# SYMMETRY OF ENTROPY IN HIGHER RANK DIAGONALIZABLE ACTIONS AND MEASURE CLASSIFICATION

MANFRED EINSIEDLER AND ELON LINDENSTRAUSS

In memory of Roy Adler

ABSTRACT. An important consequence of the theory of entropy of  $\mathbb{Z}$ -actions is that the events measurable with respect to the far future coincide (modulo null sets) with those measurable with respect to the distant past, and that measuring the entropy using the past will give the same value as measuring it using the future. In this paper we show that for measures invariant under multiparameter algebraic actions if the entropy attached to coarse Lyapunov foliations fail to display a stronger symmetry property of a similar type this forces the measure to be invariant under non-trivial unipotent groups. Some consequences of this phenomenon are noted.

### 1. INTRODUCTION

Let  $M = \prod_{\ell=1}^{m} \operatorname{SL}(d, k_{\ell})$  with  $k_{\ell}$  local fields of either zero or positive characteristic (not necessarily the same for all  $\ell$ ). Let G be a closed subgroup of M,  $\Gamma$  a lattice in G and  $a_1, a_2, \ldots, a_r$  be r elements in G so that for every i, all of the components of  $a_i$  in  $\operatorname{SL}(d, k_{\ell})$  are diagonal matrices; in particular all the  $a_i$  commute. For  $\mathbf{n} \in \mathbb{Z}^r$  we denote  $a^{\mathbf{n}} = a_1^{n_1} \ldots a_r^{n_r}$ .

Suppose  $1 \leq \ell \leq m$  and for distinct  $1 \leq i, j \leq d$  let  $E_{ij}^{\ell}$  denote the elementary unipotent subgroup of  $\operatorname{SL}(d, k_{\ell})$  with one on the diagonal, arbitrary element of  $k_{\ell}$  on the i, j entry and zero elsewhere. Then there is some linear functional  $\alpha : \mathbb{Z}^r \to \mathbb{R}$ so that for every  $h \in E_{ij}^{\ell}$ 

$$|a^{\mathbf{n}}(h-1)a^{-\mathbf{n}}| = e^{\alpha(\mathbf{n})} |h-1|.$$

These functionals will be called the Lyapunov exponents of the action of a on M, and the set of such functionals will be denoted by  $\Phi$ . Two Lyapunov exponents  $\alpha, \alpha' \in \Phi$  will be said to correspond to the same coarse exponent if  $\alpha = c\alpha'$  for some c > 0. The equivalence class of a Lyapunov exponent  $\alpha$  under this equivalence relation will be denoted by  $[\alpha]$ , and the set of equivalence classes, a.k.a. the *coarse* Lyapunov exponents will be denoted by  $[\Phi]$ .

For every  $\mathbf{n} \in \mathbb{Z}^r$  we define the corresponding expanding horospherical subgroup  $G^+_{\mathbf{n}}$  of G by

$$G_{\mathbf{n}}^{+} = \left\{ g \in G : a^{-j\mathbf{n}}ga^{j\mathbf{n}} \to 1 \text{ as } j \to \infty \right\}.$$

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We can attach a subgroup  $U_{[\alpha]}$  to every coarse Lyapunov exponent  $[\alpha] \in [\Phi]$  as follows:

$$U_{[\alpha]} = \bigcap_{\mathbf{n}:\alpha(\mathbf{n})>0} G_{\mathbf{n}}^+.$$

Somewhat more explicitly, let  $M_{[\alpha]}$  denote the subgroup of M generated by all elementary one parameter unipotent groups  $E_{ij}^{\ell}$  for which the corresponding Lyapunov exponent is in  $[\alpha]$ . Then it can be shown that  $U_{[\alpha]} = G \cap M_{[\alpha]}$ . Note that our definitions imply that if  $\alpha \in \Phi$  so is  $-\alpha$ , but it may well happen for some G that  $U_{[\alpha]}$  is nontrivial but  $U_{[-\alpha]} = \{1\}$ . We will say that a coarse Lyapunov exponent  $[\alpha]$  is a coarse Lyapunov exponent for G (or appears in G) if  $U_{[\alpha]} \neq \{1\}$ .

Let  $\mu$  be an A-invariant and ergodic probability measure on  $G/\Gamma$ . To each coarse Lyapunov exponent  $[\alpha] \in [\Phi]$ , we attach a system of leafwise measures  $\mu_x^{[\alpha]}$ . Formally,  $x \mapsto \mu_x^{[\alpha]}$  is a Borel measurable map from  $G/\Gamma$  to the space of equivalence classes up to a positive multiplicative constant of locally finite measures on  $U_{[\alpha]}$ satisfying a suitable growth condition (enforcing such a growth condition makes the space of locally finite measures up to a multiplicative constants into a compact metrizable space).

Let  $I_x^{[\alpha]}$  be the group of  $u \in U_{[\alpha]}$  satisfying that  $\mu_x^{[\alpha]}u = \mu_x^{[\alpha]}$ . As a locally compact nilpotent group,  $I_x^{[\alpha]}$  is unimodular. It would be convenient to have also a notation for the Haar measure on  $I_x^{[\alpha]}$  considered as a locally finite measure up to multiplicative constant — we denote this measure by  $\mu_{inv,x}^{[\alpha]}$ . Using Poincare recurrence and ergodicity one can easily show that if  $I_x^{[\alpha]}$  is nontrivial on a set of positive  $\mu$ -measure then it is nontrivial a.s. and moreover contains arbitrarily small and arbitrary large elements of  $U_{[\alpha]}$ , i.e. the group  $I_x^{[\alpha]}$  is neither discrete nor bounded.

We define for  $\mathbf{n} \in \mathbb{Z}^r$  and  $\alpha \in \Phi$  with  $\alpha(\mathbf{n}) > 0$  the entropy contribution of  $\alpha$  for  $\mathbf{n}$ , denoted by  $D_{\mu}(\mathbf{n}, [\alpha])$ , and the entropy contribution from the invariance group  $D_{\mu}^{\mathrm{inv}}(\mathbf{n}, [\alpha])$  by

(1.1)  
$$D_{\mu}(\mathbf{n}, [\alpha]) = \lim_{\ell \to \infty} \frac{-\log \mu_x^{[\alpha]}(a^{-\ell \mathbf{n}} \Omega_0 a^{\ell \mathbf{n}})}{\ell}$$
$$D_{\mu}^{\mathrm{inv}}(\mathbf{n}, [\alpha]) = \lim_{\ell \to \infty} \frac{-\log \mu_{\mathrm{inv}, x}^{[\alpha]}(a^{-\ell \mathbf{n}} \Omega_0 a^{\ell \mathbf{n}})}{\ell}$$

where  $\Omega_0$  is a relatively compact open neighborhood of 1 in  $U_{[\alpha]}$ . For notational convenience, we set  $D_{\mu}(\mathbf{n}, [\alpha]) = D_{\mu}^{\text{inv}}(\mathbf{n}, [\alpha]) = 0$  if  $\alpha(\mathbf{n}) \leq 0$ . Formally, these quantities depend on the choice of  $x \in G/\Gamma$  and strictly speaking to make sense of the expressions inside the limits above one needs to choose a particular measure in the proportionallity class  $\mu_x^{[\alpha]}$  and  $\mu_{\text{inv},x}^{[\alpha]}$ . Both limits in (1.1) are known to exist and moreover

$$\frac{D_{\mu}(\mathbf{n}, [\alpha])}{\alpha(\mathbf{n})} = \frac{D_{\mu}(\mathbf{m}, [\alpha])}{\alpha(\mathbf{m})}$$

for every  $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$  with  $\alpha(\mathbf{n}), \alpha(\mathbf{m}) > 0$  (and similarly for  $D^{\text{inv}}_{\mu}(\bullet, [\alpha])$ ) — see e.g. [EL1]. We will write  $a^{\bullet}$  for the group  $\{a^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^r\}$ .

**Theorem 1.1.** Let  $\mu$  be an  $a^{\bullet}$ -invariant and ergodic measure on  $G/\Gamma$ , with  $a^{\bullet}$ , G and  $\Gamma$  as above. Let  $[\alpha] \in [\Phi]$  and  $\mathbf{n} \in \mathbb{Z}^r$  satisfy  $\alpha(\mathbf{n}) > 0$ . Then

$$D_{\mu}^{\mathrm{inv}}(\mathbf{n}, [\alpha]) \ge D_{\mu}(\mathbf{n}, [\alpha]) - D_{\mu}(-\mathbf{n}, [-\alpha]).$$

In particular, if  $D_{\mu}(\mathbf{n}, [\alpha]) > D_{\mu}(-\mathbf{n}, [-\alpha])$  the invariance group  $I_x^{[\alpha]}$  contains arbitrarily small and arbitrary large elements. Moreover, if  $D_{\mu}(\mathbf{n}, [-\alpha]) = 0$ , and in particular if  $U_{[-\alpha]} = \{1\}$ , then  $\mu_x^{[\alpha]} = \mu_{\text{inv},x}^{[\alpha]}$  a.s.

For  $\alpha$  for which  $U_{[-\alpha]} = \{1\}$ , Theorem 1.1 essentially reduces to the main technical result of Katok and Spatzier's paper [KS] (see also [KK]). The symmetry of the entropy contribution has been an important component in the classification of measures invariant under a maximal split torus in semisimple groups, e.g. [EKL, Cor. 3.4], [EL2, Thm. 5.1]. In those papers, this symmetry was derived from more detailed analysis and using additional structure; the main point of this paper is that this symmetry is a rather general feature of higher rank diagonal actions. Thus one can view Theorem 1.1 as a common generalization of both the techniques of [KS] and the above auxiliary results from [EKL, EL2].

Specializing further, we obtain a sharper result in the same vein. Suppose now all the local fields  $k_{\ell}$  are either  $\mathbb{R}$  or  $\mathbb{Q}_p$  (of course, more than one prime p may be used). Suppose further that  $a_1, a_2, \ldots, a_r$  are not only diagonal but satisfy the following further assumption:

### $Class-\mathcal{A}':$

The components of all  $a_i$  over  $\mathbb{R}$  are *positive* diagonal matrices, and for every  $\mathbb{Q}_p$  there is some  $\theta_p \in \mathbb{Q}_p^{\times}$  with  $|\theta_p| > 1$  so that all the entries in the diagonal of the  $\mathbb{Q}_p$ -components of all the  $a_i$  are powers of  $\theta_p$ .

**Theorem 1.2.** Let G be as above,  $\Gamma < G$  a discrete subgroup, and assume that  $a_1, \ldots, a_r$  satisfy the Class- $\mathcal{A}'$  assumption. Let  $\mu$  be an  $a^{\bullet}$ -invariant and ergodic measure on  $G/\Gamma$ . Then there is a closed subgroup L < G containing the group  $\{a^{\bullet}\}$ , an element  $g_0 \in G$  and a closed normal subgroup (possibly trivial)  $H \triangleleft L$  so that  $\mu$  is H-invariant and supported on the single L-orbit  $L.[g_0]_{\Gamma}, g_0^{-1}Hg_0 \cap \Gamma$  is a lattice in  $g_0^{-1}Hg_0$ , and if  $\pi: L \to L/H$  is the natural projection  $\Lambda = \pi(g_0\Gamma g_0^{-1} \cap L)$  is a discrete subgroup of L/H. Moreover, the corresponding  $\bar{a}_1 = \pi(a_1), \ldots, \bar{a}_1 = \pi(a_1)$  invariant probability measure  $\bar{\mu}$  on  $(L/H)/\Lambda$  satisfies that for each coarse Lyapunov exponent  $[\alpha] \in [\Phi]$  and for every  $\mathbf{n} \in \mathbb{Z}^r$ ,

(1.2) 
$$D_{\bar{\mu}}(\mathbf{n}, [\alpha]) = D_{\bar{\mu}}(-\mathbf{n}, [-\alpha]).$$

In (1.2), the entropy contributions  $D_{\bar{\mu}}(\mathbf{n}, [\alpha])$  are defined as above using the leafwise measures for the action of  $\pi(U_{[\alpha]})$  on  $(L/H)/\Lambda$ .

To illustrate better the implications of Theorems 1.1 and 1.2 we consider the action of the full diagonal group  $A < SL(n, \mathbb{R})$  on  $SL(n, \mathbb{R}) \ltimes \mathbb{R}^n / SL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ .

**Theorem 1.3.** Let  $G = \mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$  and  $\Gamma = \mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ , and let A be the maximal diagonalizable subgroup of  $\mathrm{SL}(n, \mathbb{R}) < G$  for  $n \geq 3$ . Let  $\mu$  be an Ainvariant and ergodic measure on  $G/\Gamma$  such that for some  $a \in A$  the ergodic theoretic entropy  $h_{\mu}(a)$  is positive. Then either  $\mu$  is homogeneous or  $\mu$  is an extension of a zero entropy A-invariant measure  $\overline{\mu}$  on  $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$  (i.e.  $h_{\overline{\mu}}(a) = 0$  for any  $a \in A$ ) with Haar measure on the fibers of the extension  $G/\Gamma \to \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ .

In addition to Theorem 1.1, the proof of this theorem uses a measure classification results by A. Katok and the two authors of this paper [EKL] and a result from our

paper [EL3]. We note that the proof of Theorem 1.2 also uses a previous measure classification result — the classification of invariant measure under groups generated by one-parameter unipotent subgroups in the setting of products of real and p-adic linear algebraic groups by Ratner [R4] and Margulis and Tomanov [MT1], extending Ratner's measure classification theorem in the real case [R3].

## 2. Preliminaries on leafwise measures

We recall some basic facts about the construction of leafwise measures. These are defined in [EL1, §6] in the following general setup: let X be a locally compact second countable metric space, and U a unimodular locally compact second countable group equipped with a proper right invariant metric. Let  $B_r^U(u)$  denotes the open ball of radius r around  $u \in U$ , and  $B_r^U = B_r^U(1)$ . We assume U acts continuously on X (i.e. the map  $(u, x) \mapsto u.x$  is a continuous map  $U \times X \to X$ ) which is locally free, i.e. for every compact  $K \subset X$  there is a  $\delta > 0$  so that for all  $x \in K$  the map  $u \mapsto u.x$  is injective on  $B_{\delta}^U$ . Let  $\lambda_U$  denote the Haar measure on U normalized so that  $\lambda_U(B_1^U) = 1$ .

Given a strictly positive function  $\rho$  on U we can consider the space  $PM^*_{\infty}(U)$ of equivalence classes under proportionality of Radon measures  $\vartheta$  on U for which  $\int_U \rho(u) d\vartheta(u) < \infty$ . For a locally compact second countable group U one can introduce a metric on this space under which it is relatively compact. Indeed, one may take a sequence  $f_i \in C_c(U)$  spanning a dense subset of  $C_c(U)$ , and define

$$d(\nu,\nu') = \sum_{i} 2^{-i} \left| \frac{\int_{U} f_{i} \rho \, d\nu(u)}{\int_{U} \rho \, d\nu(u)} - \frac{\int_{U} f_{i} \rho \, d\nu'(u)}{\int_{U} \rho \, d\nu'(u)} \right|.$$

The space  $PM^*_{\infty}(U)$  depends implicitly on the choice of  $\rho$ , but we shall keep this dependence implicit in our notation. We say that a countably generated  $\sigma$ -algebra  $\mathcal{A}$  of subsets of X is subordinate to U on  $Y \subset X$  if for every  $x \in Y$  there is some  $\delta > 0$  so that

$$(2.1) B_{\delta}^{U}.x \subset [x]_{\mathcal{A}} \subset B_{\delta^{-1}}^{U}.x$$

If x satisfies (2.1), we say that  $[x]_{\mathcal{A}}$  is a U-plaque for x.

**Proposition 2.1** ([EL1, Thm. 6.3 and Thm. 6.29]). Let X, U be as above. Then there is a strictly positive function  $\rho$  on U so that for every probability measure  $\mu$ on X such that the action of U on  $(X, \mu)$  is a.e. free, we have a Borel measurable map  $x \mapsto \mu_x^U$  from X to the space of proportionality classes of Radon measures  $PM^*_{\infty}(U)$  with the following properties:

- (1) there is a co-null set X' so that for every  $x \in X'$  and  $u \in U$  for which  $u.x \in X'$  we have that  $\mu_x^U = \mu_{u.x}^U u$ , with  $\mu_{u.x}^U u$  denoting the push forward of  $\mu_{u.x}^U$  under right multiplication by u.
- (2) for a.e.  $x \in X$ , we have that  $1 \in \operatorname{supp} \mu_x^U$  (i.e. the identity is a.s. in the support of  $\mu_x^U$ ).
- (3) suppose  $\mathcal{A}$  is a countably generated  $\sigma$ -algebra subordinate to U on  $Y \subset X$ . For  $x \in Y$ , let  $V_x := \{u \in U : u.x \in [x]_{\mathcal{A}}\}$ . Then for  $\mu$ -a.e.  $x \in Y$ , the conditional measure  $(\mu|_U)_x^{\mathcal{A}}$  on  $[x]_{\mathcal{A}}$  is proportional to  $(\mu_x^U|_{V_x}).x$ , i.e. the pushforward of  $\mu_x^U|_{V_x}$  under the map  $u \mapsto u.x$ .

(4) for any  $r_n \uparrow \infty$  and  $b_n > 0$  such that  $\sum_n b_n^{-1} < \infty$  we have that

$$\overline{\lim_{n \to \infty}} \frac{\mu_x^U(B_{r_n}^U)}{b_n \lambda_U(B_{r_n+2}^U)} = 0.$$

for  $\mu$ -a.e. x.

For proof, see [EL1, §6]. Part (3) is stated in a slightly different but equivalent way in [EL1, §6], see [EL1, §7.24] for a brief discussion. Note that by (4), every choice of  $r_n \uparrow \infty$  and  $b_n > 0$  such that  $\sum_n b_n^{-1} < \infty$  it follows that the function

$$\rho(u) = \sum_{n} \frac{1}{b_n^2 \lambda_U(B_{r_n+2}^U)} \mathbf{1}_{B_{r_n}^U}(u)$$

is in  $L^1(\mu_x^U)$ , hence can be used as the  $\rho$  defining  $PM^*_{\infty}(U)$ . Moreover, the proof actually gives that for any sequence of measurable subsets  $B_n \subset U$ 

(2.2) 
$$\overline{\lim_{n \to \infty} \frac{\mu_x^U(B_n)}{b_n \lambda_U(B_1^U B_n B_1^U)}} = 0 \quad \text{For } \mu\text{-a.e. } x.$$

Constructing  $\sigma$ -algebras which are subordinate to U on a set Y of large measure is not difficult. For instance, it follows from [EL1, Cor. 6.15] that for any  $\epsilon > 0$ , one can find a countably generated  $\sigma$ -algebra  $\mathcal{A}$  which is subordinate to U on Ywith  $\mu(Y) > 1 - \epsilon$ . The following proposition gives under some extra assumptions a countably generated  $\sigma$ -algebra  $\mathcal{A}$  subordinate to U on a large subset  $Y \subset X$  which plays nicely with that *a*-action:

**Proposition 2.2** ([EL1, Prop. 7.36]). Let G,  $\Gamma$ , be as above  $a \in G$  diagonalizable, and  $U < G^+ = \{g \in G : a^{-n}ga^n \to 1\}$  closed and normalized by a. Let  $\mu$  be an *a*-invariant probability measure on  $X = G/\Gamma$  (not necessarily ergodic). Then for every  $\epsilon > 0$  there is a *a*-invariant subset  $Y \subset X$  with  $\mu(Y) > 1 - \epsilon$  and *a* countably generated  $\sigma$ -algebra  $\mathcal{A}$  which is subordinate to U on Y and which is monotonic under a in the sense that  $aB \in \mathcal{A}$  for every  $B \in \mathcal{A}$ .

Note that a-invariance of Y is not explicitly stated in [EL1, Prop. 7.36], but if  $\mathcal{A}$  is a-monotone and subordinate to U on Y then it is also subordinate to U on

$$Y' = \left(\bigcup_{k \ge 0} a^k Y\right) \cap \left(\bigcup_{k \le 0} a^k Y\right).$$

Indeed, by monotonicity of  $\mathcal{A}$ , we have that if  $\ell \leq 0 \leq k$ ,

$$a^{-k}[a^k x]_{\mathcal{A}} \subseteq [x]_{\mathcal{A}} \subseteq a^{-\ell}[a^\ell x]_{\mathcal{A}}.$$

Thus, if  $x \in a^{\ell}Y \cap a^{k}Y$ , then as  $[a^{k}x]_{\mathcal{A}}$  contains a small neighborhood of the *U*-orbit around  $a^{k}x$ , the atom  $[x]_{\mathcal{A}}$  also contains a small neighborhood of x in its *U*-orbit, and since  $[a^{\ell}x]_{\mathcal{A}}$  is a subset of a compact subset of *U* acting on  $a^{\ell}x$ , the atom  $[x]_{\mathcal{A}}$ is bounded. This shows that indeed  $\mathcal{A}$  is subordinate to *U* on *Y'*. The set *Y'* contains *Y* and is clearly *a*-invariant up to null sets.

We also note in this context the following:

**Proposition 2.3** ([EL1, Lem. 7.16]). Suppose  $\langle a \rangle \ltimes U$  acts on X with the action by a preserving a probability measure  $\mu$  on X. Then for  $\mu$ -a.e. x

$$\mu_{a.x}^U = a\mu_x^U a^{-1}.$$

Sometimes it will be convenient to work with countably generated  $\sigma$ -algebras whose atoms are a bit more general subsets of *U*-orbits. We say that a countably generated  $\sigma$ -algebra  $\mathcal{A}$  of subsets of *X* is *weakly subordinate to U on*  $Y \subset X$  *relative* to  $\mu$  if for every  $x \in Y$  we have that  $[x]_{\mathcal{A}} \subset U.x$  and that

$$(2.3) V_x = \{ u \in U : u.x \in [x]_{\mathcal{A}} \}$$

is a bounded subset of U with  $\mu_x^U(V_x) > 0$ .

**Lemma 2.4.** Let C be a countably generated  $\sigma$ -algebra that is weakly subordinate to U on a set  $Y \subset X$ . Let  $V_x$  be as in (2.3). Then for  $\mu$ -a.e.  $x \in Y$ ,

(2.4) 
$$\mu_x^{\mathcal{C}} = \frac{1}{\mu_x^U(V_x)} \left( \left. \mu_x^U \right|_{V_x} .x \right).$$

To prove this, one verifies that the right-hand side of (2.4) satsifies the defining properties of the system of conditional measures  $\mu_x^{\mathcal{C}}$ ; we leave the details to the reader.

We will mostly be focusing our attention on the special case where we have a closed subgroup  $G < M = \prod_{i=1}^{m} \operatorname{SL}(d, k_i)$  with  $k_i$  local fields of either zero or positive characteristic,  $X = G/\Gamma$ ,  $J < \mathbb{Z}^r$  a finite set, and

(2.5) 
$$U = U_J = \bigcap_{\mathbf{n} \in J} G_{\mathbf{n}}^+.$$

Clearly the coarse Lyapunov groups  $U^{[\alpha]}$  are of this form. Moreover if  $\tilde{J}$  denotes the set of triplets  $(\ell, i, j)$  with  $1 \leq \ell \leq m$  and  $1 \leq i, j \leq d$  with  $i \neq j$  so that the elementary one parameter unipotent subgroups  $E_{ij}^{\ell} < SL(d, k_{\ell})$  is contracted by  $a^{-\mathbf{n}}$  under conjugation for every  $\mathbf{n} \in J$  then  $U_J = G \cap M_J$  with

(2.6) 
$$M_J = \langle E_{i,j}^{\ell} : (\ell, i, j) \in J \rangle$$

(note that  $M_J$  is not just generated by  $E_{i,j}^{\ell}$  but is in fact equal to the image of the product set  $\prod_{(\ell,i,j)\in \tilde{J}} E_{i,j}^{\ell}$  under the multiplication map.)

We recall from [EL1, §7] that the measure theoretic entropy  $h_{\mu}(\mathbf{n})$  can be easily obtained from the entropy contributions  $D_{\mu}(\mathbf{n}, [\alpha])$  for  $[\alpha] \in [\Phi]$  as follows:

(2.7) 
$$h_{\mu}(\mathbf{n}) = \sum_{[\alpha]:\alpha(\mathbf{n})>0} D_{\mu}(\mathbf{n}, [\alpha]).$$

By the symmetry of entropy we know that  $h_{\mu}(-\mathbf{n}) = h_{\mu}(\mathbf{n})$  for every  $\mathbf{n} \in \mathbb{Z}^r$ , hence equation (2.7) implies that the entropy contributions satisfy the identity

(2.8) 
$$\sum_{[\alpha]:\alpha(\mathbf{n})>0} D_{\mu}(\mathbf{n}, [\alpha]) = \sum_{[\alpha]:\alpha(\mathbf{n})>0} D_{\mu}(-\mathbf{n}, [-\alpha]).$$

A relative version of this identity also holds and can be proved along the same lines. Explicitly, let  $\mathcal{A}$  be a countably generated  $a^{\bullet}$ -invariant  $\sigma$ -algebra (i.e. a  $\sigma$ algebra so that for every  $B \in \mathcal{A}$  we have that  $a^{\mathbf{n}}B \in \mathcal{A}$  for every  $\mathbf{n} \in \mathbb{Z}^r$ ). We decompose  $\mu$  as  $\mu = \int \mu_{\xi}^{\mathcal{A}} d\mu(\xi)$  with each  $\mu_{\xi}^{\mathcal{A}}$  a probability measure supported on the atom  $[\xi]_{\mathcal{A}}$ . Given  $[\alpha] \in [\Phi]$  and  $\xi \in G/\Gamma$ , we can construct a new system of leafwise measures  $(\mu_{\xi}^{\mathcal{A}})_x^{[\alpha]}$  (the construction of leafwise measures along  $U_{[\alpha]}$  works for any probability measure on  $G/\Gamma$ , not necessarily a  $a^{\bullet}$ -invariant one). Since the measure  $\mu_{\xi}^{\mathcal{A}}$  is supported on  $[\xi]_{\mathcal{A}}$ , for  $\mu_{\xi}^{\mathcal{A}}$ -a.e.  $x \in G/\Gamma$  we have that  $\mu_{\xi}^{\mathcal{A}} = \mu_x^{\mathcal{A}}$  hence we may define a new system of equivalence classes of up to proportionality of locally finite measures on  $U_{[\alpha]}$ , to be denoted by  $\mu_x^{\mathcal{A},[\alpha]}$ , so that for  $\mu$ -a.e.  $\xi$ , for  $\mu_{\xi}^{\mathcal{A}}$ -a.e. x,

$$\mu_x^{\mathcal{A},[\alpha]} = (\mu_\xi^{\mathcal{A}})_x^{[\alpha]}$$

For **n** such that  $\alpha(\mathbf{n}) > 0$  we define the *entropy contributions of*  $[\alpha]$  *relative to*  $\mathcal{A}$  by

(2.9) 
$$D^{\mathcal{A}}_{\mu}(\mathbf{n}, [\alpha]) = \lim_{\ell \to \infty} \frac{-\log \mu_x^{\mathcal{A}, [\alpha]}(a^{-\ell \mathbf{n}} \Omega_0 a^{\ell \mathbf{n}})}{\ell}$$

as above we set  $D^{\mathcal{A}}_{\mu}(\mathbf{n}, [\alpha]) = 0$  for **n** such that  $\alpha(\mathbf{n}) \leq 0$ . These relative entropy contributions are related to the conditional entropy  $h_{\mu}(\mathbf{n}|\mathcal{A})$  in the same way that the ordinary entropy contributions relate to the usual ergodic theoretic entropy:

(2.10) 
$$h_{\mu}(\mathbf{n}|\mathcal{A}) = \sum_{[\alpha]:\alpha(\mathbf{n})>0} D_{\mu}^{\mathcal{A}}(\mathbf{n}, [\alpha])$$

and satisfy a similar identity to (2.8) because of the identity  $h_{\mu}(\mathbf{n}|\mathcal{A}) = h_{\mu}(-\mathbf{n}|\mathcal{A})$ . For properies of conditional entropy, see e.g. [ELW, §2].

We note the following observation regarding relative leafwise measures:

**Proposition 2.5.** Let  $[\alpha]$  be a coarse Lyapunov exponent and  $\mathcal{A}$  a countably generated  $\sigma$ -algebra of  $U_{[\alpha]}$ -invariant sets. Then  $\mu$ -a.s.,

(2.11) 
$$\mu_x^{[\alpha]} = \mu_x^{\mathcal{A},[\alpha]}$$

*Proof.* Let  $Y \subset X$  and  $\mathcal{C}$  be as in Proposition 2.2 for some  $a = a^{\mathbf{n}}$  expanding  $U_{[\alpha]}$ and some  $\epsilon > 0$ . By assumption on  $\mathcal{A}$ , for every  $x \in Y$  we have that  $[x]_{\mathcal{A}} \subset [x]_{\mathcal{C}}$ . This implies that  $(\mu_x^{\mathcal{A}})_x^{\mathcal{C}} = \mu_x^{\mathcal{C}}$  for a.e.  $x \in Y$  (see e.g. [EW, Prop. 5.20]). By Proposition 2.1(3) (applied to  $\mu$  and  $\mu_x^{\mathcal{A}}$ ) and Proposition 2.2 it follows from this that for a.e.  $x \in Y$  there exists some r = r(x) > 0 such that

$$\mu_x^{[\alpha]}|_{B_x^{U^{[\alpha]}}} \propto \mu_x^{\mathcal{A},[\alpha]}|_{B_x^{U^{[\alpha]}}}.$$

By Poincaré recurrence applied to  $a^{-1}$  we have this equation for infinitely many points of the form  $a^{-n}x \in Y$  and such that the corresponding radii  $r(a^{-n}x)$  do not converge to zero. Applying  $a^n$  to  $a^{-n}x$  and using Proposition 2.3 this implies (2.11) on increasingly larger subsets of U for a.e.  $x \in Y$ , and the proposition follows.  $\Box$ 

2.1. **Product structure of leafwise measures.** An important property of how leafwise measures on different course Lyapunov exponents interact is a product structures that is due to A. Katok and the first named author [EK1, EK2].

Consider a group  $U_J < G$  constructed from a finite subset  $J \subset \mathbb{Z}^r$  as in (2.5). Let  $[\Phi]_J$  be the collection of course Lyapunov exponents  $[\alpha]$  for which  $U_{[\alpha]} \leq U$ ; clearly

$$[\Phi]_J = \{ [\alpha] \in [\Phi] : \alpha(\mathbf{n}) > 0 \text{ for all } \mathbf{n} \in J \}.$$

A coarse Lyapunov exponent  $[\alpha] \in [\Phi]_J$  is said to be *exposed* in  $U_J$  if there is an element  $\mathbf{j}_{\alpha} \in \mathbb{Z}^r$  so that  $\alpha(\mathbf{j}_{\alpha}) \leq 0$  while  $\beta(\mathbf{j}_{\alpha}) > 0$  for all other  $[\beta] \in [\Phi]_J$ .

**Lemma 2.6.** Set  $J' = J \cup \{\mathbf{j}_{\alpha}\}$ . Then  $U_{[\alpha]} \cap U_{J'} = 1$  and

$$U_J = U_{[\alpha]}U_{J'} = U_{J'}U_{[\alpha]}.$$

Proof. Since  $\alpha$  is exposed in  $U_J$  clearly  $U_{[\alpha]} \cap U_{J'} = \{1\}$ . This also implies that we can find a sequence  $\mathbf{n}_i \in \mathbb{Z}^d$  so that  $\alpha(\mathbf{n}_i)$  is bounded but  $\beta(\mathbf{n}_i) \to -\infty$  as  $i \to \infty$ . Recall that  $U_J = G \cap M_J$  with  $M_J < \prod_{\ell=1}^m \mathrm{SL}(d, k_\ell)$  as in (2.6). Let  $\tilde{J}, \tilde{J'}, \tilde{J}_{[\alpha]}$  denote the set of triplets  $(\ell, i, j)$  so that

$$M_J = \prod_{(\ell,i,j)\in\tilde{J}} E_{ij}^{\ell},$$
$$M_{J'} = \prod_{(\ell,i,j)\in\tilde{J'}} E_{ij}^{\ell},$$
$$M_{[\alpha]} = \prod_{(\ell,i,j)\in\tilde{J}_{[\alpha]}} E_{ij}^{\ell}.$$

Then as  $[\alpha]$  is exposed,  $\tilde{J} = \tilde{J}' \sqcup \tilde{J}_{[\alpha]}$  and  $M_J = M_{J'}M_{[\alpha]}$ . In particular, we may write any  $u \in U_J$  as  $u = m'm_{[\alpha]}$  with  $m' \in M_{J'}$  and  $m_{[\alpha]} \in M_{[\alpha]}$ . To prove the lemma one only needs to show  $m_{[\alpha]}$  (and hence m') is in G. For the sequence  $\mathbf{n}_i$ described above,  $a^{\mathbf{n}_i}m'a^{-\mathbf{n}_i} \to 1$ , while  $a^{\mathbf{n}_i}m_{[\alpha]}a^{-\mathbf{n}_i}$  is a sequence of elements of Mwhich can be obtained from  $m_{[\alpha]}$  by conjugating with bounded diagonal matrices (in M). Without loss of generality, by passing to a subsequence, we can assume

$$(2.12) a^{\mathbf{n}_i} m_{[\alpha]} a^{-\mathbf{n}_i} \to m_1.$$

Then  $a^{\mathbf{n}_i}ua^{-\mathbf{n}_i} \to m_1$ , so  $m_1 \in G$ . However, by equation (2.12) and the fact that while  $a^{-\mathbf{n}_i}$  is an unbounded sequence, on  $U_{[\alpha]}$  conjugation by these elements is equivalent to conjugation by bounded elements of the diagonal subgroup of M

$$a^{-\mathbf{n}_i}m_1a^{\mathbf{n}_i} \to m_{[\alpha]}.$$

Since G is closed, and both  $m_1$  and the  $a^{\mathbf{n}_i}$  are in G we conclude that  $m_{[\alpha]} \in G$ , and the lemma follows.

The basic phenomena underlying the product structure, which is a slight variation on [EK1, Prop. 5.1] and [EL1, Prop. 8.5], is the following:

**Theorem 2.7.** Let  $J \subset \mathbb{Z}^r$ ,  $U_J$ , and  $[\Phi]_J$  be as above. Suppose  $[\alpha] \in [\Phi]_J$  is exposed in  $U_J$ , and set J' as above. Then there is a set of full measure  $X' \subset X$  so that if  $x, u.x \in X'$  for  $u = u_{[\alpha]}u' \in U_J$  (with  $u_{[\alpha]} \in U_{[\alpha]}$  and  $u' \in U_{J'}$ ) one has that

$$\mu_x^{[\alpha]} = \mu_{u,x}^{[\alpha]} u_{[\alpha]}.$$

In particular, for any  $[\beta] \in [\Phi]_J \setminus \{[\alpha]\}, \text{ if } x, u.x \in X' \text{ for } u \in U_{[\beta]} \text{ then}$ 

$$\mu_x^{[\alpha]} = \mu_{u.x}^{[\alpha]}.$$

In order to prove Theorem 2.7, we should make use of the system  $F_{T,R}$  of subsets of  $\mathbb{Z}^r$  defined as follows:

(2.13) 
$$F_{T,R} = \{ \mathbf{n} \in \mathbb{Z}^r : |\mathbf{n}| < T, |\alpha(\mathbf{n})| < R, \text{ and } \beta(\mathbf{n}) > 0 \text{ for all } \beta \in [\Phi]_{J'} \}.$$

We recall from [L1] that a systems of subsets  $\{F_T\}_{T \ge T_0}$  of a discrete group  $\Lambda$  is said to be *tempered* if there is some C so that for every T

(2.14) 
$$\left| \left( \bigcup_{T' < T} F_{T'}^{-1} \right) F_T \right| < C \left| F_T \right|.$$

We leave the verification of the following easy lemma to the reader:

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**Lemma 2.8.** For any fixed R > 0, if  $T_0$  is large enough:

- (1) The collection  $\{F_{T,R}\}_{T>T_0}$  defined in (2.13) is tempered.
- (2) There is a  $c_R > 0$  so that for  $T \ge T_0$

$$c_R T^{r-1} \le |F_{T,R}| \le 1.01 c_R T^{r-1}$$

We will make use of the following maximal ergodic theorem:

**Theorem 2.9** ([L1, Thm. 3.2]). Let  $\Lambda$  be a countable amenable group acting in a measure preserving way on a measure space  $(X, \mu)$ , and let  $\{F_T\}_{T \ge T_0}$  be a tempered sequence of subsets of  $\Lambda$ . Let M[f](x) denotes the maximal function

(2.15) 
$$M[f](x) = \sup_{T \ge T_0} \frac{1}{|F_T|} \sum_{h \in F_T} |f(hx)|.$$

Then there is a constant  $C_1$  depending only on the constant C in (2.14) so that for any  $f \in L^1_{\mu}(X)$ ,

(2.16) 
$$\mu \{x : M[f](x) > \lambda\} \le C_1 \lambda^{-1} \|f\|_1$$

Note that  $\{F_T\}$  does not need to be a Følner sequence for Theorem 2.9 to hold.

Proof of Theorem 2.7. Let  $\epsilon > 0$  be arbitrary. By Lusin's Theorem, there is a compact subset  $X_{\epsilon} \subset X$  with  $\mu(X_{\epsilon}) > 1 - \epsilon$  so that the map  $x \mapsto \mu_x^{[\alpha]}$  is continuous on  $X_{\epsilon}$ . We may also assume that the subset  $X_{\epsilon}$  satisfy that if  $x, u.x \in X_{\epsilon}$  for  $u \in U_{[\alpha]}$  then  $\mu_x^{[\alpha]} = \mu_{u.x}^{[\alpha]} u$  and that moreover for every **n** 

$$\mu_{a^{\mathbf{n}}x}^{[\alpha]} = a^{\mathbf{n}}\mu_x^{[\alpha]}a^{-\mathbf{n}}.$$

Fix some R > 0, and let M[f] denote, for  $f \in L^1(\mu)$ , the maximal function for f with respect to averaging on the subsets  $\{F_{T,R}\}_{T \ge T_0}$  as in Lemma 2.8. Let  $C_1$  be as in (2.16) for this sequence. Let

$$X'_{\epsilon} = X_{\epsilon} \cap \{x : M[1_{X \smallsetminus X_{\epsilon}}](x) \le 1/8\}.$$

By Theorem 2.9, it follows that  $\mu(X'_{\epsilon}) \geq 1 - (8C_1 + 1)\epsilon$ . Suppose now that  $x, u.x \in X'_{\epsilon}$  with  $u = u_{[\alpha]}u' \in U_J$ . Then for every  $T > 2T_0$ , we have that the cardinality of  $\mathbf{n} \in F_{T,R}$  for which at least one of  $a^{\mathbf{n}}x, a^{\mathbf{n}}(ux)$  is not in  $X_{\epsilon}$  is at most  $|F_{T,R}|/4$ . Recall that by (2) of Lemma 2.8, the cardinality of  $F_{T,R} \cap \{|\mathbf{n}| < T/2\} = F_{T/2,R}$  is  $\leq 1.01 \cdot 2^{-r+1} |F_{T,R}|$ . Hence for any T large enough there is a  $\mathbf{n}_T \in \mathbb{Z}^r$  for which

- (1)  $|\alpha(\mathbf{n}_T)| < R$
- (2)  $a^{\mathbf{n}_T}x, a^{\mathbf{n}_T}(ux) \in X_{\epsilon}$
- (3) for every  $\beta \in \Phi$  for which  $[\beta] \in [\Phi]_J \setminus \{[\alpha]\}$  we have that  $\beta(\mathbf{n}_T) < -cT$  for some c independent of T.

Set  $x_T = a^{\mathbf{n}_T} x$ ,  $x'_T = a^{\mathbf{n}_T} u x$ , and suppose  $T_j \to \infty$  is such that  $(x_{T_j}, x'_{T_j})$  converges to say  $(x_{\infty}, x'_{\infty})$ . Then as  $X_{\epsilon}$  is compact,  $x_{\infty}, x'_{\infty} \in X_{\epsilon}$ . By assumption, conjugation by both  $a^{\pm \mathbf{n}_T}$  on  $U_{[\alpha]}$  is an equicontinuous sequence of maps, and on the other hand conjugation by  $a^{\mathbf{n}_T}$  contracts  $U_{J'}$ . It follows that w.l.o.g.  $a^{\mathbf{n}_T} u_{[\alpha]} a^{-\mathbf{n}_T}$  converges along the subsequence  $T_j$  to some nontrivial element  $\tilde{u}_{[\alpha]} \in U_{[\alpha]}$  while  $a^{\mathbf{n}_T} u'a^{-\mathbf{n}_T} \to 1$ , hence  $x'_{\infty} = \tilde{u}_{[\alpha]} x_{\infty}$ . Since  $x'_{\infty}, x_{\infty} \in X_{\epsilon}$  we may conclude that  $\mu^{[\alpha]}_{x_{\infty}} = \mu^{[\alpha]}_{x'_{\infty}} \tilde{u}_{[\alpha]}$ . By continuity of the map  $\mu \mapsto \mu^{[\alpha]}_{\mu}$  on  $X_U$  it follows that as  $j \to \infty$  the pairs of proportionality class of measures

$$\mu_{a^{\mathbf{n}_{T_j}}x}^{[\alpha]}, \quad \mu_{a^{\mathbf{n}_{T_j}}x}^{[\alpha]} \left( a^{\mathbf{n}_{T_j}} u_{[\alpha]} a^{-\mathbf{n}_{T_j}} \right)$$

becomes increasingly similar, hence by equicontinuity of the conjugation by  $a^{\pm \mathbf{n}_T} x$ on  $U_{[\alpha]}$ 

$$\mu_x^{[\alpha]} = a^{-\mathbf{n}_{T_j}} \mu_{a^{\mathbf{n}_{T_j}}x}^{[\alpha]} a^{\mathbf{n}_{T_j}} \approx \left( a^{-\mathbf{n}_{T_j}} \mu_{a^{\mathbf{n}_{T_j}}ux}^{[\alpha]} a^{\mathbf{n}_{T_j}} \right) u_{[\alpha]} = \mu_{ux}^{[\alpha]} u_{[\alpha]}$$

with the approximation in the middle of the above displayed equation becoming increasingly better as  $j \to \infty$ . It follows that  $\mu_x^{[\alpha]} = \mu_{ux}^{[\alpha]} u_{[\alpha]}$ . Taking  $X' = \bigcup_{\epsilon} X'_{\epsilon}$ we obtain the theorem as  $\mu(X') = 1$ .

2.2. Behavior of leafwise measures under finite-to-one extensions. The main result of this subsection is the following:

**Proposition 2.10.** Let X, X' be locally compact,  $\pi: X \to X'$  finite-to-one, with a semi-direct product  $\langle a \rangle \ltimes U$  acting on both X and X'. We assume that U is equipped with a metric  $d(\cdot, \cdot)$  inducing the topology on U such that

$$cd(u_1, u_2) \le d(au_1a^{-1}, au_2a^{-1}) \le Cd(u_1, u_2)$$

for some fixed c, C > 1 and all  $u_1, u_2 \in U$ . Furthermore we assume that  $\pi$  intertwines the action of  $\langle a \rangle \ltimes U$  on X and X'. Let  $\mu$  be an a-invariant measure on X, and let  $\mu' = \pi_* \mu$ . Then for  $\mu$ -a.e.  $x \in X$ 

$$\mu_x^U = (\mu')_{\pi(x)}^U.$$

For simplicity, we assume that the cardinality of the fibers  $\pi^{-1}(x')$  are the same, say p, for all  $x' \in X'$ . This proposition can also be viewed as a special case of the product structure of leafwise measures (cf.  $\S2.1$ ): in this case between the conditional measure  $\mu$  induces on inverse images  $\pi^{-1}(x')$  and the leafwise orbits on U-orbits.

*Proof.* Let  $\mathcal{B}$  denote the Borel  $\sigma$ -algebra on X, and  $\mathcal{B}'$  the Borel  $\sigma$ -algebra on X' which we identify with the corresponding sub  $\sigma$ -algebra of  $\mathcal{B}$ . The system of conditional measures  $\mu_x^{\mathcal{B}'}$  can be considered as a measurable map from X' to probability measures on finite subsets of cardinality p of X.

Let  $\epsilon > 0$  be arbitrary, and let  $X'_{\epsilon} \subset X'$  be a compact subset with  $\mu'(X'_{\epsilon}) > 1 - \epsilon$ on which the map  $x \mapsto \mu_x^{\mathcal{B}'}$  is continuous. Let

$$\bar{X}'_{\epsilon} = \left\{ x' \in X'_{\epsilon} : \inf_{N \ge 1} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_{X_{\epsilon}}(a^{-n}.x') \ge 0.9 \right\};$$

by the maximal ergodic theorem,  $\mu'(\bar{X}'_{\epsilon}) \geq 1 - 10\epsilon$ .

Let  $x' \in X'$  and choose  $\delta > 0$  and a small open set B' around x' so that:

(1)  $\pi^{-1}(B') = \bigsqcup_{i=1}^{p} B_i$ , and for each  $y' \in B'$  we have that  $|\pi^{-1}(y') \cap B_i| = 1$ .

- (2) if  $\{x_i\} = B_i \cap \pi^{-1}(x'), B_{\delta}^U . x_i \subset B_i$ (3) if  $y_i \in B_i$ , and  $u.y_i \in B_j$  for  $u \in U, i \neq j$  then  $d(u, 1) > 100\delta$ .

It would be convenient to denote by  $\phi_i(y')$  the unique point in  $\pi^{-1}(y') \cap B_i$  for  $y' \in B'$ . We claim that if both  $y', u.y' \in B' \cap \bar{X}'_{\epsilon}$  for  $u \in B^U_{\delta}$  then

(2.17) 
$$\mu_{y'}^{\mathcal{B}'}(\{\phi_i(y')\}) = \mu_{u,y'}^{\mathcal{B}'}(\{\phi_i(u,y')\}) \qquad i = 1, \dots, p.$$

Indeed, since  $y', u.y' \in B' \cap \bar{X}'_{\epsilon}$  it follows that there is a subsequence  $n_j \to \infty$  so that both  $a^{-n_j}.y' \in X'_{\epsilon}$  and  $a^{-n_j}.(u.y) \in X'_{\epsilon}$ ; moreover since  $X'_{\epsilon}$  is compact we may assume that  $y'' = \lim_{j \to \infty} a^{-n_j} y' = \lim_{j \to \infty} a^{-n_j} (u.y')$  exists. By continuity of  $\mu_{\bullet}^{\mathcal{B}'}$  on  $X'_{\epsilon}$  it follows that

$$\mu_{a^{-n_j}.y'}^{\mathcal{B}'}, \ \mu_{a^{-n_j}.(u.y')}^{\mathcal{B}'} \to \mu_{y''}^{\mathcal{B}'}.$$

However, by *a*-invariance of  $\mu$ , the probability vector on the *p*-preimages of a point  $z' \in X'$  given by  $\mu_{z'}^{\mathcal{B}'}$  is the same as the probability vector given by  $\mu_{a,y'}^{\mathcal{B}'}$  on the preimages of *a.z'*. This implies that up to permuting the indices on the right-hand side of (2.17), this equation holds; in view of property (3) of  $\mathcal{B}'$  necessarily this permutation has to be the identity permutation.

Let  $\mathcal{A}'_1$  be a  $\sigma$ -algebra on X' subordinate to U on a set Y' of measure  $\geq 1 - \epsilon_1$ . Refining  $\mathcal{A}'_1$  with a finite algebra of sets generated by open balls with  $\mu$ -null boundaries, we may assume that for every  $y' \in B' \cap Y'$  we have that  $[y']_{\mathcal{A}'_1} \subset B^U_{\delta}.y'$ . Now set

$$\mathcal{A}' = \{X' \smallsetminus B'\} \cup \{C' \cap B' : C' \in \mathcal{A}'_1\}$$
$$\mathcal{A} = \{X \land B\} \cup \{\pi^{-1}(C') \cap B_i : C' \in \mathcal{A}', 1 \le i \le p\}$$

For  $y' \in B'$ , we have that  $[y']_{\mathcal{A}'} = [y']_{\mathcal{A}'_1} \cap B'$ , hence since B' is open,  $\mathcal{A}'$  is subordinate to U on  $Y' \cap B'$ , and by the assumption we made on the atoms  $[y']_{\mathcal{A}'_1}$ and property (3) of B' we have that if  $y' \in B' \cap Y'$  and if we define  $C \subset U$  by  $[y']_{\mathcal{A}'} = C.y$  then

$$[\phi_i(y')]_{\mathcal{A}} = C.\phi_i(y') \subset B_i$$

hence  $\mathcal{A}$  is subordinate to U on  $\pi^{-1}(Y' \cap B')$ .

Letting  $\epsilon \to 0$  we see that (2.17) holds a.e. on X', hence for  $y' \in Y' \cap B'$  if  $\rho$  is the probability measure on  $B^U_{\delta}$  defined by

$$(\mu')_{\eta'}^{\mathcal{A}'} = \rho.y'$$

then

$$\mu^{\mathcal{A}}_{\phi_i(y')} = \rho.\phi_i(y') \qquad 1 \le i \le p.$$

By Proposition 2.1 part (3), it follows that for a sufficiently small  $\delta'$  (possibly depending on y'),

$$(\mu')_{y'}^{U}\Big|_{B_{\delta'}^{U}} = \mu_{\phi_{i}(y')}^{U}\Big|_{B_{\delta'}^{U}} \qquad 1 \le i \le p$$

and hence since a expands U and preserves  $\mu$  the proposition follows by Poincaré recurrence (cf. Proposition 2.3).

# 3. Proof of Theorem 1.1

Let  $[\alpha]$  be a coarse Lyapunov exponent, considered fixed in this section. The key to the proof of Theorem 1.1 is a careful comparison between the entropy contributions  $D_{\mu}(\mathbf{n}, [\beta])$  and the relative entropy contributions  $D_{\mu}^{\mathcal{A}}(\mathbf{n}, [\beta])$  for a specific choice of  $\mathcal{A}$ , namely the  $\sigma$ -algebra  $\mathcal{A}$  corresponding to the Borel map  $x \mapsto \mu_x^{[\alpha]}$ . To be precise, we take  $\mathcal{A}$  to be the preimage of the Borel  $\sigma$ -algebra under the map  $x \in X \mapsto \mu_x^{[\alpha]}$ , where the leafwise measure is considered as an element of the space of measures  $PM_{\infty}^*(U_{[\alpha]})$  up to proportionality. We will fix this  $\sigma$ -algebra throughout the section.

By Theorem 2.7, if  $\beta$  is linearly independent from  $\alpha$  there is a set  $X' \subset G/\Gamma$  of full  $\mu$ -measure so that for every  $x, x' \in X'$  with  $x \in U_{[\beta]}.x'$  we have that  $\mu_x^{[\alpha]} = \mu_{x'}^{[\alpha]}$ . It follows that there is a countably generated  $\sigma$ -algebra  $\mathcal{A}'$  equivalent to  $\mathcal{A}$  consisting

of  $U_{[\beta]}$ -invariant sets, and hence by Proposition 2.5, for every  $\beta$  linearly independent from  $\alpha$ , we have that

$$\mu_x^{[\beta]} = \mu_x^{\mathcal{A},[\beta]} \qquad \text{a.e}$$

It follows from the definition of entropy contribution of coarse Lyapunov exponents (1.1) and the definition of relative entropy contribution (2.9) that for  $\beta$  linearly independent from  $\alpha$  and  $\mathbf{n} \in \mathbb{Z}^r$ ,

$$D_{\mu}(\mathbf{n}, [\beta]) = D_{\mu}^{\mathcal{A}}(\mathbf{n}, [\beta]).$$

Recall that

$$I_x^{[\alpha]} = \{ u \in U_{[\alpha]} : \mu_x^{[\alpha]} u = \mu_x^{[\alpha]} \}.$$

**Lemma 3.1.** With the notations above we have  $\sup \mu_x^{\mathcal{A},[\alpha]} \subset I_x^{[\alpha]}$ .

*Proof.* By Proposition 2.1, there is a set  $X' \subset G/\Gamma$  of full  $\mu$ -measure, so that if  $x, u.x \in X'$  for  $u \in U_{[\alpha]}$  then  $\mu_x^{[\alpha]} = \mu_{u.x}^{[\alpha]} u$ . Since  $\mathcal{A}$  is the  $\sigma$ -algebra generated by the map  $x \mapsto \mu_x^{[\alpha]}$ , we may also assume that for every  $\xi \in G/\Gamma$ , the leafwise measure  $\mu_x^{[\alpha]}$  is constant on  $[\xi]_{\mathcal{A}} \cap X'$ . It follows from  $\mu(X') = 1$  that for  $\mu$ -a.e.  $\xi$  we have that  $\mu_{\mathcal{E}}^{\mathcal{A}}(X') = 1$ , and also that for  $\mu_{\mathcal{E}}^{\mathcal{A}}$ -a.e. x

(3.1) 
$$\mu_x^{\mathcal{A},[\alpha]}\left(\left\{u \in U_{[\alpha]} : u.x \notin X' \cap [x]_{\mathcal{A}}\right\}\right) = 0;$$

recall here that by definition

$$\mu_x^{\mathcal{A},[\alpha]} = (\mu_\xi^{\mathcal{A}})_x^{[\alpha]} \qquad \mu_\xi^{\mathcal{A}}\text{-a.s.}$$

hence for  $\mu$ -a.e.  $\xi$ , equation (3.1) follows for  $\mu_{\xi}^{\mathcal{A}}$ -a.e. x from the fact that X' is a conull set with respect to  $\mu_{\xi}^{\mathcal{A}}$ . Fix  $x \in X'$  for which (3.1) holds, and suppose  $u \in \operatorname{supp} \mu_x^{\mathcal{A}, [\alpha]}$ . Then there exist  $u_i \in U_{[\alpha]}$  tending to u so that  $u_i \cdot x \in X'$ .

 $u \in \operatorname{supp} \mu_x^{\mathcal{A},[\alpha]}$ . Then there exist  $u_i \in U_{[\alpha]}$  tending to u so that  $u_i \cdot x \in X'$ . By definition of X' it follows that  $\mu_{u_i \cdot x}^{\mathcal{A}}$  is simultaneously equal to  $\mu_x^{\mathcal{A}}$  and to  $\mu_x^{\mathcal{A}} u_i^{-1}$ , hence  $u_i \in I_x^{[\alpha]}$ . Since the latter is a closed subgroup of  $U_{[\alpha]}$ , it follows that  $u \in I_x^{[\alpha]}$ .

**Lemma 3.2.** Let  $\xi \in G/\Gamma$  and consider the system of leafwise measures along  $I_{\xi} := I_{\xi}^{[\alpha]}$  for the probability measure  $\mu_{\xi}^{\mathcal{A}}$ , i.e.  $\mu_{x}^{\mathcal{A},I_{\xi}} := (\mu_{\xi}^{\mathcal{A}})_{x}^{I_{\xi}}$ . Then they are a.e. equal to  $\mu_{x}^{\mathcal{A},[\alpha]}$  in the sense that for every  $V_{1}, V_{2} \subset U_{[\alpha]}$  and  $\mu_{\xi}^{\mathcal{A}}$ -a.e. x

$$\frac{\mu_x^{\mathcal{A},[\alpha]}(V_1)}{\mu_x^{\mathcal{A},[\alpha]}(V_2)} = \frac{\mu_x^{\mathcal{A},I_{\xi}}(V_1 \cap I_{\xi})}{\mu_x^{\mathcal{A},I_{\xi}}(V_2 \cap I_{\xi})}.$$

*Proof.* Let  $a_1 = a^{\mathbf{n}}$  be such that conjugation by  $a_1$  expands  $U_{[\alpha]}$ , i.e.  $\alpha(\mathbf{n}) > 0$ . Let  $\epsilon > 0$ . By Proposition 2.2 there is an  $a_1$ -invariant  $Y \subset X$  with  $\mu(Y) > 1 - \epsilon$  and an  $a_1$ -monotone countably generated  $\sigma$ -algebra  $\mathcal{C}$  subordinate to  $U_{[\alpha]}$  on Y.

Consider the  $\sigma$ -algebra  $\mathcal{C}$ 

$$\tilde{\mathcal{C}} = \mathcal{C} \lor \mathcal{A} \lor \{X', X \smallsetminus X'\},\$$

with X' as in the proof of Lemma 3.1 and fix  $\xi \in Y$  for which  $\mu_{\xi}^{\mathcal{A}}(X') = 1$ . We claim that  $\tilde{\mathcal{C}}$  is weakly subordinate to  $I_{\xi}$  on Y relative to  $\mu_{\xi}^{\mathcal{A}}$ . Note that  $\mu_{\xi}^{\mathcal{A}}(X') = 1$  implies that for  $\mu_{\xi}^{\mathcal{A}}$ -a.e. x,

(3.2) 
$$\mu_x^{\mathcal{A}, I_{\xi}} \{ u \in I_{\xi} : u.x \in X' \} = 0.$$

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We first show that for such  $\xi$ , for  $\mu_{\xi}^{\mathcal{A}}$ -a.e.  $x \in Y$ ,

$$[x]_{\tilde{\mathcal{C}}} = [x]_{\mathcal{C}} \cap I_{\xi} \cdot x \cap X'.$$

Indeed, for  $\mu_{\xi}^{\mathcal{A}}$ -a.e. x we have that  $x \in X'$  and hence  $[x]_{\tilde{\mathcal{C}}} \subset X'$ . Moreover, for  $x \in Y$  we have that  $[x]_{\mathcal{C}}$  is a U-plaque, hence every  $z \in [x]_{\tilde{\mathcal{C}}}$  is of the form u.x, and since such z are in particular in  $[x]_{\mathcal{A}}$ , we know that  $\mu_z^{[\alpha]} = \mu_x^{[\alpha]}$ . As in the proof of Lemma 3.1, this implies that  $u \in I_x^{[\alpha]} = I_{\xi}$ . Thus  $[x]_{\tilde{\mathcal{C}}} \subset [x]_{\mathcal{C}} \cap I_{\xi} x$ . On the other hand,  $I_{\xi} \cdot x \cap X' \subset [x]_{\mathcal{A}}$ , so  $I_{\xi} \cdot x \cap X' \cap [x]_{\mathcal{C}} \subset [x]_{\tilde{\mathcal{C}}}$ . This implies (3.3).

Let  $V_x = \{u \in U : u.x \in [x]_{\mathcal{C}}\}, \tilde{V}_x = \{u \in U : u.x \in [x]_{\tilde{\mathcal{C}}}\}$ . If  $x \in Y$ , then  $V_x$  contains an open neighbourhood B of 1 in  $U_{[\alpha]}$ , and by (2) of Proposition 2.1 a.s.  $\mu_x^{I_{\xi}}(B \cap I_{\xi}) > 0$ . Assuming x also satisfies (3.2) (which again happens a.s.), we have that

$$\mu_x^{I_\xi}((B \cap I_\xi) \smallsetminus \tilde{V}_x) = 0$$

hence  $\mu_x^{I_{\xi}}(\tilde{V}_x) > 0$  so  $\tilde{\mathcal{C}}$  is indeed weakly subordinate to  $I_{\xi}$  relative to  $\mu_{\xi}^{\mathcal{A}}$  on a subset of full  $\mu_x^{\mathcal{A}}$  of Y. Thus by Lemma 2.4, for any bounded  $V_1, V_2 \subset U_{[\alpha]}$ 

$$\frac{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{I_{\xi}}(V_{1}\cap V_{x}\cap I_{\xi})}{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{I_{\xi}}(V_{2}\cap V_{x}\cap I_{\xi})} = \frac{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\mathcal{C}}(V_{1}.x)}{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\tilde{\mathcal{C}}}(V_{2}.x)}.$$

However, since  $\{X', X \smallsetminus X'\}$  is a trivial  $\sigma$ -algebra  $\tilde{\mathcal{C}}$  is equivalent to the  $\sigma$ -algebra  $\mathcal{C} \lor \mathcal{A}$  hence

$$\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\tilde{\mathcal{C}}} = \left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\mathcal{C}}$$
 a.e

By Proposition 2.1 it follows that

$$\frac{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\left[\alpha\right]}(V_{1}\cap V_{x})}{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\left[\alpha\right]}(V_{2}\cap V_{x})} = \frac{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\mathcal{C}}(V_{1}.x)}{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{\mathcal{C}}(V_{2}.x)}$$

hence

$$\frac{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{[\alpha]}(V_{1}\cap V_{x})}{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{[\alpha]}(V_{2}\cap V_{x})} = \frac{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{I_{\xi}}(V_{1}\cap V_{x}\cap I_{\xi})}{\left(\mu_{\xi}^{\mathcal{A}}\right)_{x}^{I_{\xi}}(V_{2}\cap V_{x}\cap I_{\xi})}.$$

Replacing  $\mathcal{C}$  by  $a_1^k \mathcal{C}$  the sets  $V_x$  will contain (for any fixed  $x \in Y$ ) an arbitrary large ball around  $1 \in U_{[\alpha]}$ , and the lemma follows. 

**Lemma 3.3.** Let  $\psi$  be a Borel measurable map from the space of closed subgroups of  $U_{[\alpha]}$  to a symmetric compact subset  $\Omega \subset U_{[\alpha]}$  so that  $\psi(I) \in I$  for every closed  $I \leq U_{[\alpha]}$ . Then

(1) for every  $f \in L^{\infty}(\mu)$  we have that

$$\int f(x) \, d\mu = \int f(\psi(I_x^{[\alpha]}).x) \, d\mu.$$

(2) for  $\mu$ -a.e. x it holds that  $\psi(I_x^{[\alpha]}).x \in [x]_{\mathcal{A}}$ . (3) for  $\mu$ -a.e. x the measure  $\mu_x^{\mathcal{A}}$  is  $\psi(I_x^{[\alpha]})$ -invariant.

*Proof.* We start by showing (1). Let  $a_1 = a^{\mathbf{n}}$  be such that conjugation by  $a_1$  expands  $U_{[\alpha]}$ , i.e.  $\alpha(\mathbf{n}) > 0$ ,  $\epsilon > 0$  arbitrary and apply Proposition 2.2 to get an  $a_1$ -invariant  $Y \subset X$  with  $\mu(Y) > 1 - \epsilon$  and an  $a_1$ -monotone countably generated  $\sigma$ -algebra  $\mathcal{C}$  subordinate to  $U_{[\alpha]}$  on Y.

For any k,

(3.4)  

$$\int_{X} f(x) d\mu - \int_{X} f(\psi(I_{x}^{[\alpha]}).x) d\mu = \int_{X} \mathbf{E}_{x} \left( f(x) - f(\psi(I_{x}^{[\alpha]}).x) \middle| a_{1}^{k} \mathcal{C} \right)(\xi) d\mu(\xi).$$

For  $\xi \in Y$  we have that  $[\xi]_{a_1^k \mathcal{C}}$  has the form  $V(k,\xi).\xi$  for some  $V(k,\xi) \subset U_{[\alpha]}$ , and moreover by (3) of Proposition 2.1 (3.5)

$$\begin{split} \mathbf{E}_{x}\left(f(x) - f(\psi(I_{x}^{[\alpha]}).x)\Big|a_{1}^{k}\mathcal{C}\right)(\xi) &= \frac{\displaystyle\int_{V(k,\xi)} \left(f(u.\xi) - f(\psi(I_{u.\xi}^{[\alpha]})u.\xi)\right) \, d\mu_{\xi}^{[\alpha]}(u)}{\mu_{\xi}^{[\alpha]}(V(k,\xi))} \\ &= \frac{\displaystyle\int_{V(k,\xi)} \left(f(u.\xi) - f(\psi(uI_{\xi}^{[\alpha]}u^{-1})u.\xi)\right) \, d\mu_{\xi}^{[\alpha]}(u)}{\mu_{\xi}^{[\alpha]}(V(k,\xi))} \\ &= \frac{\displaystyle\int_{V(k,\xi)} \left(f(u.\xi) - f(u\psi'(\xi,u).\xi)\right) \, d\mu_{\xi}^{[\alpha]}(u)}{\mu_{\xi}^{[\alpha]}(V(k,\xi))}, \end{split}$$

with  $\psi'(\xi, \bullet)$  some measurable, right  $I_{\xi}^{[\alpha]}$ -invariant function  $V(k, \xi) \to I_{\xi}^{[\alpha]} \cap \Omega$ . Define

$$V'(k,\xi) = \{ u\psi'(\xi, u) : u \in V(k,\xi) \}.$$

Then since  $I_{\xi}^{[\alpha]}$  fixes  $\mu_{\xi}^{[\alpha]}$ 

(3.6) 
$$\int_{V(k,\xi)} f(u\psi'(\xi,u).\xi) \, d\mu_{\xi}^{[\alpha]}(u) = \int_{V'(k,\xi)} f(u.\xi) \, d\mu_{\xi}^{[\alpha]}(u).$$

 $\operatorname{Set}$ 

$$\partial(\Omega,k) = \left\{ x \in Y : \Omega.x \not\subset [x]_{a_1^k \mathcal{C}} \right\};$$

as  $a_1$  expands  $U_{[\alpha]}$  and since  $\mathcal{C}$  is subordinate to  $U_{[\alpha]}$  on Y, it follows that  $\mu(\partial(\Omega, k)) \to 0$  as  $k \to \infty$ . On the other hand, by (3.6), (3.4) and (3.5) we obtain

$$\left| \int_{X} f(x) \, d\mu - \int_{X} f(\psi(I_x^{[\alpha]}) \cdot x) \, d\mu \right| \le \|f\|_{\infty} \left( \mu(X \smallsetminus Y) + 2\mu(\partial(\Omega, k)) \le 2\epsilon \|f\|_{\infty}$$

for k large enough. Since  $\epsilon$  is arbitrary this proves (1).

Let X' be as in (1) of Proposition 2.1 applied to  $U_{[\alpha]}$ . Then by (1) we have that

$$X'' = \left\{ x \in X' : \psi(I_x^{[\alpha]}) . x \in X' \right\}$$

also has full measure. For  $x \in X''$ , we have that

$$\mu_x^{[\alpha]} \psi(I_x^{[\alpha]})^{-1} = \mu_{\psi(I_x^{[\alpha]}).x}^{[\alpha]}$$

But  $\psi(I_x^{[\alpha]}) \in I_x^{[\alpha]}$ , the groups stabilizing from the right  $\mu_x^{[\alpha]}$ . Hence  $\mu_x^{[\alpha]} = \mu_{\psi(I_x^{[\alpha]}).x}^{[\alpha]}$ , or equivalently

$$\psi(I_x^{[\alpha]}).x \in [x]_{\mathcal{A}}.$$

This proves 
$$(2)$$
.

By (1) we have that

(3.7) 
$$\int f(x) \, d\mu_{\xi}^{[\alpha]}(x) \, d\mu(\xi) = \int f(\psi(I_{\xi}^{[\alpha]}).x) \, d\mu_{\xi}^{[\alpha]}(x) \, d\mu(\xi).$$

By (2) we see that for a.e.  $\xi$ , the measure  $\psi(I_{\xi}^{[\alpha]}).\mu_{\xi}^{[\alpha]}$  is a probability measure on X giving full measure to  $[x]_{\mathcal{A}}$ . But then (3.7) shows that the system of measures  $\psi(I_{\xi}^{[\alpha]}).\mu_{\xi}^{[\alpha]}$  satisfies the defining properties of the conditional measures  $\mu_{\xi}^{[\alpha]}$ . By uniqueness of conditional measures, it follows that a.s.

$$\psi(I_{\xi}^{[\alpha]}).\mu_{\xi}^{[\alpha]} = \mu_{\xi}^{[\alpha]},$$

establishing (3).

**Corollary 3.4.** The leafwise conditional measures  $\mu_x^{\mathcal{A}, [\alpha]}$  are a.s. the proportionality class of the Haar measure on  $I_x^{[\alpha]}$ .

*Proof.* For simplicity, we again denote  $I_x = I_x^{[\alpha]}$ . On each item of  $\mathcal{A}$  the leafwise measure  $\mu_x^{[\alpha]}$  is fixed, hence also  $I_x$ . We claim that for  $\mu$ -almost every x, the conditional measures  $\mu_x^{\mathcal{A}}$  is invariant under  $I_x$ . Indeed, let B be an arbitrary open subset of  $U_{[\alpha]}$  with compact closure.

By the Borel selector theorem (e.g. [K, Thm 12.16]) there is a Borel measurable map  $\psi_B$  from the space of closed subgroups of  $U_{[\alpha]}$  to  $U_{[\alpha]}$  so that for subgroups I which are disjoint from B we have that  $\psi_B(I) = 1$  whereas for subgroups Iintersecting B we have that  $\psi_B(I) \in B$ . It follows from (3) of Lemma 3.3 that for  $\mu$ -almost every x for which  $B \cap I_x \neq \emptyset$ , left multiplication by  $\psi_B(I_x^{[\alpha]})$  preserves  $\mu_x^A$ . Thus, by letting B vary under a countable base of the topology of  $U_{[\alpha]}$ , we get that for a.e. x, the set of  $g \in I_x$  preserving  $\mu_x^A$  is dense in  $I_x$ . Thus, a.s.,  $\mu_x^A$ is  $I_x$ -invariant, hence for  $\mu_x^A$ -a.e.  $\xi$ , the leafwise conditional measure  $\mu_{\xi}^{A,I_x}$  is Haar (cf. [EL1, Prob. 6.27]). By Lemma 3.2, it follows that  $\mu_x^{A,[\alpha]}$  is a.s. Haar measure on  $I_x$ .

# Corollary 3.5.

$$D^{\mathcal{A}}_{\mu}(\mathbf{n}, [\alpha]) = D^{\text{inv}}_{\mu}(\mathbf{n}, [\alpha])$$

*Proof.* This follows immediately from the definitions and Corollary 3.4.

We can now prove Theorem 1.1:

Proof of Theorem 1.1. Let **n** be so that  $\alpha(\mathbf{n}) > 0$ . By (2.8),

$$\sum_{\beta:\beta(\mathbf{n})>0} D_{\mu}(\mathbf{n},[\beta]) = \sum_{[\beta:\beta(\mathbf{n})>0} D_{\mu}(-\mathbf{n},[-\beta]).$$

Using  $h_{\mu}(\mathbf{n}|\mathcal{A}) = h_{\mu}(-\mathbf{n}|\mathcal{A})$  we get a similar identity

$$\sum_{\beta:\beta(\mathbf{n})>0} D^{\mathcal{A}}_{\mu}(\mathbf{n},[\beta]) = \sum_{[\beta]:\beta(\mathbf{n})>0} D^{\mathcal{A}}_{\mu}(-\mathbf{n},[-\beta]).$$

But for  $[\beta] \neq [\pm \alpha]$ , we have that  $D_{\mu}(\mathbf{n}, [\beta]) = D_{\mu}^{\mathcal{A}}(\mathbf{n}[\beta])$  hence by combining the above two displayed equations we obtain

$$D_{\mu}(\mathbf{n}, [\alpha]) - D_{\mu}^{\mathcal{A}}(\mathbf{n}, [\alpha]) = D_{\mu}(-\mathbf{n}, [-\alpha]) - D_{\mu}^{\mathcal{A}}(-\mathbf{n}, [-\alpha]),$$

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hence

$$D_{\mu}^{\mathcal{A}}(\mathbf{n}, [\alpha]) = D_{\mu}(\mathbf{n}, [\alpha]) - D_{\mu}(-\mathbf{n}, [-\alpha]) + D_{\mu}^{\mathcal{A}}(-\mathbf{n}, [-\alpha])$$
$$\geq D_{\mu}(\mathbf{n}, [\alpha]) - D_{\mu}(-\mathbf{n}, [-\alpha])$$

By Corollary 3.5, we may conclude that

$$D_{\mu}^{\text{inv}}(\mathbf{n}, [\alpha]) \ge D_{\mu}(\mathbf{n}, [\alpha]) - D_{\mu}(-\mathbf{n}, [-\alpha]).$$

We concludes this section with a slight extension of Theorem 1.1 that will be useful for us in the sequel.

**Theorem 3.6.** Suppose G and  $a^{\bullet}$  are as in Theorem 1.1. Let H < F be closed subgroups of G with  $H \lhd G$  and F/H discrete. Assume furthermore that for any  $[\alpha] \in [\Phi]$ , the coarse Lyapunov subgroup  $U_{[\alpha]}$  intersects H trivially. Let  $\mu$  be an  $a^{\bullet}$ invariant and ergodic probability measure on G/F. Then for  $[\alpha] \in [\Phi]$  and  $\mathbf{n} \in \mathbb{Z}^r$ satisfying  $\alpha(\mathbf{n}) > 0$  it holds that

$$D_{\mu}^{\text{inv}}(\mathbf{n}, [\alpha]) \ge D_{\mu}(\mathbf{n}, [\alpha]) - D_{\mu}(-\mathbf{n}, [-\alpha]).$$

Note that under the assumptions of Theorem 3.6, for any  $[\alpha] \in [\Phi]$  we have that  $U_{[\alpha]}$  acts locally freely on G/F so that if  $\mu$  is an  $a^{\bullet}$ -invariant and ergodic probability measure on G/F we can define  $\mu_x^{[\alpha]}$ ,  $D_{\mu}(\mathbf{n}, [\alpha])$  and  $D_{\mu}^{\text{inv}}(\mathbf{n}, [\alpha])$  just as in the case of  $G/\Gamma$  with  $\Gamma$ -discrete. Essentially this amounts to extending slightly the class of groups we consider: not just closed subgroups of linear groups but quotients of such groups by closed normal subgroups. The proof of the theorem in this case is identical to that of Theorem 1.1. Adapting the results of [EL1, §7] to this setting, in particular the proof of (2.7) also poses no difficulty. We leave the details to the reader.

## 4. Proof of Theorem 1.2

A key difference between the positive characteristic and the zero characteristic case is that in zero characteristic unipotent groups have very few subgroups (see [ELM] for more details).

Recall that our group G is embedded in  $M = \prod_{\ell=1}^{m} \operatorname{SL}(d, k_{\ell})$ , and that  $U_{[\alpha]} = G \cap M_{[\alpha]}$  with  $M_{[\alpha]}$  a product of unipotent algebraic subgroups of  $k_{\ell}$ . Enlarging d if necessary, we may (and will) assumes that all the  $k_{\ell}$  are distinct (recall that we have already assumed that they are all either  $\mathbb{R}$  or  $\mathbb{Q}_p$  for appropriate p).

The invariance groups  $I_x^{[\alpha]}$  are closed subgroup of  $M_{[\alpha]}$ . Moreover, since the map  $x \mapsto \mu_x^{[\alpha]}$  is measurable (cf. Proposition 2.1) and since the map taking a measure to its right invariance group is also Borel measurable, we have that the map  $x \mapsto I_x^{[\alpha]}$  is measurable. It follows from Proposition 2.3 that for every  $\mathbf{n} \in \mathbb{Z}^r$ ,

(4.1) 
$$I_{a^{\mathbf{n}}.x}^{[\alpha]} = a^{\mathbf{n}} I_{x}^{[\alpha]} a^{-\mathbf{n}}$$

a.e., hence by ergodicity of  $\mu$  these invariance groups are either a.e. trivial (i.e. =  $\{1\}$ ) or a.e. non-trivial. In [EK2] the following has been shown using Poincare recurrence and the properties of unipotent groups over  $\mathbb{R}$  and  $\mathbb{Q}_p$ :

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**Lemma 4.1** ([EK2, §6]). Under the assumptions of Theorem 1.2, for any coarse Lyapunov exponent  $[\alpha] \in [\Phi]$ , for a.e. x, the group  $I_x^{[\alpha]}$  can be written as  $\prod I_{(x,\ell)}$  with each  $I_{(x,\ell)}$  a connected algebraic subgroup of  $SL(d, k_\ell) \cap M_{[\alpha]}$ .

We exploit this to show that under the assumptions of Theorem 1.2 the invariance groups of the leafwise measures on the coarse Lyapunov foliations have to be almost everywhere constant:

**Lemma 4.2.** Under the assumptions of Theorem 1.2, for any coarse Lyapunov exponent  $[\alpha] \in [\Phi]$ , there is a closed subgroup  $I^{[\alpha]} \leq U_{[\alpha]}$  so that  $I_x^{[\alpha]} = I^{[\alpha]}$  a.e. Moreover, this group is normalized by the group  $a^{\bullet}$ , and  $\mu$  is  $I^{[\alpha]}$ -invariant.

*Proof.* Fix  $1 \leq \ell \leq m$ . To the unipotent algebraic subgroup  $I_{(x,\ell)}$  of  $\mathrm{SL}(d,k_\ell)$  there corresponds a Lie subalgebra in  $\mathfrak{sl}(d,k_\ell)$ . By ergodicity and (4.1) it is clear that  $\dim(I_{(x,\ell)})$  is a.e. constant, say k. Therefore on a set of full measure there is a 1:1 correspondence between  $I_{(x,\ell)}$  and a corresponding homethety class of pure wedge vector  $\bar{v}_{(x,\ell)} \in P(\wedge^k \mathfrak{sl}(d,k_\ell))$ . The equation (4.1) implies that for every  $\mathbf{n} \in \mathbb{Z}^r$ 

$$\bar{v}_{(a^{\mathbf{n}}.x,\ell)} = (\wedge^q \mathrm{Ad})(a^{\mathbf{n}}).\bar{v}_{(x,\ell)}.$$

By ergodicity of  $\mu$ , and since A locally compact, to show that  $I_x^{[\alpha]}$  is a.e. constant, it is enough to show that for every  $a \in A$ ,

$$I_x^{[\alpha]} = I_{ax}^{[\alpha]}.$$

Recall that for every **n**, the element  $a^{\mathbf{n}}$  is of class- $\mathcal{A}'$ . This implies that  $(\wedge^q \operatorname{Ad})(a^{\mathbf{n}})$ is of class- $\mathcal{A}'$ . A basic property of elements g of class- $\mathcal{A}'$  is that for any action of the ambient group on a projective space, for any vector  $\bar{v}$  in the project to space,  $g^k \bar{v}$  tends to a g invariant point in the projective space (cf.<sup>1</sup> [MT1, Prop. 2.2]). In particular,  $(\wedge^q \operatorname{Ad})(a^{k\mathbf{n}})\bar{v}_{(x,\ell)} = \bar{v}_{(a^{k\mathbf{n}},x,\ell)}$  converges to a  $(\wedge^q \operatorname{Ad})(a^{k\mathbf{n}})$ -invariant point in the appropriate projective space. But then Poincare recurrence for the  $\mathbb{Z}$ -action generated by  $a^{\mathbf{n}}$  implies that  $\bar{v}_{(x,\ell)}$  was fixed by  $(\wedge^q \operatorname{Ad})(a^{\mathbf{n}})$  to begin with. Since  $\bar{v}_{(x,\ell)}$  uniquely determines the algebraic group  $I_{(x,\ell)}$  it follows that

$$I_{(x,\ell)} = I_{(a^{\mathbf{n}}.x,\ell)}$$

and since this is true for every  $\ell$  it follows that  $I_x^{[\alpha]} = I_{a^n.x}^{[\alpha]}$  and the first claim of the lemma follows.

To show invariance of  $\mu$  under  $I^{[\alpha]}$  we apply Corollary 3.4. Let  $\mathcal{A}$  be a  $\sigma$ -algebra corresponding to the Borel map  $x \mapsto \mu_x^{[\alpha]}$ . According to the lemma, for  $\mu$ -almost every x, the conditional measure  $\mu_x^{\mathcal{A}}$  is a probability measure with the properties that for  $\mu_x^{\mathcal{A}}$ -a.e.  $\xi$  the leafwise measure  $\mu_{\xi}^{\mathcal{A},[\alpha]} = (\mu_x^{\mathcal{A}})_{\xi}^{[\alpha]}$  is the Haar measure on  $I^{[\alpha]}$ . By Lemma 3.2 and [L2, Prop. 4.3] (or [EL1, Prob. 6.27]) it follows that  $\mu_x^{\mathcal{A}}$  is  $I^{[\alpha]}$ -invariant, hence so is  $\mu = \int \mu_x^{\mathcal{A}} d\mu(x)$ .

**Remark:** We note that the same argument will work also in the slightly more general setup of  $\mu$  an  $a^{\bullet}$ -invariant and ergodic probability measure on G/F where F is as in Theorem 3.6, as long as  $k_{\ell}$  are all  $\mathbb{R}$  or  $\mathbb{Q}_p$  (for possibly more than one choice of p) and  $a^{\bullet}$  satisfies the class- $\mathcal{A}'$  assumption.

<sup>&</sup>lt;sup>1</sup>That proposition deals with a slightly different class of elements that Margulis and Tomanov calls class- $\mathcal{A}$ , but the proof carries out without any modifications, and indeed whether one uses class- $\mathcal{A}$  or class- $\mathcal{A}'$  is purely a matter of taste.

Let  $J_u$  be the closed group generated by all one-parameter unipotent groups preserving  $\mu$ . Let J be the group generated by  $J_u$  and  $a^{\bullet}$ . Note that by Lemma 4.1 for all  $[\alpha] \in [\Phi]$  the group  $I^{[\alpha]}$  is generated by unipotent one parameter groups, hence by Lemma 4.2 we have that  $I^{[\alpha]} \leq J_u$ .

**Lemma 4.3.** Under the assumptions of Theorem 1.2, there are closed subgroups H, L of G with  $H \triangleleft L$ ,  $J_u \leq H$  and  $J \leq L$  so that

- (1)  $\mu$  is *H*-invariant
- (2) every  $x \in \operatorname{supp} \mu$  has a periodic H-orbit
- (3)  $\mu$  is supported on a single L-orbit.

*Proof.* By definition, the measure  $\mu$  is *J*-invariant. Since  $\mu$  is  $a^{\bullet}$ -ergodic and  $a^{\bullet} \leq J$ , it follows that  $\mu$  is *J*-ergodic. The statement now follows from the main result of Margulis and Tomanov's paper [MT2].

We note that the main ingredient used by Margulis and Tomanov in [MT2] is a measure classification result [MT1, R4] extending Ratner's Measure Classification Theorem [R3, R2] to the S-arithmetic setting.

Proof of Theorem 1.2. Without loss of generality we may (and will for the remainder of this section) assume  $[1]_{\Gamma} \in \operatorname{supp} \mu$ , and since  $\mu$  is supported on a single L-orbit we may as well assume L = G and  $L > \Gamma$ . Also, by Theorem 1.1 we have that if  $J_u$  is trivial, equation (1.2) holds for  $H = \{1\}$ , hence we may assume that  $J_u$  is non-trivial. By Lemma 4.3 we know that  $H.[1]_{\Gamma}$  is periodic, i.e. that  $H \cap \Gamma$  is a lattice in H. By [R1, Thm. 1.13] we have that  $H\Gamma$  is closed, hence if  $\pi : L \to L/H$  is the natural projection,  $\Lambda = \pi(H\Gamma) = \pi(\Gamma)$  is a discrete subgroup of L/H.

Let  $H_u$  denote the subgroup of H generated by one parameter unipotent groups. It is clear from the definition that since  $H \triangleleft L$  conjugation by elements of L preserves the class of unipotent one parameter subgroups of H, hence  $H_u \triangleleft L$ . Recall that  $J_u \leq H_u$ , in particular by assumption  $H_u$  is nontrivial. From the definition it is clear  $H_u$  has the form of a product of (possibly trivial) subgroups  $H_{u,\ell} < SL(d, k_\ell)$  for  $1 \leq \ell \leq m$ , with each  $H_{u,\ell}$  generated by one parameter unipotent groups. These  $H_{u,\ell}$  are essentially algebraic: if  $M_\ell$  is the Zariski closure of  $H_{u,\ell}$ , then the radical of  $M_\ell$  equals its unipotent radical (otherwise  $M_\ell$  would have contained an algebraic proper subgroup that contain all the unipotent elements of  $M_\ell$  — in contradiction to the definition of  $M_\ell$ ). Moreover this also implies that the Lie algebras of  $H_{u,\ell}$  and  $M_\ell$  coincide, hence we have that  $H_{u,\ell} = M_\ell^+$  — the closed subgroup of  $M_\ell$  generated by one parameter unipotent subgroups<sup>2</sup>. It follows from [M, Thm. I.2.3.1] that  $H_{u,\ell}$  has finite index in  $M_\ell$ .

Let  $M = \prod_{\ell} M_{\ell}$ . Since L normalizes  $H_u$  it also normalizes its Zariski closure M, hence L is a subgroup of the normalizer  $N = \prod_{\ell} N_{\ell}$  of M in  $\prod_{\ell=1}^{m} \operatorname{SL}(d, k_{\ell})$ . As quotients of algebraic groups we can embed for every  $\ell$  the group  $N_{\ell}/M_{\ell}$  into some  $\operatorname{SL}(d_{\ell}, k_{\ell})$ , and taking  $d' = \max(d_{\ell})$  we can therefor view L/M as a closed subgroup of  $\prod_{\ell} \operatorname{SL}(d', k_{\ell})$ . Let  $\pi_u$  denote the natural projection map  $L \to L/H_u$  and  $\pi_M$  the natural projection  $N \to N/M$ . Then the induced  $a^{\bullet}$ -action on  $(L/H)/\Lambda$  is isomorphic to the induced  $a^{\bullet}$ -action on  $(L/H_u)/\Lambda'$  with  $\Lambda'$  the closed subgroup of  $L/H_u$  given by  $\pi_u(\Gamma H)$ . The group  $\Lambda'$  is not necessarily discrete, but is discrete modulo the normal subgroup  $H/H_u$  of  $L/H_u$ .

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<sup>&</sup>lt;sup>2</sup>We use the notation  $M^+$  only for algebraic groups M.

Since  $H_u$  has finite index in M, the space  $(L/H_u)/\Lambda'$  is in turn a finite extension of  $(L/M)/\Lambda''$  with  $\Lambda'' = \pi_M(\Gamma H)$ . The induced action of  $a^{\bullet}$  on  $(L/M)/\Lambda''$ , equipped with the measure  $\mu'' = (\pi_M)_*\mu$  satisfies the condition of Theorem 3.6, and the coarse Lyapunov subgroups of L/M coincide with the images of the coarse Lyapunov subgroups  $U_{[\alpha]}$  of L under  $\pi_M$ .

The measure  $\mu''$  cannot be invariant under any one parameter unipotent subgroup of L/M, for if  $u'_t = \exp(\mathbf{u}'t)$  were such a subgroup with  $\mathbf{u}' \in \operatorname{Lie}(L/M)$ nilpotent then in view of the fact that M is algebraic we can find a nilpotent  $\mathbf{u}$  so that  $d\pi_M(\mathbf{u}) = \mathbf{u}'$ . The invariance of  $\mu''$  under  $u'_t$  implies the same for  $\mu' = (\pi_u)_*\mu$ on  $(L/H_u)/\Lambda'$ . Since  $\mu$  is  $H_u$ -invariant, it follows that  $\mu$  will be invariant under the one parameter unipotent subgroup  $u_t$  of L. But if this were so,  $u_{\bullet}$  would have been contained in  $J_u$ , hence in  $H_u$ , hence  $\mathbf{u}'$  would be zero — in contradiction.

Therefore applying Theorem 3.6, we conclude that for every coarse Lyapunov exponent  $[\alpha]$ 

$$D_{\mu^{\prime\prime}}(\mathbf{n}, [\alpha]) = D_{\mu^{\prime\prime}}(-\mathbf{n}, [-\alpha]).$$

By Proposition 2.10 this property is preserved by finite-to-one extensions, hence

$$D_{\mu'}(\mathbf{n}, [\alpha]) = D_{\mu'}(-\mathbf{n}, [-\alpha]).$$

This identity implies the theorem in view of the isomorphism between the  $\pi \circ a^{\bullet}$ -action on  $(L/H)/\Lambda$  and the  $\pi_u \circ a^{\bullet}$ -action on  $(L/H_u)/\Lambda'$ .

## 5. Proof of Theorem 1.3

Recall the notations in Theorem 1.3; in particular,  $G = \operatorname{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ ,  $\Gamma = \operatorname{SL}(n, \mathbb{Z}) \ltimes \prod \mathbb{Z}^n$ , A is the maximal diagonalizable subgroup of  $\operatorname{SL}(n, \mathbb{R}) < G$ , and  $\mu$  is an A-invariant and ergodic measure on  $G/\Gamma$ . Let  $\pi$  denote the natural map  $G \to \operatorname{SL}(n, \mathbb{R})$  as well as the induced map  $G/\Gamma \to \operatorname{SL}(n, \mathbb{R})/\operatorname{SL}(n, \mathbb{Z})$ . Let  $\bar{\mu} = \pi_*\mu$ . There are two cases:

- Positive base entropy: there is some  $a \in A$  for which  $h_{\bar{\mu}}(a) > 0$
- Zero base entropy:  $h_{\bar{\mu}}(a) = 0$  for every  $a \in A$ .

The first case has essentially been already been taken care of in [EKL] and [EL3]. Indeed, by [EKL] in this case the measure  $\bar{\mu}$  is homogeneous. Moreover one can explicitly list the possible A-invariant and ergodic homogeneous measures ([LW, §6]; cf. also [ELMV] for a related discussion): when n is a prime, the only possibility is that  $\bar{\mu}$  is Haar measure on  $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ . When n is not a prime, there can be additional intermediate cases, corresponding to degree d totally real extensions K of  $\mathbb{Q}$  for  $d \mid n$ . More explicitly, it follows from the results of [LW, §6], that such measures are supported on an orbit  $L.[g_1]$  of the reductive group

$$L = \left(\prod_{i=1}^{n} \operatorname{GL}(n/d, \mathbb{R})\right) \cap \operatorname{SL}(n, \mathbb{R})$$

with  $g_1\mathbb{Z}^d$  homothetic to a finite index sublattice of the lattice  $\mathcal{O}_K \otimes \mathbb{Z}^{n/d}$ , where  $\mathcal{O}_K$  is the ring of integers of the totally real field K, and we view  $\mathcal{O}_K$  as a lattice in  $\mathbb{R}^d$  in the usual way — i.e. if  $\tau_1, \ldots, \tau_d$  are the d distinct field embeddings of K in  $\mathbb{R}$ , we identify  $\mathcal{O}_K$  with the lattice

$$\{(\tau_1(\mathbf{n}),\ldots,\tau_d(\mathbf{n})):\mathbf{n}\in\mathcal{O}_K\}.$$

Note that the case d = n is also meaningful, but corresponds to the case of  $\bar{\mu}$  being the natural measure on an A-periodic orbit which is excluded according to our entropy assumption.

Let  $L_1 = [L, L] = \prod_{i=1}^{n/d} \operatorname{SL}(n/d, \mathbb{R})$ , and let  $A_1 = A \cap U_1$  and  $A_2 = A \cap C_G(L)$ (with  $C_G(L)$  the centralizer of L in G). Then for every  $x \in L.[g_1] = \operatorname{supp} \bar{\mu}$ , we have that  $A_2.x$  is periodic. The stabilizer  $\Delta$  of x in  $A_2$  does not depend on x indeed, it is given by  $g_1^{-1}A_2g_1 \cap \operatorname{SL}(n,\mathbb{Z})$  and is commensurable to the image of  $\mathcal{O}_K^{\times}$ under the map  $\mathbf{n} \mapsto \operatorname{diag}(\tau_1(\mathbf{n}), \ldots, \tau_d(\mathbf{n}))$ . Moreover  $\bar{\mu}$  has a very simple ergodic decomposition with respect to  $A_1$ : if  $\bar{\mu}_1$  is the uniform measure on the periodic orbit  $L_1[g_1]$  then  $\bar{\mu} = \int_{A_2/\Delta} h.\bar{\mu}_1 dh$ ; by the Howe-Moore ergodicity theorem  $\bar{\mu}_1$  is  $A_1$ -ergodic.

Since the case of d = n has been excluded, the group  $\tilde{L}_1 = g_1^{-1}L_1g_1 \ltimes \mathbb{R}^n$  is a perfect algebraic group. Since  $L_1[g_1]$  was periodic in  $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ , the group  $\tilde{L}_1$  is also defined over  $\mathbb{Q}$ , and  $\Gamma \cap L_1$  is an arithmetic lattice in it. Letting  $\mu = \int \mu_{\xi} d\xi$ denote the ergodic decomposition of  $\mu$  with respect to  $A_1$ , it is clear that for a.e.  $\xi$ there would be some  $h \in A_2$  (unique up to  $\Delta$ ) so that  $\pi_*(\mu_{\xi}) = h.\bar{\mu}_1$ . We can now apply [EL3, Thm. 1.6] to conclude that the measures  $\mu_{\xi}$  are all homogeneous, i.e. supported on a single orbit of a group  $M \leq \mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$  with  $\pi(M) = L_1$ . Moreover e.g. by Poincaré recurrence (or by noting that  $g_1^{-1}L_1g_1$  acts irreducibly on  $\mathbb{R}^n/\mathbb{Z}^n$ , hence either  $\pi$  is injective on M or  $M = \mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ ) the group M is normalized by  $A_2$ , so  $\bar{\mu}$  is a homogeneous measure supported on a single  $A_2M$ -orbit. Note that we have essentially classified which homogeneous measures may occur in the positive base entropy case.

There remains the zero base entropy case. We will parameterize the acting group by vectors  $\mathbf{m} \in \mathbb{Z}^n$  with  $\sum_i m_i = 0$ , and set

$$a^{\mathbf{m}} = \begin{pmatrix} e^{m_1} & & & \\ & e^{m_2} & & \\ & & \ddots & \\ & & & e^{m_n} \end{pmatrix}.$$

There are exactly  $n^2$  coarse Lyapunov exponents  $[\alpha]$  that play a role for the action of  $a^{\bullet}$  on  $G/\Gamma$  (i.e. for which  $U_{[\alpha]}$  is nontrivial), for each of which  $\dim(U_{[\alpha]}) = 1$ : n(n-1) coarse Lyapunov exponents for which  $U_{[\alpha]} < \operatorname{SL}(n, \mathbb{R})$ , corresponding to the functionals  $\phi_{i,j} : \mathbf{m} \mapsto m_i - m_j$  for  $i \neq j$ , and n coarse Lyapunov exponents for which  $U_{[\alpha]}$  is a subgroup of the unipotent radical of G corresponding to the functionals  $\phi_i : \mathbf{m} \mapsto m_i$ .

By [EL3, Prop. 6.4], if  $D_{\mu}(\mathbf{m}, [\phi_{i,j}]) \neq 0$  for some  $i \neq j$  then  $D_{\bar{\mu}}(\mathbf{m}, [\phi_{i,j}]) \neq 0$ and hence we have positive base entropy. Therefore we may assume  $D_{\mu}(\mathbf{m}, [\phi_{i,j}]) = 0$  for all i, j.

It follows that only  $D_{\mu}(\mathbf{m}, [\phi_i])$  for  $i = 1, \ldots, n$  can be nonzero, and since for some **m** we have that  $h_{\mu}(\mathbf{m}) > 0$ , by (2.7) for at least one *i* we have that  $D_{\mu}(\mathbf{m}, [\phi_i]) > 0$ . However, the only way to satisfy (2.8) for all relevant<sup>3</sup> parameters **m** is if for all *i* we have  $D_{\mu}(\mathbf{m}, [\phi_i]) > 0$  (indeed these contributions must equal to one another, though we will not need to make use of that). The key point is that unlike the  $[\phi_{i,j}]$ , for  $[\phi_i]$  the opposite coarse Lyapunov exponent  $[-\phi_i]$  does

<sup>&</sup>lt;sup>3</sup>Recall that for convenience we take  $\mathbf{m} \in \mathbb{Z}^n$  with  $\sum m_i = 0$ .

not appear in G (in other words,  $U_{[-\phi_i]}$  is trivial), hence by Theorem 1.1 the invariance group  $I_{\mu}^{[\phi_i]}$  is nontrivial<sup>4</sup>. In this case the only nontrivial closed subgroup of  $U_{[\phi_i]}$  with arbitrarily small and arbitrarily large elements is  $U_{[\phi_i]}$  itself, hence as in the proof of Theorem 1.2 the measure  $\mu$  is invariant under the group generated by all the  $U_{[\phi_i]}$ , i.e. by the unipotent radical of G, which implies the second case of Theorem 1.3.

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<sup>&</sup>lt;sup>4</sup>In Theorem 1.1 the measure  $\mu$  is assumed to be ergodic under  $a^{\bullet}$ , whereas we only know  $\mu$  is ergodic under A. Since  $a^{\bullet}$  is cocompact in A this makes very little difference; in particular, a reader worried about this slight discrepancy may take the ergodic decomposition of  $\mu$  with respect to the action of  $a^{\bullet}$ , and prove invariance of  $\mu$  under the unipotent radical of G by showing it on each ergodic component separately.

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(M. E.) ETH ZÜRICH, RÄMISTRASSE 101 CH-8092 ZÜRICH SWITZERLAND *E-mail address*: manfred.einsiedler@math.ethz.ch

(E. L.) The Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem Jerusalem, 91904, Israel

*E-mail address*: elon@math.huji.ac.il

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