maximal antipodal set if and only if it is a maximal elementary abelian 2-subgroup of G. When $G = \mathrm{U}(1)$, then $X = \{\pm 1\}$. When G is of adjoint type, maximal elementary abelian 2-subgroups of G are classified in [5] and [17]. The other connected compact simple Lie groups fall into the following list:

(i) $\mathrm{SU}(n)/\langle e^{\frac{2\pi i}{m}} I\rangle$ $(m|n,\ m \neq n)$.
(ii) $\mathrm{Sp}(n)$ $(n \geq 2)$.
(iii) $\mathrm{Spin}(n)$ $(n \geq 7)$.
(iv) $\mathrm{SO}(n)$ $(n \geq 8$, even$)$.
(v) $\mathrm{Spin}(4m)/\langle c\rangle$ $(m \geq 2)$.
(vi) E_6^{sc}.
(vii) E_7^{sc}.

In item (i), let $G = \mathrm{SU}(n)/\langle e^{\frac{2\pi i}{m}} I\rangle$. When m is odd, any maximal elementary abelian 2-subgroup is conjugate to the subgroup consisting of diagonal matrices with entries ± 1; when m is even, the map $X \mapsto \pi(X)$ with π the projection $G \to \mathrm{PSU}(n)$ gives a bijection between conjugacy classes of maximal elementary abelian 2-subgroups in G and that in $\mathrm{PSU}(n)$. The latter is classified in [17, Proposition 2.4].

In item (ii) or item (iv), there is a unique conjugacy class of maximal elementary abelian 2-subgroups, i.e., those conjugate to the subgroup consisting of diagonal matrices with entries ± 1.

In item (vi), due to $Z(\mathrm{E}_6^{sc}) \cong \mathbb{Z}/3\mathbb{Z}$ is of odd degree, the map $X \mapsto \pi(X)$ with π the projection $\mathrm{E}_6^{sc} \to \mathrm{E}_6^{ad}$ gives a bijection between conjugacy classes of maximal elementary abelian 2-subgroups in E_6^{sc} and that in E_6^{ad}. There are two conjugacy classes, corresponding to the subgroups $F'_{2,3}$ and $F'_{0,1,0,2}$ in [17, Pages 272-273].

In item (vii), $X \sim \pi^{-1}(X')$ with π the projection $\mathrm{E}_7^{sc} \to \mathrm{E}_7^{ad}$, and $X' = F'''_{0,3}$ (rank 6) or F''_2 (rank 5) in [17, Page 284].

We do not know yet a complete classification of maximal elementary 2-subgroups for groups in item (iii) and item (v).

4.2. **Grassmannians.** In this subsection we classify maximal antipodal sets in an irreducible compact symmetric space which is isogenous to a Grassmannian. As stated in Section 3, there are six cases to consider: item (i) is the adjoint type case, which is treated in Remark 2.4; for item (v) and item (vi), we do not have a full classification yet. Below we treat items (ii)-(iv).

Example 4.1. *Let $M = \mathrm{SU}(2p)/S(\mathrm{U}(p) \times \mathrm{U}(p))$ and let $X \subset M$ be a maximal antipodal set containing the origin o. Write $G = \mathrm{SU}(2p)$. Define $\theta \in \mathrm{Aut}(G)$ by*

$$\theta(g) = I_{p,p} g I_{p,p}^{-1},\ \forall g \in G.$$

Then, $M = G/G^\theta$. Set $\bar{G} = G \rtimes \langle\bar{\theta}\rangle$, where $\bar{\theta}^2 = 1$ and $\mathrm{Ad}(\bar{\theta})|_G = \theta$.

When p is odd, we may identity $\bar{G}$ with $\mathrm{SU}^{\pm}(2p)$ and identify $\bar{\theta}$ with $I_{p,p}$. Then, $F_2(X)$ is diagonalizable. Without loss of generality we assume that $F_2(X)$ is contained in the subgroup F of $\bar{G} = \mathrm{SU}^{\pm}(2p)$ consisting of diagonal matrices with entries

± 1. *Then,*

$$|X| = |F_2(X) \cap C_{\bar{\theta}}| = |F \cap C_{\bar{\theta}}| = \binom{2p}{p}.$$

When p is even, we may identify θ with $I_{p,p}$. Then, $\bar{G} = G \times \langle \bar{\theta}\theta^{-1} \rangle$, $F_2(X) = F_1(X) \times \langle \bar{\theta} \rangle$ and $F_2(X)$ is diagonalizable. Without loss of generality we assume that $F_1(X)$ is contained in the subgroup F of $\mathrm{SU}(2p)$ consisting of diagonal matrices with entries ± 1. Then,

$$|X| = |F_2(X) \cap C_{\bar{\theta}}| = |F \cap C_{\theta}| = \binom{2p}{p}.$$

Example 4.2. *When $M = \mathrm{Sp}(2p)/(\mathrm{Sp}(p) \times \mathrm{Sp}(p))$, the classification proceeds the same as in Example 4.1: there is a unique maximal antipodal set X in M up to conjugacy, and $|X| = \binom{2p}{p}$.*

Example 4.3. *Let $M = \mathrm{SO}(2p)/S(\mathrm{O}(p) \times \mathrm{O}(p))$ ($p \geq 3$), the classification proceeds the same as in Example 4.1: there is a unique maximal antipodal set X in M up to conjugacy, and $|X| = \binom{2p}{p}$.*

4.3. **Types AI and AII.** As stated in § 3, any irreducible compact symmetric space M which is of type AI or AII in Cartan's notation is isomorphic to G/G^{θ} where $G = G_{n,m}$ and $\theta = \tau$ or τ'. When $m = n$, M is of adjoint type and it is treated in Remark 2.4. According to [17, Propositions 2.12 and 2.16], there are $k+1$ conjugacy classes of maximal elementary abelian 2-subgroups in $\mathrm{PO}(n) = G^{\tau}$ (or $\mathrm{PSp}(n/2) = G^{\tau'}$), where k is the 2-power index of n (or $\frac{n}{2}$).

When $m = 1$, we have $M \cong \mathrm{SU}(n)/\mathrm{SO}(n)$ or $\mathrm{SU}(n)/\mathrm{Sp}(n/2)$.

Example 4.4. *Let $M = \mathrm{SU}(n)/\mathrm{SO}(n)$ and let $X \subset M$ be a maximal antipodal set containing the origin o. Write $G = \mathrm{SU}(n)$ and $\theta = \tau \in \mathrm{Aut}(G)$. Then, $M = G/G^{\theta}$. Set $\bar{G} = G \rtimes \langle \bar{\theta} \rangle$, where $\bar{\theta}^2 = 1$ and $\mathrm{Ad}(\bar{\theta})|_G = \theta$. Taking similar study as in Example 4.1, we have $F_1(X) \subset G^{\theta} = \mathrm{SO}(n)$. Then, $F_1(X)$ is conjugate to the subgroup of $\mathrm{SO}(n)$ consisting of diagonal matrices with entries ± 1 and*

$$|X| = |F_2(X) \cap C_{\bar{\theta}}| = |F_1(X)| = 2^{n-1}.$$

Example 4.5. *When $M = \mathrm{SU}(n)/\mathrm{Sp}(\frac{n}{2})$ ($n \geq 4$, even), the classification is along the same line as in Example 4.4 by replacing τ, $\mathrm{SO}(n)$ with τ', $\mathrm{Sp}(\frac{n}{2})$ respectively. The result is: there is a unique maximal antipodal set X in M up to conjugacy and $|X| = 2^{\frac{n}{2}-1}$.*

In general, when m is odd, the classification is the same as in the case of $m = 1$. When n/m is odd, the classification is the same as in the adjoint type case. When m and n/m are both even, we have $G^{\tau} \cong \mathrm{PSO}(n) \times \mathbb{Z}/2\mathbb{Z}$ and $G^{\tau'} \cong \mathrm{PSp}(n/2) \times \mathbb{Z}/2\mathbb{Z}$. Using the classification of elementary abelian 2-subgroups of $\mathrm{PSO}(n)$ and of $\mathrm{PSp}(\frac{n}{2})$ given in [17], one can classify maximal antipodal sets.

4.4. **Types CI and DIII.** Let M be a compact symmetric space of type CI or DIII. Item (i) as listed in § 3 is treated in Remark 2.4. We treat items (ii)-(iii) below.

Example 4.6. *Let $M = \mathrm{SO}(2n)/\mathrm{U}(n)$ and let $X \subset M$ be a maximal antipodal set containing the origin o. Write $G = \mathrm{SO}(2n)$ and $\theta = \mathrm{Ad}(J_n) \in \mathrm{Aut}(G)$. Then, $M = G/G^\theta$. Set $\bar{G} = G \rtimes \langle \bar{\theta} \rangle$, where $\bar{\theta}^2 = 1$ and $\mathrm{Ad}(\bar{\theta})|_G = \theta$. Taking similar study as in Example 4.1, we have $F_1(X) \subset G^\theta = \mathrm{U}(n)$. Then, $F_1(X)$ is conjugate to the subgroup of $\mathrm{U}(n)$ consisting of diagonal matrices with entries ± 1 and with determinant 1 (this condition is forced by $F_2(X)$ is generated by elements in $C_{\bar{\theta}}$) and*

$$|X| = |F_2(X) \cap C_{\bar{\theta}}| = |F_1(X)| = 2^{n-1}.$$

Example 4.7. *Let $M = \mathrm{Sp}(n)/\mathrm{U}(n)$ and let $X \subset M$ be a maximal antipodal set containing the origin o. The classification is similar to Example 4.6: there is a unique maximal antipodal set X in M up to conjugacy and $|X| = 2^n$.*

4.5. **Exceptional type.** Let M be an irreducible compact symmetric space of exceptional type. Item (i) as listed in § 3 is treated in Remark 2.4. We treat items (ii)-(iii) below.

Example 4.8. *Let $M = \mathrm{E}_6^{sc}/(\mathrm{E}_6^{sc})^\theta$ for θ an outer involution, and let $X \subset M$ be a maximal antipodal set containing the origin o. Write $G = \mathrm{E}_6^{sc}$. Set $\bar{G} = G \rtimes \langle \bar{\theta} \rangle$, where $\bar{\theta}^2 = 1$ and $\mathrm{Ad}(\bar{\theta})|_G = \theta$. Let $\pi : \bar{G} \to \mathrm{Aut}(\mathfrak{e}_6)$ be the adjoint homomorphism.*

When $\theta \sim \sigma_3$, by [17, Proposition 6.3] *one shows that $\pi(F_2(X))$ is conjugate to the subgroup $F_{2,0}$ of $\mathrm{F}_4 = G^\theta$. Then, $|X| = 4$.*

When $\theta \sim \sigma_4$ and $\pi(F_2(X))$ contains no element conjugate to σ_3, by [17, Proposition 6.5] *one shows that $\pi(F_2(X))$ is conjugate to the subgroup $F_{0,1,0,2}$ of* $\mathrm{Aut}(\mathfrak{e}_6)$. *Then, $|X| = 64$. When $\theta \sim \sigma_4$ and $\pi(F_2(X))$ contains an element conjugate to σ_3, by* [17, Proposition 6.3] *one shows that $\pi(F_2(X))$ is conjugate to the subgroup $F_{2,3}$ of* $\mathrm{Aut}(\mathfrak{e}_6)$. *Then, $|X| = 2^5 - 2^2 = 28$.*

Example 4.9. *Let $M = \mathrm{E}_7^{sc}/(\mathrm{E}_7^{sc})^\theta$ for $\theta \sim \sigma_2$ or $\theta \sim \sigma_3$, and let $X \subset M$ be a maximal antipodal set containing the origin o. Write $G = \mathrm{E}_7^{sc}$. Set $\bar{G} = G \rtimes \langle \bar{\theta} \rangle$, where $\bar{\theta}^2 = 1$ and $\mathrm{Ad}(\bar{\theta})|_G = \theta$. Let $\pi : \bar{G} \to \mathrm{Aut}(\mathfrak{e}_7)$ be the adjoint homomorphism.*

When $\theta \sim \sigma_2$, taking similar study as in Example 4.1 we have

$$F_1(X) \subset G^\theta \cong (\mathrm{E}_6 \times \mathrm{U}(1))/\langle (c, e^{\frac{2\pi i}{3}}) \rangle.$$

Write $c = [(1,-1)] \in (\mathrm{E}_6 \times \mathrm{U}(1))/\langle (c, e^{\frac{2\pi i}{3}}) \rangle$. As shown in [17, §7.1], *for any element $x \in F_1(X)$, $x\bar{\theta} \in C_{\bar{\theta}}$ if and only if $x \sim 1$, c, τ_2 or $c\tau_2$ in $(\mathrm{E}_6 \times \mathrm{U}(1))/\langle (c, e^{\frac{2\pi i}{3}}) \rangle$. Then, $F_1(X)$ is of the form $F_1(X) = J \times \langle (1,-1) \rangle$, where J is an elementary abelian 2-subgroup of E_6. By* [17, Proposition 6.5] *one can show that $\pi(F_2(X))$ is conjugate to $F'_{0,1,0,2} \subset \mathrm{E}_7^{ad}$. Counting conjugacy classes of elements in $F'_{0,1,0,2}$ we get $|X| = 2 \times \frac{2^6 - 2^3}{2} = 56$.*

When $\theta \sim \sigma_3$, taking similar study as in Example 4.1 we have

$$F_1(X) \subset G^\theta \cong \mathrm{SU}(8)/\langle -I \rangle.$$

For any element $x \in F_1(X)$, $x\bar{\theta} \in C_{\bar{\theta}}$ *if and only if* x *is conjugate to* $[I]$, $[\mathbf{i}I]$, $[I_{4,4}]$ *or* $[\mathbf{i}I_{4,4}]$ *in* $\mathrm{SU}(8)/\langle -I\rangle$. *Choose a maximal elementary abelian 2-subgroup* F *of* $\mathrm{PSU}(8)$ *containing the image of projection* F *of* $F_1(X)$ *in it. As in* [17, §2], *it is associated with a multiplicative function* $m : F \times F \to \{\pm 1\}$. *Put* $r = \operatorname{rank}\ker m + 1$ *and* $s = \frac{1}{2}\operatorname{rank}(F/\ker m)$. *Then,* $r \cdot 2^s = 8$. *Then,* $(r,s) = (8,0)$, $(4,1)$ *or* $(1,3)$. *When* $(r,s) = (8,0)$, *we have* $|X| = 72$*; when* $(r,s) = (4,1)$, *we have* $|X| = 2^4 \cdot 3 + 2^3 = 56$*; when* $(r,s) = (1,3)$, *we have* $|X| = 128$.

5. Supplements

5.1. **Characterization of polars.** Let M be a compact symmetric space. Connected components of the fixed point set of the geodesic symmetry s_x at a point $x \in M$ are called polars by Chen and Nagano and are classified in [3] and [10]. Now let $M = G/G^\theta$ for G a connected compact simple Lie group and θ an involutive automorphism of G. We remark here that results in [7] also apply to classify G^θ orbits in the fixed point set of s_o. When G^θ is connected, this is equivalent to the classification of polars. In general, $\pi_0(G^\theta) = (\mathbb{Z}/2\mathbb{Z})^r$ ($r = 0,1,2$) and $r = 2$ happens only when $M = \mathrm{PSO}(8)/\mathrm{PSO}(8)^{[\mathrm{Ad}(I_{4,4})]}$ ([7, Table 2, p. 408]). Thus, a G^θ orbit is the union of 1,2 or 4 polars. Set $\bar{G} = G \rtimes \langle\bar{\theta}\rangle$ where $\bar{\theta}^2 = 1$ and $\mathrm{Ad}(\bar{\theta})|_G = \theta$. Write $C_{\bar{\theta}} = \{g\bar{\theta}g^{-1} : g \in G\}$. The classification is based on the following lemma, which is easy and we omit the proof.

Lemma 5.1. *A point* $x = gG^\theta \in G/G^\theta$ *is in the fixed point set of* s_o *if and only if* $\phi(x) \in G^\theta$ *and* $\phi(x)^2 = 1$. *The* G^θ *orbits in the fixed point set of* s_o *are in one-to-one correspondence with* G *orbits of ordered pairs* $(\theta_1, \theta_2) \in \bar{G} \times \bar{G}$ *such that* $\theta_1, \theta_2 \in C_{\bar{\theta}}$ *and* $\theta_1\theta_2 = \theta_2\theta_1$.

When G is of adjoint type, ordered pairs of commuting involutions in $\bar{G}$ are classified in [7]. When G is not of adjoint type, the classification can be made by considering the projection $\pi : G \to \mathrm{Int}(\mathfrak{u}_0)$ and using the classification in [7]. For any $o \neq x = gG^\theta \in G/G^\theta$, put $\theta_1 = \bar{\theta}$ and $\theta_2 = g\bar{\theta}g^{-1}$. Then,

$$\mathrm{Stab}_{G^\theta}(x) = G^\theta \cap gG^\theta g^{-1} = Z_G(\langle\theta_1, \theta_2\rangle).$$

The group $\langle\theta_1, \theta_2\rangle$ is a Klein four subgroup of G, the centralizers $Z_G(\langle\theta_1, \theta_2\rangle)$ are calculated in [7, Table 6, p. 420] when G is of adjoint type. When G is not of adjoint type, one can apply the method in [7] to calculate the centralizers $Z_G(\langle\theta_1, \theta_2\rangle)$ as well.

5.2. **Some corrections to** [17]**.** Here I would like to make several corrections to [17]. In [17, p. 273, lines 7-9], the correct definition for the groups $F_{\epsilon,\delta,r,s}$ ($\epsilon + \delta \leq 1$, $r + s \leq 2$) should be

$$F_{\epsilon,\delta,r,s} = \begin{cases} \langle x_0, x_1, \ldots, x_{\epsilon+2\delta}, x_3, \ldots, x_{2+r+2s}\rangle \text{ if } (r,s) \neq (2,0) \\ \langle x_0, x_1, \ldots, x_{\epsilon+2\delta}, x_3, x_5\rangle \text{ if } (r,s) = (2,0). \end{cases}$$

Accordingly, $F'_{\epsilon,\delta,r,s}$ ($\epsilon+\delta\leq 1$, $r+s\leq 2$) should be defined by

$$F'_{\epsilon,\delta,r,s}=\begin{cases}\langle x_1,\dots,x_{\epsilon+2\delta},x_3,\dots,x_{2+r+2s}\rangle \text{ if } (r,s)\neq(2,0)\\ \langle x_1,\dots,x_{\epsilon+2\delta},x_3,x_5\rangle \text{ if } (r,s)=(2,0).\end{cases}$$

I would like to thank Alastair Litterick, Heiko Dietrich, Haian He for pointing out these two mistakes.

In [17, p. 291, lines -1], it should be $C=\Gamma_3$. In [17, p. 291, lines -12 - -11], it should be

$$F''_{r,0}=A^r\times B,$$
$$F''_{r,1}=A^r\times C,$$
$$F''_{r,2}=A^r\times D,$$

where B,C,D are elementary abelian 2-subgroups with rank equal to 1,2,3 respectively and each has a unique element conjugate to σ_1.

These mistakes do not affect the statement of any result in [17].

5.3. **Open cases.** In summary, the only irreducible compact symmetric spaces for which we do not have a complete classification of maximal antipodal sets yet are in the following list:

(i) $M=\mathrm{Spin}(n)$ ($n\geq 7$).
(ii) $M=\mathrm{Spin}(4n)/\langle c\rangle$ ($n\geq 3$).
(iii) $M=\mathrm{Spin}(p+q)/\mathrm{Spin}(p)\cdot\mathrm{Spin}(q)$ ($p\geq q\geq 1$ and $p+q\geq 7$).
(iv) $M=G/G^\theta$ where $G=\mathrm{Spin}(4n)/\langle c\rangle$ and $\theta=\mathrm{Ad}(e_1e_2\dots e_{2n})$.

For any $k\geq 1$, identify the $\mathbb{Z}/2\mathbb{Z}$-vector space $V_k=(\mathbb{Z}/2\mathbb{Z})^k$ with the set of subsets of $\{1,\dots,k\}$ and denote by $e_I\in V_k$ for an element corresponding to a subset I of $\{1,\dots,k\}$. Define an anti-symmetric form on V_k by $(e_I,e_J)=|I\cap J|$ (mod 2). Let V'_k be the subspace of $e_I\in V_k$ such that $\sharp I$ is even. A sub-vector space W of V'_k is said to be an isotropic subspace if $(e_I,e_J)=0$ for any $e_I,e_J\in W$; an isotropic subspace W of V'_k is called a Lagrangian if it is not properly contained in any other isotropic subspace. Write X_k for the set of Lagrangians in V'_k and write X'_k for the subset of X_k consisting of Lagrangians $W\subset V'_k$ such that $|I|\neq 2$ for any $e_I\in W$. Then, both X_k and X'_k admit actions of the permutation group S_k. Write $X_k/S_k, X'_k/S_k$ for the corresponding orbit sets. For any $W\in X_k$ (or $W\in X'_k$), write $[W]\in X_k/S_k$ (or $[W]\in X'_k/S_k$) for the S_k orbit containing W.

The following proposition says something for maximal antipodal sets in $\mathrm{Spin}(n)$.

Proposition 5.2. *Let $n\geq 1$. Then:*

(1) *the cardinality of each maximal antipodal set in* $\mathrm{Spin}(n)$ *is equal to* $2^{\lfloor\frac{n+2}{2}\rfloor}$*;*
(2) *the orbit set of maximal antipodal sets in* $\mathrm{Spin}(n)$ *can be parametrized by the set X_n/S_n;*
(3) *there is a decomposition*

$$X_n/S_n\cong\bigsqcup_{0\leq r\leq\lfloor\frac{n}{2}\rfloor}X'_{n-2r}/S_{n-2r}.$$

(4) $X'_k = \emptyset$ *if* $k \in \{2, 3, 4, 5, 6\}$.

Sketch of proof. We show a correspondence between maximal antipodal sets in $\mathrm{Spin}(n)$ and Lagrangians in V'_n. Let F be a maximal antipodal set in $\mathrm{Spin}(n)$. Without loss of generality we assume that $1 \in F$. Then, F is a maximal elementary abelian 2-subgroup of $\mathrm{Spin}(n)$. Thus, $Z(\mathrm{Spin}(n)) \subset F$. Consider the natural projection $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. Since any elementary abelian 2-subgroup of $\mathrm{SO}(n)$ is conjugate to a diagonal one, we assume that $\pi(F)$ is contained in the subgroup F'_0 of diagonal matrices in $\mathrm{SO}(n)$. Identify F'_0 with the $\mathbb{Z}/2\mathbb{Z}$-vector space $V'_n \subset V_n = (\mathbb{Z}/2\mathbb{Z})^n$, and also the set of subsets I of $\{1, \ldots, n\}$ with $\sharp I$ even. Let $W \subset V'_n$ correspond to $\pi(F)$. We have $[e_I, e_J] = (-1)^{I \cap J} \in \mathrm{Spin}(n)$ (the repetition of the notation e_I to mean either an element in V_I or an element in $\mathrm{Spin}(n)$ is cute, here e_I, e_J means elements in $\mathrm{Spin}(n)$) for any two subsets I, J of $\{1, \ldots, n\}$ with $e_I, e_J \in W$. Then, F is a maximal elementary abelian 2-subgroup if and only if W is a Langrangian. This shows the assertion (1).

The assertion (2) can be shown in an inductive way using two facts: (a)$\pi(e_I)$ and $\pi(e_J)$ are conjugate in $\mathrm{SO}(n)$ if and only if I and J and in the same S_n orbit; (b)for any set of elements $e_{I_1}, \ldots, e_{I_s}$ of F, the centralizer of $\langle \pi(e_{I_1}), \ldots, \pi(e_{I_s}) \rangle$ in $\mathrm{O}(n)$ is a product of $\mathrm{O}(n_j)$ $(1 \leq j \leq t)$ where $\sum_{1 \leq j \leq t} n_j = n$.

The assertion (3) is easy to show. The assertion (4) can be shown by a case by case verification. □

For items (ii)-(iv), I even don't know cardinalities of maximal antipodal sets except when n or $\min\{p, q\}$ is small.

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