

FIRST EIGENVALUES OF GEOMETRIC OPERATORS UNDER THE YAMABE FLOW

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ABSTRACT. Suppose (M, g_0) is a compact Riemannian manifold without boundary of dimension $n \geq 3$. Using the Yamabe flow, we obtain estimate for the first nonzero eigenvalue of the Laplacian of g_0 with negative scalar curvature in terms of the Yamabe metric in its conformal class. On the other hand, we prove that the first eigenvalue of some geometric operators on a compact Riemannian manifold is nondecreasing along the unnormalized Yamabe flow under suitable curvature assumption. Similar results are obtained for manifolds with boundary and for CR manifold.

1. INTRODUCTION

Suppose (M, g_0) is a compact n -dimensional manifold without boundary where $n \geq 3$. As a generalization of Uniformization theorem, the Yamabe problem is to find a metric g conformal to g_0 such that its scalar curvature R_g is constant. The Yamabe problem was first studied by Yamabe in [50]. Note that if we write $g = u^{\frac{4}{n-2}} g_0$ where u is a positive smooth function in M , then the scalar curvature R_g of g can be written as

$$(1.1) \quad R_g = u^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_{g_0} u + R_{g_0} u \right).$$

Therefore, the Yamabe problem is to solve (1.1) such that R_g is constant. This was solved by Trudinger [48], Aubin [1], and Schoen [45]. See the survey article [34] of Lee and Parker for more details.

Yamabe flow was introduced by Hamilton in [21] to study the Yamabe problem, which is defined as the evolution of the metric $g = g(t)$:

$$\frac{\partial}{\partial t} g = -(R_g - \bar{R}_g)g \text{ for } t \geq 0, \quad g|_{t=0} = g_0,$$

where $\bar{R}_g = \frac{\int_M R_g dV_g}{\int_M dV_g}$ is the average of the scalar curvature R_g of the Riemannian metric g . In [15], Chow proved that the Yamabe flow approaches a metric of constant scalar curvature provided that the initial metric is locally conformally flat and has positive Ricci curvature. In [51], Ye proved the convergence of the Yamabe flow by assuming only that the initial metric is locally conformally flat.

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Later, Schwetlick and Struwe [46] proved the convergence of the Yamabe flow for the case when $3 \leq n \leq 5$ under the assumption that the initial metric has large energy. Finally Brendle [4, 5] showed that the Yamabe flow converges to a metric of constant scalar curvature by using positive mass theorem. See also [2, 16, 41, 42, 47] for results related to the Yamabe flow.

In this paper, we prove the following theorem by using the Yamabe flow.

Theorem 1.1. *Suppose (M, g_0) is a compact Riemannian manifold of dimension $n \geq 3$ without boundary such that $\max_M R_{g_0} < 0$, and g_Y is the Yamabe metric conformal to g_0 which has same volume as g_0 . Then the first nonzero eigenvalues of the Laplacian of g_0 and g_Y satisfy*

$$(1.2) \quad e^{-c} \lambda_1(g_Y) \leq \lambda_1(g_0) \leq e^c \lambda_1(g_Y)$$

where $c = 2(n-1) \left(\frac{\min_M R_{g_0}}{\max_M R_{g_0}} - 1 \right)$.

Recall that g_Y is a Yamabe metric in the conformal class of g_0 if g_Y is a Riemannian metric conformal to g_0 such that its scalar curvature is constant. Theorem 1.1 can be applied to estimate the first nonzero eigenvalue of a metric with negative scalar curvature in terms of the Yamabe metric in its conformal class. See Theorem 2.10.

The Yamabe problem was also studied in the context of manifolds with boundary. Suppose (M, g_0) is a compact n -dimensional manifold with smooth boundary ∂M where $n \geq 3$. The Yamabe problem is to find a metric g conformal to g_0 such that its scalar curvature R_g is constant in M and its mean curvature H_g is zero on ∂M . This has been studied by Escobar in [18]. See also [6, 22] for results in this direction. In particular, the Yamabe flow was introduced on manifolds with boundary by Brendle in [3]: given a metric g_0 with vanishing mean curvature on the boundary, i.e. $H_{g_0} = 0$ on ∂M , we can define the Yamabe flow as follows:

$$\frac{\partial}{\partial t} g = -(R_g - \bar{R}_g)g \text{ in } M \text{ and } H_g = 0 \text{ on } \partial M \text{ for } t \geq 0, \quad g|_{t=0} = g_0.$$

Using the Yamabe flow, we obtain estimate for the first nonzero eigenvalue of the Laplacian with Dirichlet boundary condition when (M, g_0) has negative scalar curvature. See Theorem 4.4.

One can consider the following CR analogue of the Yamabe problem, the CR Yamabe problem. Suppose (M, θ_0) is a compact strictly pseudoconvex CR manifold of real dimension $2n+1$ with a contact form θ_0 . The CR Yamabe problem is to find a contact form θ conformal to θ_0 such that its Webster scalar curvature is constant. Jerison and Lee [28, 29, 30] solved the CR Yamabe problem when $n \geq 2$ and M is not locally CR equivalent to the sphere. The remaining cases, namely when $n = 1$ or M is locally CR equivalent to the sphere, were studied respectively by Gamara and Yacoub in [20] and by Gamara in [19]. See also the recent work of Cheng-Chiu-Yang in [13] and Cheng-Malchiodi-Yang in [14].

The CR Yamabe flow was introduced to study the CR Yamabe problem, which is defined as:

$$\frac{\partial}{\partial t} \theta = -(R_\theta - \bar{R}_\theta)\theta \text{ for } t \geq 0, \quad \theta|_{t=0} = \theta_0,$$

where R_θ is the Webster scalar curvature of the contact form θ , and \bar{R}_θ is the average of the Webster scalar curvature. See [10, 11, 24, 25, 26, 27, 52] for results

related to the CR Yamabe flow. Using the CR Yamabe flow, we obtain estimate for the first nonzero eigenvalue of the sub-Laplacian of a contact form θ_0 with negative Webster scalar curvature. See Theorem 6.6.

In another direction, we consider eigenvalues of some geometric operators under the unnormalized Yamabe flow. In recent years, there has been increasing attentions on the study of eigenvalues of geometric operators under different kinds of geometric flow. In [43], Perelman proved that the first eigenvalue of $-4\Delta_g + R_g$ is nondecreasing along the Ricci flow

$$\frac{\partial}{\partial t}g = -2Ric_g,$$

where Ric_g and R_g are the Ricci curvature and scalar curvature of g respectively. As an application, he showed that there is no nontrivial steady or expanding breathers on closed manifolds. In [7], Cao showed that the eigenvalues of $-\Delta_g + \frac{1}{2}R_g$ on Riemannian manifolds with nonnegative curvature operator are nondecreasing under the Ricci flow. See [35, 36, 40, 53] for related results.

In [8], Cao proved that the first eigenvalue of $-\Delta_g + aR_g$ on a closed manifold M , where $a > 1/4$, is nondecreasing along the Ricci flow. In [9], Cao-Hou-Ling showed that the first eigenvalue of $-\Delta_g + aR_g$, where $0 < a < 1/2$, on a closed surface with nonnegative scalar curvature is nondecreasing under the Ricci flow. Combining these results, we have the following: (see Theorem 2.2 in [9])

Theorem 1.2. *On a closed surface with nonnegative scalar curvature, for all $a > 0$, the first eigenvalue of $-\Delta_g + aR_g$ is nondecreasing under the Ricci flow.*

Note that when the dimension $n = 2$, we have $Ric_g = \frac{1}{2}R_gg$. Therefore, the Ricci flow becomes the unnormalized Yamabe flow

$$\frac{\partial}{\partial t}g = -R_gg.$$

We would like to generalize Theorem 1.2 to higher dimension by considering the unnormalized Yamabe flow. In particular, we prove the following:

Theorem 1.3. *Along the unnormalized Yamabe flow, the first eigenvalue of $-\Delta_g + aR_g$ is nondecreasing*

- (i) if $0 \leq a < \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq \frac{n-2}{n} \max_M R_g \geq 0$;
- (ii) if $a \geq \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq 0$.

Corresponding results are also obtained for manifolds with boundary and for CR manifolds. See Theorem 5.2, Theorem 5.3 and Theorem 7.3. Note that Theorem 1.3 was proved in [25] for the cases when $a = 0$ and $a = \frac{n-2}{4(n-1)}$ (c.f. Theorem 6.1 and Theorem 6.2 in [25]). Note also that the condition $\min_M R_g \geq 0$ is preserved by the Yamabe flow (see Proposition 6.2 in [25]). Therefore, Theorem 1.3 implies that following corollary, which can be considered as a generalization of Theorem 1.2 in higher dimensions.

Corollary 1.4. *On a closed Riemannian manifold with nonnegative scalar curvature, the first eigenvalue of $-\Delta_g + aR_g$, where $a \geq \frac{n-2}{4(n-1)}$, is nondecreasing along the unnormalized Yamabe flow.*

We would like to point out the following main difference between the proof of Theorem 1.1 and Theorem 1.3. By the eigenvalue perturbation theory (c.f. [8, 31, 33, 44]), we know that there is a family of the first eigenvalue and its corresponding eigenfunction of the geometric operator, which is C^1 in t along the flow in Theorem 1.3. However, in Theorem 1.1, we only know that the first nonzero eigenvalue (that is, the second eigenvalue since the first eigenvalue is always zero) of the Laplacian is Lipschitz continuous in t . But we are able to overcome the difficulty by following the ideas of Wu-Wang-Zheng in [49].

2. THE YAMABE FLOW ON MANIFOLDS WITHOUT BOUNDARY

In this section, we let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary. We consider the Yamabe flow, which is defined as

$$(2.1) \quad \frac{\partial}{\partial t} g = -(R_g - \bar{R}_g)g \text{ for } t \geq 0, \quad g|_{t=0} = g_0.$$

Here R_g is the scalar curvature of g and \bar{R}_g is the average of the scalar curvature given by

$$(2.2) \quad \bar{R}_g = \frac{\int_M R_g dV_g}{\int_M dV_g},$$

where dV_g is the volume form of g . Since the Yamabe flow preserves the conformal structure, we can write $g = u^{\frac{4}{n-2}} g_0$ for some positive function u , where u satisfies the following evolution equation:

$$(2.3) \quad \frac{\partial u}{\partial t} = -\frac{n-2}{4}(R_g - \bar{R}_g)u \text{ for } t \geq 0, \quad u|_{t=0} = 1.$$

Hence, the volume form dV_g of g satisfies

$$(2.4) \quad \frac{\partial}{\partial t}(dV_g) = \frac{\partial}{\partial t}(u^{\frac{2n}{n-2}} dV_{g_0}) = \frac{2n}{n-2} u^{\frac{2n}{n-2}-1} \frac{\partial u}{\partial t} dV_{g_0} = -\frac{n}{2}(R_g - \bar{R}_g)dV_g.$$

On the other hand, the scalar curvature R_g of g satisfies the following evolution equation: (see [4])

$$(2.5) \quad \frac{\partial}{\partial t} R_g = (n-1)\Delta_g R_g + R_g(R_g - \bar{R}_g).$$

We have the following proposition, which is inspired by Proposition 3.1 in [49].

Proposition 2.1. *Let $g = g(t)$ be the solution of the the Yamabe flow (2.1) and $\lambda_1(t)$ be the corresponding first nonzero eigenvalue of the Laplacian. Then for any $t_2 \geq t_1$, there exists a C^∞ function f on $M \times [t_1, t_2]$ satisfying*

$$(2.6) \quad \int_M f^2 dV_g = 1 \quad \text{and} \quad \int_M f dV_g = 0 \text{ for all } t \in [t_1, t_2],$$

and

$$\lambda_1(t_2) \geq \lambda_1(t_1) - \frac{n-2}{2} \int_{t_1}^{t_2} \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g dt - 2 \int_{t_1}^{t_2} \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g dt$$

such that at time t_2 , $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$.

Proof. At time t_2 , we let $f_2 = f(t_2)$ be the eigenfunction for the first nonzero eigenvalue $\lambda_1(t_2)$ of $g(t_2)$. We define the following smooth function on M :

$$h(t) = \left(\frac{u(t_2)}{u(t)} \right)^{\frac{2n}{n-2}} f_2$$

where $u(t)$ is the solution of (2.3). We normalize this smooth function on M by

$$f(t) = \frac{h(t)}{\left(\int_M h(t)^2 dV_{g(t)} \right)^{\frac{1}{2}}}.$$

Then we can easily check that $f(t)$ satisfies (2.6).

Set

$$G(g(t), f(t)) := \int_M |\nabla_{g(t)} f(t)|^2 dV_{g(t)}.$$

Note that $G(g(t), f(t))$ is a smooth function in t . Since $g = u^{\frac{4}{n-2}} g_0$, we have

$$(2.7) \quad \langle \nabla_g f_1, \nabla_g f_2 \rangle_g = u^{-\frac{4}{n-2}} \langle \nabla_{g_0} f_1, \nabla_{g_0} f_2 \rangle_{g_0}$$

for any functions f_1, f_2 in M , which implies that

$$G(g(t), f(t)) = \int_M u^2 |\nabla_{g_0} f|_{g_0}^2 dV_{g_0}.$$

Differentiating it with respect to t , we get

$$(2.8) \quad \begin{aligned} \mathcal{G}(g(t), f(t)) &:= \frac{d}{dt} G(g(t), f(t)) \\ &= \int_M 2u \frac{\partial u}{\partial t} |\nabla_{g_0} f|_{g_0}^2 dV_{g_0} + 2 \int_M u^2 \langle \nabla_{g_0} f, \nabla_{g_0} \left(\frac{\partial f}{\partial t} \right) \rangle_{g_0} dV_{g_0} \\ &= -\frac{n-2}{2} \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g + 2 \int_M \langle \nabla_g f, \nabla_g \left(\frac{\partial f}{\partial t} \right) \rangle_g dV_g \\ &= -\frac{n-2}{2} \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g - 2 \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g \end{aligned}$$

where the second last equality follows from (2.3) and (2.7), the last equality follows from integration by parts. It follows from the definition of $\mathcal{G}(g(t), f(t))$ in (2.8) that

$$(2.9) \quad G(g(t_2), f(t_2)) - G(g(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(g(t), f(t)) dt.$$

Since $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$, we have

$$(2.10) \quad G(g(t_2), f(t_2)) = \lambda_1(t_2) \int_M f(t_2)^2 dV_{g(t_2)} = \lambda_1(t_2)$$

by (2.6). On the other hand, it follows from (2.6) and the definition of $G(g(t), f(t))$ that

$$(2.11) \quad G(g(t_1), f(t_1)) \geq \lambda_1(t_1) \int_{\partial M} f(t_1)^2 dV_{g(t_1)} = \lambda_1(t_1).$$

Now the result follows from (2.8)-(2.11). \square

Hence, we have the following:

Proposition 2.2. *The first nonzero eigenvalue λ_1 of the Laplacian along the Yamabe flow (2.1) satisfies*

$$\frac{d}{dt} \log \lambda_1 \geq -\frac{n-2}{2} \left(\max_M R_g - \bar{R}_g \right) + \frac{n}{2} \left(\min_M R_g - \bar{R}_g \right).$$

Here the derivative on the left hand side is in the sense of the lim inf of backward difference quotients.

Proof. Differentiate the first equation in (2.6) with respect to t , we have

$$(2.12) \quad \int_M f \frac{\partial f}{\partial t} dV_g = \frac{n}{4} \int_M f^2 (R_g - \bar{R}_g) dV_g$$

by (2.4). On the other hand, since $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$, we have

$$(2.13) \quad - \int_M \frac{\partial f(t_2)}{\partial t} \Delta_{g(t_2)} f(t_2) dV_{g(t_2)} = \lambda_1(t_2) \int_M f(t_2) \frac{\partial f(t_2)}{\partial t} dV_{g(t_2)}.$$

Combining (2.12) and (2.13), we get

$$(2.14) \quad - \int_M \frac{\partial f(t_2)}{\partial t} \Delta_{g(t_2)} f(t_2) dV_{g(t_2)} = \frac{n}{4} \lambda_1(t_2) \int_M f(t_2)^2 (R_{g(t_2)} - \bar{R}_{g(t_2)}) dV_{g(t_2)} \\ \geq \frac{n}{4} \lambda_1(t_2) \left(\min_M R_{g(t_2)} - \bar{R}_{g(t_2)} \right),$$

where we have used (2.6) in the last inequality. Since $\lambda(t_2)$ is positive, we have for any $\epsilon > 0$

$$(2.15) \quad - \int_M \frac{\partial f(t_2)}{\partial t} \Delta_{g(t_2)} f(t_2) dV_{g(t_2)} > \frac{n}{4} \lambda_1(t_2) \left(\min_M R_{g(t_2)} - \bar{R}_{g(t_2)} - \epsilon \right),$$

by (2.14). By the definition of f , the function $s \mapsto \int_M \frac{\partial f(s)}{\partial t} \Delta_{g(s)} f(s) dV_{g(s)}$ is continuous in s . On the other hand, the function $s \mapsto \left(\min_M R_{g(s)} - \bar{R}_{g(s)} \right)$ is continuous in s . Hence, it follows from (2.15) that for any $\epsilon > 0$

$$(2.16) \quad - \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g \geq \frac{n}{4} \lambda_1(t_2) \left(\min_M R_g - \bar{R}_g - \epsilon \right)$$

when t is sufficiently closed to t_2 . On the other hand, we have

$$(2.17) \quad - \int_M (R_{g(t_2)} - \bar{R}_{g(t_2)}) |\nabla_{g(t_2)} f|_{g(t_2)}^2 dV_{g(t_2)} \\ \geq - \left(\max_M R_{g(t_2)} - \bar{R}_{g(t_2)} \right) \int_M |\nabla_{g(t_2)} f|_{g(t_2)}^2 dV_{g(t_2)} = - \left(\max_M R_{g(t_2)} - \bar{R}_{g(t_2)} \right) \lambda_1(t_2)$$

by (2.6) and the fact that $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$. Since $\lambda(t_2)$ is positive, we have for any $\epsilon > 0$

$$(2.18) \quad - \int_M (R_{g(t_2)} - \bar{R}_{g(t_2)}) |\nabla_{g(t_2)} f|_{g(t_2)}^2 dV_{g(t_2)} > - \left(\max_M R_{g(t_2)} - \bar{R}_{g(t_2)} + \epsilon \right) \lambda_1(t_2)$$

By the definition of f , the function $s \mapsto \int_M (R_{g(s)} - \bar{R}_{g(s)}) |\nabla_{g(s)} f|_{g(s)}^2 dV_{g(s)}$ is continuous in s . On the other hand, the function $s \mapsto \left(\max_M R_{g(s)} - \bar{R}_{g(s)} \right)$ is

continuous in s . Hence, it follows from (2.18) that for any $\epsilon > 0$

$$(2.19) \quad - \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g \geq -\lambda_1(t_2) \left(\max_M R_g - \bar{R}_g + \epsilon \right)$$

when t is sufficiently closed to t_2 . Substituting (2.16) and (2.19) into the inequality in Proposition 2.1, we obtain

$$\begin{aligned} & \lambda_1(t_2) - \lambda_1(t_1) \\ & \geq -\frac{n-2}{2} \lambda_1(t_2) \int_{t_1}^{t_2} \left(\max_M R_g - \bar{R}_g + \epsilon \right) dt + \frac{n}{2} \lambda_1(t_2) \int_{t_1}^{t_2} \left(\min_M R_g - \bar{R}_g - \epsilon \right) dt \end{aligned}$$

for $t_1 < t_2$ and t_1 sufficiently closed to t_2 . Dividing $t_2 - t_1$ in the above inequality and letting t_1 go to t_2 , we obtain

$$(2.20) \quad \begin{aligned} & \liminf_{t_1 \rightarrow t_2} \frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \\ & \geq -\frac{n-2}{2} \lambda_1(t_2) \left(\max_M R_{g(t_2)} - \bar{R}_{g(t_2)} + \epsilon \right) + \frac{n}{2} \lambda_1(t_2) \left(\min_M R_{g(t_2)} - \bar{R}_{g(t_2)} - \epsilon \right). \end{aligned}$$

Note that

$$(2.21) \quad \liminf_{t_1 \rightarrow t_2} \frac{\log \lambda_1(t_2) - \log \lambda_1(t_1)}{t_2 - t_1} \geq \frac{1}{\lambda_1(t_2)} \liminf_{t_1 \rightarrow t_2} \frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \text{ for } t_1 < t_2.$$

To see this, note that

$$\log \lambda_1(t_2) - \log \lambda_1(t_1) = \log \left(\left(\frac{\lambda_1(t_2)}{\lambda_1(t_1)} - 1 \right) + 1 \right) = \left(\frac{\lambda_1(t_2)}{\lambda_1(t_1)} - 1 \right) + O \left(\left(\frac{\lambda_1(t_2)}{\lambda_1(t_1)} - 1 \right)^2 \right)$$

which implies that

$$\begin{aligned} & \liminf_{t_1 \rightarrow t_2} \frac{\log \lambda_1(t_2) - \log \lambda_1(t_1)}{t_2 - t_1} \\ & \geq \liminf_{t_1 \rightarrow t_2} \frac{1}{\lambda_1(t_1)} \left(\frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \right) + \liminf_{t_1 \rightarrow t_2} O \left(\frac{1}{t_2 - t_1} \left(\frac{\lambda_1(t_2)}{\lambda_1(t_1)} - 1 \right)^2 \right). \end{aligned}$$

Note that

$$\begin{aligned} \liminf_{t_1 \rightarrow t_2} \frac{1}{\lambda_1(t_1)} \left(\frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \right) & \geq \left(\liminf_{t_1 \rightarrow t_2} \frac{1}{\lambda_1(t_1)} \right) \left(\liminf_{t_1 \rightarrow t_2} \frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \right) \\ & = \frac{1}{\lambda_1(t_2)} \liminf_{t_1 \rightarrow t_2} \frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \end{aligned}$$

and

$$\lim_{t_1 \rightarrow t_2} \frac{1}{t_2 - t_1} \left(\frac{\lambda_1(t_2)}{\lambda_1(t_1)} - 1 \right)^2 = 0$$

since $\lambda_1(t)$ is Lipschitz continuous in t , (2.21) follows from combining all these. Now, combining (2.20) and (2.21), we have

$$\begin{aligned} & \liminf_{t_1 \rightarrow t_2} \frac{\log \lambda_1(t_2) - \log \lambda_1(t_1)}{t_2 - t_1} \\ & \geq -\frac{n-2}{2} \left(\max_M R_{g(t_2)} - \bar{R}_{g(t_2)} + \epsilon \right) + \frac{n}{2} \left(\min_M R_{g(t_2)} - \bar{R}_{g(t_2)} - \epsilon \right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, Proposition 2.2 follows from letting $\epsilon \rightarrow 0$. \square

Similarly, we can prove the following:

Proposition 2.3. *Let $g = g(t)$ be the solution of the the Yamabe flow (2.1) and $\lambda_1(t)$ be the corresponding first nonzero eigenvalue of the Laplacian. Then for any $t_2 \geq t_1$, there exists a C^∞ function f on $M \times [t_1, t_2]$ satisfying*

$$(2.22) \quad \int_M f^2 dV_g = 1 \quad \text{and} \quad \int_M f dV_g = 0 \quad \text{for all } t \in [t_1, t_2],$$

and

$$\lambda_1(t_2) \leq \lambda_1(t_1) - \frac{n-2}{2} \int_{t_1}^{t_2} \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g dt - 2 \int_{t_1}^{t_2} \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g dt$$

such that at time t_1 , $f(t_1)$ is the corresponding eigenfunction of $\lambda_1(t_1)$.

Proof. We only sketch the proof since it is almost the same as the proof of Proposition 2.1. At time t_1 , we let $f_1 = f(t_1)$ be the eigenfunction for the first nonzero eigenvalue $\lambda_1(t_1)$ of $g(t_1)$. We define

$$h(t) = \left(\frac{u(t_1)}{u(t)} \right)^{\frac{2n}{n-2}} f_1$$

where $u(t)$ is the solution of (2.3). Then

$$f(t) = \frac{h(t)}{\left(\int_M h(t)^2 dV_{g(t)} \right)^{\frac{1}{2}}}$$

satisfies (2.22). As in the proof of Proposition 2.1, we define

$$G(g(t), f(t)) := \int_M |\nabla_{g(t)} f(t)|^2 dV_{g(t)}.$$

Then it is clear that (2.8) and (2.9) are still true. Since $f(t_1)$ is the corresponding eigenfunction of $\lambda_1(t_1)$, we have

$$G(g(t_1), f(t_1)) = \lambda_1(t_1) \int_M f(t_1)^2 dV_{g(t_1)} = \lambda_1(t_1)$$

by (2.22). On the other hand, it follows from (2.22) and the definition of $G(g(t), f(t))$ that

$$G(g(t_2), f(t_2)) \geq \lambda_1(t_2) \int_{\partial M} f(t_2)^2 dV_{g(t_2)} = \lambda_1(t_2).$$

Now Lemma 2.3 follows from combining all these. \square

We also have the following:

Proposition 2.4. *The first nonzero eigenvalue λ_1 of the Laplacian along the Yamabe flow (2.1) satisfies*

$$\frac{d}{dt} \log \lambda_1 \leq -\frac{n-2}{2} \left(\min_M R_g - \bar{R}_g \right) + \frac{n}{2} \left(\max_M R_g - \bar{R}_g \right).$$

Here the derivative on the left hand side is in the sense of the limsup of forward difference quotients.

Proof. Again we only sketch the proof since it is almost the same as the proof of Proposition 2.2. As in the proof of Proposition 2.2, by (2.22) and the fact that $f(t_1)$ is the corresponding eigenfunction of $\lambda_1(t_1)$, we have

$$\begin{aligned} - \int_M \frac{\partial f(t_1)}{\partial t} \Delta_{g(t_1)} f(t_1) dV_{g(t_1)} &= \lambda_1(t_1) \int_M f(t_1) \frac{\partial f(t_1)}{\partial t} dV_{g(t_1)} \\ &= \frac{n}{4} \lambda_1(t_1) \int_M f(t_1)^2 (R_{g(t_1)} - \bar{R}_{g(t_1)}) dV_{g(t_1)} \\ &\leq \frac{n}{4} \lambda_1(t_1) \left(\max_M R_{g(t_1)} - \bar{R}_{g(t_1)} \right). \end{aligned}$$

By continuity and the fact that $\lambda(t_1) > 0$, we can conclude that for any $\epsilon > 0$

$$(2.23) \quad - \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g \leq \frac{n}{4} \lambda_1(t_1) \left(\max_M R_g - \bar{R}_g + \epsilon \right)$$

when t is sufficiently closed to t_1 . Similarly,

$$(2.24) \quad - \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g \leq -\lambda_1(t_1) \left(\min_M R_g - \bar{R}_g - \epsilon \right)$$

when t is sufficiently closed to t_1 . Now putting (2.23) and (2.24) into the inequality in Proposition 2.3, we obtain

$$\begin{aligned} &\lambda_1(t_2) - \lambda_1(t_1) \\ &\leq -\frac{n-2}{2} \lambda_1(t_1) \int_{t_1}^{t_2} \left(\min_M R_g - \bar{R}_g - \epsilon \right) dt + \frac{n}{2} \lambda_1(t_1) \int_{t_1}^{t_2} \left(\max_M R_g - \bar{R}_g + \epsilon \right) dt \end{aligned}$$

for $t_2 > t_1$ and t_2 sufficiently closed to t_1 . Dividing the last inequality by $t_2 - t_1$ and letting t_2 go to t_1 , we get

$$(2.25) \quad \begin{aligned} &\limsup_{t_2 \rightarrow t_1} \frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \\ &\leq -\frac{n-2}{2} \lambda_1(t_1) \left(\min_M R_{g(t_1)} - \bar{R}_{g(t_1)} - \epsilon \right) + \frac{n}{2} \lambda_1(t_1) \left(\max_M R_{g(t_1)} - \bar{R}_{g(t_1)} + \epsilon \right). \end{aligned}$$

Replacing $\liminf_{t_1 \rightarrow t_2}$ by $\limsup_{t_2 \rightarrow t_1}$ and reversing the sign of the inequality, one can follow the arguments of proving (2.21) to prove that

$$(2.26) \quad \limsup_{t_2 \rightarrow t_1} \frac{\log \lambda_1(t_2) - \log \lambda_1(t_1)}{t_2 - t_1} \leq \frac{1}{\lambda_1(t_1)} \limsup_{t_2 \rightarrow t_1} \frac{\lambda_1(t_2) - \lambda_1(t_1)}{t_2 - t_1} \text{ for } t_2 > t_1.$$

Combining (2.25) and (2.26), we have

$$\begin{aligned} &\limsup_{t_2 \rightarrow t_1} \frac{\log \lambda_1(t_2) - \log \lambda_1(t_1)}{t_2 - t_1} \\ &\leq -\frac{n-2}{2} \left(\min_M R_{g(t_1)} - \bar{R}_{g(t_1)} - \epsilon \right) + \frac{n}{2} \left(\max_M R_{g(t_1)} - \bar{R}_{g(t_1)} + \epsilon \right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, Proposition 2.4 follows from letting $\epsilon \rightarrow 0$. \square

Using the maximum principle, we can prove the following:

Proposition 2.5. *If $\max_M R_{g_0} < 0$, then $\max_M R_g < 0$ for all $t \geq 0$ under the Yamabe flow (2.1).*

Proof. Consider the function F on $M \times [0, \infty)$ given by

$$F(x, t) = R_g - \frac{1}{2} \max_M R_{g_0}.$$

We claim that $F < 0$. By contradiction, we suppose that

$$(2.27) \quad F(x_0, t_0) \geq 0 \text{ for some } (x_0, t_0) \in M \times [0, \infty).$$

Since $g = g_0$ at $t = 0$ and

$$F(x, 0) = R_{g_0} - \frac{1}{2} \max_M R_{g_0} \leq \frac{1}{2} \max_M R_{g_0} < 0$$

by assumption, we must have $t_0 > 0$. We assume that t_0 is the smallest time satisfying (2.27), i.e.

$$(2.28) \quad F(x, t) < 0 \text{ for all } t \in [0, t_0).$$

By continuity, we have

$$(2.29) \quad F(x, t_0) \leq 0 \text{ for all } x \in M.$$

Combining (2.27) and (2.29), we have

$$(2.30) \quad 0 = F(x_0, t_0) = \max_{x \in M} F(x, t_0),$$

which implies

$$(2.31) \quad R_g(x_0, t_0) = \max_{x \in M} R_g(x, t_0)$$

by the definition of F .

Therefore, at (x_0, t_0) , we have

$$(2.32) \quad 0 \leq \frac{\partial F}{\partial t} = \frac{\partial R_g}{\partial t} = (n-1)\Delta_g R_g + R_g(R_g - \bar{R}_g) \leq R_g(R_g - \bar{R}_g),$$

where the first inequality follows from the fact that $F(x_0, t)$ is increasing at t_0 by (2.28)-(2.30), and the second equality follows from (2.5), and the last inequality follows from (2.31). But this is a contradiction, since the last term of (2.32) is negative. To see this, it follows from (2.30) and the definition of F that

$$(2.33) \quad R_g(x_0, t_0) = \frac{1}{2} \max_M R_{g_0},$$

which implies together with (2.31) that at (x_0, t_0)

$$(2.34) \quad R_g(R_g - \bar{R}_g) = \frac{1}{2} \max_M R_{g_0} (R_g - \bar{R}_g) = \frac{1}{2} \max_M R_{g_0} \left(\max_{x \in M} R_g(x, t_0) - \bar{R}_g \right)$$

Since $\max_M R_{g_0} < 0$ by assumption, (2.34) is nonpositive and is equal to zero if and only if $\max_{x \in M} R_g(x, t_0) = \bar{R}_g$, or equivalently, $g(t_0)$ has constant scalar curvature. Hence, it follows from (2.33) that

$$\bar{R}_{g(t_0)} = \frac{\int_M R_{g(t_0)} dV_{g(t_0)}}{\int_M dV_{g(t_0)}} = \frac{1}{2} \max_M R_{g_0} > \max_M R_{g_0} \geq \bar{R}_{g_0},$$

which is a contradiction, since $t \mapsto \bar{R}_{g(t)}$ is nonincreasing along the Yamabe flow (see (9) in [4]). This shows that (2.34) must be negative, as we claimed.

This contradiction shows that (2.27) is impossible. This proves that $F < 0$, or equivalently, $R_g < \frac{1}{2} \max_M R_{g_0} < 0$, which proves the assertion. \square

Similarly, we have the following:

Proposition 2.6. *If $\max_M R_{g_0} < 0$, then $\max_M R_g \leq \max_M R_{g_0}$ for all $t \geq 0$ under the Yamabe flow (2.1).*

Proof. The proof is similar to Proposition 2.5. For $\epsilon > 0$, we define the function $F(x, t) = R_g(x, t) - \epsilon(t + 1)$ on $M \times [0, \infty)$. We claim that $\max_M F < \max_M R_{g_0}$ on $[0, \infty)$. By contradiction, we suppose that

$$(2.35) \quad F(x_0, t_0) \geq \max_M R_{g_0} \text{ for some } (x_0, t_0) \in M \times [0, \infty).$$

Since $g = g_0$ at $t = 0$, we have $\max_M F = \max_M R_{g_0} - \epsilon < \max_M R_{g_0}$. Therefore, we must have $t_0 > 0$. We may assume that t_0 is the smallest time satisfying (2.35), i.e.

$$(2.36) \quad F(x, t) < \max_M R_{g_0} \text{ for all } (x, t) \in M \times [0, t_0).$$

By continuity, we have

$$(2.37) \quad F(x, t_0) \leq \max_M R_{g_0} \text{ for all } x \in M.$$

Combining (2.35) and (2.37), we have

$$(2.38) \quad F(x_0, t_0) = \max_{x \in M} F(x, t_0),$$

which implies

$$(2.39) \quad R_g(x_0, t_0) = \max_{x \in M} R_g(x, t_0),$$

by the definition of F .

Therefore, at (x_0, t_0) , we have

$$\begin{aligned} 0 &\leq \frac{\partial F}{\partial t} = -\epsilon + \frac{\partial R_g}{\partial t} = -\epsilon + (n-1)\Delta_g R_g + R_g(R_g - \bar{R}_g) \\ &\leq -\epsilon + R_g(R_g - \bar{R}_g) \\ &= -\epsilon + \max_{x \in M} R_g(x, t_0) \left(\max_{x \in M} R_g(x, t_0) - \bar{R}_g \right) \leq -\epsilon, \end{aligned}$$

where the first inequality follows from (2.36)-(2.38), the second equality follows from (2.5), the second inequality and the third equality follows from (2.39), and the last inequality follows from the fact that $\max_M R_g < 0$ by Proposition 2.5. This contradicts the assumption that $\epsilon > 0$, which proves the claim.

By the claim, for any $\epsilon > 0$, we have $\max_M F < \max_M R_{g_0}$ on $[0, \infty)$. By letting $\epsilon \rightarrow 0$, we get the required estimate. \square

We can also prove the following:

Proposition 2.7. *If $\max_M R_{g_0} < 0$, then $\min_M R_g \geq \min_M R_{g_0}$ for all $t \geq 0$ under the Yamabe flow (2.1).*

Proof. For $\epsilon > 0$, we define the function $F(x, t) = R_g(x, t) + \epsilon(t + 1)$ on $M \times [0, \infty)$. We claim that $\min_M F > \min_M R_{g_0}$ on $[0, \infty)$. By contradiction, we suppose that

$$(2.40) \quad F(x_0, t_0) \leq \min_M R_{g_0} \text{ for some } (x_0, t_0) \in M \times [0, \infty).$$

Since $g = g_0$ at $t = 0$, we have $\min_M F = \min_M R_{g_0} + \epsilon > \min_M R_{g_0}$. Therefore, we must have $t_0 > 0$. We may assume that t_0 is the smallest time satisfying (2.40), i.e.

$$(2.41) \quad F(x, t) > \min_M R_{g_0} \text{ for all } (x, t) \in M \times [0, t_0).$$

By continuity, we have

$$(2.42) \quad F(x, t_0) \geq \max_M R_{g_0} \text{ for all } x \in M.$$

Combining (2.40) and (2.42), we have

$$(2.43) \quad F(x_0, t_0) = \min_{x \in M} F(x, t_0),$$

which implies

$$(2.44) \quad R_g(x_0, t_0) = \min_{x \in M} R_g(x, t_0),$$

by the definition of F .

Therefore, at (x_0, t_0) , we have

$$\begin{aligned} 0 &\geq \frac{\partial F}{\partial t} = \epsilon + \frac{\partial R_g}{\partial t} = \epsilon + (n-1)\Delta_g R_g + R_g(R_g - \overline{R}_g) \\ &\geq \epsilon + R_g(R_g - \overline{R}_g) \\ &= \epsilon + \min_{x \in M} R_g(x, t_0) \left(\min_{x \in M} R_g(x, t_0) - \overline{R}_g \right) \geq \epsilon, \end{aligned}$$

where the first inequality follows from (2.41)-(2.43), the second equality follows from (2.5), the second inequality and the third equality follows from (2.44), and the last inequality follows from the fact that $\min_M R_g \leq \max_M R_g < 0$ by Proposition 2.5. This contradicts the assumption that $\epsilon > 0$, which proves the claim.

By the claim, for any $\epsilon > 0$, we have $\min F > \min_M R_{g_0}$ on $[0, \infty)$. By letting $\epsilon \rightarrow 0$, we get the required estimate. \square

Using the maximum principle, we can also prove the following:

Lemma 2.8. *If $\max_M R_{g_0} < 0$, then*

$$R_{g(t)} \leq \overline{R}_{g_0} + \left(\max_M R_{g_0} - \min_M R_{g_0} \right) + \left(\max_M R_{g_0} \right) \int_0^t \left(\max_M R_{g(s)} - \overline{R}_{g(s)} \right) ds.$$

for all $t \geq 0$ under the Yamabe flow (2.1).

Proof. For $\epsilon > 0$, we let

$$\begin{aligned} F(x, t) &= R_g(x, t) - \left(\max_M R_{g_0} - \min_M R_{g_0} \right) \\ &\quad - \left(\max_M R_{g_0} \right) \int_0^t \left(\max_M R_{g(s)} - \overline{R}_{g(s)} \right) ds - \epsilon(t+1) \end{aligned}$$

be a function defined on $M \times [0, \infty)$. We claim that $F(x, t) < \overline{R}_{g_0}$. By contradiction, we suppose that

$$(2.45) \quad F(x_0, t_0) \geq \overline{R}_{g_0} \text{ for some } (x_0, t_0) \in M \times [0, \infty).$$

Since $g = g_0$ at $t = 0$, we have

$$-\overline{R}_{g_0} + F(x, 0) = -\overline{R}_{g_0} + R_{g_0} - \left(\max_M R_{g_0} - \min_M R_{g_0} \right) - \epsilon \leq -\epsilon < 0.$$

Therefore, we must have $t_0 > 0$. We may assume that t_0 is the smallest time satisfying (2.45), i.e.

$$(2.46) \quad F(x, t) < \overline{R}_{g_0} \text{ for all } (x, t) \in M \times [0, t_0).$$

By continuity, we have

$$(2.47) \quad F(x, t_0) \leq \overline{R}_{g_0} \text{ for all } x \in M.$$

Combining (2.45)-(2.47), we have

$$(2.48) \quad F(x_0, t_0) = \max_{x \in M} F(x, t_0),$$

which implies

$$(2.49) \quad R_g(x_0, t_0) = \max_{x \in M} R_g(x, t_0)$$

by the definition of F .

Therefore, at (x_0, t_0) , we have

$$\begin{aligned} 0 &\leq \frac{\partial F}{\partial t} = \frac{\partial}{\partial t} R_g - \left(\max_M R_{g_0} \right) \left(\max_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) - \epsilon \\ &= (n-1)\Delta_g R_g + R_g(R_g - \overline{R}_g) - \left(\max_M R_{g_0} \right) \left(\max_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) - \epsilon \\ &\leq R_g(R_g - \overline{R}_g) - \left(\max_M R_{g_0} \right) \left(\max_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) - \epsilon \\ &= \left(\max_M R_{g(t_0)} \right) \left(\max_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) - \left(\max_M R_{g_0} \right) \left(\max_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) - \epsilon \\ &\leq -\epsilon, \end{aligned}$$

where the first inequality follows from (2.46)-(2.48), the second equality follows from (2.5), the second inequality and the third equality follows from (2.49), and the last inequality follows from Proposition 2.6. This contradicts to the assumption that $\epsilon > 0$. This contradiction shows that $F(x, t) < \overline{R}_{g_0}$. Letting $\epsilon \rightarrow 0$, we get the desired result. \square

Similarly, we can prove the following:

Lemma 2.9. *If $\max_M R_{g_0} < 0$, then*

$$R_{g(t)} \geq \overline{R}_{g_0} - \left(\max_M R_{g_0} - \min_M R_{g_0} \right) + \left(\max_M R_{g_0} \right) \int_0^t \left(\min_M R_{g(s)} - \overline{R}_{g(s)} \right) ds.$$

for all $t \geq 0$ under the Yamabe flow (2.1)

Proof. We only sketch the proof since it is essentially the same as the proof of Lemma 2.8. For $\epsilon > 0$, we define the function

$$\begin{aligned} F(x, t) &= R_g(x, t) + \left(\max_M R_{g_0} - \min_M R_{g_0} \right) \\ &\quad - \left(\max_M R_{g_0} \right) \int_0^t \left(\min_M R_{g(s)} - \overline{R}_{g(s)} \right) ds + \epsilon(t+1) \end{aligned}$$

on $M \times [0, \infty)$. We claim that $F(x, t) > \overline{R}_{g_0}$. If it were not true, then we could find (x_0, t_0) such that at (x_0, t_0)

$$\begin{aligned} 0 &\geq \frac{\partial F}{\partial t} = \frac{\partial}{\partial t} R_g - \left(\max_M R_{g_0} \right) \left(\min_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) + \epsilon \\ &= (n-1)\Delta_g R_g + R_g(R_g - \overline{R}_g) - \left(\max_M R_{g_0} \right) \left(\min_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) + \epsilon \\ &\geq R_g(R_g - \overline{R}_g) - \left(\max_M R_{g_0} \right) \left(\min_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) + \epsilon \\ &= \left(\min_M R_{g(t_0)} \right) \left(\min_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) - \left(\max_M R_{g_0} \right) \left(\min_M R_{g(t_0)} - \overline{R}_{g(t_0)} \right) + \epsilon \\ &\geq \epsilon, \end{aligned}$$

where the last inequality follows from Proposition 2.6. This contradicts to the assumption that $\epsilon > 0$. This contradiction shows that $F(x, t) > \overline{R}_{g_0}$. Letting $\epsilon \rightarrow 0$, we get the desired result. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It was proved by Ye (see Theorem 2 in [51]) that $g \rightarrow g_\infty$ as $t \rightarrow \infty$ under the Yamabe flow (2.1) such that g_∞ is conformal to g_0 and has constant negative scalar curvature. Along the Yamabe flow (2.1), we have

$$\frac{d}{dt} \left(\int_M dV_g \right) = \int_M \frac{\partial}{\partial t} (dV_g) = -\frac{n}{2} \int_M (R_g - \overline{R}_g) dV_g = 0$$

by (2.2) and (2.4), which implies that $\int_M dV_g = \int_M dV_{g_0}$ for all $t \geq 0$. In particular, we have

$$(2.50) \quad \int_M dV_{g_\infty} = \int_M dV_{g_0}.$$

On the other hand, note that $R_{g_Y} = c^{\frac{4}{n-2}} R_{g_\infty}$ for some constant $c > 0$. Indeed, one can take c to be $(R_{g_Y}/R_{g_\infty})^{\frac{n-2}{4}}$. This together with (1.1) implies that the metric $c^{\frac{4}{n-2}} g_Y$ has scalar curvature being equal to

$$R_{c^{\frac{4}{n-2}} g_Y} = c^{-\frac{4}{n-2}} R_{g_Y} = R_{g_\infty}.$$

Hence, we can conclude that

$$(2.51) \quad c^{\frac{4}{n-2}} g_Y = g_\infty$$

using the result of Kazdan-Warner in [32] (see also [39]), which says that if g_1 and g_2 are two metrics conformal to g_0 such that $R_{g_1} = R_{g_2} < 0$, then $g_1 = g_2$. Therefore, by (2.50) and (2.51), we have

$$\int_M dV_{g_0} = \int_M dV_{g_\infty} = \int_M dV_{c^{\frac{4}{n-2}} g_Y} = c^{\frac{2n}{n-2}} \int_M dV_{g_Y} = c^{\frac{2n}{n-2}} \int_M dV_{g_0}$$

where the last equality follows from the assumption that g_Y and g_0 have the same volume. This implies that $c = 1$, or equivalently,

$$(2.52) \quad g_Y = g_\infty.$$

Note that by Proposition 2.6 and Proposition 2.7, we have

$$(2.53) \quad \min_M R_{g_0} \leq \overline{R}_{g(t)} \leq \max_M R_{g_0} \text{ for all } t \geq 0.$$

It follows from (2.53) and Lemma 2.8 that

$$\begin{aligned}
(2.54) \quad & \left(\max_M R_{g_0} \right) \int_0^t \left(\max_M R_{g(s)} - \overline{R}_{g(s)} \right) ds \\
& \geq \left(\overline{R}_{g(t)} - \overline{R}_{g_0} \right) + \left(\max_M R_{g(t)} - \overline{R}_{g(t)} \right) - \left(\max_M R_{g_0} - \min_M R_{g_0} \right) \\
& \geq \left(\max_M R_{g(t)} - \overline{R}_{g(t)} \right) - 2 \left(\max_M R_{g_0} - \min_M R_{g_0} \right).
\end{aligned}$$

Therefore, as $t \rightarrow \infty$, by (2.54) and Ye's result stated above that $g(t) \rightarrow g_\infty$ as $t \rightarrow \infty$, we get

$$(2.55) \quad -2 \left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}} \right) \geq \int_0^\infty \left(\max_M R_{g(s)} - \overline{R}_{g(s)} \right) ds.$$

Similarly, we obtain from (2.53) and Lemma 2.9 that

$$\begin{aligned}
& - \left(\max_M R_{g_0} \right) \int_0^t \left(\min_M R_{g(s)} - \overline{R}_{g(s)} \right) ds \\
& \geq \left(\overline{R}_{g_0} - \overline{R}_{g(t)} \right) + \left(\overline{R}_{g(t)} - \min_M R_{g(t)} \right) - \left(\max_M R_{g_0} - \min_M R_{g_0} \right) \\
& \geq \left(\overline{R}_{g(t)} - \min_M R_{g(t)} \right) - 2 \left(\max_M R_{g_0} - \min_M R_{g_0} \right).
\end{aligned}$$

Letting $t \rightarrow \infty$, we obtain

$$(2.56) \quad -2 \left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}} \right) \geq - \int_0^\infty \left(\min_M R_{g(s)} - \overline{R}_{g(s)} \right) ds.$$

Integrating the inequality in Proposition 2.2 and using (2.52), (2.55) and (2.56), we get

$$\begin{aligned}
\log \frac{\lambda_1(g_Y)}{\lambda_1(g_0)} &= \log \frac{\lambda_1(g_\infty)}{\lambda_1(g_0)} \\
&\geq -\frac{(n-2)}{2} \int_0^\infty \left(\max_M R_{g(s)} - \overline{R}_{g(s)} \right) ds + \frac{n}{2} \int_0^\infty \left(\min_M R_{g(s)} - \overline{R}_{g(s)} \right) ds \\
&\geq 2(n-1) \left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}} \right)
\end{aligned}$$

which gives the upper bound for $\lambda_1(g_0)$ in (1.2). We remark that the integration holds since the Dini derivative is finite (see [23] for example). Similarly, integrating the inequality in Proposition 2.4 and using (2.52), (2.55) and (2.56), we obtain

$$\begin{aligned}
\log \frac{\lambda_1(g_Y)}{\lambda_1(g_0)} &= \log \frac{\lambda_1(g_\infty)}{\lambda_1(g_0)} \\
&\leq -\frac{(n-2)}{2} \int_0^\infty \left(\min_M R_{g(s)} - \overline{R}_{g(s)} \right) ds + \frac{n}{2} \int_0^\infty \left(\max_M R_{g(s)} - \overline{R}_{g(s)} \right) ds \\
&\leq -2(n-1) \left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}} \right)
\end{aligned}$$

which gives the lower bound for $\lambda_1(g_0)$ in (1.2). This proves the assertion. \square

One can apply Theorem 1.1 to obtain estimate of the first eigenvalue. It was proved by Li and Yau (see Theorem 7 in [37]) that if (M, g) is an n -dimensional compact Riemannian manifold without boundary such that its Ricci curvature satisfies

$Ric_g \geq (n-1)\kappa g$ where $\kappa < 0$, then its first eigenvalue satisfies

$$\lambda_1(g) \geq \frac{1}{(n-1)d^2} \exp\{-1 - \sqrt{1 + 4(n-1)^2 d^2 |\kappa|}\}$$

where d is the diameter of (M, g) .

Theorem 2.10. *Suppose M is an n -dimensional compact manifold without boundary, and g_E is an Einstein metric on M with $Ric_{g_E} = (n-1)\kappa g_E$ where $\kappa < 0$. If g_0 is a Riemannian metric conformal to g_E which has negative scalar curvature and same volume as g_E , then the first eigenvalue of (M, g_0) satisfies*

$$\lambda_1(g_0) \geq \frac{1}{(n-1)e^{\frac{nc}{n-1}} d(M, g_0)^2} \exp\left\{-1 - \sqrt{1 + 4(n-1)^2 e^{\frac{nc}{n-1}} d(M, g_0)^2 |\kappa|}\right\}$$

where $c = 2(n-1) \left(\frac{\min_M R_{g_0}}{\max_M R_{g_0}} - 1 \right)$.

Proof. As in the proof of Theorem 1.1, we can show that $g \rightarrow g_\infty = g_E$ as $t \rightarrow \infty$ under the Yamabe flow (2.1). We claim that

$$(2.57) \quad e^{-\frac{c}{2(n-1)}} d(M, g_E) \leq d(M, g_0) \leq e^{\frac{c}{2(n-1)}} d(M, g_E),$$

where $c = 2(n-1) \left(\frac{\min_M R_{g_0}}{\max_M R_{g_0}} - 1 \right)$ as in Theorem 1.1, $d(M, g_0)$ and $d(M, g_E)$ are the diameter of M with respect to the initial metric g_0 and the Einstein metric g_E respectively. To see this, we let $\gamma : [s_0, s_1] \rightarrow M$ be a differentiable curve joining x and y in M . Consider the solution g of the Yamabe flow (2.1) with g_0 as the initial metric. Then the length of γ with respect to the metric g is given by

$$L_g(\gamma) = \int_{s_0}^{s_1} \sqrt{g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)} ds.$$

Differentiate it with respect t , we obtain

$$(2.58) \quad \begin{aligned} \frac{dL_g(\gamma)}{dt} &= \frac{d}{dt} \left(\int_{s_0}^{s_1} u^{\frac{2}{n-2}} \sqrt{g_0\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)} ds \right) \\ &= \int_{s_0}^{s_1} \frac{2}{n-2} u^{\frac{2}{n-2}-1} \frac{\partial u}{\partial t} \sqrt{g_0\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)} ds \\ &= \frac{1}{2} \int_{s_0}^{s_1} (\bar{R}_g - R_g) \sqrt{g\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)} ds \\ &\leq \frac{1}{2} \left(\bar{R}_g - \min_M R_g \right) L_g(\gamma) \end{aligned}$$

where we have used (2.3) and the fact that $g = u^{\frac{4}{n-2}} g_0$. Similarly, we can get

$$(2.59) \quad \frac{dL_g(\gamma)}{dt} \geq \frac{1}{2} \left(\bar{R}_g - \max_M R_g \right) L_g(\gamma).$$

Integrating (2.58) and (2.59) from 0 to ∞ , we obtain

$$\frac{1}{2} \int_0^\infty \left(\bar{R}_{g(t)} - \max_M R_{g(t)} \right) dt \leq \log \frac{L_{g_E}(\gamma)}{L_{g_0}(\gamma)} \leq \frac{1}{2} \int_0^\infty \left(\bar{R}_{g(t)} - \min_M R_{g(t)} \right) dt.$$

Combining this with (2.55) and (2.56), we obtain

$$\left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right) \leq \log \frac{L_{g_E}(\gamma)}{L_{g_0}(\gamma)} \leq -\left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right).$$

This implies

$$\left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right) \leq \log \frac{d(M, g_E)}{d(M, g_0)} \leq -\left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right)$$

which proves the claim (2.57).

By the assumption and the result of Li and Yau mentioned above, we have

$$\lambda_1(g_E) \geq \frac{1}{(n-1)d(M, g_E)^2} \exp \left\{ -1 - \sqrt{1 + 4(n-1)^2 d(M, g_\infty)^2 |\kappa|} \right\}.$$

Combining this with (2.57) and Theorem 1.1, we obtain

$$e^c \lambda_1(g_0) \geq \frac{1}{(n-1)e^{\frac{c}{n-1}} d(M, g_0)^2} \exp \left\{ -1 - \sqrt{1 + 4(n-1)^2 e^{\frac{c}{n-1}} d(M, g_0)^2 |\kappa|} \right\}.$$

This proves the assertion. \square

3. THE UNNORMALIZED YAMABE FLOW ON MANIFOLDS WITHOUT BOUNDARY

Now we consider the unnormalized Yamabe flow on an n -dimensional compact Riemannian manifold (M, g_0) without boundary, which is defined as

$$(3.1) \quad \frac{\partial}{\partial t} g = -R_g g \text{ for } t \geq 0, \quad g|_{t=0} = g_0.$$

If we write $g = u^{\frac{4}{n-2}} g_0$ for some $0 < u \in C^\infty(M)$, then u satisfies the following evolution equation:

$$(3.2) \quad \frac{\partial u}{\partial t} = -\frac{n-2}{4} R_g u \text{ for } t \geq 0, \quad u|_{t=0} = 1.$$

Hence, the volume form dV_g and the scalar curvature R_g of g satisfy (see [15])

$$(3.3) \quad \frac{\partial}{\partial t} dV_g = -\frac{n}{2} R_g dV_g,$$

$$(3.4) \quad \frac{\partial}{\partial t} R_g = (n-1) \Delta_g R_g + R_g^2.$$

Let λ_1 be the first eigenvalue of $-\Delta_g + aR_g$ where a is a constant, i.e.

$$(3.5) \quad -\Delta_g f + aR_g f = \lambda_1 f$$

for some function f . By the eigenvalue perturbation theory (see [44] and also [8, 31, 33]), we may assume that there is a family of the first eigenvalue and the corresponding eigenfunction which is C^1 in t . By rescaling, we may assume that the eigenfunction f satisfies

$$(3.6) \quad \int_M f^2 dV_g = 1.$$

Proposition 3.1. *Along the unnormalized Yamabe flow (3.1), we have*

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \left(2(n-1)a - \frac{n-2}{2}\right) \int_M R_g (|\nabla_g f|_g^2 + aR_g f^2) dV_g \\ &\quad - \left(2(n-1)a - \frac{n}{2}\right) \lambda_1 \int_M R_g f^2 dV_g. \end{aligned}$$

Proof. Differentiate (3.6) with respect to t , we have

$$(3.7) \quad \int_M f \frac{\partial f}{\partial t} dV_g = \frac{n}{4} \int_M R_g f^2 dV_g$$

by (3.3). Multiply (3.5) by f and integrate it over M , we obtain

$$(3.8) \quad \lambda_1 = \int_M (|\nabla_g f|_g^2 + a R_g f^2) dV_g$$

by (3.6) and integration by parts. Since $g = u^{\frac{4}{n-2}} g_0$, (2.7) holds. Combining (2.7) and (3.8), we have

$$\lambda_1 = \int_M u^2 |\nabla_{g_0} f|_{g_0}^2 dV_{g_0} + a \int_M R_g f^2 dV_g.$$

Differentiate it with respect to t , we obtain

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \int_M 2u \frac{\partial u}{\partial t} |\nabla_{g_0} f|_{g_0}^2 dV_{g_0} + 2 \int_M u^2 \langle \nabla_{g_0} f, \nabla_{g_0} \left(\frac{\partial f}{\partial t} \right) \rangle_{g_0} dV_{g_0} \\ &\quad + a \int_M f^2 \frac{\partial R_g}{\partial t} dV_g + 2a \int_M R_g f \frac{\partial f}{\partial t} dV_g + a \int_M R_g f^2 \frac{\partial}{\partial t} (dV_g) \\ &= -\frac{n-2}{2} \int_M R_g |\nabla_g f|_g^2 dV_g + 2 \int_M \langle \nabla_g f, \nabla_g \left(\frac{\partial f}{\partial t} \right) \rangle_g dV_g \\ &\quad + a \int_M f^2 ((n-1)\Delta_g R_g + R_g^2) dV_g + 2a \int_M R_g f \frac{\partial f}{\partial t} dV_g - \frac{n}{2} a \int_M R_g^2 f^2 dV_g \\ &= -\frac{n-2}{2} \int_M R_g |\nabla_g f|_g^2 dV_g + 2 \int_M \frac{\partial f}{\partial t} (-\Delta_g f + a R_g f) dV_g \\ &\quad + (n-1)a \int_M R_g \Delta_g (f^2) dV_g - \frac{n-2}{2} a \int_M R_g^2 f^2 dV_g \\ &= \left(2(n-1)a - \frac{n-2}{2} \right) \int_M R_g |\nabla_g f|_g^2 dV_g + 2\lambda_1 \int_M f \frac{\partial f}{\partial t} dV_g \\ &\quad - 2(n-1)a \int_M R_g f (-\Delta_g f + a R_g f) dV_g \\ &\quad + \left(2(n-1)a^2 - \frac{n-2}{2} a \right) \int_M R_g^2 f^2 dV_g \\ &= \left(2(n-1)a - \frac{n-2}{2} \right) \int_M R_g (|\nabla_g f|_g^2 + a R_g f^2) dV_g \\ &\quad - \left(2(n-1)a - \frac{n}{2} \right) \lambda_1 \int_M R_g f^2 dV_g \end{aligned}$$

where the second equality follows from (2.7) and (3.2)-(3.4), the third equality follows from integration by parts, and the last two equalities follow from (3.5) and (3.7). This proves the assertion. \square

Proposition 3.2. *If $0 \leq a \leq \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq \frac{n-2}{n} \max_M R_g \geq 0$, then*

$\frac{d\lambda_1}{dt} \geq 0$, and equality holds if and only if R_g is constant.

Proof. If $a \leq \frac{n-2}{4(n-1)}$, then $\frac{n}{2} - 2(n-1)a \geq \frac{n-2}{2} - 2(n-1)a \geq 0$. Combining this with Proposition 3.1, we obtain

$$\begin{aligned} \frac{d\lambda_1}{dt} &\geq -\left(\frac{n-2}{2} - 2(n-1)a\right) \left(\max_M R_g\right) \int_M (|\nabla_g f|_g^2 + aR_g f^2) dV_g \\ &\quad + \left(\frac{n}{2} - 2(n-1)a\right) \left(\min_M R_g\right) \lambda_1 \int_M f^2 dV_g \\ &= \lambda_1 \left(\frac{n}{2} \min_M R_g - \frac{n-2}{2} \max_M R_g\right) \end{aligned}$$

where we have used (3.6) and (3.8) in the last equality. From this, the assertion follows. \square

Proposition 3.3. *If $\frac{n-2}{4(n-1)} \leq a \leq \frac{n}{4(n-1)}$ and $\min_M R_g \geq 0$, then $\frac{d\lambda_1}{dt} \geq 0$ and equality holds if and only if $R_g \equiv 0$.*

Proof. If $\frac{n-2}{4(n-1)} \leq a \leq \frac{n}{4(n-1)}$, then $2(n-1)a - \frac{n-2}{2} \geq 0$ and $\frac{n}{2} - 2(n-1)a \geq 0$. Combining this with Proposition 3.1, we get

$$\begin{aligned} \frac{d\lambda_1}{dt} &\geq \left(2(n-1)a - \frac{n-2}{2}\right) \left(\min_M R_g\right) \int_M (|\nabla_g f|_g^2 + aR_g f^2) dV_g \\ &\quad + \left(\frac{n}{2} - 2(n-1)a\right) \left(\min_M R_g\right) \lambda_1 \int_M f^2 dV_g \\ &= \lambda_1 \left(\min_M R_g\right) \end{aligned}$$

where we have used (3.6) and (3.8). From this, the assertion follows. \square

Proposition 3.4. *If $a \geq \frac{n}{4(n-1)}$ and $\min_M R_g \geq 0$, then $\frac{d\lambda_1}{dt} \geq 0$ and equality holds if and only if $R_g \equiv 0$.*

Proof. If $a \geq \frac{n}{4(n-1)}$, then $2(n-1)a - \frac{n-2}{2} \geq 2(n-1)a - \frac{n}{2} \geq 0$. Combining this with Proposition 3.1, we obtain

$$\begin{aligned} \frac{d\lambda_1}{dt} &\geq \left(2(n-1)a - \frac{n-2}{2}\right) \left(\min_M R_g\right) \int_M (|\nabla_g f|_g^2 + aR_g f^2) dV_g \\ &\quad - \left(2(n-1)a - \frac{n}{2}\right) \left(\max_M R_g\right) \lambda_1 \int_M f^2 dV_g \\ &= \lambda_1 \left(\frac{n}{2} \max_M R_g - \frac{n-2}{2} \min_M R_g\right) \end{aligned}$$

where the last equality follows from (3.6) and (3.8). From this, the assertion follows. \square

Proof of Theorem 1.3. This follows from Proposition 3.2-3.4. \square

4. THE YAMABE FLOW ON MANIFOLDS WITH BOUNDARY

Throughout this section, we assume that (M, g_0) is a compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M . Up to a conformal change, we may assume that the mean curvature of g_0 on ∂M vanishes. See Lemma 2.1 in [3] for the proof. We consider the Yamabe flow, which is defined as

$$(4.1) \quad \frac{\partial}{\partial t} g = -(R_g - \bar{R}_g)g \text{ in } M \text{ and } H_g = 0 \text{ on } \partial M \text{ for } t \geq 0, \quad g|_{t=0} = g_0.$$

Here H_g is the mean curvature of g with respect to the outward unit normal ν_g . If we write $g = u^{\frac{4}{n-2}} g_0$, then u satisfies

$$(4.2) \quad \frac{\partial u}{\partial t} = -\frac{n-2}{4}(R_g - \bar{R}_g)u \text{ in } M \text{ for } t \geq 0.$$

Under the Yamabe flow (4.1), the volume form dV_g and the scalar curvature R_g of g satisfy (see [3])

$$(4.3) \quad \frac{\partial}{\partial t}(dV_g) = -\frac{n}{2}(R_g - \bar{R}_g)dV_g,$$

$$(4.4) \quad \frac{\partial}{\partial t} R_g = (n-1)\Delta_g R_g - R_g(\bar{R}_g - R_g).$$

We have the following:

Lemma 4.1. *Let $g = g(t)$ be the solution of the Yamabe flow (4.1) and $\lambda_1(t)$ be the corresponding first nonzero Dirichlet eigenvalue of the Laplacian. Then for any $t_2 \geq t_1$, there exists a C^∞ function f on $M \times [t_1, t_2]$ satisfying*

$$(4.5) \quad f = 0 \text{ on } \partial M \quad \text{and} \quad \int_M f^2 dV_g = 1 \text{ for all } t,$$

and

$$\lambda_1(t_2) \geq \lambda_1(t_1) - \frac{n-2}{2} \int_{t_1}^{t_2} \int_M (R_g - \bar{R}_g) |\nabla_g f|_g^2 dV_g dt - 2 \int_{t_1}^{t_2} \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g dt$$

such that at time t_2 , $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$.

Proof. As in the proof of Lemma 2.1, we choose $f_2 = f(t_2)$ to be the eigenfunction for the first nonzero Dirichlet eigenvalue $\lambda_1(t_2)$ of $g(t_2)$. Then f_2 satisfies the Dirichlet boundary condition, i.e. $f_2 = 0$ on ∂M . Thus if we define

$$h(t) = \left(\frac{u(t_2)}{u(t)} \right)^{\frac{2n}{n-2}} f_2$$

where $u(t)$ is the solution of (4.2), then the normalized function

$$f(t) = \frac{h(t)}{\left(\int_M h(t)^2 dV_{g(t)} \right)^{\frac{1}{2}}}$$

satisfies (4.5). Now we can follow the same proof of Lemma 2.1 to finish the proof, except we have to use the fact that $f = 0$ on ∂M when we do the integration by parts in the last equality in (2.8). This proves the assertion. \square

Once Lemma 4.1 is proved, we can follow the same proof of Proposition 2.2 to prove the following:

Proposition 4.2. *The first nonzero Dirichlet eigenvalue λ_1 of g along the Yamabe flow (4.1) satisfies*

$$\frac{d}{dt} \log \lambda_1 \geq -\frac{n-2}{2} \left(\max_M R_g - \overline{R}_g \right) + \frac{n}{2} \left(\min_M R_g - \overline{R}_g \right).$$

Here the derivative on the left hand side is in the sense of the lim inf of backward difference quotients.

Similar to the case when the manifold has no boundary, we can prove the corresponding version of Lemma 4.1. Then we can follow the same proof of Proposition 2.4 to prove the following:

Proposition 4.3. *The first nonzero Dirichlet eigenvalue λ_1 of g along the Yamabe flow (2.1) satisfies*

$$\frac{d}{dt} \log \lambda_1 \leq -\frac{n-2}{2} \left(\min_M R_g - \overline{R}_g \right) + \frac{n}{2} \left(\max_M R_g - \overline{R}_g \right).$$

Here the derivative on the left hand side is in the sense of the lim sup of forward difference quotients.

On the other hand, one can apply the maximum principle to prove Proposition 2.5-2.7 and Lemma 2.8-2.9 for the Yamabe flow (4.1) when the manifold has boundary.

Therefore, if (M, g_0) is a compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary such that $\max_M R_{g_0} < 0$, and g_Y is the Yamabe metric in the conformal class of g_0 , i.e. g_Y is the Riemannian metric conformal to g_0 such that its scalar curvature is constant in M and its mean curvature is zero on ∂M , then we have the following:

Theorem 4.4. *Suppose (M, g_0) is a compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M which has negative scalar curvature in M and vanishing mean curvature on ∂M , and g_Y is the Yamabe metric conformal to g_0 which has same volume as g_0 . Then the first nonzero Dirichlet eigenvalue of g_0 and g_Y satisfy*

$$e^{-c} \lambda_1(g_Y) \leq \lambda_1(g_0) \leq e^c \lambda_1(g_Y)$$

where c is the constant in Theorem 1.1.

Proof. We only sketch the proof since it is essentially the same as the proof of Theorem 1.1. Brendle has proved in [3] that $g \rightarrow g_\infty$ as $t \rightarrow \infty$ under the Yamabe flow (4.1) such that g_∞ has constant scalar curvature in M and vanishing mean curvature on ∂M (c.f. Theorem 1.1 in [3]). As in the proof of Theorem 1.1, we can prove that $g_\infty = g_Y$ by using the result of Escobar (see Corollary in [17]), which says that if g_1 and g_2 are two metrics conformal to g_0 such that $R_{g_1} = R_{g_2} < 0$ in M and $H_{g_1} = H_{g_2} = 0$ on ∂M , then $g_1 = g_2$. On the other hand, we can follow the same arguments as in the case without boundary to get (2.55) and (2.56). The remaining arguments are the same as the proof of Theorem 1.1. This proves the assertion. \square

One can apply Theorem 4.4 to obtain estimate of the first nonzero Dirichlet eigenvalue. In [38], Ling proved the following: Let (M, g_0) be an n -dimensional compact Riemannian manifold with boundary. Suppose that the boundary ∂M has nonnegative mean curvature with respect to the outward normal and that the

Ricci curvature of M has lower bound $Ric(M) \geq (n-1)\kappa$. Then the first nonzero Dirichlet eigenvalue λ_1 of the Laplacian of M satisfies

$$\lambda_1 \geq \frac{1}{2}(n-1)\kappa + \frac{\pi^2}{d^2},$$

where d is the diameter of the largest interior ball in M .

Theorem 4.5. *Suppose M is an n -dimensional compact manifold with smooth boundary ∂M , and g_E is an Einstein metric on M with $Ric_{g_E} = (n-1)\kappa g_E$ where $\kappa < 0$ and vanishing mean curvature on ∂M . If g_0 is a Riemannian metric conformal to g_E which has negative scalar curvature in M , vanishing mean curvature on ∂M , and same volume as g_E , then the first nonzero Dirichlet eigenvalue of (M, g_0) satisfies*

$$\lambda_1(g_0) \geq \frac{1}{2e^c}(n-1)\kappa + \frac{\pi^2}{e^{\frac{nc}{n-1}}d(M, g_0)^2}$$

where $c = 2(n-1) \left(\frac{\min_M R_{g_0}}{\max_M R_{g_0}} - 1 \right)$ and $d(M, g_0)$ is the diameter of the largest interior ball in M .

Proof. As in the proof of Theorem 2.10, we can prove that

$$(4.6) \quad e^{-\frac{c}{2(n-1)}}d(M, g_E) \leq d(M, g_0) \leq e^{\frac{c}{2(n-1)}}d(M, g_E),$$

where $c = 2(n-1) \left(1 - \frac{\min_M R_{g_0}}{\max_M R_{g_0}} \right)$, and $d(M, g_0)$ and $d(M, g_E)$ are respectively the diameter of the largest interior ball in M with respect to the initial metric g_0 and the Einstein metric g_E . On the other hand, by the assumptions and the above result of Ling, we have

$$\lambda_1(g_E) \geq \frac{1}{2}(n-1)\kappa + \frac{\pi^2}{d(M, g_E)^2}.$$

Combining this with (4.6) and Theorem 4.4, we obtain

$$e^c \lambda_1(g_0) \geq \frac{1}{2}(n-1)\kappa + \frac{\pi^2}{e^{\frac{c}{n-1}}d(M, g_0)^2}.$$

This proves the assertion. \square

5. THE UNNORMALIZED YAMABE FLOW ON MANIFOLDS WITH BOUNDARY

In this section, we study the unnormalized Yamabe flow on a compact Riemannian manifold (M, g_0) with smooth boundary ∂M , which is defined as

$$(5.1) \quad \frac{\partial}{\partial t}g = -R_g g \text{ in } M \text{ and } H_g = 0 \text{ on } \partial M \text{ for } t \geq 0, \quad g|_{t=0} = g_0.$$

If we write $g = u^{\frac{4}{n-2}}g_0$, then

$$(5.2) \quad \frac{\partial u}{\partial t} = -\frac{n-2}{4}R_g u \text{ in } M \text{ for } t \geq 0.$$

Note that the volume form dV_g and the scalar curvature R_g of g satisfy

$$(5.3) \quad \frac{\partial}{\partial t}(dV_g) = -\frac{n}{2}R_g dV_g,$$

$$(5.4) \quad \frac{\partial}{\partial t}R_g = (n-1)\Delta_g R_g + R_g^2.$$

along the unnormalized Yamabe flow (5.1).

Let λ_1 be the first eigenvalue of $-\Delta_g + aR_g$ with Dirichlet boundary condition, i.e.

$$(5.5) \quad -\Delta_g f + aR_g f = \lambda_1 f \text{ in } M \quad \text{and } f = 0 \text{ on } \partial M$$

for some function f . Again we assume that there is a family of the first eigenvalue and the corresponding eigenfunction which is C^1 in t . By rescaling, we may assume that the eigenfunction f satisfies

$$(5.6) \quad \int_M f^2 dV_g = 1.$$

Proposition 5.1. *Along the unnormalized Yamabe flow (5.1), we have*

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \left(2(n-1)a - \frac{n-2}{2}\right) \int_M R_g (|\nabla_g f|_g^2 + aR_g f^2) dV_g \\ &\quad - \left(2(n-1)a - \frac{n}{2}\right) \lambda_1 \int_M R_g f^2 dV_g. \end{aligned}$$

Proof. The proof is almost identical to the proof of Proposition 3.1 except we have to take care of the boundary term when we integrate by parts. More precisely, we multiply the first equation in (5.5) by f and integrate it over M , we obtain

$$(5.7) \quad \begin{aligned} \lambda_1 &= \int_M (-f \Delta_g f + aR_g f^2) dV_g = \int_M (|\nabla_g f|_g^2 + aR_g f^2) dV_g - \int_{\partial M} f \frac{\partial f}{\partial \nu_g} dV_g \\ &= \int_M (|\nabla_g f|_g^2 + aR_g f^2) dV_g \end{aligned}$$

by (5.5). On the other hand, we have

$$(5.8) \quad \int_M \langle \nabla_g f, \nabla_g \left(\frac{\partial f}{\partial t}\right) \rangle_g dV_g = - \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g + \int_{\partial M} \frac{\partial f}{\partial t} \frac{\partial f}{\partial \nu_g} dV_g = - \int_M \frac{\partial f}{\partial t} \Delta_g f dV_g$$

since $\frac{\partial f}{\partial t} = 0$ on ∂M by (5.5), and

$$(5.9) \quad \begin{aligned} \int_M f^2 \Delta_g R_g dV_g - \int_M R_g \Delta_g (f^2) dV_g &= \int_{\partial M} f^2 \frac{\partial R_g}{\partial \nu_g} dV_g - \int_{\partial M} R_g \frac{\partial}{\partial \nu_g} (f^2) dV_g \\ &= \int_{\partial M} f^2 \frac{\partial R_g}{\partial \nu_g} dV_g - 2 \int_{\partial M} R_g f \frac{\partial f}{\partial \nu_g} dV_g = 0 \end{aligned}$$

by (5.5). Except these, all the other steps are the same as the proof of Proposition 3.1. This proves the assertion. \square

Now, by the same proof of Proposition 3.2-3.4, we have the following:

Theorem 5.2. *Along the unnormalized Yamabe flow (5.1), the first eigenvalue of $-\Delta_g + aR_g$ with Dirichlet boundary condition is nondecreasing*

(i) if $0 \leq a < \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq \frac{n-2}{n} \max_M R_g \geq 0$;

(ii) if $a \geq \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq 0$.

5.1. Neumann boundary condition. Let μ_1 be the first eigenvalue of $-\Delta_g + aR_g$ with Neumann boundary condition, i.e.

$$(5.10) \quad -\Delta_g f + aR_g f = \mu_1 f \text{ in } M \quad \text{and} \quad \frac{\partial f}{\partial \nu_g} = 0 \text{ on } \partial M.$$

Again we may assume that there is a family of the first eigenvalue and the corresponding eigenfunction which is C^1 in t . By rescaling, we may further assume that the eigenfunction f satisfies

$$\int_M f^2 dV_g = 1.$$

Note that (5.7)-(5.9) are still true thanks to (5.10). Thus, following the proof of Proposition 5.1, we get

$$\begin{aligned} \frac{d\mu_1}{dt} &= \left(2(n-1)a - \frac{n-2}{2}\right) \int_M R_g (|\nabla_g f|_g^2 + aR_g f^2) dV_g \\ &\quad - \left(2(n-1)a - \frac{n}{2}\right) \mu_1 \int_M R_g f^2 dV_g. \end{aligned}$$

along the unnormalized Yamabe flow (5.1). As for the Dirichlet boundary condition, we have the following:

Theorem 5.3. *Along the unnormalized Yamabe flow (5.1), the first eigenvalue of $-\Delta_g + aR_g$ with Neumann boundary condition is nondecreasing*

- (i) if $0 \leq a < \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq \frac{n-2}{n} \max_M R_g \geq 0$;
- (ii) if $a \geq \frac{n-2}{4(n-1)}$ and $\min_M R_g \geq 0$.

6. THE CR YAMABE FLOW

Throughout this section, we suppose that (M, θ_0) is a compact strictly pseudoconvex CR manifold of real dimension $2n+1$. We consider the CR Yamabe flow, which is defined as

$$(6.1) \quad \frac{\partial}{\partial t} \theta = -(R_\theta - \bar{R}_\theta) \theta \text{ for } t \geq 0, \quad \theta|_{t=0} = \theta_0.$$

Here R_θ is the Webster scalar curvature of the contact form θ , and \bar{R}_θ is the average of the Webster scalar curvature given by

$$(6.2) \quad \bar{R}_\theta = \frac{\int_M R_\theta dV_\theta}{\int_M dV_\theta},$$

where $dV_\theta = \theta \wedge (d\theta)^n$ is the volume form of θ . Since the CR Yamabe flow preserves the conformal structure, we can write $\theta = u^{\frac{2}{n}} \theta_0$ for some positive function u , and u satisfies the following evolution equation:

$$(6.3) \quad \frac{\partial u}{\partial t} = -\frac{n}{2} (R_\theta - \bar{R}_\theta) u \text{ for } t \geq 0, \quad u|_{t=0} = 1.$$

Hence, the volume form dV_θ of θ satisfies

$$(6.4) \quad \frac{\partial}{\partial t} (dV_\theta) = \frac{\partial}{\partial t} (u^{\frac{2n+2}{n}} dV_{\theta_0}) = \frac{2n+2}{n} u^{\frac{2n+2}{n}-1} \frac{\partial u}{\partial t} dV_{\theta_0} = -(n+1)(R_\theta - \bar{R}_\theta) dV_\theta.$$

On the other hand, the Webster scalar curvature R_θ of θ satisfies the following evolution equation: (see Proposition 3.2 in [25] or Lemma 2.4 in [24])

$$(6.5) \quad \frac{\partial}{\partial t} R_\theta = (n+1)\Delta_\theta R_\theta - R_\theta(\overline{R}_\theta - R_\theta).$$

Here Δ_θ is the sub-Laplacian of the contact form θ .

We have the following lemma, which is again inspired by the Proposition 3.1 in [49].

Lemma 6.1. *Let $\theta = \theta(t)$ be the solution of the the CR Yamabe flow (6.1) and $\lambda_1(t)$ be the corresponding first nonzero eigenvalue of the sub-Laplacian. Then for any $t_2 \geq t_1$, there exists a C^∞ function f on $M \times [t_1, t_2]$ satisfying*

$$(6.6) \quad \int_M f^2 dV_\theta = 1 \quad \text{and} \quad \int_M f dV_\theta = 0 \quad \text{for all } t,$$

and

$$\lambda_1(t_2) \geq \lambda_1(t_1) - n \int_{t_1}^{t_2} \int_M (R_\theta - \overline{R}_\theta) |\nabla_\theta f|_\theta^2 dV_\theta dt - 2 \int_{t_1}^{t_2} \int_M \frac{\partial f}{\partial t} \Delta_\theta f dV_\theta dt$$

such that at time t_2 , $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$.

Proof. At time t_2 , we let $f_2 = f(t_2)$ be the eigenfunction for the first nonzero eigenvalue $\lambda_1(t_2)$ of $\theta(t_2)$. We define the following smooth function on M :

$$h(t) = \left(\frac{u(t_2)}{u(t)} \right)^{2 + \frac{2}{n}} f_2$$

where $u(t)$ is the solution of (6.3). We normalize this smooth function on M by

$$f(t) = \frac{h(t)}{\left(\int_M h(t)^2 dV_{\theta(t)} \right)^{\frac{1}{2}}}.$$

Then we can easily check that $f(t)$ satisfies (6.6).

Set

$$G(\theta(t), f(t)) := \int_M |\nabla_{\theta(t)} f(t)|_{\theta(t)}^2 dV_{\theta(t)}.$$

Note that $G(\theta(t), f(t))$ is a smooth function in t . Since $\theta = u^{\frac{2}{n}} \theta_0$, we have

$$(6.7) \quad \langle \nabla_\theta f_1, \nabla_\theta f_2 \rangle_\theta = u^{-\frac{2}{n}} \langle \nabla_{\theta_0} f_1, \nabla_{\theta_0} f_2 \rangle_{\theta_0}$$

for any functions f_1, f_2 in M , which implies that

$$G(g(t), f(t)) = \int_M u^2 |\nabla_{\theta_0} f|_{\theta_0}^2 dV_{\theta_0}.$$

Differentiating it with respect to t , we get

$$(6.8) \quad \begin{aligned} \mathcal{G}(\theta(t), f(t)) &:= \frac{d}{dt} G(\theta(t), f(t)) \\ &= \int_M 2u \frac{\partial u}{\partial t} |\nabla_{\theta_0} f|_{\theta_0}^2 dV_{\theta_0} + 2 \int_M u^2 \langle \nabla_{\theta_0} f, \nabla_{\theta_0} \left(\frac{\partial f}{\partial t} \right) \rangle_{\theta_0} dV_{\theta_0} \\ &= -n \int_M (R_\theta - \overline{R}_\theta) |\nabla_\theta f|_\theta^2 dV_\theta - 2 \int_M \frac{\partial f}{\partial t} \Delta_\theta f dV_\theta \end{aligned}$$

where we have used integration by parts, (6.3) and (6.7). It follows from the definition of $\mathcal{G}(\theta(t), f(t))$ in (6.8) that

$$(6.9) \quad G(\theta(t_2), f(t_2)) - G(\theta(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{G}(\theta(t), f(t)) dt.$$

Since $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$, we have

$$(6.10) \quad G(\theta(t_2), f(t_2)) = \lambda_1(t_2) \int_M f(t_2)^2 dV_{\theta(t_2)} = \lambda_1(t_2)$$

by (6.6). On the other hand, it follows from (6.6) and the definition of $G(\theta(t), f(t))$ that

$$(6.11) \quad G(\theta(t_1), f(t_1)) \geq \lambda_1(t_1) \int_{\partial M} f(t_1)^2 dV_{\theta(t_1)} = \lambda_1(t_1).$$

Now the result follows from (6.8)-(6.11). \square

Hence, we have the following:

Proposition 6.2. *The first nonzero eigenvalue λ_1 of the sub-Laplacian along the CR Yamabe flow (6.1) satisfies*

$$\frac{d}{dt} \log \lambda_1 \geq -n \left(\max_M R_\theta - \bar{R}_\theta \right) + (n+1) \left(\min_M R_\theta - \bar{R}_\theta \right).$$

Here the derivative on the left hand side is in the sense of the lim inf of backward difference quotients.

Proof. Differentiate the first equation in (6.6) with respect to t , we have

$$(6.12) \quad \int_M f \frac{\partial f}{\partial t} dV_\theta = \frac{n+1}{2} \int_M f^2 (R_\theta - \bar{R}_\theta) dV_\theta$$

by (6.4). On the other hand, since $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$, we have

$$(6.13) \quad - \int_M \frac{\partial f(t_2)}{\partial t} \Delta_{\theta(t_2)} f(t_2) dV_{\theta(t_2)} = \lambda_1(t_2) \int_M f(t_2) \frac{\partial f(t_2)}{\partial t} dV_{\theta(t_2)}.$$

Combining (6.12) and (6.13), we get

$$\begin{aligned} - \int_M \frac{\partial f(t_2)}{\partial t} \Delta_{\theta(t_2)} f(t_2) dV_{\theta(t_2)} &= \frac{n+1}{2} \lambda_1(t_2) \int_M f(t_2)^2 (R_{\theta(t_2)} - \bar{R}_{\theta(t_2)}) dV_{\theta(t_2)} \\ &\geq \frac{n+1}{2} \lambda_1(t_2) \left(\min_M R_{\theta(t_2)} - \bar{R}_{\theta(t_2)} \right), \end{aligned}$$

where we have used (6.6) in the last inequality. Since $\lambda_1(t_2) > 0$, we have for any $\epsilon > 0$

$$- \int_M \frac{\partial f(t_2)}{\partial t} \Delta_{\theta(t_2)} f(t_2) dV_{\theta(t_2)} > \frac{n+1}{2} \lambda_1(t_2) \left(\min_M R_{\theta(t_2)} - \bar{R}_{\theta(t_2)} - \epsilon \right).$$

Hence, by continuity, we can conclude that for any $\epsilon > 0$

$$(6.14) \quad - \int_M \frac{\partial f}{\partial t} \Delta_\theta f(t_2) dV_\theta \geq \frac{n+1}{2} \lambda_1(t_2) \left(\min_M R_\theta - \bar{R}_\theta - \epsilon \right)$$

when t is sufficiently closed to t_2 . On the other hand,

$$\begin{aligned} & \int_M (R_{\theta(t_2)} - \overline{R}_{\theta(t_2)}) |\nabla_{\theta(t_2)} f|_{\theta(t_2)}^2 dV_{\theta(t_2)} \\ & \geq - \left(\max_M R_{\theta(t_2)} - \overline{R}_{\theta(t_2)} \right) \int_M |\nabla_{\theta(t_2)} f|_{\theta(t_2)}^2 dV_{\theta(t_2)} = -\lambda_1(t_2) \left(\max_M R_{\theta(t_2)} - \overline{R}_{\theta(t_2)} \right) \end{aligned}$$

by (6.6) and the fact that $f(t_2)$ is the corresponding eigenfunction of $\lambda_1(t_2)$. Since $\lambda_1(t_2) > 0$, we have for any $\epsilon > 0$

$$\int_M (R_{\theta(t_2)} - \overline{R}_{\theta(t_2)}) |\nabla_{\theta(t_2)} f|_{\theta(t_2)}^2 dV_{\theta(t_2)} > -\lambda_1(t_2) \left(\max_M R_{\theta(t_2)} - \overline{R}_{\theta(t_2)} + \epsilon \right).$$

Hence, by continuity, we can conclude that for any $\epsilon > 0$

$$(6.15) \quad \int_M (R_{\theta} - \overline{R}_{\theta}) |\nabla_{\theta} f|_{\theta}^2 dV_{\theta} \geq -\lambda_1(t_2) \left(\max_M R_{\theta} - \overline{R}_{\theta} + \epsilon \right)$$

when t is sufficiently closed to t_2 . Substituting (6.14) and (6.15) into the inequality in Lemma 6.1, we obtain

$$\begin{aligned} & \lambda_1(t_2) - \lambda_1(t_1) \\ & \geq -n\lambda_1(t_2) \int_{t_1}^{t_2} \left(\max_M R_{\theta} - \overline{R}_{\theta} + \epsilon \right) dt + (n+1)\lambda_1(t_2) \int_{t_1}^{t_2} \left(\min_M R_{\theta} - \overline{R}_{\theta} - \epsilon \right) dt \end{aligned}$$

for $t_1 < t_2$ and t_1 sufficiently closed to t_2 . Now the assertion follows from dividing the last inequality by $t_2 - t_1$, letting t_1 go to t_2 , and then letting $\epsilon \rightarrow 0$, as we have done in the proof of Proposition 2.2. \square

Similar to Proposition 6.2, we can prove the following:

Proposition 6.3. *The first nonzero eigenvalue λ_1 of the sub-Laplacian along the CR Yamabe flow (6.1) satisfies*

$$\frac{d}{dt} \log \lambda_1 \leq -n \left(\min_M R_{\theta} - \overline{R}_{\theta} \right) + (n+1) \left(\max_M R_{\theta} - \overline{R}_{\theta} \right).$$

Here the derivative on the left hand side is in the sense of the lim sup of forward difference quotients.

To prove Proposition 6.3, we first prove the corresponding version of Lemma 2.3 for the CR Yamabe flow. Then we can prove Proposition 6.3 as we have done in the proof of Proposition 2.4. We omit the proof and leave it to the readers.

We will omit the proof of the following proposition, since the idea is the same as the proof of Proposition 2.5-2.7. More precisely, one can use the fact that $t \mapsto \overline{R}_{\theta}$ is nonincreasing along the CR Yamabe flow (c.f. see Proposition 3.3 in [25]) and apply the maximum principle to (6.5) to prove the following:

Proposition 6.4. *If $\max_M R_{\theta_0} < 0$, then*

$$\min_M R_{\theta_0} \leq \min_M R_{\theta} \leq \max_M R_{\theta} \leq \max_M R_{\theta_0} < 0 \text{ for all } t \geq 0$$

under the CR Yamabe flow (6.1).

We will also omit the proof of the following lemma, which is to apply the maximum principle to (6.5) and is the same as the proof of Lemma 2.8 and Lemma 2.9.

Lemma 6.5. *If $\max_M R_{\theta_0} < 0$, then*

$$R_{\theta(t)} \leq \overline{R}_{\theta_0} + \left(\max_M R_{\theta_0} - \min_M R_{\theta_0} \right) + \left(\max_M R_{\theta_0} \right) \int_0^t \left(\max_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds.$$

and

$$R_{\theta(t)} \geq \overline{R}_{\theta_0} - \left(\max_M R_{\theta_0} - \min_M R_{\theta_0} \right) + \left(\max_M R_{\theta_0} \right) \int_0^t \left(\min_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds.$$

for all $t \geq 0$ under the CR Yamabe flow (6.1)

Given a contact form θ , we let $\lambda_1(\theta)$ be the first nonzero eigenvalue of the sub-Laplacian of θ . We have the following:

Theorem 6.6. *Suppose (M, θ_0) is a compact strictly pseudoconvex manifold of real dimension $2n + 1$ such that $\max_M R_{\theta_0} < 0$, and θ_Y is the contact form conformal to θ_0 such that its Webster scalar curvature is constant and*

$$(6.16) \quad \int_M dV_{\theta_Y} = \int_M dV_{\theta_0}.$$

Then we have

$$(6.17) \quad e^{-c} \lambda_1(\theta_Y) \leq \lambda_1(\theta_0) \leq e^c \lambda_1(\theta_Y)$$

where $c = 2(2n + 1) \left(\frac{\min_M R_{\theta_0}}{\max_M R_{\theta_0}} - 1 \right)$.

Proof. It was proved by Zhang in [52] that $\theta \rightarrow \theta_\infty$ as $t \rightarrow \infty$ under the CR Yamabe flow (6.1) such that θ_∞ has constant Webster scalar curvature. Along the CR Yamabe flow (6.1), we have

$$\frac{d}{dt} \left(\int_M dV_\theta \right) = \int_M \frac{\partial}{\partial t} (dV_\theta) = -(n + 1) \int_M (R_\theta - \overline{R}_\theta) dV_\theta = 0$$

by (6.2) and (6.4), which implies that $\int_M dV_\theta = \int_M dV_{\theta_0}$ for all $t \geq 0$. In particular, we have

$$(6.18) \quad \int_M dV_{\theta_\infty} = \int_M dV_{\theta_0}.$$

On the other hand, note that $R_{\theta_Y} = c^{\frac{2}{n}} R_{\theta_\infty}$ for some constant $c > 0$. Indeed, one can take c to be $(R_{\theta_Y}/R_{\theta_\infty})^{\frac{n}{2}}$. This implies that the metric $c^{\frac{2}{n}} \theta_Y$ has scalar curvature being equal to

$$R_{c^{\frac{2}{n}} \theta_Y} = c^{-\frac{2}{n}} R_{\theta_Y} = R_{\theta_\infty}.$$

Hence, we can conclude that

$$(6.19) \quad c^{\frac{2}{n}} \theta_Y = \theta_\infty$$

using the result of Jerison and Lee (see Theorem 7.1 in [30] and also Theorem 1.3 in [24]), which says that if θ_1 and θ_2 are two contact forms conformal to θ_0 such that their Webster scalar curvatures satisfy $R_{\theta_1} = R_{\theta_2} < 0$, then $\theta_1 = \theta_2$. Therefore, by (6.18) and (6.19), we have

$$\int_M dV_{\theta_0} = \int_M dV_{\theta_\infty} = \int_M dV_{c^{\frac{2}{n}} \theta_Y} = c^{\frac{2n+2}{n}} \int_M dV_{\theta_Y} = c^{\frac{2n+2}{n}} \int_M dV_{\theta_0}$$

where the last equality follows from (6.16). This implies that $c = 1$, or equivalently,

$$(6.20) \quad \theta_Y = \theta_\infty.$$

Note that

$$(6.21) \quad \min_M R_{\theta_0} \leq \overline{R}_{\theta(t)} \leq \max_M R_{\theta_0} \text{ for all } t \geq 0$$

by Proposition 6.4. Therefore, it follows from (6.21) and the first inequality of Lemma 6.5 that

$$(6.22) \quad \begin{aligned} & \left(\max_M R_{\theta_0} \right) \int_0^t \left(\max_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds \\ & \geq \left(\overline{R}_{\theta(t)} - \overline{R}_{\theta_0} \right) + \left(\max_M R_{\theta(t)} - \overline{R}_{\theta(t)} \right) - \left(\max_M R_{\theta_0} - \min_M R_{\theta_0} \right) \\ & \geq \left(\max_M R_{\theta(t)} - \overline{R}_{\theta(t)} \right) - 2 \left(\max_M R_{\theta_0} - \min_M R_{\theta_0} \right), \end{aligned}$$

and it follows from (6.21) and the second inequality of Lemma 6.5 that

$$(6.23) \quad \begin{aligned} & - \left(\max_M R_{\theta_0} \right) \int_0^t \left(\min_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds \\ & \geq \left(\overline{R}_{\theta_0} - \overline{R}_{\theta(t)} \right) + \left(\overline{R}_{\theta(t)} - \min_M R_{\theta(t)} \right) - \left(\max_M R_{\theta_0} - \min_M R_{\theta_0} \right) \\ & \geq \left(\overline{R}_{\theta(t)} - \min_M R_{\theta(t)} \right) - 2 \left(\max_M R_{\theta_0} - \min_M R_{\theta_0} \right). \end{aligned}$$

Since $\theta(t) \rightarrow \theta_\infty$ as $t \rightarrow \infty$ by Zhang's result stated above, by letting t go to infinity, we obtain from (6.22) and (6.23) respectively that

$$(6.24) \quad -2 \left(1 - \frac{\min_M R_{\theta_0}}{\max_M R_{\theta_0}} \right) \geq \int_0^\infty \left(\max_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds$$

and

$$(6.25) \quad -2 \left(1 - \frac{\min_M R_{\theta_0}}{\max_M R_{\theta_0}} \right) \geq - \int_0^\infty \left(\min_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds.$$

Integrating the inequality in Proposition 6.3 and using (6.20), (6.24) and (6.25), we get

$$\begin{aligned} \log \frac{\lambda_1(\theta_Y)}{\lambda_1(\theta_0)} &= \log \frac{\lambda_1(\theta_\infty)}{\lambda_1(\theta_0)} \\ &\leq -n \int_0^\infty \left(\min_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds + (n+1) \int_0^\infty \left(\max_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds \\ &\leq -2(2n+1) \left(1 - \frac{\min_M R_{\theta_0}}{\max_M R_{\theta_0}} \right), \end{aligned}$$

which gives the lower bound for $\lambda_1(\theta_0)$ in (6.17). On the other hand, integrating the inequality in Proposition 6.2 and using (6.20), (6.24) and (6.25), we get

$$\begin{aligned} \log \frac{\lambda_1(\theta_Y)}{\lambda_1(\theta_0)} &= \log \frac{\lambda_1(\theta_\infty)}{\lambda_1(\theta_0)} \\ &\geq -n \int_0^\infty \left(\max_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds + (n+1) \int_0^\infty \left(\min_M R_{\theta(s)} - \overline{R}_{\theta(s)} \right) ds \\ &\geq 2(2n+1) \left(1 - \frac{\min_M R_{\theta_0}}{\max_M R_{\theta_0}} \right), \end{aligned}$$

which gives the upper bound for $\lambda_1(\theta_0)$ in (6.17). This proves the assertion. \square

7. THE UNNORMALIZED CR YAMABE FLOW

In this section, we consider the unnormalized CR Yamabe flow:

$$(7.1) \quad \frac{\partial}{\partial t} \theta = -R_\theta \theta \text{ for } t \geq 0, \quad \theta|_{t=0} = \theta_0.$$

If we write $\theta = u^{\frac{2}{n}} \theta_0$, then

$$(7.2) \quad \frac{\partial u}{\partial t} = -\frac{n}{2} R_\theta u \text{ for } t \geq 0, \quad u|_{t=0} = 1.$$

Hence, the volume form dV_θ and the Webster scalar curvature R_θ of θ satisfy (see (6.3) in [24])

$$(7.3) \quad \frac{\partial}{\partial t} (dV_\theta) = -(n+1) R_\theta dV_\theta,$$

$$(7.4) \quad \frac{\partial}{\partial t} R_\theta = (n+1) \Delta_\theta R_\theta + R_\theta^2.$$

Let λ_1 be the first eigenvalue of $-\Delta_\theta + aR_\theta$ where a is a constant, i.e.

$$(7.5) \quad -\Delta_\theta f + aR_\theta f = \lambda_1 f$$

for some function f . Again we assume that there is a family of the first eigenvalue and the corresponding eigenfunction which is C^1 in t . By rescaling, we may assume that the eigenfunction f satisfies

$$(7.6) \quad \int_M f^2 dV_\theta = 1.$$

Proposition 7.1. *Along the unnormalized CR Yamabe flow (6.1), we have*

$$\frac{d\lambda_1}{dt} = (2(n+1)a - n) \int_M R_\theta (|\nabla_\theta f|_\theta^2 + aR_\theta f^2) dV_\theta - (n+1)(2a-1)\lambda_1 \int_M R_\theta f^2 dV_\theta.$$

Proof. Differentiate (7.6) with respect to t , we have

$$(7.7) \quad \int_M f \frac{\partial f}{\partial t} dV_\theta = \frac{n+1}{2} \int_M R_\theta f^2 dV_\theta$$

by (7.3). Multiply (7.5) by f and integrate it over M , we obtain

$$(7.8) \quad \lambda_1 = \int_M (|\nabla_\theta f|_\theta^2 + aR_\theta f^2) dV_\theta$$

by (7.6) and integration by parts. Since $\theta = u^{\frac{2}{n}} \theta_0$, (6.7) holds. Combining (6.7) and (7.8), we have

$$\lambda_1 = \int_M u^2 |\nabla_{\theta_0} f|_{\theta_0}^2 dV_{\theta_0} + a \int_M R_\theta f^2 dV_\theta.$$

Differentiate it with respect to t , we obtain

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= \int_M 2u \frac{\partial u}{\partial t} |\nabla_{\theta_0} f|_{\theta_0}^2 dV_{\theta_0} + 2 \int_M u^2 \langle \nabla_{\theta_0} f, \nabla_{\theta_0} \left(\frac{\partial f}{\partial t} \right) \rangle_{\theta_0} dV_{\theta_0} \\
&\quad + a \int_M f^2 \frac{\partial R_\theta}{\partial t} dV_\theta + 2a \int_M R_\theta f \frac{\partial f}{\partial t} dV_\theta + a \int_M R_\theta f^2 \frac{\partial}{\partial t} (dV_\theta) \\
&= -n \int_M R_\theta |\nabla_g f|_\theta^2 dV_\theta + 2 \int_M \langle \nabla_\theta f, \nabla_\theta \left(\frac{\partial f}{\partial t} \right) \rangle_\theta dV_\theta \\
&\quad + a \int_M f^2 ((n+1)\Delta_\theta R_\theta + R_\theta^2) dV_\theta + 2a \int_M R_\theta f \frac{\partial f}{\partial t} dV_\theta - (n+1)a \int_M R_\theta^2 f^2 dV_\theta \\
&= -n \int_M R_\theta |\nabla_\theta f|_\theta^2 dV_\theta + 2 \int_M \frac{\partial f}{\partial t} (-\Delta_\theta f + aR_\theta f) dV_\theta \\
&\quad + (n+1)a \int_M R_\theta \Delta_\theta (f^2) dV_\theta - na \int_M R_\theta^2 f^2 dV_\theta \\
&= (2(n+1)a - n) \int_M R_\theta |\nabla_\theta f|_\theta^2 dV_\theta + 2\lambda_1 \int_M f \frac{\partial f}{\partial t} dV_\theta \\
&\quad - 2(n+1)a \int_M R_\theta f (-\Delta_\theta f + aR_\theta f) dV_\theta + (2(n+1)a^2 - na) \int_M R_\theta^2 f^2 dV_\theta \\
&= (2(n+1)a - n) \int_M R_\theta (|\nabla_\theta f|_\theta^2 + aR_\theta f^2) dV_\theta - (n+1)(2a-1)\lambda_1 \int_M R_\theta f^2 dV_\theta
\end{aligned}$$

where the second equality follows from (6.7) and (7.2)-(7.4), the third equality follows from integration by parts, and the last two equalities follow from (7.5) and (7.7). This proves the assertion. \square

Now, by the same proof of Proposition 3.2-3.4, we have the following:

Proposition 7.2. *Along the unnormalized CR Yamabe flow (7.1),*

(i) *if $0 \leq a \leq \frac{n}{2n+2}$ and $\min_M R_\theta \geq \frac{n}{n+1} \max_M R_\theta \geq 0$, then $\frac{d\lambda_1}{dt} \geq 0$, and equality holds if and only if R_θ is constant;*

(ii) *if $\frac{n}{2n+2} \leq a \leq \frac{1}{2}$ and $\min_M R_\theta \geq 0$, then $\frac{d\lambda_1}{dt} \geq 0$ and equality holds if and only if $R_\theta \equiv 0$; and*

(iii) *if $a \geq \frac{1}{2}$ and $\min_M R_\theta \geq 0$, then $\frac{d\lambda_1}{dt} \geq 0$ and equality holds if and only if $R_\theta \equiv 0$.*

Proposition 7.2 implies the following:

Theorem 7.3. *Along the unnormalized CR Yamabe flow (7.1), the first eigenvalue of $-\Delta_\theta + aR_\theta$ is nondecreasing*

(i) *if $0 \leq a < \frac{n}{2n+2}$ and $\min_M R_\theta \geq \frac{n}{n+1} \max_M R_\theta \geq 0$;*

(ii) *if $a \geq \frac{n}{2n+2}$ and $\min_M R_\theta \geq 0$.*

Note that Theorem 7.3 was obtained in [25] for the cases when $a = 0$ and $a = \frac{n}{2n+2}$. See Theorem 1.4 and Theorem 1.5 in [25]. Note also that Chang-Lin-Wu [12] has obtained a result similar to Theorem 7.3 for the case when $n = 1$.

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