

# Lagrange's Theorem For Hom-Groups

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## Abstract

Hom-groups are nonassociative generalizations of groups where the unitality and associativity are twisted by a map. We show that a Hom-group  $(G, \alpha)$  is a pointed idempotent quasigroup (pique). We use Cayley table of quasigroups to introduce some examples of Hom-groups. Introducing the notions of Hom-subgroups and cosets we prove Lagrange's theorem for finite Hom-groups. This states that the order of any Hom-subgroup  $H$  of a finite Hom-group  $G$  divides the order of  $G$ . We linearize Hom-groups to obtain a class of nonassociative Hopf algebras called Hom-Hopf algebras. As an application of our results, we show that the dimension of a Hom-sub-Hopf algebra of the finite dimensional Hom-group Hopf algebra  $\mathbb{K}G$  divides the order of  $G$ . The new tools introduced in this paper could potentially have applications in theories of quasigroups, nonassociative Hopf algebras, Hom-type objects, combinatorics, and cryptography.

## 1 Introduction

Nonassociative objects such as quasigroups, loops, non-associative algebras, and Hopf algebras have many applications in several contexts. Among all of these, Hom-type objects have been under intensive research in the last decade. Hom-Lie algebras have appeared in quantum deformations of Witt and Virasoro algebras [AS], [CKL], [CZ]. Hom-Lie algebras, [HLS], are generalizations of Lie algebras where Jacobi identity is twisted by a linear map. The Witt algebra is the complexification of the Lie algebra of polynomial vector fields on a circle with a basis  $L_n = -z^{n+1} \frac{\partial}{\partial z}$  and the Lie bracket which is given by  $[L_m, L_n] = (m - n)L_{m+n}$ . This Lie algebra can also be viewed as the Lie algebra of derivations  $D$  of the ring  $\mathbb{C}[z, z^{-1}]$ , where  $D(ab) = D(a)b + aD(b)$ . The Lie bracket of two derivations  $D$  and  $D'$  is given by  $[D, D'] = D \circ D' - D' \circ D$ . This algebra has a central extension, called the Virasoro algebra, which appears in two-dimensional conformal field theory and string theory. One can define the quantum deformation of  $D$  given by  $D_q(f)(z) = \frac{f(qz) - f(z)}{qz - z}$ . These linear operators are different from the usual derivations and they satisfy  $\sigma$ -derivation property  $D_q(fg) = gD_q(f) + \sigma(f)D_q(g)$  where  $\sigma(f)(z) = f(qz)$ . An example of a  $\sigma$ -derivation is the Jackson derivative on polynomials in one variable.

The set of  $\sigma$ -derivations with the classical bracket is a new type of algebra so called  $\sigma$ -deformations of the Witt algebra. This algebra does not satisfy the Jacobi identity. Instead, it satisfies Hom-Jacobi identity and it is called Hom-Lie algebra. The corresponding associative algebras, called Hom-associative algebras, were introduced in [MS1]. Any Hom-associative algebra with the bracket  $[a, b] = ab - ba$  is a Hom-Lie algebra. Later, other nonassociative objects such as Hom-coalgebras [MS2], Hom-bialgebras [MS2], [MS3], [Ya2], [GMMP], and Hom-Hopf algebras [MS2], [Ya3], [Ya4], were introduced and studied.

Hom-groups are nonassociative objects which were recently appeared in the study of group-like elements of Hom-Hopf algebras [LMT]. Studying Hom-groups gives us more information about Hom-Hopf algebras. One knows that groups and Lie algebras have important rules to develop many concepts related to Hopf algebras. During the last years, Hom-Lie algebras had played important rules to understand the structures of Hom-Hopf algebras. However, a lack of the notion of Hom-groups can affect to miss some concepts which potentially can be extended from Hom-groups to Hom-group Hopf algebras and therefore possibly to all Hom-Hopf algebras. A Hom-group  $(G, \alpha)$  is a set  $G$  with a bijective map  $\alpha : G \rightarrow G$  which is endowed with a multiplication that satisfies the Hom-associativity property  $\alpha(a)(bc) = (ab)\alpha(c)$ . Furthermore  $G$  has the Hom-unit element  $1$  which satisfies  $a1 = 1a = \alpha(a)$ . Every element  $g \in G$  has an inverse  $g^{-1}$  satisfying  $gg^{-1} = g^{-1}g = 1$ . If  $\alpha = \text{Id}$ , then  $G$  is a group. Although the twisting map  $\alpha$  of a Hom-group  $(G, \alpha)$  does not need to be invertible in the original works [LMT], [H1], however, more interesting results, including the main results of this paper, are obtained if  $\alpha$  is invertible. As a result through the paper, we assume that the twisting map  $\alpha$  is bijective. Since any Hom-group gives rise to a Hom-Hopf algebra, called Hom-group Hopf algebra [H2], it is interesting to know what properties will be enforced by the invertibility of  $\alpha$  on Hom-Hopf algebras. It is shown in [CG] that the category of modules over a Hom-Hopf algebra with invertible twisting map is monoidal. For this reason, they called them monoidal Hom-Hopf algebras. Also many interesting properties of Hom-Hopf algebras, such as integrals, modules, comodules, and Hopf representations are obtained when  $\alpha$  is bijective, see [CWZ], [H2], [PSS] [ZZ]. The author in [H1] introduced some basics of Hom-groups, their representations, and Hom-group (co)homology. They showed that the Hom-group (co)homology is related to the Hochschild (co)homology, [HSS], of Hom-group algebras.

Lagrange-type's theorem for nonassociative objects is a nontrivial problem. For instance, whether Lagrange's theorem holds for Moufang loops was an open problem in the theory of Moufang loops for more than four decades [CKRV]. In fact, not every loop satisfies the Lagrange property and the problem was finally answered in [GZ]. The authors in [BS] proved a version of Lagrange's theorem for Bruck loops. The strong Lagrange property was shown for left Bol loops of odd order in [FKP]. However, it is still an open problem whether Bol loops satisfy the Lagrange property. The authors in [SW] proved Lagrange's theorem for gyrogroups which are a class of Bol loops. In this paper, we prove Lagrange's theorem for Hom-groups which are an interesting class of quasigroups. More precisely, in Section 2, we introduce some fundamental concepts of Hom-groups such as Hom-subgroups, cosets,

center, and the centralizer of an element. In Theorem 2.11, we show that any Hom-group is a quasigroup. This means division is always possible to solve the equations  $ax = b$  and  $ya = b$ . The unit element 1 is an idempotent element and in fact, this class of interesting quasigroups is known as pointed idempotent quasigroups (piques) [BH]. Indeed, a Hom-group is a special case of piques which satisfies certain twisted associativity condition given by the idempotent element 1. If a Hom-group  $G$  is a loop then  $1x = \alpha(x) = x$  which means  $\alpha = \text{Id}$ , and therefore  $G$  should be a group. We use properties of Cayley table of Hom-groups to present some examples. The Cayley tables of quasigroups have been used in combinatorics and cryptography, see [CPS], [DK], [SC], [BBW]. In Section 3, we use cosets to partition a Hom-group  $G$ . Then we prove Lagrange's theorem for Hom-groups which states that for any finite Hom-group, the order of any Hom-subgroup divides the order of the Hom-group. In Section 4, we apply Lagrange's theorem to show that for a Hom-sub-Hopf algebra  $A$  of  $\mathbb{K}G$ ,  $\dim(A)$  divides  $|G|$ . We finish the paper by raising some conjectures about Hom-groups.

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## 2 Hom-groups

In this section, we introduce basic notions for a Hom-group  $(G, \alpha)$  such as Hom-subgroups, cosets, and the center. Our definition of a Hom-group is a special case of the one discussed in [LMT] and [H1]. Through the paper, we assume the map  $\alpha$  is invertible. Therefore our axioms will be different from the ones in the original definition. We show that some of the axioms can be obtained by Hom-associativity when  $\alpha$  is invertible.

**Definition 2.1.** *A Hom group consists of a set  $G$  together with a distinguished member  $1 \in G$ , a bijective set map:  $\alpha : G \rightarrow G$ , a binary operation  $\mu : G \times G \rightarrow G$ , where these pieces of structure are subject to the following axioms:*

*i) The product map  $\mu : G \times G \rightarrow G$  is satisfying the Hom-associativity property*

$$\mu(\alpha(g), \mu(h, k)) = \mu(\mu(g, h), \alpha(k)).$$

*For simplicity when there is no confusion we omit the multiplication sign  $\mu$ .*

*ii) The map  $\alpha$  is multiplicative, i.e.,  $\alpha(gk) = \alpha(g)\alpha(k)$ .*

iii) The element 1 is called unit and it satisfies the Hom-unitality conditions

$$g1 = 1g = \alpha(g), \quad \alpha(1) = 1.$$

v) For every element  $g \in G$ , there exists an element  $g^{-1} \in G$  which

$$gg^{-1} = g^{-1}g = 1.$$

Based on the definition of a Hom-group in [LMT], [H1] and [H2], for any element  $g \in G$  there exists a natural number  $n$  satisfying the Hom-invertibility condition  $\alpha^n(gg^{-1}) = \alpha^n(g^{-1}g) = 1$ , where the smallest such  $n$  is called the invertibility index of  $g$ . Clearly if  $\alpha$  is invertible this condition will be simplified to our condition (v). One also notes that the condition (iii) implies  $\mu(1, 1) = 1$ . This shows that 1 is an idempotent element. The following lemma is crucial for our studies of Hom-groups.

**Lemma 2.2.** *Let  $(G, \alpha)$  be a Hom-group. Then*

i) *The inverse of any element is unique.*

ii)  *$(ab)^{-1} = b^{-1}a^{-1}$  for all  $a, b \in G$ .*

*Proof.* i) First we show that the right inverse is unique. Let us assume an element  $g \in G$  has right inverses  $a, b \in G$ . So  $ga = 1$  and  $gb = 1$ . Since there exists  $g^{-1} \in G$  where  $g^{-1}g = 1$ , then

$$\alpha(g^{-1})(ga) = \alpha(g^{-1})1.$$

By Hom-associativity we have

$$(g^{-1}g)\alpha(a) = \alpha^2(g^{-1}).$$

By Hom-unitality we obtain  $\alpha^2(a) = \alpha^2(g^{-1})$ . Since  $\alpha$  is invertible then  $a = g^{-1}$ . Similarly  $b = g^{-1}$  and therefore  $a = b$ . Likewise we can prove that left inverse is unique. Now we show that left and right inverses are the same. Let  $g \in G$ ,  $ag = 1$  and  $gb = 1$ . Therefore

$$(ag)\alpha(b) = 1\alpha(b).$$

By Hom-associativity we have  $\alpha(a)(gb) = \alpha^2(b)$ . Then  $\alpha(a)1 = \alpha^2(b)$ . So  $\alpha^2(a) = \alpha^2(b)$ . By invertibility of  $\alpha$  we obtain  $a = b$ . Therefore the inverse element is unique.

ii) The following computations shows that  $b^{-1}a^{-1}$  is the inverse of  $ab$ .

$$\begin{aligned} & (ab)(b^{-1}a^{-1}) \\ &= \alpha(\alpha^{-1}(ab))[b^{-1}a^{-1}] \\ &= [\alpha^{-1}(ab)b^{-1}]\alpha(a^{-1}) \\ &= ([\alpha^{-1}(a)\alpha^{-1}(b)]b^{-1})\alpha(a^{-1}) \\ &= ([\alpha^{-1}(a)\alpha^{-1}(b)]\alpha(\alpha^{-1}(b^{-1})))\alpha(a^{-1}) \\ &= [a(\alpha^{-1}(b)\alpha^{-1}(b^{-1}))]\alpha(a^{-1}) \\ &= (a1)\alpha(a^{-1}) \\ &= \alpha(a)\alpha(a^{-1}) = 1. \end{aligned}$$

We used the Hom-associativity in the third equality, multiplicity of  $\alpha$  in the fourth equality, and Hom-associativity in the fifth equality.  $\square$

The following proposition which was introduced in [H1] provides a source of examples for Hom-groups.

**Proposition 2.3.** *Let  $(G, \mu)$  be a group and  $\alpha : G \rightarrow G$  a group automorphism. Then  $(G, \alpha \circ \mu, \alpha)$  is a Hom-group.*

**Definition 2.4.** *A subset  $H$  of a Hom-group  $(G, \alpha)$  is called a Hom-subgroup [H1] of  $G$  if  $(H, \alpha)$  is itself a Hom-group. We denote a Hom-subgroup  $H$  of  $G$  by  $H \preceq G$ .*

**Definition 2.5.** *The set  $gH = \{gh, \quad h \in H\}$  is called the left coset of the Hom-subgroup  $H$  in  $G$  with respect to the element  $g$ . Similarly the set  $Hg = \{hg, \quad h \in H\}$  is called the right coset of  $H$  in  $G$ .*

We denote the number of elements of a Hom-group  $G$  by  $|G|$ .

**Definition 2.6.** *The Center  $Z(G)$  of a Hom-group  $(G, \alpha)$  is the set of all  $x \in G$  where  $xy = yx$  for all  $y \in G$ .*

**Proposition 2.7.** *Let  $(G, \alpha)$  be a Hom-group. Then  $Z(G) \preceq G$ .*

*Proof.* Let  $x, y \in Z(G)$ . Then for all  $a \in G$  we have

$$\begin{aligned} (xy)a &= (xy)(\alpha(\alpha^{-1}(a))) = \alpha(x)(y\alpha^{-1}(a)) = \alpha(x)[\alpha^{-1}(a)y] \\ &= [x\alpha^{-1}(a)]\alpha(y) = [\alpha^{-1}(a)x]\alpha(y) = a(xy). \end{aligned}$$

Thus  $Z(G)$  is closed under multiplication of  $G$ . Also if  $x \in Z(G)$  and  $a \in G$  then  $xa^{-1} = a^{-1}x$ . Then  $(xa^{-1})^{-1} = (a^{-1}x)^{-1}$ . Therefore  $ax^{-1} = x^{-1}a$ . So  $x^{-1} \in Z(G)$ . Therefore  $Z(G)$  is a Hom-subgroup of  $G$ .  $\square$

**Definition 2.8.** *The centralizer of an element  $x \in G$  is the set of all elements  $g \in G$  where  $gx = xg$  and it is denoted by  $C_G(x)$ .*

**Proposition 2.9.** *Let  $(G, \alpha)$  be a Hom-group. Then  $C_G(x) \preceq G$  for all  $x \in G$ .*

*Proof.* The proof is similar to previous Proposition.  $\square$

Let  $(G, \alpha)$  and  $(H, \beta)$  be two Hom-groups. The morphism  $f : G \rightarrow H$  is called a morphism of Hom-groups [H1], if  $\beta(f(g)) = f(\alpha(g))$  and  $f(gk) = f(g)f(k)$  for all  $g, k \in G$ . Two Hom-groups  $G$  and  $H$  are called isomorphic if there exist a bijective morphism of Hom-groups  $f : G \rightarrow H$ .

**Example 2.10.** Let  $(G, \alpha)$  and  $(G', \alpha')$  be two Hom-groups. Then  $(G \times G', \alpha \times \alpha')$  is a Hom-group by the multiplication given by  $(g, h)(g', h') = (gg', hh')$ .

Here we recall the definition of a quasigroup. A quasigroup  $(Q, *)$  is a set  $Q$  with a multiplication  $* : Q \times Q \rightarrow Q$ , where for all  $a, b \in Q$ , there exist unique elements  $x, y \in Q$  such that

$$a * x = b, \quad y * a = b.$$

In the following theorem we show that Hom-groups are a class of quasigroups.

**Theorem 2.11.** *Every Hom-group  $(G, \alpha)$  is a quasigroup.*

*Proof.* First we show that the  $x = \alpha^{-1}(a^{-1})\alpha^{-2}(b)$  satisfies the equation  $ax = b$ .

$$\begin{aligned} ax &= a(\alpha^{-1}(a^{-1})\alpha^{-2}(b)) \\ &= \alpha(\alpha^{-1}(a))(\alpha^{-1}(a^{-1})\alpha^{-2}(b)) \\ &= (\alpha^{-1}(a)\alpha^{-1}(a^{-1}))\alpha^{-1}(b) \\ &= \alpha^{-1}(aa^{-1})\alpha^{-1}(b) \\ &= \alpha^{-1}(1)\alpha^{-1}(b) \\ &= 1\alpha^{-1}(b) = b. \end{aligned}$$

We used the Hom-associativity in the third equality, invertibility of  $a$  in the fifth equality and the Hom-unitality in the last equality. Similarly  $y = \alpha^{-2}(b)\alpha^{-1}(a^{-1})$  satisfies the equation  $ya = b$ . Therefore  $G$  is a quasigroup.  $\square$

Since the element  $1 \in G$  is an idempotent element then any Hom-group is a pointed idempotent quasigroup (piques). This is an interesting class of quasigroups which have been under intensive research [S], [CPS].

**Remark 2.12. (Cayley table of finite Hom-groups)**

Since every Hom-group is a quasi group, then the Cayley table of a Hom-group has all the properties of the one for quasigroup. However having invertibility and Hom-unitality conditions, one obtains more properties. We put all the different elements of  $G$  in the first row and column such that the Hom-unit 1 is in the first place. For simplicity we use the matrix notation  $[c_{ij}]$  for the Cayley matrix. Some of the properties of Hom-groups are as follows:

- i) Every row and column of the Cayley table of a Hom-group  $(G, \alpha)$  is a permutation of the set  $G$ . This is because Hom-groups have the cancelation property. In fact the Cayley table of a Hom-group is an example of a Latin square.
- ii) Rows and columns can not be the identity permutation of  $G$  except  $\alpha = \text{Id}$ , which means  $G$  is a group.

- iii) If the element in the  $i^{th}$  row and  $j^{th}$  column is 1 then the element in the  $j^{th}$  row and  $i^{th}$  column also should be 1. This is because of the invertibility condition.
- iv) The Cayley table is symmetric if and only if  $(G, \alpha)$  is abelian.
- v) The first row and first column are the same, because  $1a = a1 = \alpha(a)$ .

**Example 2.13. (Classification of Hom-groups of order 3).**

In this example we show that there is only one Hom-group of order 3. We use the Cayley table to classify all the Hom-groups of order 3. Let  $G = \{1, a, b\}$ . To fill out the Cayley table, we start from the first row. Since by property (ii), the first row can not be the identity permutation of  $G$ , therefore the only possibility is  $c_{12} = b$  and  $c_{13} = a$ . Since the first column is the same as the first row therefore there will be only 4 spots  $c_{22}, c_{23}, c_{32}, c_{33}$  to find. Now we argue on the place of the unit 1 in the second row. We note that  $c_{22} \neq 1$  because otherwise  $c_{23} = a$  which is a contradiction as the third column will have two copies of  $a$ . Therefore  $c_{22} = a$  and  $c_{23} = 1$ . Now since the second and the third columns should be a permutation of  $G$  then there is only one case left which is the following Cayley table:

$G$	1	$a$	$b$
1	1	$b$	$a$
$a$	$b$	$a$	1
$b$	$a$	1	$b$

This defines the twisting map by  $\alpha(a) = b$  and  $\alpha(b) = a$ . One can check that  $(G, \alpha)$  with above Cayley table satisfies the Hom-associativity and therefore it is an abelian Hom-group. This Hom-group is isomorphic to  $Z_3^\alpha$  where the multiplication is obtained by twisting the multiplication of the additive cyclic group  $Z_3$  by the group automorphism  $\alpha : Z_3 \rightarrow Z_3$  given by  $\alpha(1) = 2$ . See Proposition 2.3.

We finish this section by introducing a non abelian Hom-group of order 6.

**Example 2.14. (Hom-Dihedral group  $D_3^\alpha$ )**

We recall that the Dihedral group  $D_3$  is the smallest non-abelian group which is given by  $\{r, s, \mid r^3 = s^2 = 1, \quad srs = r^{-1}\}$ . In fact the elements of  $D_3$  are  $\{1, r, r^2, s, sr, rs\}$ . We consider the conjugation automorphism  $\varphi_s(x) = sxs^{-1}$ . Now we twist the multiplication of  $D_3$  by  $\varphi_s$ , as explained in Proposition 2.3, to obtain a Hom-group  $D_3^\alpha$  where  $\alpha = \varphi_s$ . The Cayley table is given by

$D_3^\alpha$	1	$r$	$r^2$	$s$	$rs$	$sr$
1	1	$r^2$	$r$	$s$	$sr$	$rs$
$r$	$r^2$	$r$	1	$sr$	$rs$	$s$
$r^2$	$r$	1	$r^2$	$rs$	$s$	$sr$
$s$	$s$	$rs$	$sr$	1	$r$	$r^2$
$rs$	$sr$	$s$	$rs$	$r^2$	1	$r$
$sr$	$rs$	$sr$	$s$	$r$	$r^2$	1

### 3 Lagrange's theorem for a class of quasigroups

Lagrange-type's theorem for nonassociative structures (magmas) is a challenging problem due to nonassociativity. In this section we focus on finite Hom-groups. We prove Lagrange's theorem for this interesting class of quasigroups. This in fact generalizes the theorem for groups. The Hom-associativity condition plays an important role in our proof. First we need the following lemma which shows that the number of elements of a Hom-subgroup and its cosets are the same.

**Lemma 3.1.** *Let  $(G, \alpha)$  be a finite Hom-group. If  $H \preceq G$ , then  $|gH| = |H|$  for all  $g \in G$ .*

*Proof.* It is enough to show that for  $h_i \neq h_j$ , the elements  $gh_i$  and  $gh_j$  are different in  $gH$ . Suppose  $gh_i = gh_j = b$ . By Theorem 2.11 we have

$$h_i = \alpha^{-1}(g^{-1})\alpha^{-2}(b) = h_j.$$

□

**Lemma 3.2.** *Let  $(G, \alpha)$  be a finite Hom-group and  $H \preceq G$ . Then  $gH = H$  if and only if  $g \in H$ .*

*Proof.* If  $g \in H$  then  $gH \subseteq H$ . However by Lemma 3.1 we have  $|gH| = |H|$ . Therefore  $gH = H$ . Conversely if  $gH = H$  then  $g1 \in H$ . So  $\alpha(g) \in H$ . Since  $\alpha$  is invertible and  $\alpha(H) = H$  then  $g \in H$ . □

**Lemma 3.3.** *Let  $(G, \alpha)$  be a finite Hom-group and  $H \preceq G$ . For all  $x, y \in G$  if  $xH \cap yH \neq \emptyset$  then  $xH = yH$ .*

*Proof.* Since  $xH \cap yH \neq \emptyset$ , there exists  $h_1, h_2 \in H$  such that  $xh_1 = yh_2$ . Then by Theorem 2.11 we have

$$x = \alpha^{-2}(yh_2)\alpha^{-1}(h_1^{-1}).$$

By invertibility of  $\alpha$  we obtain

$$x = [\alpha^{-2}(y)\alpha^{-2}(h_2)]\alpha(\alpha^{-2}(h_1^{-1})).$$

By Hom-associativity we have

$$x = \alpha^{-1}(y)[\alpha^{-2}(h_2)\alpha^{-2}(h_1^{-1})] = \alpha^{-1}(y)[\alpha^{-2}(h_2h_1^{-1})].$$

Now we show that  $xH \subseteq yH$ . Let  $xh \in xH$ . Then

$$\begin{aligned} xh &= [\alpha^{-1}(y)[\alpha^{-2}(h_2h_1^{-1})]] h \\ &= [\alpha^{-1}(y)[\alpha^{-2}(h_2h_1^{-1})]] \alpha(\alpha^{-1}(h)) \\ &= y[\alpha^{-2}(h_2h_1^{-1})\alpha^{-1}(h)]. \end{aligned}$$



We used invertibility of  $\alpha$  in the second equality, and Hom-associativity in the third equality. Since  $\alpha(H) = H$  then

$$x = y[\alpha^{-2}(h_2h_1^{-1})\alpha^{-1}(h)] \in yH.$$

Therefore  $xH \subseteq yH$ . By Lemma 3.1 we have  $|xH| = |H| = |yH|$ . Therefore  $xH = yH$ .  $\square$

As a consequence of the previous results we obtain the following proposition.

**Proposition 3.4.** *Let  $(G, \alpha)$  be a finite Hom-group and  $H \preceq G$ . Then the set of all cosets of  $H$  in  $G$  gives a partition of the set  $G$ .*

**Theorem 3.5. (Lagrange's theorem for Hom-groups)**

*Let  $(G, \alpha)$  be a finite Hom-group and  $H \preceq G$ . Then  $|H|$  divides  $|G|$ .*

*Proof.* By the previous Proposition the cosets of  $H$  in  $G$  gives a partition of  $G$ . By Lemma 3.1 the size of all cosets are the same as the size of  $H$ . Since  $G = \cup_{x \in G} xH$ , then  $|G|$  is a multiplication of  $|H|$ .  $\square$

**Example 3.6.** Let  $G$  be a group,  $\alpha : G \rightarrow G$  a group automorphism and  $H \preceq G$  which is preserved by  $\alpha$ , i.e,  $\alpha(H) = H$ . We twist the multiplication of  $G$  by  $\alpha$  to obtain the Hom-group  $G_\alpha$  as we explained in Proposition 2.3. Clearly we have  $H \preceq G_\alpha$ . One notes that if  $\alpha$  does not preserve  $H$  then  $H$  will not be a Hom-subgroup of  $G_\alpha$  because  $\alpha(h) = 1h \in H$ . Therefore studying group of  $Aut(G)$  has an important rule to have examples of Hom-(sub)groups of  $G_\alpha$ . As an example if  $\alpha \in Inn(G)$  and  $N \triangleleft G$  then  $\alpha(N) = N$  and therefore  $N \preceq G_\alpha$ .

**Example 3.7.** All cyclic groups of order 6 are isomorphic to  $Z_6$ . We define the group automorphism  $\alpha : Z_6 \rightarrow Z_6$  given by

$$\alpha(1) = 5, \alpha(2) = 4, \alpha(3) = 3, \alpha(4) = 2, \alpha(5) = 1, \alpha(0) = 0$$

Now we twist the multiplication of  $Z_6$  by  $\alpha$  as we explained in Proposition 2.3 to obtain a Hom-group  $Z_6^\alpha$  given by the following Cayley table of multiplication,

$Z_6^\alpha$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	5	4	3	2	1	0
2	4	3	2	1	0	5
3	3	2	1	0	5	4
4	2	1	0	5	4	3
5	1	0	5	4	3	2

It can be verified that  $Z_2^\alpha = \{0, 3\}$  and  $Z_3^\alpha = \{0, 2, 4\}$  are the only non-trivial Hom-subgroups of the Hom-group  $Z_6^\alpha$ . They are of orders 2 and 3 which both divides the order of  $Z_6^\alpha$ . One notes that the Hom-subgroup  $Z_3^\alpha$  is not cyclic in the usual sense. In fact  $2+2$  in  $Z_3^\alpha$  is 2 and  $4+4$  is 4. Therefore 2 and 4 can not be the generators of  $Z_3^\alpha$  in the usual sense. Therefore the proper notion of power of an element in Hom-groups is not clear to us.

## 4 Linearization of Hom-groups

In this section first we recall the linearization of Hom-groups from [H1], [H2] to obtain some examples of an interesting class of nonassociative Hopf algebras called Hom-Hopf algebras. This linearization is called Hom-group Hopf algebras. Then we apply Lagrange's theorem for finite Hom-groups to find out about dimensions of Hom-sub-Hopf algebras of Hom-group Hopf algebras. First we recall the definitions of Hom-algebras, Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras. By [MS1], a Hom-associative algebra  $A$  over a field  $\mathbb{K}$  is a  $\mathbb{K}$ -vector space with a bilinear map  $\mu : A \otimes A \rightarrow A$ , called multiplication, and a linear homomorphism  $\alpha : A \rightarrow A$  satisfying the Hom-associativity condition

$$\mu(\alpha(a), \mu(b, c)) = \mu(\mu(a, b), \alpha(c)),$$

for all elements  $a, b, c \in A$ . A Hom-associative algebra  $A$  is called unital with unit 1 if  $\alpha(1) = 1$ , and  $a1 = 1a = \alpha(a)$ . By [MS2], [MS3], a Hom-coalgebra is a triple  $(C, \Delta, \varepsilon, \beta)$ , where  $C$  is a  $\mathbb{K}$ -vector space,  $\Delta : C \rightarrow C \otimes C$  a linear map, called comultiplication, with a Sweedler notation  $\Delta(c) = c^{(1)} \otimes c^{(2)}$ , counit  $\varepsilon : \mathbb{K} \rightarrow C$ , and  $\beta : C \rightarrow C$  a linear map satisfying the Hom-coassociativity condition,

$$\beta(c^{(1)}) \otimes c^{(2)(1)} \otimes c^{(2)(2)} = c^{(1)(1)} \otimes c^{(1)(2)} \otimes \beta(c^{(2)}),$$

and

$$c^{(1)}\varepsilon(c^{(2)}) = \varepsilon(c^{(1)})c^{(2)} = \beta(c), \quad \varepsilon(\beta(c)) = \varepsilon(c).$$

A  $(\alpha, \beta)$ -Hom-bialgebra is a tuple  $(B, m, 1, \alpha, \Delta, \varepsilon, \beta)$  where  $(B, m, 1, \alpha)$  is a unital Hom-algebra and  $(B, \varepsilon, \Delta, \beta)$  is a counital Hom-coalgebra where  $\Delta$  and  $\varepsilon$  are morphisms of Hom-algebras, that is

- i)  $\Delta(hk) = \Delta(h)\Delta(k)$ .
- ii)  $\Delta(1) = 1 \otimes 1$ .
- iii)  $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ .
- iv)  $\varepsilon(1) = 1$ .
- v)  $\varepsilon(\alpha(x)) = \varepsilon(x)$ .

Here we recall the definition of Hom-Hopf algebras from [MS2] and [MS3]. A Hom-bialgebra  $(B, m, \eta, \alpha, \Delta, \varepsilon, \beta)$  is called a  $(\alpha, \beta)$ -Hom-Hopf algebra if it is endowed with a morphism  $S : B \rightarrow B$ , called antipode, satisfying

- a)  $S \circ \eta = \eta$  and  $\varepsilon \circ S = \varepsilon$ .
- b)  $S$  is an inverse of the identity map  $\text{Id} : B \rightarrow B$  for the convolution product, i.e, for any  $x \in B$ ,

$$S(x^{(1)})x^{(2)} = x^{(1)}S(x^{(2)}) = \varepsilon(x)1_B. \quad (4.1)$$

This definition of a Hom-Hopf algebra and specially antipodes is different from the one in [LMT]. For more details see [H2]. However if  $\alpha$  is invertible, both definitions will be equivalent.

**Example 4.1.** For any Hom-group  $(G, \alpha)$ , the Hom-group algebra  $\mathbb{K}G$  is a  $(\alpha, \text{Id})$ -Hom-Hopf algebra. It is a free algebra on  $G$  where the coproduct is given by  $\Delta(g) = g \otimes g$ , counit by  $\varepsilon(g) = 1$ , the antipode by  $S(g) = g^{-1}$ , and  $\beta = \text{Id}$  with  $\alpha$  which is linearly extended from  $G$  to  $\mathbb{K}G$ . One notes that elements  $g \in \mathbb{K}G$  are group-like elements. Also  $\mathbb{K}G$  is a cocommutative Hom-Hopf algebra. If  $G$  is an abelian Hom-group then  $\mathbb{K}G$  is a commutative Hom-Hopf algebra.

**Lemma 4.2.** *Let  $(G, \alpha)$  be a Hom-group. The vector space  $A$  is a Hom-sub Hopf algebra of  $\mathbb{K}G$  if and only if there exists  $H \preceq G$  where  $A = \mathbb{K}H$ .*

*Proof.* Let  $A$  be a Hom-sub-Hopf algebra of  $\mathbb{K}G$ . Then the set of group-like elements of  $A$  forms a Hom-group  $H$ , see [LMT], and clearly  $A = \mathbb{K}H$ . Conversely if  $H \preceq G$  then by the structure of the product, coproduct and the antipode explained in the previous example,  $\mathbb{K}H$  is a Hom-sub Hopf algebra of  $\mathbb{K}G$ .  $\square$

**Theorem 4.3.** *Let  $(G, \alpha)$  be a Hom-group and  $\mathbb{K}G$  be the Hom-group Hopf algebra. If  $A$  is a Hom-sub-Hopf algebra of  $\mathbb{K}G$ , then  $\dim(A)$  divides  $|G|$ .*

*Proof.* By previous Lemma there exists  $H \preceq G$  where  $A = \mathbb{K}H$ . Since  $\dim(A) = |H|$ , then by Lagrange's theorem  $\dim(A)$  divides  $|G|$ .  $\square$

**Example 4.4.** Let us consider the cyclic group  $Z_5$ . We define a group automorphism  $\alpha : Z_5 \rightarrow Z_5$  given by

$$\alpha(1) = 2, \alpha(2) = 4, \alpha(3) = 1, \alpha(4) = 3, \alpha(0) = 0.$$

One twists the multiplication of  $Z_5$  by  $\alpha$  as explained in Proposition 2.3 to obtain a Hom-group  $Z_5^\alpha$  given by the following table of multiplication

$Z_5^\alpha$	0	1	2	3	4
0	0	2	4	1	3
1	2	4	1	3	0
2	4	1	3	0	2
3	1	3	0	2	4
4	3	0	2	4	1

Since the order of  $\mathbb{K}Z_5^\alpha$  is prime, by previous theorem it does not have any non-trivial Hom-sub Hopf algebra.

**Example 4.5.** Consider the Hom-group Hopf algebra  $\mathbb{K}G$ . Since  $Z(G) \preceq G$  then the center of the Hom-Hopf algebra  $\mathbb{K}G$  is the same as  $\mathbb{K}Z(G)$  and its dimension divides  $|G|$ .

#### **Remark 4.6. Conjectures**

A challenge in studying a Hom-group  $(G, \alpha)$  is defining a proper notion of power of an element. Since  $G$  is not associative we can define two different types of powers called left

and right powers. Following the contexts of nonassociative objects such as quasigroups, an approach to define a right power of an element  $x$  in a Hom-group  $(G, \alpha)$  is as follows. We set  $x^1 = x$ ,  $x^2 = xx$ . Now  $x^3 = (x^2)x$  and inductively we can define other right powers. In fact one can define the right multiplication function  $R_a(x) = xa$ . So  $x^2 = R_x(x)$ ,  $x^3 = R_x(x^2)$  and generally  $x^n = R_x(x^{n-1})$ . Similarly if  $L_a(x) = x$  then the left powers of  $x$  can inductively be defined by  $x^n = L_x(x^{n-1})$ . However, this method has some problems such as defining cyclic Hom-subgroups. The notions of power and order of an element of  $G$  are not clear for Hom-groups. Consequently, some fundamental theorems of group theory such as Cauchy's theorem will be left as a conjecture for Hom-groups; if  $(G, \alpha)$  is a finite Hom-group and  $p$  is a prime number dividing the order of  $G$ , then  $G$  contains an element, and therefore a Hom-subgroup of order  $p$ .

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