

# The Menger and projective Menger properties of function spaces with the set-open topology

Alexander V. Osipov

*Krasovskii Institute of Mathematics and Mechanics, Ural Federal University,  
Ural State University of Economics, 620219, Yekaterinburg, Russia*

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## Abstract

For a Tychonoff space  $X$  and a family  $\lambda$  of subsets of  $X$ , we denote by  $C_\lambda(X)$  the space of all real-valued continuous functions on  $X$  with the set-open topology. In this paper, we study the Menger and projective Menger properties of a Hausdorff space  $C_\lambda(X)$ . Our main results state that

if  $\lambda$  is a  $\pi$ -network of  $X$ , then

(1)  $C_\lambda(X)$  is Menger space, if and only, if  $C_\lambda(X)$  is  $\sigma$ -compact,

and, if  $Y$  is a dense subset of  $X$ , then

(2)  $C_p(Y|X)$  is projective Menger space, if and only, if  $C_p(Y|X)$  is  $\sigma$ -pseudocompact.

*Keywords:* Menger, projective Menger, set-open topology,  $\sigma$ -compact,  $\sigma$ -pseudocompact,  $\sigma$ -bounded, basically disconnected space, function space

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## 1. Introduction

Throughout this paper  $X$  will be a Tychonoff space. Let  $\lambda$  be a family non-empty subsets of  $X$ ,  $C(X)$  the set of all continuous real-valued function on  $X$ . Denote by  $C_\lambda(X)$  the set  $C(X)$  is endowed with the  $\lambda$ -open topology. The elements of the standard subbases of the  $\lambda$ -open topology will be denoted as follows:  $[F, U] = \{f \in C(X) : f(F) \subseteq U\}$ , where  $F \in \lambda$ ,  $U$  is an open subset of  $\mathbb{R}$ . Note that if  $\lambda$  consists of all finite subsets of  $X$  then the  $\lambda$ -open topology is equal to the topology of pointwise convergence, that

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*Email address:* OAB@list.ru (Alexander V. Osipov)

is  $C_\lambda(X) = C_p(X)$ . Denote by  $C_p(Y|X) = \{h \in C_p(Y) : h = f|_Y \text{ for } f \in C(X)\}$  for  $Y \subset X$ .

Recall that, if  $X$  is a space and  $\mathcal{P}$  a topological property, we say that  $X$  is  $\sigma$ - $\mathcal{P}$  if  $X$  is the countable union of subspaces with the property  $\mathcal{P}$ .

So a space  $X$  is called  $\sigma$ -compact ( $\sigma$ -pseudocompact,  $\sigma$ -bounded), if  $X = \bigcup_{i=1}^{\infty} X_i$ , where  $X_i$  is a compact (pseudocompact, bounded) for every  $i \in \mathbb{N}$ . N.V. Velichko proved that  $C_p(X)$  is  $\sigma$ -compact, if and only, if  $X$  is finite. In [20], V.V. Tkachuk clarified when  $C_p(X)$  is  $\sigma$ -pseudocompact and when  $C_p(X)$  is  $\sigma$ -bounded, and considered similar questions for the space  $C_p^*(X)$  of bounded continuous functions on  $X$ .

A space  $X$  is said to be Menger [9] (or, [17]) if for every sequence  $\{\mathcal{U}_n : n \in \omega\}$  of open covers of  $X$ , there are finite subfamilies  $\mathcal{V}_n \subset \mathcal{U}_n$  such that  $\bigcup \{\mathcal{V}_n : n \in \omega\}$  is a cover of  $X$ .

Every  $\sigma$ -compact space is Menger, and a Menger space is Lindelöf. The Menger property is closed hereditary, and it is preserved by continuous maps. It is well known that the Baire space  $\mathbb{N}^{\mathbb{N}}$  (hence,  $\mathbb{R}^{\omega}$ ) is not Menger.

In [2], A.V. Arhangel'skii proved that  $C_p(X)$  is Menger, if and only, if  $X$  is finite.

Let  $\mathcal{P}$  be a topological property. A.V. Arhangel'skii calls  $X$  *projectively*  $\mathcal{P}$  if every second countable image of  $X$  is  $\mathcal{P}$ . Arhangel'skii consider projective  $\mathcal{P}$  for  $\mathcal{P} = \sigma$ -compact, analytic [3], and other properties.

Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in  $\mathbb{R}^{\omega}$ . The projective selection principles were introduced and first time considered in [11].

Every Menger space is projectively Menger. It is known (Theorem 2.2 in [11]) that a space is Menger, if and only, if it is Lindelöf and projectively Menger.

Characterizations of projectively Menger spaces  $X$  in terms a selection principle restricted to countable covers by cozero sets are given in [5].

In [16], M. Sakai proved that  $C_p(X)$  is projectively Menger, if and only, if  $X$  is pseudocompact and  $b$ -discrete.

In this paper we study the Menger property of Hausdorff space  $C_\lambda(X)$ , and the projective Menger property of  $C_p(Y|X)$  where  $Y$  is dense subset of  $X$ .

## 2. Main definitions and notation

Recall that a family  $\lambda$  of non-empty subsets of a topological space  $(X, \tau)$  is called a  $\pi$ -network for  $X$  if for any nonempty open set  $U \in \tau$  there exists  $A \in \lambda$  such that  $A \subset U$ .

Throughout this paper, a family  $\lambda$  of nonempty subsets of the set  $X$  is a  $\pi$ -network. This condition is equivalent to the space  $C_\lambda(X)$  being a Hausdorff space [12].

We will also need the following assertion [1], [4].

**Proposition 2.1.** *If  $\mathbb{I}_\alpha = \mathbb{I} = [0, 1]$  for  $\alpha \in A$  and  $Y$  is a subspace of the Tychonoff cube  $\mathbb{I}^A = \prod\{\mathbb{I}_\alpha : \alpha \in A\}$  which, whatever the countable set  $B \subset A$ , projects under the canonical projection  $\pi_B : \mathbb{I}^A \mapsto \mathbb{I}^B$  onto the whole cube  $\mathbb{I}^B = \prod\{\mathbb{I}_\alpha : \alpha \in B\}$  of  $\mathbb{I}^A$ , then  $Y$  is pseudocompact.*

**Theorem 2.2.** (Nokhrin [12]) *For a Tychonoff space  $X$  the following statements are equivalent:*

1.  $C_\lambda(X)$  is a  $\sigma$ -compact;
2.  $X$  is a pseudocompact,  $D(X)$  is a dense  $C^*$ -embedded set in  $X$  and family  $\lambda$  consists of all finite subsets of  $D(X)$ , where  $D(X)$  is an isolated points of  $X$ .

The closure of a set  $A$  will be denoted by  $\overline{A}$  (or  $cl(A)$ ); the symbol  $\emptyset$  stands for the empty set. As usual,  $f(A)$  and  $f^{-1}(A)$  are the image and the complete preimage of the set  $A$  under the mapping  $f$ , respectively.

A subset  $A$  of a space  $X$  is said to be *bounded* in  $X$  if for every continuous function  $f : X \mapsto \mathbb{R}$ ,  $f|_A : A \mapsto \mathbb{R}$  is a bounded function. Every  $\sigma$ -bounded space is projectively Menger (Proposition 1.1 in [3]).

## 3. Main results

In order to prove the main theorem we need to prove some statements that we call Lemmas, but note their self-importance.

Recall that a space  $X$  is called basically disconnected ([8]), if every cozero-set has an open closure. Clearly, every basically disconnected (Tychonoff) space is zero-dimensional space.

**Lemma 3.1.** *If  $C_\lambda(X)$  is Menger, then  $X$  is a basically disconnected space.*

*Proof.* Let  $U \subseteq X$  be a cozero set in  $X$ . Claim that  $\overline{U} = \text{Int}\overline{U}$ . Suppose that  $\overline{U} \setminus \text{Int}\overline{U} \neq \emptyset$ . Since  $U$  is a cozero set, there are open sets  $U_n$  of  $X$  such that for each  $n \in \mathbb{N}$ ,  $\overline{U_n} \subseteq U_{n+1}$  and  $\bigcup_{n=1}^{\infty} U_n = U$ . For each  $n, m \in \mathbb{N}$ , we put  $Z_{n,m} = \{f \in C_\lambda(X, [0, 1]) : f|(X \setminus \text{Int}\overline{U}) \equiv 0 \text{ and } f(U_n) \subset [\frac{1}{2^m}, 1]\}$ .

Note that  $Z_{n,m}$  is closed subset of  $C_\lambda(X)$  for each  $n, m \in \mathbb{N}$ . Let  $h \notin Z_{n,m}$ .

If  $x \in X \setminus \text{Int}\overline{U}$  such that  $h(x) \neq 0$ . Since  $\lambda$  is  $\pi$ -network of  $X$ , there is  $A \in \lambda$  such that  $A \subset h^{-1}(h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2}) \cap \text{Int}(X \setminus \text{Int}\overline{U})$ . Then  $h \in [A, (h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2})]$  and  $[A, (h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2})] \cap Z_{n,m} = \emptyset$ .

If  $x \in U_n$  and  $h(x) \notin [\frac{1}{2^m}, 1]$ . Let  $d = \frac{\text{diam}(h(x), [\frac{1}{2^m}, 1])}{2}$ . Since  $\lambda$  is a  $\pi$ -network of  $X$ , there is  $A \in \lambda$  such that  $A \subset h^{-1}((h(x) - d, h(x) + d) \cap U_n)$ . Then  $h \in [A, (h(x) - d, h(x) + d)]$  and  $[A, (h(x) - d, h(x) + d)] \cap Z_{n,m} = \emptyset$ .

Assume that  $\bigcap\{Z_{n,m} : n \in \mathbb{N}\} = \emptyset$  for all  $m \in \mathbb{N}$ . Using the Menger property of  $C_\lambda(X)$ , we can take some  $\varphi \in \mathbb{N}^{\mathbb{N}}$  such that  $\bigcap\{Z_{\varphi(m),m} : m \in \mathbb{N}\} = \emptyset$ . For each  $m \in \mathbb{N}$ , take any  $g_m \in C_\lambda(X)$  satisfying  $g_m(X \setminus \text{Int}(\overline{U})) \equiv 0$  and  $g_m(U_{\varphi(m)}) = \{1\}$ . Let  $g = \sum_{j=1}^{\infty} 2^{-j} g_j$ . Then,  $g \in C_\lambda(X)$  and  $g(X \setminus \text{Int}(\overline{U})) \equiv 0$ . Fix any  $m \in \mathbb{N}$ ,  $1 \leq k \leq \varphi(m)$  and  $x \in U_k$ . Then we have

$$g(x) = \sum_{j=1}^{\infty} 2^{-j} g_j(x) \geq 2^{-m} g_m(x) = 2^{-m}.$$

Hence,  $g \in \bigcap\{Z_{\varphi(m),m} : m \in \mathbb{N}\}$ . This is a contradiction. Thus, there is some  $m \in \mathbb{N}$  such that  $\bigcap\{Z_{n,m} : n \in \mathbb{N}\} \neq \emptyset$ . Let  $p \in \bigcap\{Z_{n,m} : n \in \mathbb{N}\}$ . Then  $p(U) \subset [\frac{1}{2^m}, 1]$  and  $p|(X \setminus \text{Int}\overline{U}) \equiv 0$ . It follows that  $\overline{U} \setminus \text{Int}\overline{U} = \emptyset$ .  $\square$

A subset  $G \subset \omega^\omega$  is *dominating* if for every  $f \in \omega^\omega$  there is a  $g \in G$  such that  $f(n) \leq g(n)$  for all but finitely many  $n$ .

**Theorem 3.2.** (Hurewicz [10]) *A second countable space  $X$  is Menger iff for every continuous mapping  $f : X \mapsto \mathbb{R}^\omega$ ,  $f(X)$  is not dominating.*

”Second countable” can be extended to ”Lindelöf”:

**Theorem 3.3.** (Kočinac [11], Theorem 2.2) *A Lindelöf space  $X$  is Menger iff for every continuous mapping  $f : X \mapsto \mathbb{R}^\omega$ ,  $f(X)$  is not dominating.*

**Lemma 3.4.** *If  $C_\lambda(X)$  is Menger. Then  $X$  is pseudocompact.*

*Proof.* Assume that  $X$  is not pseudocompact and  $f \in C(X)$  is not bounded function. Without loss of generality we can assume that  $\mathbb{N} \subset f(X)$ . For each  $n \in \mathbb{N}$  we choose  $A_n \in \lambda$  such that  $A_n \subset f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$ . By

Lemma 3.1,  $F_n = \overline{f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))}$  is clopen set for each  $n \in \mathbb{N}$ . Let  $K = \{f \in C(X) : f|_{F_n} \equiv s_{f,n} \text{ for each } n \in \mathbb{N} \text{ and } s_{f,n} \in \mathbb{R}\}$ . Then  $K$  is closed subset of  $C_\lambda(X)$  and, hence, it is Menger. Fix  $a_n \in A_n$  for every  $n \in \mathbb{N}$ . Note that  $D = \{a_n : n \in \mathbb{N}\}$  is a  $C$ -embedded copy of  $\mathbb{N}$  (3L (1) in [8]). So we have a continuous mapping  $F : K \mapsto \mathbb{R}^D$  the space  $K$  onto  $\mathbb{R}^D$ . But  $F(K) = \mathbb{R}^D = \mathbb{R}^\omega$  is dominating, contrary to the Theorem 3.3.  $\square$

**Lemma 3.5.** *If  $C_\lambda(X)$  is Menger, then  $\mu = \{A \in \lambda : A \text{ is finite subset of } X\}$  is a  $\pi$ -network of  $X$ .*

*Proof.* Assume that there exist an open set  $U$  of  $X$  such that  $B \not\subset U$  for every  $B \in \mu$ . Fix a family  $\{V_n : n \in \mathbb{N}\}$  of open subsets of  $X$  such that  $V_n \subset U$  for every  $n \in \mathbb{N}$  and  $V_{n'} \cap V_{n''} = \emptyset$  for  $n' \neq n''$ . Fix  $x_n \in V_n$  and  $\epsilon > 0$ . For every  $f \in C_\lambda(X)$  and  $n \in \mathbb{N}$  consider  $B_{f,n} \in \lambda$  such that  $B_{f,n} \subset f^{-1}((f(x_n) - \epsilon, f(x_n) + \epsilon)) \cap V_n$ . Then  $\mathcal{U}_n = \{[B_{f,n}, (f(x_n) - \epsilon, f(x_n) + \epsilon)] : f \in C_\lambda(X)\}$  is an open cover of  $C_\lambda(X)$  for every  $n \in \mathbb{N}$ . Using the Menger property of  $C_\lambda(X)$ , for sequence  $\{\mathcal{U}_n : n \in \omega\}$  of open covers of  $C_\lambda(X)$ , there are finite subfamilies  $\mathcal{S}_n \subset \mathcal{U}_n$  such that  $\bigcup\{\mathcal{S}_n : n \in \omega\}$  is a cover of  $C_\lambda(X)$ . Let  $\mathcal{S}_n = \{[B_{f_1,n}, (f_{1,n}(x_n) - \epsilon, f_{1,n}(x_n) + \epsilon)], \dots, [B_{f_{k(n),n}, (f_{k(n),n}(x_n) - \epsilon, f_{k(n),n}(x_n) + \epsilon)]\}$  for every  $n \in \mathbb{N}$ . Since  $B_{f_s,n}$  is an infinite subset of  $X$ , we fix  $z_{s',n} \in B_{f_s,n}$  for every  $s \in \overline{1, k(n)}$  and  $n \in \mathbb{N}$  such that  $z_{s',n} \neq z_{s'',n}$  for  $s' \neq s''$ . Let  $Z = \{z_{s,n} : s \in \overline{1, k(n)} \text{ and } n \in \mathbb{N}\}$ .

Define the function  $q : Z \mapsto \mathbb{R}$  such that  $q(z_{s,n}) = 0$  if  $0 \notin (f_{s,n}(x_n) - \epsilon, f_{s,n}(x_n) + \epsilon)$ , else  $q(z_{s,n}) = 2\epsilon$  for  $s \in \overline{1, k(n)}$  and  $n \in \mathbb{N}$ . By Lemma 3.1,  $X$  is a basically disconnected space.

Recall that (14N p.215 in [8]) every countable set in a basically disconnected space is  $C^*$ -embedded.

Hence, there is  $t \in C_\lambda(X)$  such that  $t|_Z = q$ . But  $t \notin \bigcup\{\mathcal{S}_n : n \in \omega\}$ . This is a contradiction.  $\square$

Denote  $D(X)$  a set of isolated points of  $X$ .

**Lemma 3.6.** *If  $C_\lambda(X)$  is Menger, then  $D(X)$  is dense set in  $X$ .*

*Proof.* Assume that there exist an open set  $W \neq \emptyset$  such that  $W \cap D(X) = \emptyset$ . By Lemma 3.5,  $\mu$  is  $\pi$ -network of  $X$ , hence, there is  $A \in \mu$  such that  $A \subset W$ . Note that  $X \setminus A$  is dense set in  $X$ . The constant zero function defined on

$X$  is denoted by  $f_0$ . For every  $f \in C(X) \setminus \{f_0\}$  there is  $x_f \in X \setminus A$  such that  $f(x_f) \neq 0$ . For every  $f \in C(X) \setminus \{f_0\}$ , consider  $B_f \in \mu$  such that  $B_f \subset f^{-1}((f(x_f) - \frac{|f(x_f)|}{2}, f(x_f) + \frac{|f(x_f)|}{2})) \cap (X \setminus A)$ . Let  $\epsilon > 0$ . Then  $\mathcal{V} = \{[B_f, (f(x_f) - \frac{|f(x_f)|}{2}, f(x_f) + \frac{|f(x_f)|}{2})] : f \in C(X) \setminus \{f_0\}\} \cup [A, (-\epsilon, \epsilon)]$  is an open cover of  $C_\lambda(X)$ . Since  $C_\lambda(X)$  is Menger and, hence,  $C_\lambda(X)$  is Lindelöf, there is a countable subcover  $\mathcal{V}' = \{[B_{f_n}, (f_n(x_f) - \frac{|f_n(x_f)|}{2}, f_n(x_f) + \frac{|f_n(x_f)|}{2})] : \text{for } n \in \mathbb{N}\} \cup [A, (-\epsilon, \epsilon)] \subset \mathcal{V}$  of  $C_\lambda(X)$ . Since  $X$  is a basically disconnected space and every countable set in a basically disconnected space is  $C^*$ -embedded, there is  $h \in C(X)$  such that  $h|_{\bigcup_{n \in \mathbb{N}} B_{f_n}} \equiv 0$  and  $h(a) = \epsilon$  for some  $a \in A$ . Note that  $h \notin \bigcup \mathcal{V}'$ , to contradiction.  $\square$

**Lemma 3.7.** *If  $C_\lambda(X)$  is Menger, then  $D(X)$  is  $C^*$ -embedded.*

*Proof.* Let  $f$  be a bounded continuous function from  $D(X)$  into  $\mathbb{R}$ , and  $F_A = \{g \in C(X) : g|_A = f|_A\}$  for  $A \in D(X)^\omega$ . Note that  $F_A$  is closed subset of  $C_\lambda(X)$  and, by Lemma 3.1,  $F_A \neq \emptyset$ . So  $\xi = \{F_A : A \in D(X)^\omega\}$  is family of closed subspaces with the countable intersection property. Since  $C_\lambda(X)$  is Menger, hence, it is Lindelöf, and every family of closed subspaces of with the countable intersection property has non-empty intersection. It follows that  $\bigcap \xi \neq \emptyset$ . We thus get that  $\tilde{f} \in \bigcap \xi$  such that  $\tilde{f} \in C(X)$  and  $\tilde{f}|_{D(X)} = f$ .  $\square$

**Proposition 3.8.** *Let  $X = \mathbb{N}$  and let  $\lambda = \{X\} \cup \{\{x\} : x \in X\}$ . Then  $C_\lambda^*(X)$  is not Menger.*

*Proof.* Assume that  $C_\lambda^*(X)$  is Menger. For every  $i \in \mathbb{N}$  consider an open cover  $\mathcal{V}_i = \{[\mathbb{N}, (-2 + \frac{1}{i+1}, 2 - \frac{1}{i+1})]\} \cup \{[x, (-\infty, -2 + \frac{2i+1}{2i(i+1)}) \cup (2 - \frac{2i+1}{2i(i+1)}, +\infty)] : x \in X\}$  of  $C_\lambda^*(X)$ . Using the Menger property of  $C_\lambda^*(X)$ , for sequence  $\{\mathcal{V}_i : i \in \mathbb{N}\}$  of open covers of  $C_\lambda^*(X)$ , there are finite subfamilies  $\mathcal{S}_i \subset \mathcal{V}_i$  such that  $\bigcup \{\mathcal{S}_i : i \in \mathbb{N}\}$  is a cover of  $C_\lambda^*(X)$ .

Without loss of generality we can assume that  $[\mathbb{N}, (-2 + \frac{1}{i+1}, 2 - \frac{1}{i+1})] \in \mathcal{S}_i$  for each  $i \in \mathbb{N}$ .

By using induction, for each  $i \in \mathbb{N}$ , determine the values of the function  $f$  at some points, depending on the  $\mathcal{S}_i$ , as follows:

for  $i = 1$  and

$\mathcal{S}_1 = \{[\mathbb{N}, (-2 + \frac{1}{2}, 2 - \frac{1}{2})], [x_1^1, (-\infty, -2 + \frac{3}{4}) \cup (2 - \frac{3}{4}, +\infty)], \dots, [x_k^1, (-\infty, -2 + \frac{3}{4}) \cup (2 - \frac{3}{4}, +\infty)]\}$ , define

$f(x_n^1) = 0$  for  $n \in \overline{1, k}$  and  
 $f(s_1) = p_1$  where  $p_1 \in [-2 + \frac{5}{12}, 2 - \frac{5}{12}] \setminus (-2 + \frac{1}{2}, 2 - \frac{1}{2})$  for some  $s_1 \in X \setminus \{x_n^1 : n \in \overline{1, k}\}$ . Denote  $P_1 = \bigcup_{n \in \overline{1, k}} x_n^1 \cup s_1$ .

for  $i = m$

$\mathcal{S}_m = \{[\mathbb{N}, (-2 + \frac{1}{m+1}, 2 - \frac{1}{m+1})], [x_1^m, (-\infty, -2 + \frac{2m+1}{2m(m+1)}) \cup (2 - \frac{2m+1}{2m(m+1)}, +\infty)], \dots, [x_{k(m)}^m, (-\infty, -2 + \frac{2m+1}{2m(m+1)}) \cup (2 - \frac{2m+1}{2m(m+1)}, +\infty)]\}$ , define

$f(x_n^m) = 0$  where  $x_n^m \notin P_{m-1}$  for  $n \in \overline{1, k(m)}$  and

$f(s_m) = p_m$  where  $p_m \in [-2 + \frac{2(m+1)+1}{2(m+1)(m+2)}, 2 - \frac{2(m+1)+1}{2(m+1)(m+2)}] \setminus (-2 + \frac{1}{m+1}, 2 - \frac{1}{m+1})$  for some  $s_m \in X \setminus P_{m-1}$ . Denote  $P_m = \bigcup_{n \in \overline{1, k(m)}} x_n^m \cup s_m \cup P_{m-1}$  and

$P = \bigcup_{m \in \mathbb{N}} P_m$ .

If  $X \setminus P \neq \emptyset$ , then let  $f(x) = 1$  for  $x \in X \setminus P$ .

By construction of  $f$ ,  $f \notin \mathcal{S}_i$  for every  $i \in \mathbb{N}$ , to contradiction.  $\square$

**Lemma 3.9.** *If  $C_\lambda(X)$  is Menger, then each  $A \in \lambda$  is finite subset of  $D(X)$ .*

*Proof.* Suppose that  $C_\lambda(X)$  is Menger,  $\tilde{\lambda} = \{A\} \cup \{\{x\}, x \in D(X)\}$  and  $A \in \lambda$  is an infinite subset of  $X$ . Then  $C_{\tilde{\lambda}}(X)$  is Menger, too. Note that if  $A$  is countable and  $A \subset D(X)$ , then we have a continuous mapping  $g : C_{\tilde{\lambda}}(X) \mapsto C_{p \cup \{\mathbb{N}\}}^*(\mathbb{N})$ . Hence,  $C_{p \cup \{\mathbb{N}\}}^*(\mathbb{N})$  is Menger, contrary to Proposition 3.8.

Let  $V = (-1, 1) \cup (\mathbb{R} \setminus [-4, 4])$ . Consider  $\mathcal{U} = \{[A, V]\} \cup \{[x, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]] : x \in D(X)\}$ . Since  $D(X)$  is dense subset of  $C_{\tilde{\lambda}}(X)$  (Lemma 3.6),  $\mathcal{U}$  is an open cover of  $C_{\tilde{\lambda}}(X)$  and, hence, there is a countable subcover  $\mathcal{U}' \subset \mathcal{U}$  of  $C_{\tilde{\lambda}}(X)$ . Let  $\mathcal{U}' = \{[A, V], [x_1, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]], \dots, [x_n, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]], \dots\}$ . Let  $z \in A \setminus \bigcup_{n \in \mathbb{N}} \{x_n\}$  (note that either  $z \in A \setminus D(X)$  or  $A \subset D(X)$  and  $|A| > \aleph_0$ ). Since every countable set in a basically disconnected space is  $C^*$ -embedded, there is  $h \in C_{\tilde{\lambda}}(X)$  such that  $h|_{\bigcup_{n \in \mathbb{N}} \{x_n\}} = 0$  and  $h(z) = 2$ . It follows that  $h \notin \bigcup \mathcal{U}'$ , to contradiction. It follows that  $A$  is finite subset of  $D(X)$ .  $\square$

**Theorem 3.10.** *Let  $X$  be a Tychonoff space and let  $\lambda$  be a  $\pi$ -network of  $X$ . Then a space  $C_\lambda(X)$  is Menger, if and only if,  $C_\lambda(X)$  is  $\sigma$ -compact.*

*Proof.* By Lemma 3.4,  $X$  is pseudocompact. By Lemmas 3.5 and 3.9, the family  $\lambda$  consists of all finite subsets of  $D(X)$ , where  $D(X)$  is an isolated

points of  $X$ . By Lemma 3.7,  $D(X)$  is a dense  $C^*$ -embedded set in  $X$ . It follows that  $C_\lambda(X)$  is  $\sigma$ -compact (Theorem 2.2). □

Various properties between  $\sigma$ -compactness and Menger are investigated in the papers [19, 6]. We can summarize the relationships between considered notions in ([19], see Figure 1), Theorems 3.10 and 2.2. Then we have the next

**Theorem 3.11.** *For a Tychonoff space  $X$  and a  $\pi$ -network  $\lambda$  of  $X$ , the following statements are equivalent:*

1.  $C_\lambda(X)$  is  $\sigma$ -compact;
2.  $C_\lambda(X)$  is Alster;
3. (CH)  $C_\lambda(X)$  is productively Lindelöf;
4. "TWO wins  $M$ -game" for  $C_\lambda(X)$ ;
5.  $C_\lambda(X)$  is projectively  $\sigma$ -compact and Lindelöf;
6.  $C_\lambda(X)$  is Hurewicz;
7.  $C_\lambda(X)$  is Menger;
8.  $X$  is a pseudocompact,  $D(X)$  is a dense  $C^*$ -embedded set in  $X$  and family  $\lambda$  consists of all finite subsets of  $D(X)$ , where  $D(X)$  is an isolated points of  $X$ .

#### 4. Projectively Menger space

According to Tkačuk [20], a space  $X$  said to be *b-discrete* if every countable subset of  $X$  is closed (equivalently, closed and discrete) and  $C^*$ -embedded in  $X$ .

**Lemma 4.1.** *(Lemma 2.1 in [16]) The following are equivalent for a space  $X$ :*

1.  $X$  is *b-discrete*;
2. For any disjoint countable subsets  $A$  and  $B$  in  $X$ , there are disjoint zero-sets  $Z_A$  and  $Z_B$  in  $X$  such that  $A \subset Z_A$  and  $B \subset Z_B$ ;
3. For any disjoint countably subsets  $A$  and  $B$  in  $X$  such that  $A$  is closed in  $X$ , there are disjoint zero-sets  $Z_A$  and  $Z_B$  in  $X$  such that  $A \subset Z_A$  and  $B \subset Z_B$ .

**Definition 4.2.** For  $A \subset X$ , a space  $X$  will be called  $b_A$ -discrete if every countable subset of  $A$  is closed in  $A$  and  $C^*$ -embedded in  $X$ .

**Lemma 4.3.** *The following are equivalent for a space  $X$  and  $A \subset X$ :*

1.  $X$  is  $b_A$ -discrete;
2. For any disjoint countable subsets  $D$  and  $B$  in  $A$ , there are disjoint zero-sets  $Z_D$  and  $Z_B$  in  $X$  such that  $D \subset Z_D$  and  $B \subset Z_B$ ;
3. For any disjoint countably subsets  $D$  and  $B$  in  $A$  such that  $D$  is closed in  $A$ , there are disjoint zero-sets  $Z_A$  and  $Z_B$  in  $X$  such that  $D \subset Z_D$  and  $B \subset Z_B$ .

Similarly to the proof of implication ( $C_p(X, \mathbb{I})$  is projectively Menger  $\Rightarrow X$  is  $b$ -discrete) of Theorem 2.4 in [16], we claim the next

**Lemma 4.4.** *Let  $C_\lambda(X)$  be a projectively Menger space, then  $X$  is  $b_A$ -discrete where  $A = \bigcup \lambda$ .*

*Proof.* Let  $C_\lambda(X)$  be a projectively Menger. We show the statement (3) in Lemma 4.3. Let  $D$  and  $B$  be a disjoint countable subsets in  $A$  such that  $D$  is closed in  $A$ . Let  $B = \{b_n : n \in \mathbb{N}\}$ , and let  $B_n = \{b_1, \dots, b_n\}$ .

For each  $n, m \in \mathbb{N}$ , we put  $Z_{n,m} = \{f \in C_\lambda(X) : f(D) = \{0\} \text{ and } f(B_m) \subset [\frac{1}{2^n}, 1]\}$ . Since  $D$  and  $B_m$  are countable and  $\lambda$  is a  $\pi$ -network of  $X$ , each  $Z_{n,m}$  is a zero-set in  $C_\lambda(X)$ . Assume that  $\bigcap \{Z_{n,m} : m \in \mathbb{N}\} = \emptyset$  for all  $n \in \mathbb{N}$ . Using the projective Menger property of  $C_\lambda(X)$ , Theorem 6 in [5], we can take some  $\varphi \in \mathbb{N}^{\mathbb{N}}$  such that  $\bigcap \{Z_{n,\varphi(n)} : n \in \mathbb{N}\} = \emptyset$ . For each  $n \in \mathbb{N}$ , take any  $g_n \in C_\lambda(X)$  satisfying  $g_n(D) = \{0\}$  and  $g_n(B_{\varphi(n)}) = \{1\}$ . Let  $g = \sum_{j=1}^{\infty} 2^{-j} g_j$ . Then,  $g \in C_\lambda(X)$  and  $g(D) \equiv 0$ . Fix any  $n \in \mathbb{N}$ ,

$1 \leq k \leq \varphi(m)$ . Then we have

$$g(b_k) = \sum_{j=1}^{\infty} 2^{-j} g_j(b_k) \geq 2^{-n} g_n(b_k) = 2^{-n}.$$

Hence,  $g \in \bigcap \{Z_{n,\varphi(n)} : n \in \mathbb{N}\}$ . This is a contradiction. Thus, there is some  $n \in \mathbb{N}$  such that  $\bigcap \{Z_{n,m} : m \in \mathbb{N}\} \neq \emptyset$ . Let  $h \in \bigcap \{Z_{n,m} : m \in \mathbb{N}\}$ . Then  $D \subset Z_A = h^{-1}(0)$  and  $B \subset Z_B = h^{-1}([\frac{1}{2^n}, 1])$ . □

**Theorem 4.5.** *Let  $X$  be a Tychonoff space and let  $Y$  be a dense subset of  $X$ . Then the following statements are equivalent:*

1.  $C_p(Y|X)$  is projectively Menger;

2.  $C_p(Y|X)$  is  $\sigma$ -bounded;
3.  $C_p(Y|X)$  is  $\sigma$ -pseudocompact;
4.  $X$  is pseudocompact and  $b_Y$ -discrete.

*Proof.* Note that  $C_p(Y|X)$  is homeomorphic to  $C_\lambda(X)$  for  $\lambda = [Y]^{<\omega}$ .

(1)  $\Rightarrow$  (4). By Lemma 4.4,  $X$  is  $b_Y$ -discrete. Assume that  $X$  is not pseudocompact and  $f \in C(X)$  is not bounded function. Without loss of generality we can assume that  $\mathbb{N} \subset f(X)$ . For each  $n \in \mathbb{N}$  we choose  $a_n \in Y$  such that  $a_n \in f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$ . Note that  $D = \{a_n : n \in \mathbb{N}\}$  is a  $C$ -embedded copy of  $\mathbb{N}$  (3L (1) in [8]). So we have a continuous mapping  $F : C_p(Y|X) \mapsto \mathbb{R}^D$  the Menger space  $C_p(Y|X)$  onto  $\mathbb{R}^D$ . But  $F(C_p(Y|X)) = \mathbb{R}^D = \mathbb{R}^\omega$  is dominating, contrary to the Theorem 3.3.

(4)  $\Rightarrow$  (3). Since  $C_p(Y|X, \mathbb{I})$  is a dense subset of  $\mathbb{I}^Y$  and  $X$  is  $b_Y$ -discrete, by Proposition 2.1,  $C_p(Y|X, \mathbb{I})$  is pseudocompact. Hence,  $C_p(Y|X)$  is  $\sigma$ -pseudocompact.

Note that every  $\sigma$ -pseudocompact space is  $\sigma$ -bounded, and every  $\sigma$ -bounded space is projectively Menger (Proposition 1.1 in [3]).  $\square$

## 5. Examples

Using Theorem 3.10 and Theorem 4.5, we can construct example of projective Menger topological group  $C_\lambda(X)$  such that it is not Menger.

Note that if  $\lambda = [\bigcup \lambda]^{<\omega}$ , then  $C_\lambda(X)$  is a topological group (locally convex topological vector space, topological algebra) ([14], [15]).

**Example 5.1.** (*Example 1 in [13]*) Let  $T$  be a  $P$ -space without isolated points,  $X = \beta(T)$  and let  $\lambda$  be a family of all finite subsets of  $T$ . Then  $C_\lambda(X)$  is  $\sigma$ -countably compact (Theorem 1.2 in [13]), hence, the topological group  $C_\lambda(X)$  is projective Menger. But the space  $X$  does not contain isolated points, hence,  $C_\lambda(X)$  is not Menger.

**Example 5.2.** (*Example 2 in [13]*) Let  $D$  be an uncountable discrete space and  $\lambda = D^{<\omega}$ . Consider  $F = \beta(D) \setminus \bigcup \{\overline{S} : S \subset D, \text{ and } S \text{ countable}\}$ . Denote by  $b(D)$  a quotient space obtained from  $\beta(D)$  by identifying the set  $F$  with the point  $\{F\}$ . Then the topological group  $C_\lambda(b(D))$  is projective Menger ( $\sigma$ -countably compact), but is not Menger.

**Example 5.3.** ([18]) D.B.Shahmatov has constructed for an arbitrary cardinal  $\tau \geq 2^{\aleph_0}$  an everywhere dense pseudocompact space  $X_\tau$  in  $\mathbb{I}^\tau$  such that

$X_\tau$  is a  $b$ -discrete. Hence, the topological group  $C_p(X_\tau)$  is projective Menger ( $\sigma$ -pseudocompact and is not  $\sigma$ -countably compact), but is not Menger for an arbitrary cardinal  $\tau \geq 2^{\aleph_0}$ .

**Remark 5.4.** By Theorems 2.2 and 3.10, if  $X$  is compact,  $\lambda$  is a  $\pi$ -network of  $X$  and  $C_\lambda(X)$  is Menger, then  $X$  is homeomorphic to  $\beta(D)$ , where  $\beta(D)$  is Stone-Čech compactification of a discrete space  $D$ , and  $\lambda = [D]^{<\omega}$ .

## References

- [1] A.V. Arhangel'skii, *Continuous maps, factorization theorems, and function spaces*, Trudy Moskovsk. Mat. Obshch., 47, (1984), 3–21.
- [2] A.V. Arhangel'skii, *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Sov. Math. Dokl., 33, (1986), 396–399.
- [3] A.V. Arhangel'skii, *Projective  $\sigma$ -compactness,  $\omega_1$ -caliber, and  $C_p$ -spaces*, Topology and its Applications, 157, (2000), 874–893.
- [4] A.V. Arhangel'skii, V.I. Ponomarev, *Fundamentals of general topology: problems and exercises*, Reidel, 1984. (Translated from the Russian.)
- [5] M. Bonanzinga, F. Cammaroto, M. Matveev, *Projective versions of selection principles*, Topology and its Applications, 157, (2010), 874–893.
- [6] H. Duanmu, F.D. Tall, L. Zdomskyy, *Productively Lindelöf and indestructibly Lindelöf spaces*, Topology and its Applications 160:18 (2013), 2443-2453.
- [7] R. Engelking, *General Topology*, PWN, Warsaw, (1977); Mir, Moscow, (1986).
- [8] L. Gillman, M. Jerison, *Rings of continuous functions*, The University Series in Higher Mathematics. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1960. 300 p.
- [9] W. Hurewicz, *Über eine verallgemeinerung des Borelschen Theorems*, Math. Z. 24 (1925) 401-421.
- [10] W. Hurewicz, *Über folger stetiger funktionen*, Fund. Math. 9 (1927) 193-204.

- [11] Lj.D.R. Kočinac, *Selection principles and continuous images*, Cubo Math.J. 8 (2) (2006) 23–31.
- [12] S.E. Nokhrin, *Some properties of set-open topologies*, Jurnal of Mathematical Sciences, issue 144, n 3, (2007) 4123–4151.
- [13] A.V. Osipov, E.G. Pytkeev, *On the  $\sigma$ -countable compactness of spaces of continuous functions with the set-open topology*, Proceedings of the Steklov Institute of Mathematics, issue 285, n. S1, (2014) 153–162.
- [14] A.V. Osipov, *Topological-algebraic properties of function spaces with set-open topologies*, Topology and its Applications, issue 3, n. 159, (2012) 800–805.
- [15] A.V. Osipov, *Group structures of a function spaces with the set-open topology*, Sib. Elektron. Mat. Izv., 14, (2017) 1440-1446.
- [16] M. Sakai, *The projective Menger property and an embedding of  $S_\omega$  into function spaces*, Topology and its Applications, Vol. 220 (2017) 118–130.
- [17] M. Sakai, M. Scheepers, *The combinatorics of open covers* in: K.P. Hart, J. van Mill, P.Simon (Eds.), Recent Progress in General Topology III, Atlantic Press, 2014, pp. 751–799.
- [18] D.B. Shahmatov, *A pseudocompact Tychonoff space all countable subsets of which are closed and  $C^*$ -embedded*, Topology and its Applications, 22:2, (1986), 139–144.
- [19] F.D. Tall, *Productively Lindelöf spaces may all be  $D$* , Canadian Mathematical Bulletin 56:1 (2013), 203–212.
- [20] V.V. Tkačuk, *The spaces  $C_p(X)$ : decomposition into a countable union of bounded subspaces and completeness properties*, Topology and its Applications, n 22, (1986), 241–253.