HARMONIC MEASURE AND QUANTITATIVE CONNECTIVITY: GEOMETRIC CHARACTERIZATION OF THE L^p SOLVABILITY OF THE DIRICHLET PROBLEM. PART II

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ABSTRACT. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with *n*-AD-regular boundary. In this paper we prove that if the harmonic measure for Ω satisfies the so-called weak-A_∞ condition, then $Ω$ satisfies a suitable connectivity condition, namely the weak local John condition. Together with other previous results by Hofmann and Martell, this implies that the weak- A_{∞} condition for harmonic measure holds if and only if $\partial\Omega$ is uniformly n-rectifiable and the weak local John condition is satisfied. This yields the first geometric characterization of the weak- A_{∞} condition for harmonic measure, which is important because of its connection with the Dirichlet problem for the Laplace equation.

CONTENTS

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2 AZZAM, MOURGOGLOU, AND TOLSA

1. INTRODUCTION

The weak- A_{∞} condition for harmonic measure of an open set $\Omega \subset \mathbb{R}^{n+1}$ is a quantitative version of absolute continuity of harmonic measure with respect to the surface measure. In this paper we complete one of the fundamental steps for the characterization of the weak- A_{∞} condition for harmonic measure in terms of quantitative rectifiability of the boundary $\partial\Omega$ and a quantitative connectivity property of Ω . More precisely, we show that if the weak- A_{∞} condition holds, then the so-called local John condition is satisfied. Together with previous results by Hofmann and Martell, this yields the aforementioned characterization.

The fact that rectifiability plays a fundamental role in the characterization of absolute continuity of harmonic measure with respect to surface measure has been well known since 1916 by the classical theorem of F. and M. Riesz [\[RR\]](#page-44-0). Recall that this asserts that, given a simply connected domain $\Omega \subset \mathbb{C}$, the rectifiability of $\partial\Omega$ implies that harmonic measure for Ω is absolutely continuous with respect to arc-length measure of the boundary. A local version of this theorem was obtained much later, in 1990, by Bishop and Jones [\[BiJo\]](#page-43-1). For related results in higher dimensions see [\[AAM\]](#page-43-2). On the other hand, in the converse direction, it was shown recently in [\[AHM](#page-43-3)³TV] that, for arbitrary open sets $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, the mutual absolute continuity of harmonic measure and surface measure (i.e. *n*-dimensional Hausdorff measure, which we will denote by \mathcal{H}^n) in a subset $E \subset \partial \Omega$ implies the *n*-rectifiability of E.

To describe other results of more quantitative nature we need now to introduce some notation and definitions. A set $E \subset \mathbb{R}^{n+1}$ is called *n-AD-regular* if there exists some constant $C_0 > 0$ such that

$$
C_0^{-1}r^n \le \mathcal{H}^n(E \cap B(x,r)) \le C_0 r^n \quad \text{ for all } x \in E \text{ and } 0 < r \le \text{diam}(E).
$$

The set $E \subset \mathbb{R}^{n+1}$ is *uniformly n-rectifiable* if it is *n*-AD-regular and there exist constants $\theta, M > 0$ such that for all $x \in E$ and all $0 < r \leq$ diam(E) there is a Lipschitz mapping g from the ball $B_n(0,r)$ in \mathbb{R}^n to \mathbb{R}^d with $\text{Lip}(g) \leq M$ such that

$$
\mathcal{H}^n(E \cap B(x,r) \cap g(B_n(0,r))) \ge \theta r^n.
$$

Uniform *n*-rectifiability is a quantitative version of *n*-rectifiability introduced by David and Semmes (see $[DS1]$ and $[DS2]$).

Let $\Omega \subset \mathbb{R}^{n+1}$ be open. One says that this satisfies the corkscrew condition if for every $x \in \partial\Omega$ and $0 < \rho \leq \text{diam}(\Omega)$ there exists a ball $B \subset B(x, \rho) \cap \Omega$ with radius $r(B) \geq c \rho$, for some fixed $c > 0$.

Given $p \in \Omega$, we denote by ω^p the harmonic measure for Ω with pole at p. Assume that $\partial\Omega$ has locally finite \mathcal{H}^n -measure. We say that the harmonic measure for Ω satisfies the *weak-A_∞ condition* if for every $\varepsilon_0 \in (0,1)$ there exists $\delta_0 \in (0,1)$ such that for every ball B centered at $\partial\Omega$ and all $p \in \Omega \setminus 4B$ the following holds: for any subset $E \subset B \cap \partial \Omega$,

(1.1) if
$$
\mathcal{H}^n(E) \leq \delta_0 \mathcal{H}^n(B \cap \partial \Omega)
$$
, then $\omega^p(E) \leq \varepsilon_0 \omega^p(2B)$.

In the case when the harmonic measure is doubling, that is, there is some constant $C > 0$ such that

$$
\omega^p(2B) \le C \,\omega^p(B) \qquad \text{for any ball } B \text{ centered at } \Omega \text{ and all } p \in \Omega,
$$

the weak- A_{∞} condition coincides with the more familiar A_{∞} condition for ω^p (uniform on p). Both the A_{∞} and weak- A_{∞} condition should be understood as quantitative versions of the notion of absolute continuity. We will write $\omega \in A_\infty(\mathcal{H}^1|_{\partial\Omega})$ and $\omega \in \text{weak-}\mathcal{A}_\infty(\mathcal{H}^1|_{\partial\Omega})$ to indicate that the harmonic measure satisfies the A_{∞} and weak- A_{∞} conditions, respectively.

The weak- A_{∞} condition is particularly important from a PDE perspective. In fact, Hofmann and Le showed in [\[HLe\]](#page-44-3) that, if we assume Ω to satisfy the corkscrew condition and $\partial\Omega$ to be n-ADregular, then the Dirichlet problem is BMO-solvable for the Laplace equation if and only if the harmonic measure is in weak- A_{∞} . So a geometric description of the domains Ω such that $\omega \in \text{weak-}A_{\infty}$ is particularly desirable.

The first result of quantitative nature involving harmonic measure and rectifiability was obtained by Lavrentiev [\[Lav\]](#page-44-4) in 1936 for planar domains. He showed that if $\Omega \subset \mathbb{C}$ is a simply connected domain which is bounded by a chord-arc curve, then $\omega \in A_\infty(\mathcal{H}^1|_{\partial\Omega})$. A fundamental result in arbitrary dimensions was obtained much later by Dahlberg [\[Dah\]](#page-43-4). He showed that if $\Omega \subset \mathbb{R}^{n+1}$ is a bounded Lipschitz domain, then the harmonic measure satisfies the reverse Hölder condition B_2 and thus it belongs to $A_\infty(\mathcal{H}^1|_{\partial\Omega})$. This result was extended to chord-arc domains by David and Jerison [\[DJ\]](#page-43-5), and independently by Semmes [\[Se\]](#page-44-5). They proved that chord-arc domains in \mathbb{R}^{n+1} (i.e., NTA domains with *n*-AD regular boundaries) have interior big pieces of Lipschitz, implying that $\omega \in A_{\infty}(\mathcal{H}^n|_{\partial\Omega})$.

In connection with harmonic measure, the weak- A_{∞} condition first appeared in the work by Ben-newitz and Lewis in [\[BL\]](#page-43-6), where it was shown that if the boundary of $\Omega \subset \mathbb{R}^{n+1}$ is *n*-AD-regular and Ω has interior big pieces of Lipschitz domains, then $\omega \in \text{weak-}A_{\infty}(\mathcal{H}^n|_{\partial\Omega})$. They also showed that this is the best one can expect under these assumptions on the geometry of the domain. One can also show by the arugments in [\[DJ\]](#page-43-5) that this still holds if we replace Lipschitz with chord-arc subdomains.

Later, Hofmann and Martell [\[HM1\]](#page-44-6), and in collaboration with Uriarte-Tuero [\[HMU\]](#page-44-7), showed that for a uniform domain with *n*-AD regular boundary, $\omega \in \text{weak-} A_\infty(\mathcal{H}^n|_{\partial\Omega})$ if and only if $\partial\Omega$ is uniformly n -rectifiable. This was further improved in $[AHMNT]$ where it was shown that any uniform domain with uniformly *n*-rectifiable boundary is in fact NTA and thus $\omega \in A_{\infty}(\mathcal{H}^n|_{\partial\Omega})$. In [\[HM2\]](#page-44-8)^{[1](#page-2-0)} Hofmann and Martell removed the uniformity assumption entirely by showing that for a domain with n -ADregular boundary that satisfies the corkscrew condition, if $\omega \in \text{weak-} A_\infty(\mathcal{H}^n|_{\partial\Omega})$, then $\partial\Omega$ is uniformly n -rectifiable. This result was later extended to the case when the surface measure is non-doubling in [\[MT\]](#page-44-9).

Also note that according to Bishop and Jones' example in [\[BiJo\]](#page-43-1), there exists an infinitely connected planar domain whose boundary is uniformly 1-rectifiable but ω is not absolutely continuous to arc-length. In fact, by [\[GMT\]](#page-44-10) and [\[HMM\]](#page-44-11), the uniform rectifiability of $\partial\Omega$ is equivalent to the existence of a suitable corona type decomposition of $\partial\Omega$ in terms of harmonic measure (and also equivalent to a Carleson type condition for the gradient of bounded harmonic functions). So uniform rectifiability alone cannot characterize the weak- A_{∞} condition for harmonic measure.

The first named author of the current manuscript recently showed in [\[Azz2\]](#page-43-8) that if a domain is semiuniform and has uniformly rectifiable boundary, then harmonic measure is in A_{∞} . Aikawa and Hirata had shown previously in [\[AH\]](#page-43-9) that a domain is semi-uniform if and only if the harmonic measure is doubling, which happens, in particular, if harmonic measure is in A_{∞} (they also assumed the domains were John but this assumption was removed in [\[Azz2\]](#page-43-8)). This and [\[HM2\]](#page-44-8) show that the A_{∞} condition

¹This result was published in [\[HLMN\]](#page-44-12).

implies semi-uniformity of the domain and uniform rectifiability of the boundary. Thus, the combination of these works yields a geometric characterization of the A_{∞} property.

Hofmann and Martell, however, introduced an a priori weaker connectivity condition than interior big pieces of chord-arc domains that is sufficient for the weak- A_{∞} condition. Given $x \in \Omega$, $y \in \partial\Omega$, and $\lambda > 0$, a λ -carrot curve (or just *carrot curve*) from x to y is a curve $\gamma \subset \Omega \cup \{y\}$ with end-points x and y such that $\delta_{\Omega}(z) := \text{dist}(z, \partial \Omega) \geq \kappa H^1(\gamma(y, z))$ for all $z \in \gamma$, where $\gamma(y, z)$ is the arc in γ between y and z .

One says that Ω satisfies the *weak local John condition* (with parameters λ, θ, Λ) if there are constants $\lambda, \theta \in (0, 1)$ and $\Lambda \geq 2$ such that for every $x \in \Omega$ there is a Borel subset $F \subset B(x, \Lambda \delta_{\Omega}(x)) \cap \partial \Omega$ with $\mathcal{H}^n(F) \geq \theta \mathcal{H}^n(B(x, \Lambda \delta_{\Omega}(x)) \cap \partial \Omega)$ such that every $y \in F$ can be joined to x by a λ -carrot curve. Note that the weak local John condition is weaker than semi-uniformity: rather than requiring nice carrot curves to every point on the boundary, there are only nice curves to points in a big piece.

In [\[HM3\]](#page-44-13) Hofmann and Martell showed that if $\Omega \subset \mathbb{R}^{n+1}$ is open (not necessarily connected), with a uniformly rectifiable boundary, and Ω satisfies the weak local John condition, then harmonic measure is in weak- A_{∞} . In the same work they conjectured that, conversely, if the harmonic measure is in weak- A_{∞} , then the weak local John condition holds.

Our main result confirms this conjecture:

Theorem 1.1. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with n-AD-regular boundary. If the harmonic *measure for* Ω *satisfies the weak-*A[∞] *condition, then* Ω *satisfies the weak local John condition.*

After the publication of a first version of our paper in Arxiv, Hofmann and Martell also updated their paper [\[HM3\]](#page-44-13) to show that the weak local John condition implies interior big pieces of chord-arc domains. See [\[HM3\]](#page-44-13) for the precise definition of "interior big pieces of chord-arc domains". Thus, combining our results with the main result of [\[HM3\]](#page-44-13), we can conclude the following.

Corollary 1.2. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with n-AD-regular boundary satisfying the *corkscrew condition. The harmonic measure for* Ω *is in weak-* A_{∞} *if and only if* $\partial\Omega$ *is uniformly nrectifiable and* Ω *satisfies the weak local John condition, if and only if* Ω *has interior big pieces of chord-arc domains.*

Some of the difficulties that we have to overcome to prove Theorem [1.1](#page-3-0) arise from the fact that the weak- A_{∞} -condition does not imply any doubling condition on harmonic measure. Roughly speaking, given a ball B centered at in $\partial\Omega$ and $x \in \Omega$, if $\omega_{\Omega}^x(B)$ is large, then x should be well connected to a big piece of $\partial\Omega \cap B$ (though not necessarily any point in B). If we knew that the doubling property holds for each ball and also for different choices of x , then we would be able to piece together nice Harnack chains between different base points and the boundary. The weak A_{∞} -condition, however, at best implies that ω_{Ω}^x is doubling on balls centered on some large subset of the boundary, and this large subset may change as one changes the pole. So it is difficult to compare harmonic measure with respect to different poles in Ω (in fact, they may be mutually singular when Ω is not connected).

Because of the reasons above, to prove Theorem [1.1](#page-3-0) we cannot use arguments similar to the ones in [\[AH\]](#page-43-9) or [\[Azz2\]](#page-43-8). In fact, we have to prove a local result which involves only one pole and one ball which has its own interest. See the Main Lemma [2.13](#page-11-2) for more details. Two essential ingredients of the proof are a corona type decomposition (whose existence is ensured by the uniform n -rectifiability of the boundary) and the Alt-Caffarelli-Friedman monotonicity formula [\[ACF\]](#page-43-10). This formula is used in some of the connectivity arguments in this paper. This allows to connect by carrot curves corkscrew points where the Green function is not too small to other corkscrews at a larger distance from the boundary where the Green function is still not too small (see Lemma [3.2](#page-12-0) for the precise statement). See also

the work [\[AGMT\]](#page-43-11) for another related application of the Alt-Caffarelli-Friedman formula in connection with elliptic measure.

Two important steps of the proof of the Main Lemma [2.13](#page-11-2) (and so of Theorem [1.1\)](#page-3-0) are the Geometric Lemma [6.3](#page-23-0) and the Key Lemma [7.1.](#page-30-1) An essential idea consists of distinguishing cubes with "two well separated big corkscrews" (see Subsection [5.4](#page-20-2) for the precise definition). In the Geometric Lemma [2.13](#page-11-2) we construct two disjoint open sets satisfying a John condition associated to trees involving this type of cubes, so that the boundaries of the open sets are located in places where the Green function is very small. This construction is only possible because the associated tree involves only cubes with two well separated big corkscrews. The existence of these cubes is an obstacle for the construction of carrot curves. However, in a sense, in the Key Lemma [7.1](#page-30-1) we take advantage of their existence to obtain some delicate estimates for the Green function on some corkscrew points.

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2. PRELIMINARIES

We will write $a \leq b$ if there is $C > 0$ so that $a \leq Cb$ and $a \leq t$ b if the constant C depends on the parameter t. We write $a \approx b$ to mean $a \leq b \leq a$ and define $a \approx_b b$ similarly. Sometimes, given a measure ν , we will also use the notation $\int g \, d\nu$ for the average $\nu(F)^{-1} \int_F g \, d\nu$.

In the whole paper, Ω will be an open set in \mathbb{R}^{n+1} , with $n \geq 2$.

2.1. The dyadic lattice \mathcal{D}_{μ} . Given an *n*-AD-regular measure μ in \mathbb{R}^{n+1} we consider the dyadic lattice of "cubes" built by David and Semmes in [\[DS2,](#page-44-2) Chapter 3 of Part I]. The properties satisfied by \mathcal{D}_{μ} are the following. Assume first, for simplicity, that $\text{diam}(\text{supp }\mu) = \infty$). Then for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_{\mu,j}$ of Borel subsets of supp μ (the dyadic cubes of the j-th generation) such that:

- (a) each $\mathcal{D}_{\mu,j}$ is a partition of supp μ , i.e. supp $\mu = \bigcup_{Q \in \mathcal{D}_{\mu,j}} Q$ and $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{D}_{\mu,j}$ and $Q \neq Q'$;
- (b) if $Q \in \mathcal{D}_{\mu,j}$ and $Q' \in \mathcal{D}_{\mu,k}$ with $k \leq j$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;
- (c) for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_{\mu,j}$, we have $2^{-j} \lesssim \text{diam}(Q) \le 2^{-j}$ and $\mu(Q) \approx 2^{-jn}$;
- (d) there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_{\mu,j}$, and $0 < \tau < 1$,

(2.1)
$$
\mu\big(\lbrace x \in Q : \text{dist}(x, \text{supp}\,\mu \setminus Q) \leq \tau 2^{-j}\rbrace\big) +\mu\big(\lbrace x \in \text{supp}\,\mu \setminus Q : \text{dist}(x, Q) \leq \tau 2^{-j}\rbrace\big) \leq C\tau^{1/C}2^{-jn}.
$$

This property is usually called the *small boundaries condition*. From [\(2.1\)](#page-4-2), it follows that there is a point $z_Q \in Q$ (the center of Q) such that $\text{dist}(z_Q, \text{supp}\,\mu \setminus Q) \gtrsim 2^{-j}$ (see [\[DS2,](#page-44-2) Lemma 3.5 of Part I]).

We set $\mathcal{D}_{\mu} := \bigcup_{j \in \mathbb{Z}} \mathcal{D}_{\mu,j}$, and for $Q \in \mathcal{D}_{\mu}$, we denote write $J(Q) = j$ if $Q \in \mathcal{D}_{\mu,j}$.

In case that $\text{diam}(\text{supp}\,\mu) < \infty$, the families $\mathcal{D}_{\mu,j}$ are only defined for $j \geq j_0$, with $2^{-j_0} \approx$ $\text{diam}(\text{supp }\mu)$, and the same properties above hold for $\mathcal{D}_{\mu} := \bigcup_{j \geq j_0} \mathcal{D}_{\mu,j}$.

Given a cube $Q \in \mathcal{D}_{\mu,j}$, we say that its side length is 2^{-j} , and we denote it by $\ell(Q)$. Notice that $diam(Q) \leq \ell(Q)$. We also denote

$$
(2.2) \t\t B_Q := B(z_Q, 4\ell(Q)),
$$

and for $\lambda > 1$, we write

$$
\lambda Q = \{ x \in \mathrm{supp} \,\mu : \, \mathrm{dist}(x, Q) \le (\lambda - 1) \,\ell(Q) \}.
$$

Given $R \in \mathcal{D}_{\mu}$, we set $\mathcal{D}_{\mu}(R) := \{Q \in \mathcal{D}_{\mu} : Q \subset R\}$. We also let $\mathcal{D}_{\mu,j}(R)$ be the family of cubes $Q \in \mathcal{D}_{\mu}(R)$ such that $\ell(Q) = 2^{-j}\ell(R)$.

2.2. Uniform *n*-rectifiability. A set $E \subset \mathbb{R}^{n+1}$ is called *n-rectifiable* if there are Lipschitz maps $f_i: \mathbb{R}^n \to \mathbb{R}^d, i = 1, 2, \ldots$, such that

(2.3)
$$
\mathcal{H}^n\bigg(E\setminus\bigcup_i f_i(\mathbb{R}^n)\bigg)=0.
$$

Recall that the notion of uniform *n*-rectifiability is a quantitative version of *n*-rectifiability. It is very easy to check that uniform *n*-rectifiability implies *n*-rectifiability.

Given a ball $B \subset \mathbb{R}^{n+1}$, we denote

(2.4)
$$
b\beta_E(B) = \inf_L \frac{1}{r(B)} \Big(\sup_{y \in E \cap B} \text{dist}(y, L) + \sup_{y \in L \cap B} \text{dist}(y, E) \Big),
$$

where the infimum is taken over all the affine n -planes that intersect B . The following result is due to David and Semmes:

Theorem 2.1. Let $E \subset \mathbb{R}^{n+1}$ be n-AD-regular. Denote $\mu = \mathcal{H}^n|_E$ and let \mathcal{D}_μ be the associated dyadic *lattice. Then, E is uniformly n-rectifiable if and only if, for any* $\varepsilon > 0$ *,*

$$
\sum_{\substack{Q \in \mathcal{D}_{\mu}: Q \subset R, \\ b\beta(3B_Q) > \varepsilon}} \mu(Q) \le C(\varepsilon) \mu(R) \quad \text{for all } R \in \mathcal{D}_{\mu}.
$$

The constant 3 multiplying B_Q in the estimate above can be replaced by any number larger than 1. For the proof, see [\[DS2,](#page-44-2) Chapter II-2].

Recall also the following result (see [\[HLMN\]](#page-44-12) or [\[MT\]](#page-44-9)).

Theorem 2.2. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-AD-regular boundary such that the harmonic *measure in* Ω *belongs to weak-*A∞*. Then* ∂Ω *is uniformly* n*-rectifiable.*

2.3. **Harmonic measure.** From now on we assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with *n*-AD-regular boundary such that the harmonic measure in Ω belongs to weak A_{∞} . We denote by μ the surface measure in $\partial\Omega$. That is, $\mu = \mathcal{H}^n|_{\partial\Omega}$. We also consider the dyadic lattice \mathcal{D}_{μ} associated with μ . The AD-regularity constant of $\partial\Omega$ is denoted by C_0 .

We denote by ω^p the harmonic measure with pole at p of Ω , and by $g(\cdot, \cdot)$ the Green function. We write $\delta_{\Omega}(x) = \text{dist}(x, \partial \Omega)$.

The following well known result is sometimes called "Bourgain's estimate":

Lemma 2.3. *Let* $\Omega \subseteq \mathbb{R}^{n+1}$ *be open with n-AD-regular boundary,* $x \in \partial \Omega$ *, and* $0 < r \leq \text{diam}(\partial \Omega)/2$ *. Then*

(2.5)
$$
\omega^y(B(x, 2r)) \ge c > 0, \text{ for all } y \in \Omega \cap \overline{B}(x, r)
$$

where c *depends on* n *and the* n*-AD-regularity constant of* ∂Ω*.*

The following is also well known.

Lemma 2.4. *Let* $p, q \in \Omega$ *be such* $|p - q| \geq 4 \delta_{\Omega}(q)$ *. Then,*

$$
g(p,q) \le C \frac{\omega^p(B(q, 4\delta_{\Omega}(q)))}{\delta_{\Omega}(q)^{n-1}}.
$$

The following lemma is also known. See [\[HLMN,](#page-44-12) Lemma 3.14], for example.

Lemma 2.5. *Let* $\Omega \subsetneq \mathbb{R}^{n+1}$ *be open with n-AD-regular boundary and let* $p \in \Omega$ *. Let* B *be a ball centered at* $\partial\Omega$ *such that* $p \notin 8B$ *. Then*

$$
\int_B g(p,x) dx \le C \, \frac{\omega^p(4B)}{r(B)^{n-1}}.
$$

Lemma 2.6. *Let* $\Omega \subseteq \mathbb{R}^{n+1}$ *be open with n-AD-regular boundary. Let* $x \in \partial \Omega$ *and* $0 < r < \text{diam}(\Omega)$ *. Let* u *be a non-negative harmonic function in* $B(x, 4r) \cap \Omega$ *and continuous in* $B(x, 4r) \cap \overline{\Omega}$ *such that* $u \equiv 0$ *in* $\partial\Omega \cap B(x, 4r)$ *. Then extending* u *by* 0 *in* $B(x, 4r) \setminus \overline{\Omega}$ *, there exists a constant* $\alpha > 0$ *such that, for all* $y, z \in B(x, r)$ *,*

$$
|u(y) - u(z)| \le C \left(\frac{|y-z|}{r}\right)^\alpha \sup_{B(x,2r)} u \le C \left(\frac{|y-z|}{r}\right)^\alpha \int_{B(x,4r)} u,
$$

where C *and* α *depend on* n *and the AD-regularity of* ∂Ω*. In particular,*

$$
u(y) \le C \left(\frac{\delta_{\Omega}(y)}{r}\right)^{\alpha} \sup_{B(x,2r)} u \le C \left(\frac{\delta_{\Omega}(y)}{r}\right)^{\alpha} \int_{B(x,4r)} u.
$$

The next result provides a partial converse to Lemma [2.3](#page-5-2)

Lemma 2.7. *Let* $\Omega \subsetneq \mathbb{R}^{n+1}$ *be open with n*-AD-regular boundary. Let $p \in \Omega$ and let $Q \in \mathcal{D}_{\mu}$ be such that $p \notin 2Q$. Suppose that $\omega^p(Q) \approx \omega^p(2Q)$. Then there exists some $q \in \Omega$ such that

$$
\ell(Q) \lesssim \delta_{\Omega}(q) \approx \text{dist}(q, Q) \le 4 \text{diam}(Q)
$$

and

$$
\frac{\omega^p(2Q)}{\ell(Q)^{n-1}} \le c \, g(p,q).
$$

Proof. For a given $k_0 \geq 2$ to be fixed below, let $P \in \mathcal{D}_{\mu}$ be a cube contained in Q with $\ell(P) =$ $2^{-k_0}\ell(Q)$ such that

$$
\omega_p(P) \approx_{k_0} \omega^p(Q).
$$

Let φ_P be a C^{∞} function supported in B_P which equals 1 on P and such that $\|\nabla \varphi_P\|_{\infty} \lesssim 1/\ell(P)$. Then, choosing k_0 small enough so that $p \notin 50B_P$, say, and applying Caccioppoli's inequality,

$$
\omega^{p}(2Q) \approx_{k_{0}} \omega^{p}(P) \leq \int \varphi_{P} d\omega^{p} = -\int \nabla_{y} g(p, y) \nabla \varphi_{P}(y) dy
$$

$$
\lesssim \frac{1}{\ell(P)} \int_{B_{P}} |\nabla_{y} g(p, y)| dy \lesssim \ell(P)^{n} \left(\int_{B_{P}} |\nabla_{y} g(p, y)|^{2} dy \right)^{1/2}
$$

$$
\lesssim \ell(P)^{n-1} \left(\int_{2B_{P}} |g(p, y)|^{2} dy \right)^{1/2} \lesssim \ell(P)^{n-1} \int_{3B_{P}} g(p, y) dy.
$$

Applying now Lemmas [2.6](#page-6-0) and [2.5](#page-6-1) and taking k_0 small enough so that $24B_P \cap \partial\Omega \subset 2Q$, for any $a \in (0,1)$ we get

$$
\int_{y\in 3B_P: \delta_\Omega(y)\leq a\ell(P)} g(p,y)\,dy\lesssim a^\alpha \int_{6B_P} g(p,y)\,dy\lesssim a^\alpha\,\frac{\omega^p(24B_P)}{\ell(P)^{n-1}}\lesssim a^\alpha\,\frac{\omega^p(2Q)}{\ell(P)^{n-1}}.
$$

8 AZZAM, MOURGOGLOU, AND TOLSA

From the estimates above we infer that

$$
\omega^p(2Q) \lesssim_{k_0} \ell(P)^{n-1} \int_{y \in 3B_P:\delta_{\Omega}(y) \ge a\ell(P)} g(p,y) \, dy + a^\alpha \, \omega^p(2Q).
$$

Hence, for a small enough, we derive

$$
\omega^p(2Q) \lesssim_{k_0} \ell(P)^{n-1} \int_{y \in 3B_P:\delta_\Omega(y) \ge a\ell(P)} g(p,y) \, dy,
$$

which implies the existence of the point q required in the lemma.

2.4. **Harnack chains and carrots.** It will be more convenient for us to work with Harnack chains instead of curves. The existence of a carrot curve is equivalent to having what we call a good chain between points.

Let $x \in \Omega$, $y \in \overline{\Omega}$ be such that $\delta_{\Omega}(y) \leq \delta_{\Omega}(x)$, and let $C > 1$. A *C*-good chain (or *C*-good Harnack chain) from x to y is a sequence of balls $B_1, B_2, ...$ (finite or infinite) contained in Ω such that $x \in B_1$ and either

• $\lim_{j\to\infty} dist(y, B_j) = 0$ if $y \in \partial\Omega$, or

• $y \in B_N$ if $y \in \Omega$, where N is the number of elements of the sequence if this is finite,

and moreover the following holds:

- $B_j \cap B_{j+1} \neq \emptyset$ for all j,
- C^{-1} dist $(B_j, \partial \Omega) \le r(B_j) \le C$ dist $(B_j, \partial \Omega)$ for all j,
- $r(B_i) \leq C r(B_i)$ if $j > i$,
- for each $t > 0$ there are at most C balls B_i such that $t < r(B_i) \leq 2t$.

Abusing language, sometimes we will omit the constant C and we will just say "good chain" or "good Harnack chain".

Observe that in the definitions of carrot curves and good chains, the order of x and y is important: having a carrot curve from x to y is not equivalent to having one from y to x, and similarly with good chains.

Lemma 2.8. *There is a carrot curve from* $x \in \Omega$ *to* $y \in \overline{\Omega}$ *if and only if there is a good Harnack chain from* x *to* y*.*

Proof. Let γ be a carrot curve from x to y. We can assume $y \in \Omega$, since if $y \in \partial\Omega$, we can obtain this case by taking a limit of points $y_j \in \Omega$ converging to y. Let ${B_j}_{j=1}^N$ be a Vitali subcovering of the family $\{B(z, \delta_{\Omega}(z)/10): z \in \gamma\}$ and let r_{B_j} stand for the radius and x_{B_j} for the center of B_j . So the balls B_i are disjoint and $3B_i$ cover γ . Note that for $t > 0$, if $t < r_{B_i} \leq 2t$,

$$
|x_{B_j} - y| \le \mathcal{H}^1(\gamma(x_{B_j}, y)) \lesssim \delta_{\Omega}(x_{B_j}) \approx r_{B_j} \le 2t.
$$

In particular, since the B_i 's are disjoint, by volume considerations, there can only be boundedly many B_j of radius between $t/2$ and t, say. Moreover, we may order the balls B_j so that $x \in 5B_1$ and B_{j+1} is a ball B_k such that $5B_k \cap 5B_j \neq \emptyset$ and $5\overline{B_k}$ contains the point from $\gamma \cap \bigcup_{h:5B_h \cap 5B_j \neq \emptyset} 5\overline{B_h}$ which is maximal in the natural order induced by γ (so that x is the minimal point in γ). Then for $j > i$,

$$
r_{B_j} \approx \delta_\Omega(x_{B_j}) \le |x_{B_j}-x_{B_i}| + \delta_\Omega(x_{B_i}) \le \mathcal{H}^1(\gamma(x_{B_i},y)) + \delta_\Omega(x_{B_i}) \lesssim r_{B_i}.
$$

This implies $5B_1$, $5B_2$, ... is a C-good chain for a sufficiently big C.

Now suppose that we can find a good chain from x to y, call it $B_1, ..., B_N$. Let γ be the path obtained by connecting their centers in order. Let $z \in \gamma$. Then there is a j such that $z \in [x_{B_j}, x_{B_{j+1}}]$. Since ${B_i}_i$ is a good chain,

$$
\mathcal{H}^1(\gamma(z,y)) \leq |z - x_{B_{j+1}}| + \mathcal{H}^1(\gamma(x_{B_{j+1}},y)) \leq r_{B_{j+1}} + \sum_{i=j}^N 2r_{B_i} \lesssim r_{B_j} \approx \delta_{\Omega}(z).
$$

Thus, γ is a carrot curve from x to y.

2.5. The Alt-Caffarelli-Friedman formula.

Theorem 2.9. Let $B(x, R)$ ⊂ \mathbb{R}^{n+1} , and let $u_1, u_2 \in W^{1,2}(B(x, R))$ ∩ $C(B(x, R))$ be nonnegative *subharmonic functions. Suppose that* $u_1(x) = u_2(x) = 0$ *and that* $u_1 \cdot u_2 \equiv 0$ *. Set*

$$
J_i(x,r) = \frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_i(y)|^2}{|y-x|^{n-1}} dy,
$$

and

$$
(2.6) \t\t J(x,r) = J_1(x,r) J_2(x,r).
$$

Then $J(x, r)$ *is a non-decreasing function of* $r \in (0, R)$ *and* $J(x, r) < \infty$ *for all* $r \in (0, R)$ *. That is,*

(2.7)
$$
J(x,r_1) \leq J(x,r_2) < \infty \quad \text{for} \quad 0 < r_1 \leq r_2 < R.
$$

Further,

(2.8)
$$
J_i(x,r) \lesssim \frac{1}{r^2} \|u_i\|_{\infty, B(x,2r)}^2.
$$

In the case of equality we have the following result (see [\[PSU,](#page-44-14) Theorem 2.9]).

Theorem 2.10. Let $B(x, R)$ and u_1, u_2 be as in Theorem [2.9.](#page-8-1) Suppose that $J(x, r_a) = J(x, r_b)$ for *some* $0 < r_a < r_b < R$. Then either one or the other of the following holds:

(a) $u_1 = 0$ *in* $B(x, r_b)$ *or* $u_2 = 0$ *in* $B(x, r_b)$ *;*

(b) there exists a unit vector e and constants $k_1, k_2 > 0$ *such that*

 $u_1(y) = k_1 ((y-x) \cdot e)^+, \qquad u_2(y) = k_2 ((y-x) \cdot e)^-, \qquad \text{in } B(x, r_b).$

We will also need the following auxiliary lemma.

Lemma 2.11. Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $\{u_i\}_{i\geq 1} \subset W^{1,2}(B(x,R)) \cap C(B(x,R))$ a sequence *of functions which are nonnegative, subharmonic, such that each* u_i *is harmonic in* $\{y \in B(x,R)$: $u_i(y) > 0$ *and* $u_i(x) = 0$ *. Suppose also that*

$$
||u_i||_{\infty,B(x,R)} \leq C_1 R \quad \text{and} \quad ||u_i||_{\text{Lip}^{\alpha},B(x,R)} \leq C_1 R^{1-\alpha}
$$

for all $i \geq 1$. Then, for every $0 < r < R$ there exists a subsequence $\{u_{i_k}\}_{k \geq 1}$ which converges uniformly in $B(x,r)$ and weakly in $W^{1,2}(B(x,r))$ to some function $u \in W^{1,2}(B(x,r)) \cap C(B(x,r))$, *and moreover,*

(2.9)
$$
\lim_{k \to \infty} \int_{B(x,r)} \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}} dy = \int_{B(x,r)} \frac{|\nabla u(y)|^2}{|y-x|^{n-1}} dy.
$$

Proof. The existence of a subsequence $\{u_{i_k}\}_{k\geq 1}$ converging weakly in $W^{1,2}(B(x,r))$ and uniformly in $B(x,r)$ to some function $u \in W^{1,2}(B(x,r)) \cap C(B(x,r))$ is an immediate consequence of the Arzelà-Ascoli and the Banach-Alaoglu theorems. Quite likely, the identity [\(2.9\)](#page-8-2) is also well known. However, for completeness, we will show the details.

Consider a non-negative subharmonic function $v \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ which is harmonic in $\{y \in B(x, R) : v(y) > 0\}$ so that $v(x) = 0$. For $0 < r < R$ and $0 < \delta < R - r$, let φ be a radial C^{∞} function such that $\chi_{B(x,r)} \leq \varphi \leq \chi_{B(x,r+\delta)}$. Let $\mathcal{E}(y) = c_n^{-1} |y|^{1-n}$ be the fundamental solution of the Laplacian. For $\varepsilon > 0$, denote $v_{\varepsilon} = \max(v, \varepsilon) - \varepsilon$. Then we have

$$
\int \frac{|\nabla v_{\varepsilon}(y)|^2}{|y-x|^{n-1}} \varphi(y) dy = c_n \int \nabla v_{\varepsilon}(y) \nabla (\mathcal{E}(x - \cdot) v_{\varepsilon} \varphi)(y) dy
$$

$$
- c_n \int \nabla v_{\varepsilon}(y) \mathcal{E}(x - y) v_{\varepsilon}(y) \nabla \varphi(y) dy
$$

$$
- c_n \int \nabla v_{\varepsilon}(y) \nabla y \mathcal{E}(x - y) v_{\varepsilon}(y) \varphi(y) dy = c_n (I_1 - I_2 - I_3).
$$

Using the fact that v_{ε} is harmonic in $\{v_{\varepsilon} > 0\}$ and that $\mathcal{E}(x - \cdot) v_{\varepsilon} \varphi \in W_0^{1,2}$ $\bigcap_{0}^{1,2}(\{v_{\varepsilon} > 0\} \cap B(x,R))$ since φ is compactly supported in $B(x, R)$, $v_{\varepsilon} = 0$ on $\partial \{v_{\varepsilon} > 0\}$, and x is far away from $\{v_{\varepsilon} > 0\}$, it follows easily that $I_1 = 0$. On the other hand, we have

$$
2I_3 = \int \nabla (v_\varepsilon^2 \varphi)(y) \nabla_y \mathcal{E}(x - y) dy - \int v_\varepsilon(y)^2 \nabla_y \mathcal{E}(x - y) \nabla \varphi(y) dy
$$

= $-v_\varepsilon(x)^2 - \int v_\varepsilon(y)^2 \nabla_y \mathcal{E}(x - y) \nabla \varphi(y) dy.$

Thus,

$$
\int \frac{|\nabla v_{\varepsilon}(y)|^2}{|y-x|^{n-1}} \varphi(y) dy = -c_n \int \nabla v_{\varepsilon}(y) \mathcal{E}(x-y) v_{\varepsilon}(y) \nabla \varphi(y) dy - \frac{c_n}{2} \int v_{\varepsilon}(y)^2 \nabla_y \mathcal{E}(x-y) \nabla \varphi(y) dy.
$$

Taking into account that supp $\nabla \varphi$ is far away from x, letting $\varepsilon \to 0$, we obtain

$$
\int \frac{|\nabla v(y)|^2}{|y-x|^{n-1}} \varphi(y) dy = -c_n \int \nabla v(y) \mathcal{E}(x-y) v(y) \nabla \varphi(y) dy -\frac{c_n}{2} \int v(y)^2 \nabla_y \mathcal{E}(x-y) \nabla \varphi(y) dy.
$$

Using the preceding identity, it follows easily that

$$
\lim_{k \to \infty} \int \frac{|\nabla u_{i_k}(y)|^2}{|y - x|^{n-1}} \varphi(y) dy = \int \frac{|\nabla u(y)|^2}{|y - x|^{n-1}} \varphi(y) dy.
$$

Indeed, $\lim_{k \to \infty} u_{i_k}(x)^2 = u(x)^2$. Also, it is clear that

$$
\lim_{k \to \infty} \int u_{i_k}(y)^2 \nabla_y \mathcal{E}(x - y) \nabla \varphi(y) dy = \int u(y)^2 \nabla_y \mathcal{E}(x - y) \nabla \varphi(y) dy.
$$

Further,

$$
\int \nabla u_{i_k}(y) \mathcal{E}(x - y) u_{i_k}(y) \nabla \varphi(y) dy = \int \nabla u_{i_k}(y) \mathcal{E}(x - y) u(y) \nabla \varphi(y) dy
$$

+
$$
\int \nabla u_{i_k}(y) \mathcal{E}(x - y) (u_{i_k}(y) - u(y)) \nabla \varphi(y) dy
$$

$$
\stackrel{k \to \infty}{\to} \int \nabla u(y) \mathcal{E}(x - y) u(y) \nabla \varphi(y) dy,
$$

by the weak convergence of u_{i_k} in $W^{1,2}(B(x,R))$ and the uniform convergence in $B(x, r + \delta)$, since supp $\nabla \varphi$ is far away from x.

Let ψ be a radial C^{∞} function such that $\chi_{B(x,r-\delta)} \leq \psi \leq \chi_{B(x,r)}$. The same argument as above shows that

$$
\lim_{k \to \infty} \int \frac{|\nabla u_{i_k}(y)|^2}{|y - x|^{n-1}} \psi(y) \, dy = \int \frac{|\nabla u(y)|^2}{|y - x|^{n-1}} \psi(y) \, dy.
$$

Consequently,

$$
\limsup_{k \to \infty} \int_{B(x,r)} \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}} dy \le \lim_{k \to \infty} \int \frac{|\nabla u_{i_k}(y)|^2}{|y-x|^{n-1}} \varphi(y) dy = \int \frac{|\nabla u(y)|^2}{|y-x|^{n-1}} \varphi(y) dy,
$$

and also

$$
\liminf_{k \to \infty} \int_{B(x,r)} \frac{|\nabla u_{i_k}(y)|^2}{|y - x|^{n-1}} dy \ge \lim_{k \to \infty} \int \frac{|\nabla u_{i_k}(y)|^2}{|y - x|^{n-1}} \psi(y) dy = \int \frac{|\nabla u(y)|^2}{|y - x|^{n-1}} \psi(y) dy.
$$

Since $\delta > 0$ can be taken arbitrarily small, [\(2.9\)](#page-8-2) follows.

Lemma 2.12. Let $B(x, 2R)$ ⊂ \mathbb{R}^{n+1} , and let $u_1, u_2 \in W^{1,2}(B(x, 2R)) \cap C(B(x, 2R))$ be nonnegative *subharmonic functions such that each* u_i *is harmonic in* $\{y \in B(x, 2R) : u_i(y) > 0\}$ *. Suppose that* $u_1(x) = u_2(x) = 0$ and that $u_1 \cdot u_2 \equiv 0$. Assume also that

$$
||u_i||_{\infty,B(x,2R)} \le C_1 R \quad \text{and} \quad ||u_i||_{\text{Lip}^{\alpha},B(x,2R)} \le C_1 R^{1-\alpha} \quad \text{for } i=1,2.
$$

For any $\varepsilon > 0$ *, there exists some* $\delta > 0$ *such that if*

$$
J(x,R) \le (1+\delta) J(x,\tfrac{1}{2}R),
$$

with $J(\cdot, \cdot)$ *defined in* [\(2.6\)](#page-8-3)*, then either one or the other of the following holds:*

- *(a)* $||u_1||_{\infty,B(x,R)} \leq \varepsilon R$ *or* $||u_2||_{\infty,B(x,R)} \leq \varepsilon R$;
- *(b) there exists a unit vector e and constants* $k_1, k_2 > 0$ *such that*

$$
||u_1 - k_1 ((\cdot - x) \cdot e)^+||_{\infty, B(x,R)} \le \varepsilon R,
$$
 $||u_2 - k_2 ((\cdot - x) \cdot e)^-||_{\infty, B(x,R)} \le \varepsilon R.$

The constant δ *depends only on* $n, \alpha, C_1, \varepsilon$ *.*

Proof. Suppose that the conclusion of the lemma fails. By replacing $u_i(y)$ by $\frac{1}{R}u_i(R(y + x))$, we can assume that $x = 0$ and $R = 1$. Let $\varepsilon > 0$, and for each $\delta = 1/k$ and $i = 1, 2$, consider functions $u_{i,k}$ satisfying the assumptions of the lemma and such that neither (a) nor (b) holds for them. By Lemma [2.11,](#page-8-4) there exist subsequences (which we still denote by $\{u_{i,k}\}_k$) which converge uniformly in $B(0, \frac{3}{2})$ $\frac{3}{2}$ and weakly in $W^{1,2}(B(0, \frac{3}{2}))$ $(\frac{3}{2}))$ to some functions $u_i \in W^{1,2}(B(0, \frac{3}{2}))$ $(\frac{3}{2})\big) \cap C(B(0, \frac{3}{2}))$ $\left(\frac{3}{2}\right)$), and moreover,

$$
\lim_{k \to \infty} \int_{B(0,r)} \frac{|\nabla u_{i,k}(y)|^2}{|y|^{n-1}} dy = \int_{B(0,r)} \frac{|\nabla u_i(y)|^2}{|y|^{n-1}} dy
$$

both for $r = 1$ and $r = 1/2$. Clearly, the functions u_i are non-negative, subharmonic, and $u_1 \cdot u_2 = 0$. Hence, by Theorem [2.10,](#page-8-5) one of the following holds:

(a') $u_1 = 0$ in $B(0, 1)$ or $u_2 = 0$ in $B(0, 1)$;

12 AZZAM, MOURGOGLOU, AND TOLSA

(b') there exists a unit vector e and constants $k_1, k_2 > 0$ such that

$$
u_1(y) = k_1 (y \cdot e)^+, \qquad u_2(y) = k_2 (y \cdot e)^-, \qquad \text{in } B(0, 1).
$$

However, the fact that neither (a) nor (b) holds for any pair $u_{1,k}$, $u_{2,k}$, together with the uniform convergence of $\{u_{i,k}\}_k$, implies that neither (a') nor (b') can hold, and thus we get a contradiction.

2.6. The Main Lemma. Let $B \subset \mathbb{R}^{n+1}$ be a ball centerer at $\partial\Omega$ and let $p \in \Omega$. We say that ω^p satisfies the *weak-A_∞ condition in* B if for every $\varepsilon_0 \in (0,1)$ there exists $\delta_0 \in (0,1)$ such that the following holds: for any subset $E \subset B \cap \partial \Omega$,

$$
\text{if } \mathcal{H}^n(E) \le \delta_0 \mathcal{H}^n(B \cap \partial \Omega), \quad \text{then } \omega^p(E) \le \varepsilon_0 \,\omega^p(2B).
$$

In the next sections we will prove the following.

Main Lemma 2.13. Let $\Omega \subset \mathbb{R}^{n+1}$ have n-AD-regular boundary. Let $R_0 \in \mathcal{D}_\mu$ and let $p \in \Omega \setminus 4B_{R_0}$ *be a point such that*

$$
c\,\ell(R_0) \leq \text{dist}(p,\partial\Omega) \leq \text{dist}(p,R_0) \leq c^{-1}\,\ell(R_0)
$$

and $\omega^p(R_0) \geq c' > 0$. Suppose that ω^p satisfies the weak- A_∞ condition in B_{R_0} . Then there exists *a* subset $Con(R_0) \subset R_0$ and a constant $c'' > 0$ with $\mu(\text{Con}(R_0)) \ge c'' \mu(R_0)$ such that each point $x \in \text{Con}(R_0)$ can be joined to p by a carrot curve. The constant c'' and the constants involved in the *carrot condition only depend on* c, c', n , the weak- A_{∞} *condition, and the* n-AD-regularity of μ *.*

The notation $Con(\cdot)$ stands for "connectable".

It is easy to check that Theorem [1.1](#page-3-0) follows from this result. Indeed, given any $x \in \Omega$, we take a point $\xi \in \partial\Omega$ such that $|x - \xi| = \delta_{\Omega}(x)$. Then we consider the point p in the segment $[x, \xi]$ such that $|p-\xi| = \frac{1}{16} \delta_{\Omega}(x)$. By Lemma [2.5,](#page-5-3) we have

$$
\omega^p(B(\xi, \frac{1}{8}\delta_{\Omega}(x))) \gtrsim 1,
$$

because $p \in \frac{1}{2}B(\xi, \frac{1}{8}\delta_{\Omega}(x))$. Hence, by covering $B(\xi, \frac{1}{8}\delta_{\Omega}(x)) \cap \Omega$ with cubes $R \in \mathcal{D}_{\mu}$ contained in $B(\xi, \frac{1}{4}\delta_{\Omega}(x)) \cap \partial \Omega$ with side length comparable to $\delta_{\Omega}(x)$ we deduce that at least one these cubes, call it R_0 , satisfies $\omega^p(R_0) \gtrsim 1$. Further, by taking the side length small enough, we may also assume that $p \notin 4B_{R_0}$. So by applying Lemma [2.13](#page-11-2) above we infer that there exists a subset $F := \text{Con}(R_0) \subset R_0$ with $\mu(F) \ge c' \mu(R_0) \gtrsim \delta_{\Omega}(x)^n$ such that all $y \in F$ can be joined to x by a carrot curve, which proves that Ω satisfies the weak local John condition and concludes the proof of Theorem [1.1.](#page-3-0)

For simplicity, in the next sections we will assume that $\Omega = \mathbb{R}^{n+1} \setminus \partial\Omega$. At the end of the paper we will sketch the necessary changes for the general case.

3. SHORT PATHS

Let $p \in \Omega$ and $\Lambda > 1$. For $x \in \partial\Omega$, we write $x \in \text{WA}(p, \Lambda)$ if

•
$$
x \in B(p, 10\delta_{\Omega}(p)) \cap \partial\Omega
$$
, and

• for all $0 < r \leq \delta_{\Omega}(p)$,

$$
\Lambda^{-1} \frac{\mu(B(x,r))}{\mu(B(x,\delta_{\Omega}(p)))} \le \omega^p(B(x,r)) \le \Lambda \frac{\mu(B(x,r))}{\mu(B(x,\delta_{\Omega}(p)))}.
$$

We will see in Section [4](#page-15-0) that, under the assumptions of the Main Lemma [2.13,](#page-11-2) for some Λ big enough,

(3.1)
$$
\mu(\text{WA}(p,\Lambda) \cap R_0) \gtrsim \mu(R_0).
$$

Lemma 3.1. *Let* $p \in \Omega$, $x_0 \in \text{WA}(p, \Lambda)$, and $r \in (0, \delta_{\Omega}(p))$. Then there exists $q \in B(x_0, r)$ *such that, for some constant* $\kappa \in (0, 1/10)$ *,*

(a) $δΩ(q) ≥ κr, and$ *(b)*

$$
\kappa \frac{\omega^p(B(x_0,r))}{r^{n-1}} \le g(p,q) \le \kappa^{-1} \frac{\omega^p(B(x_0,r))}{r^{n-1}}.
$$

The constant κ *depends only on* Λ *, n, and* C_0 *, the AD-regularity constant of* $\partial\Omega$ *.*

Proof. This follows easily from Lemmas [2.4](#page-5-4) and [2.7.](#page-6-2) □

Lemma 3.2 (Short paths). Let $p \in \Omega$, $x_0 \in \text{WA}(p, \Lambda)$, and for $0 < r_0 \le \delta_{\Omega}(p)/4$, $0 < \tau_0$, $\lambda_0 \le 1$, let $q \in \Omega$ *be such that*

(3.2)
$$
q \in B(x_0, r_0), \quad \delta_{\Omega}(q) \geq \tau_0 r_0, \quad g(p, q) \geq \lambda_0 \frac{\delta_{\Omega}(q)}{\delta_{\Omega}(p)^n}.
$$

Then there exist constants $A_1 > 1$ *and* $0 < a_1, \lambda_1 < 1$ *such that for every* $r \in (r_0, \delta_{\Omega}(p)/2)$ *, there exists some point* $q' \in \Omega$ *such that*

(3.3)
$$
q' \in B(x_0, A_1r), \quad \delta_{\Omega}(q') \geq \kappa |x_0 - q'| \geq \kappa r, \quad g(p, q') \geq \lambda_1 \frac{\delta_{\Omega}(q')}{\delta_{\Omega}(p)^n},
$$

and such that q *and* q ′ *can be joined by a curve* γ *such that*

$$
\gamma \subset \{y \in B(x_0, A_1r) : \text{dist}(y, \partial \Omega) > a_1 r_0\}.
$$

The parameters λ_1 , A_1 , a_1 *depend only on* C_0 , Λ , λ_0 , τ_0 *and the ratio* r/r_0 .

Proof. All the parameters in the lemma will be fixed along the proof. We assume that $A_1 \gg \kappa^{-1} > 1$. First note that we may assume that $r < 2A_1^{-1}|x_0 - p|$. Otherwise, we just take a point $q' \in \Omega$ such that $|p - q'| = \delta_{\Omega}(p)/2$, which clear satisfies the properties in [\(3.3\)](#page-12-1). Further, both q and q' belong to the open connected set

$$
U := \{ x \in \Omega : g(p, x) > c_2 r_0 \, \delta_{\Omega}(p)^{-n} \}
$$

for a sufficiently small $c_2 > 0$. The fact that U is connected is well known. This follows from the fact that, for any $\lambda > 0$, any connected component of $\{g(p, \cdot) > \lambda\}$ should contain p. Otherwise there would be a connected component where $g(p, \cdot) - \lambda$ is positive and harmonic with zero boundary values. So, by maximum principle, $g(p, \cdot) - \lambda$ should equal λ in the whole component, which is a contradiction. So there is only one connected component.

We just let γ be a curve contained in U. Note that

$$
\text{dist}(U, \partial \Omega) \ge c r_0^{\frac{1}{\alpha}} \delta_{\Omega}(p)^{1-\frac{1}{\alpha}} \ge a r_0,
$$

for a sufficiently small $a > 0$ because, by boundary Hölder continuity,

$$
g(p,x) \lesssim \left(\frac{\delta_{\Omega}(x)}{\delta_{\Omega}(p)}\right)^{\alpha} \frac{1}{\delta_{\Omega}(p)^{n-1}}
$$

if $dist(x, \partial \Omega) \le \delta_{\Omega}(p)/2$. Further, the fact that $g(p, x) \le c|x - p|^{1-n}$ ensures that $U \subset B(p, C\delta_{\Omega}(p)),$ for a sufficiently big constant C depending on r/r_0 .

14 AZZAM, MOURGOGLOU, AND TOLSA

So from now on we assume that $r < 2A_1^{-1}|x_0 - p|$. By Lemma [3.1](#page-12-2) we know there exists some point $\widetilde{q} \in \Omega$ such that

$$
(3.4) \qquad \tilde{q} \in B(x_0, \kappa^{-1}r), \quad \delta_{\Omega}(\tilde{q}) \ge r \ge \kappa |x_0 - \tilde{q}| \ge \kappa \delta_{\Omega}(\tilde{q}) \ge \kappa r, \quad g(p, \tilde{q}) \ge c \frac{\delta_{\Omega}(\tilde{q})}{\delta_{\Omega}(p)^n},
$$

with c depending on κ and Λ .

Assume that q and \tilde{q} cannot be joined by a curve γ as in the statement of the lemma. Otherwise, we are done. For $t > 0$, consider the open set

$$
V^t = \{ x \in B(x_0, \frac{1}{4}A_1r) : g(p, x) > t r_0 \,\delta_{\Omega}(p)^{-n} \}.
$$

We fix $t > 0$ small enough such that $q, \tilde{q} \in V^{2t} \subset V^t$. Such t exists by [\(3.2\)](#page-12-3) and [\(3.4\)](#page-13-0), and it may depend on Λ , λ , r/r_0 .

Let V_1 and V_2 be the respective components of V^t to which q and \tilde{q} belong. We have

$$
V_1 \cap V_2 = \varnothing,
$$

because otherwise there is a curve contained in $V^t \subset B(x_0, \frac{1}{4}A_1r)$ which connects q and \tilde{q} , and further this is far away from $\partial\Omega$. Indeed, we claim that

$$
\text{(3.5)} \quad \text{dist}(V^t, \partial \Omega) \gtrsim_{A_1, \Lambda, t, r/r_0} r_0.
$$

To see this, note that by the Hölder continuity of $g(p, \cdot)$ in $B(x_0, \frac{1}{2}A_1r)$, for all $x \in V^t$, we have

$$
t \frac{r_0}{\delta_{\Omega}(p)^n} \le g(p, x) \lesssim \sup_{y \in B(x_0, \frac{1}{2}A_1r)} g(p, y) \left(\frac{\delta_{\Omega}(x)}{A_1r}\right)^{\alpha}
$$

$$
\le \int_{B(x_0, \frac{3}{4}A_1r)} g(p, y) dy \left(\frac{\delta_{\Omega}(x)}{A_1r}\right)^{\alpha}
$$

$$
\lesssim_{A_1, \Lambda} \frac{A_1r}{\delta_{\Omega}(p)^n} \left(\frac{\delta_{\Omega}(x)}{A_1r}\right)^{\alpha},
$$

where in the last inequality we used Lemma [2.5](#page-6-1) and that $x_0 \in WA(p, \Lambda)$. This yields our claim.

Next we wish to apply the Alt-Caffarelli-Friedman formula with

$$
u_1(x) = \chi_{V_1} (\delta_{\Omega}(p)^n g(p, x) - t r_0)^+,
$$

$$
u_2(x) = \chi_{V_2} (\delta_{\Omega}(p)^n g(p, x) - t r_0)^+.
$$

It is clear that both satisfy the hypotheses of Theorem [2.9.](#page-8-1) For $i = 1, 2$ and $0 < s < A_1r$, we denote

$$
J_i(x_0, s) = \frac{1}{s^2} \int_{B(x_0, s)} \frac{|\nabla u_i(y)|^2}{|y - x_0|^{n-1}} dy,
$$

so that $J(x_0, s) = J_1(x_0, s) J_2(x_0, s)$. We claim that:

- (i) $J_i(x_0, s) \leq_{\Lambda} 1$ for $i = 1, 2$ and $0 < s < \frac{1}{4}A_1r$.
- (ii) $J_i(x_0, 2r) \gtrsim_{\Lambda, \lambda, r/r_0} 1$ for $i = 1, 2$.

The condition (i) follows from [\(2.8\)](#page-8-6) and the fact that

(3.6)
$$
g(p,y) \lesssim \frac{s}{\delta_{\Omega}(p)^n} \quad \text{for all } y \in B(x_0,s),
$$

which holds by Lemma [2.5](#page-6-1) and subharmonicity, since $x_0 \in \text{WA}(p, \Lambda)$. Concerning (ii), note first that

$$
|\nabla u_1(y)| \lesssim \delta_{\Omega}(p)^n \frac{g(p,y)}{\delta_{\Omega}(y)} \lesssim_{\tau_0} \delta_{\Omega}(p)^n \frac{r_0}{\delta_{\Omega}(p)^n} = 1 \quad \text{ for all } y \in B(q, \tau_0 r_0/2),
$$

where we first used Cauchy estimates and then the pointwise bounds of $g(\cdot, \cdot)$ in [\(3.6\)](#page-13-1) with $s \approx \delta_{\Omega}(y)$. Thus, using also that $q \in V^{2t}$, we infer that $u_1(y) > 1.5t r_0$ in some ball $B(q, c t r_0)$ with c possibly depending on Λ , λ , r/r_0 . Analogously, we deduce that $u_2(y) > 1.5t r_0$ in some ball $B(\tilde{q}, ctr_0)$. Let B be the largest open ball centered at q not intersecting ∂V_1 and let $y_0 \in \partial V_1 \cap \partial B$. Then, by considering the convex hull $H \subset B$ of $B(q, ctr_0)$ and y₀ and integrating in spherical coordinates (with the origin in y_0), one can check that

$$
\int_H |\nabla u_1| \, dy \gtrsim_t r_0^{n+1}.
$$

An analogous estimate holds for u_2 , and then it easily follows that

$$
J_i(x_0, 2r_0) \gtrsim_t 1,
$$

which implies (ii). We leave the details for the reader.

From the conditions (i) and (ii) and the fact that $J(x, r)$ is non-decreasing we infer that

$$
J(x_0, s) \approx_{\Lambda, \lambda, r/r_0} 1 \quad \text{for } 2r < s < \frac{1}{4}A_1r.
$$

and also

(3.7)
$$
J_i(x_0, s) \approx_{\Lambda, \lambda, r/r_0} 1 \quad \text{for } i = 1, 2 \text{ and } 2r < s < \frac{1}{4} A_1 r.
$$

Assume that $\frac{1}{4}A_1 = 2^m$ for some big $m > 1$. Since $J(x_0, s)$ is non-decreasing we infer that there exists some $h \in [1, m-1]$ such that

$$
J(x_0, 2^{h+1}r) \le C(\Lambda, \lambda, r/r_0)^{1/m} J(x_0, 2^h r),
$$

because otherwise, by iterating the reverse inequality, we get a contradiction. Now from Lemma [2.12](#page-10-0) we deduce that, given any $\varepsilon > 0$, for m big enough, there are constant $k_i \approx_{\Lambda,\lambda,r/r_0} 1$ and a unit vector e such that

$$
(3.8) \t\t ||u_1 - k_1 ((\cdot - x_0) \cdot e)^+||_{\infty, B(x_0, 2^h r)} + ||u_2 - k_2 ((\cdot - x_0) \cdot e)^-||_{\infty, B(x_0, 2^h r)} \leq \varepsilon 2^h r.
$$

Indeed, $||u_i||_{\infty,B(x_0,2^hr)} \approx_{\Lambda,\lambda,r/r_0} 2^hr$ by [\(2.8\)](#page-8-6) and [\(3.7\)](#page-14-0); $||u_i||_{\text{Lip}^{\alpha},\text{B}(x_0,2^{h+r})} \lesssim_{\Lambda,\lambda,r/r_0} (2^hr)^{1-\alpha}$ by Lemma [2.6;](#page-6-0) and the option (a) in Lemma [2.12](#page-10-0) cannot hold (since $||u_i||_{\infty,B(x_0,2^{h_r})} \approx_{\Lambda,\lambda,r/r_0} 2^{h_r}$).

In particular, for ε small, [\(3.8\)](#page-14-1) implies that if $q' := x_0 + 2^{h-1}re$, then $u_1(q') \approx_{\Lambda,\lambda,r/r_0} 2^{h-1}r$, and also that

$$
u_1(y) \approx_{\Lambda,\lambda,r/r_0} 2^{h-1}r > 0 \quad \text{ for all } y \in B(q', 2^{h-2}r).
$$

Thus $B(q', 2^{h-2}r) \subset \Omega$ and so q' is at a distance at least $2^{h-2}r$ from $\partial\Omega$, and also

$$
g(p,q') \ge \frac{u_1(q')}{\delta_{\Omega}(p)^n} \approx_{\Lambda,\lambda,r/r_0} \frac{2^h r}{\delta_{\Omega}(p)^n}.
$$

Further, since q and q' are both in V_1 by definition, there is a curve γ which joins q and q' contained in V_1 satisfying

$$
dist(\gamma, \partial \Omega) \gtrsim_{A_1, \Lambda, t, r/r_0} r_0,
$$

by (3.5) .

4. TYPES OF CUBES

From now on we fix $R_0 \in \mathcal{D}_{\mu}$ and $p \in \Omega$ and we assume that we are under the assumptions of the Main Lemma [2.13.](#page-11-2)

We need now to define two families HD and LD of high density and low density cubes, respectively. Let $A \gg 1$ be some fixed constant. We denote by HD (high density) the family of maximal cubes $Q \in \mathcal{D}_{\mu}$ which are contained in R_0 and satisfy

$$
\frac{\omega^p(2Q)}{\mu(2Q)} \ge A \frac{\omega^p(2R_0)}{\mu(2R_0)}.
$$

We also denote by LD (low density) the family of maximal cubes $Q \in \mathcal{D}_{\mu}$ which are contained in R_0 and satisfy

$$
\frac{\omega^p(Q)}{\mu(Q)} \le A^{-1} \frac{\omega^p(R_0)}{\mu(R_0)}
$$

(notice that $\omega^p(R_0) \approx \omega^p(2R_0) \approx 1$ by assumption). Observe that the definition of the family HD involves the density of 2Q, while the one of LD involves the density of Q.

We denote

$$
B_H = \bigcup_{Q \in \mathsf{HD}} Q \quad \text{and} \quad B_L = \bigcup_{Q \in \mathsf{LD}} Q.
$$

Lemma 4.1. *We have*

$$
\mu(B_H) \lesssim \frac{1}{A} \mu(R_0) \quad \text{and} \quad \omega^p(B_L) \le \frac{1}{A} \omega^p(R_0).
$$

Proof. By Vitali's covering theorem, there exists a subfamily $I \subset HD$ so that the cubes $2Q, Q \in I$, are pairwise disjoint and

$$
\bigcup_{Q\in \mathsf{HD}} 2Q \subset \bigcup_{Q\in I} 6Q.
$$

Then, since μ is doubling, we obtain

$$
\mu(B_H) \lesssim \sum_{Q \in I} \mu(2Q) \le \frac{1}{A} \sum_{Q \in I} \frac{\omega^p(2Q)}{\omega^p(2R_0)} \mu(2R_0) \lesssim \frac{1}{A} \mu(R_0).
$$

Next we turn our attention to the low density cubes. Since the cubes from LD are pairwise disjoint, we have

$$
\omega^p(B_L) = \sum_{Q \in \mathsf{LD}} \omega^p(Q) \le \frac{1}{A} \sum_{Q \in \mathsf{LD}} \frac{\mu(Q)}{\mu(R_0)} \omega^p(R_0) \le \frac{1}{A} \omega^p(R_0).
$$

From the above estimates and the fact that the harmonic measure belongs to weak- A_{∞} , we infer that if A is chosen big enough, then

$$
\omega^p(B_H) \le \varepsilon_0 \,\omega^p(2B_{R_0}) \le \frac{1}{4} \,\omega^p(R_0)
$$

and thus

$$
\omega^p(B_H \cup B_L) \le \frac{1}{4} \omega^p(R_0) + \frac{1}{A} \omega^p(R_0) \le \frac{1}{2} \omega^p(R_0).
$$

As a consequence, denoting $G_0 = R_0 \setminus (B_H \cup B_L)$, we deduce that

$$
\omega^p(G_0) \ge \frac{1}{2} \omega^p(R_0) \approx \omega^p(2B_{R_0}),
$$

which implies that

$$
\mu(G_0) \gtrsim \mu(2B_{R_0}) \approx \mu(R_0),
$$

again using the fact that ω^p belongs to weak- A_{∞} in B_{R_0} . So we have:

Lemma 4.2. Assuming A big enough, the set $G_0 := R_0 \setminus (B_H \cup B_L)$ satisfies

$$
\omega^p(G_0) \approx 1 \quad \text{and} \quad \mu(G_0) \approx \mu(R_0),
$$

with the implicit constants depending on C_0 and the weak- A_∞ condition in $B_{R_0}.$

We denote by G the family of those cubes $Q \in \mathcal{D}_{\mu}(R_0)$ which are not contained in $\bigcup_{P \in \text{HD} \cup \text{LD}} P$. In particular, such cubes $Q \in \mathsf{G}$ do not belong to HD ∪ LD and thus

(4.1)
$$
A^{-1}\frac{\omega^p(R_0)}{\mu(R_0)} \leq \frac{\omega^p(Q)}{\mu(Q)} \lesssim \frac{\omega^p(2Q)}{\mu(2Q)} \leq A \frac{\omega^p(2R_0)}{\mu(2R_0)}.
$$

From this fact, it follows easily that G_0 is contained in the set WA(p, Λ) defined in Section [3,](#page-11-1) assuming Λ big enough, and so Lemma [4.2](#page-16-0) ensures that [\(3.1\)](#page-11-3) holds.

The following lemma is an immediate consequence of Lemma [3.1.](#page-12-2)

Lemma 4.3. *For every cube* $Q \in G$ *there exists some point* $x_Q \in 2B_Q \cap \Omega$ *such that* $\delta_{\Omega}(x_Q) \ge \kappa_0 \ell(Q)$ *and*

(4.2)
$$
g(p, x_Q) > c_3 \frac{\ell(Q)}{\mu(R_0)},
$$

for some κ_0 , $c_3 > 0$, which depend on A and on the weak- A_∞ constants in B_{R_0} .

If $x_Q \in 2B_Q \cap \Omega$ and $\delta_{\Omega}(x_Q) \ge \kappa_0 \ell(Q)$, we say that x_Q is κ_0 -corkscrew for Q. If [\(4.2\)](#page-16-1) holds, we say that x_Q is a c_3 -good corkscrew for Q . Abusing notation, quite often we will not write "for Q ".

We will need the following auxiliary result:

Lemma 4.4. Let $Q \in \mathcal{D}_{\mu}$ and let x_Q be a λ -good c_4 -corkscrew, for some $\lambda, c_4 > 0$. Suppose that $\ell(Q) \geq c_5 \ell(R_0)$. Then there exists some C-good Harnack chain that joins x_Q and p, with C depending *on* λ , c_5 .

Proof. Consider the open set $U = \{x \in \Omega : g(p, x) > \lambda \ell(Q)/\mu(R_0)\}\)$. This is connected and thus there exists a curve $\gamma \subset U$ that connects x_Q and p. By Hölder continuity, any point $x \in \Omega$ such that $\delta_{\Omega}(x) \leq \delta_{\Omega}(p)/2$, satisfies

$$
g(p,x) \le c \left(\frac{\delta_{\Omega}(x)}{\ell(R_0)}\right)^{\alpha} \frac{1}{\ell(R_0)^{n-1}}.
$$

Since $g(p, x) > \lambda \ell(Q)/\mu(R_0) \gtrsim_{c_5, \lambda} \ell(R_0)^{1-n}$ for all $x \in U$, we deduce that $dist(U, \partial \Omega) \ge c_6 \ell(R_0)$ for some $c_6 > 0$ depending on λ and c_5 . Thus,

$$
dist(\gamma, \partial \Omega) \ge c_6 \ell(R_0).
$$

From the fact that $g(p, x) \le |p - x|^{1-n}$ for all $x \in \Omega$, we infer that any $x \in U$ satisfies

$$
\lambda \frac{\ell(Q)}{\mu(R_0)} < g(p, x) \le \frac{1}{|p - x|^{n-1}}.
$$

Therefore,

$$
|p-x|<\left(\frac{\mu(R_0)}{\lambda \ell(Q)}\right)^{1/(n-1)}\lesssim_{c_5,\lambda} \ell(R_0).
$$

So $U \subset B(p, C_2 \ell(R_0))$ for some C_2 depending on λ and c_5 . Next we consider a Besicovitch covering of γ with balls B_i of radius $c_6\ell(R_0)/2$. By volume considerations, it easily follows that the number of balls B_i is bounded above by some constant C_3 depending on λ and c_5 , and thus this is a C-good Harnack chain, with $C = C(\lambda, c_5)$.

Lemma 4.5. *There exists some constant* κ_1 *with* $0 < \kappa_1 \leq \kappa_0$ *such that the following holds for all* $\lambda > 0$ *. Let* $Q \in G$ *,* $Q \neq R_0$ *, and let* x_Q *be a* λ *-good* κ_1 *-corkscrew. Then there exists some cube* $R \in G$ with $Q \subsetneq R \subset R_0$ and $\ell(R) \leq C \ell(Q)$ and a λ' -good κ_1 -corkscrew x_R such that x_Q and x_R can be *joined by a* $C'(\lambda)$ -good Harnack chain, with $\lambda' > 0$ and C depending on λ *.*

The proof below yields a constant $\lambda' < \lambda$. On the other hand, the lemma ensures that x_R is still a κ_1 -corkscrew, which will be important for the arguments to come.

Proof. This follows easily from Lemma [3.2.](#page-12-0) For completeness we will show the details.

By choosing $\Lambda = \Lambda(A) > 0$ big enough, $G_0 \cap Q \subset \text{WA}(p, \Lambda)$ and thus there exists some $x_0 \in$ $Q \cap WA(p, \Lambda)$. We let

$$
\kappa_1 = \min(\kappa_0, \kappa),
$$

where κ_0 is defined in Lemma [4.3](#page-16-2) and κ in Lemma [3.1](#page-12-2) (and thus it depends only on A and C_0). We apply Lemma [3.2](#page-12-0) to x_0 , q, with $r_0 = 3r(B_Q)$, $\lambda_0 \approx \lambda$, and $r = 4r(B_Q)$. To this end, note that

$$
\delta_{\Omega}(q) \geq \kappa_1 \ell(Q) = \kappa_1 \frac{1}{4} \ell(r(B_Q)) = \kappa_1 \frac{1}{12} r_0.
$$

Hence there exists $q' \in B(x_0, A_1r)$ such that

(4.3)
$$
\delta_{\Omega}(q') \geq \kappa |x_0 - q'| \geq \kappa r, \qquad g(p,q') \geq \lambda_1 \frac{\delta_{\Omega}(q')}{\delta_{\Omega}(p)^n},
$$

and such that q and q' can be joined by a curve γ such that

(4.4)
$$
\gamma \subset \{y \in B(x_0, A_1r) : \text{dist}(y, \partial \Omega) > a_1 r_0\},\
$$

with λ_1, A_1, a_1 depending on on $C_0, A, \lambda, \kappa_1$. Now let $R \in \mathcal{D}_{\mu}$ be the cube containing x_0 such that

$$
\frac{1}{2} r(B_R) < |x_0 - q'| \le r(B_R).
$$

Observe that

$$
r(B_R) \ge |x_0 - q'| \ge r = 4r(B_Q)
$$
 and $r(B_R) < 2|x_0 - q'| \le 2A_1 r \lesssim_{\lambda} \ell(Q).$

Also, we may assume that $\ell(R) \leq \ell(R_0)$ because otherwise we have $\ell(Q) \geq A_1 \delta_{\Omega}(p)$ and then the statement in the lemma follows from Lemma [4.4.](#page-16-3) So we have $Q \subsetneq R \subset R_0$.

From [\(4.3\)](#page-17-0) we get

$$
\delta_{\Omega}(q') \ge \kappa |x_0 - q'| \ge \frac{1}{2} \kappa r(B_R) > \kappa_1 \ell(R)
$$

and

$$
g(p,q') \ge c \lambda_1 \frac{2\kappa \ell(R)}{\mu(R_0)}.
$$

From [\(4.4\)](#page-17-1) and arguing as in the end of the proof of Lemma [4.4](#page-16-3) we infer that x_Q and x_R can be joined by a $C(\lambda)$ -good Harnack chain.

From now on we will assume that all corkscrew points for cubes $Q \in G$ are κ_1 -corkscrews, unless otherwise stated.

5. THE CORONA DECOMPOSITION AND THE KEY LEMMA

5.1. The corona decomposition. Recall that the $b\beta$ coefficient of a ball was defined in [\(2.4\)](#page-5-5). For each $Q \in \mathcal{D}_{\mu}$, we denote

$$
b\beta(Q) = b\beta_{\partial\Omega}(100B_Q).
$$

Now we fix a constant $0 < \varepsilon \ll \min(1, \kappa_1)$. Given $R \in \mathcal{D}_{\mu}(R_0)$, we denote by Stop (R) the maximal family of cubes $Q \in \mathcal{D}_{\mu}(R) \setminus \{R\}$ satisfying that either $Q \notin \mathsf{G}$ or $b\beta(\widehat{Q}) > \varepsilon$, where \widehat{Q} is the parent of Q. Recall that the family G was defined in [\(4.1\)](#page-16-4). Note that, by maximality, $\text{Stop}(R)$ is a family of pairwise disjoint cubes.

We define

$$
\mathsf{Tree}(R) := \{ Q \in \mathcal{D}_{\mu}(R) : \nexists \ S \in \mathsf{Stop}(R) \text{ such that } Q \subset S \}.
$$

In particular, note that $\text{Stop}(R) \not\subset \text{Tree}(R)$.

We now define the family of the top cubes with respect to R_0 as follows: first we define the families Top_k for $k \geq 1$ inductively. We set

Top₁ = {
$$
R \in \mathcal{D}_{\mu}(R_0) \cap G : \ell(R) = 2^{-10} \ell(R_0)
$$
 }.

Assuming that Top_k has been defined, we set

$$
\mathsf{Top}_{k+1}=\bigcup_{R\in \mathsf{Top}_k}(\mathsf{Stop}(R)\cap \mathsf{G}),
$$

and then we define

$$
\mathsf{Top} = \bigcup_{k \geq 1} \mathsf{Top}_k.
$$

Notice that the family of cubes $Q \in \mathcal{D}_{\mu}(R_0)$ with $\ell(Q) \leq 2^{-10} \ell(R_0)$ which are not contained in any cube $P \in \text{HD} \cup \text{LD}$ is contained in $\bigcup_{R \in \text{Top}} \text{Tree}(R)$, and this union is disjoint. Also, all the cubes in that union belong to G.

The following lemma is an easy consequence of our construction. Its proof is left for the reader.

Lemma 5.1. *We have*

$$
\mathsf{Top} \subset \mathsf{G}.
$$

Also, for each $R \in \text{Top}$,

Tree(R) \subset G.

Further, for all $Q \in \mathsf{Tree}(R) \cup \mathsf{Stop}(R)$ *,*

$$
\omega^p(2Q) \le C A \frac{\mu(Q)}{\mu(R_0)}.
$$

Remark that the last inequality holds for any cube $Q \in Stop(R)$ because its parent \hat{Q} belongs to Tree(R) and so $\widehat{Q} \notin \mathsf{HD}$, which implies that $\omega^p(2Q) \leq \omega^p(2\widehat{Q}) \lesssim A \frac{\mu(\widehat{Q})}{\mu(R_0)} \approx A \frac{\mu(Q)}{\mu(R_0)}$ $\frac{\mu(Q)}{\mu(R_0)}$.

Using that μ is uniformly rectifiable, it is easy to prove that the cubes from Top satisfy a Carleson packing condition. This is shown in the next lemma.

Lemma 5.2. *We have*

$$
\sum_{R \in \mathsf{Top}} \mu(R) \leq M(\varepsilon) \, \mu(R_0).
$$

Proof. For each $Q \in \text{Top}$ we have

$$
\mu(Q) = \sum_{P \in \mathsf{Stop}(Q) \cap \mathsf{G}} \mu(P) + \sum_{P \in \mathsf{Stop}(Q) \backslash \mathsf{G}} \mu(P) + \mu\bigg(Q \setminus \bigcup_{P \in \mathsf{Stop}(Q)} P\bigg).
$$

Then we get

(5.1)
$$
\sum_{Q \in \text{Top}} \mu(Q) \leq \sum_{Q \in \text{Top}} \sum_{P \in \text{Stop}(Q) \cap \text{G}} \mu(P)
$$

$$
+ \sum_{Q \in \text{Top}} \sum_{P \in \text{Stop}(Q) \setminus \text{G}} \mu(P) + \sum_{Q \in \text{Top}} \mu\left(Q \setminus \bigcup_{P \in \text{Stop}(Q)} P\right).
$$

Note now that, because of the stopping conditions, for all $Q \in Top$, if $P \in Stop(Q) \cap G$, then the parent \hat{P} of P satisfies $b\beta_{\partial\Omega}(100B_{\hat{P}}) > \varepsilon$. Hence, by Theorems [2.1](#page-5-6) and [2.2,](#page-5-7)

$$
\sum_{Q \in \mathsf{Top}} \sum_{P \in \mathsf{Stop}(Q) \cap \mathsf{G}} \mu(P) \leq \sum_{P \in \mathcal{D}_{\mu}(R_0): b\beta_{\partial\Omega}(100B_{\widehat{P}}) > \varepsilon} \mu(P) \leq C(\varepsilon) \, \mu(R_0).
$$

On the other hand, the cubes $P \in \text{Stop}(Q) \setminus G$ with $Q \in \text{Top}$ do not contain any cube from Top, by construction. Hence, they are disjoint and thus

$$
\sum_{Q \in \text{Top }P \in \text{Stop}(Q) \backslash \text{G}} \mu(P) \le \mu(R_0).
$$

By an analogous reason,

$$
\sum_{Q \in \mathsf{Top}} \mu\bigg(Q \setminus \bigcup_{P \in \mathsf{Stop}(Q)} P\bigg) \leq \mu(R_0).
$$

By (5.1) and the estimates above, the lemma follows.

Given a constant $K \gg 1$, next we define

(5.2)
$$
G_0^K = \left\{ x \in G_0 : \sum_{R \in \mathsf{Top}} \chi_R(x) \leq K \right\},\
$$

By Chebyshev and the preceding lemma, we have

$$
\mu(G_0 \setminus G_0^K) \le \mu(R_0 \setminus G_0^K) \le \frac{1}{K} \int_{R_0} \sum_{R \in \mathsf{Top}} \chi_R d\mu \le \frac{M(\varepsilon)}{K} \mu(R_0).
$$

Therefore, if K is chosen big enough (depending on $M(\varepsilon)$ and the constants on the weak- A_{∞} condition), by Lemma [4.2](#page-16-0) we get

$$
\mu(G_0 \setminus G_0^K) \le \frac{1}{2} \mu(G_0),
$$

and thus

$$
\mu(G_0^K) \ge \frac{1}{2}\,\mu(G_0) \gtrsim \mu(R_0).
$$

 $\ddot{}$

We distinguish now two types of cubes from Top. We denote by Top_a the family of cubes $R \in \text{Top}$ such that $Tree(R) = \{R\}$, and we set $Top_b = Top \setminus Top_a$. Notice that, by construction, if $R \in Top_b$, then $b\beta(R) \leq \varepsilon$. On the other hand, this estimate may fail if $R \in \mathsf{Top}_a$.

5.2. The truncated corona decomposition. For technical reasons, we need now to define a truncated version of the previous corona decomposition. We fix a big natural number $N \gg 1$. Then we let Top^(N) be the family of the cubes from Top with side length larger than $2^{-N}\ell(R_0)$. Given $R \in \text{Top}^{(N)}$ we let Tree $_{b}^{(N)}(R)$ be the subfamily of the cubes from Tree(R) with side length larger than $2^{-N}\ell(R_0)$, and we let Stop $^{(N)}(R)$ be a maximal subfamily from Stop $(R)\cup\mathcal{D}_{\mu,N}(R_0),$ where $\mathcal{D}_{\mu,N}(R_0)$ is the subfamily of the cubes from $\mathcal{D}_{\mu}(R_0)$ with side length $2^{-N}\ell(R_0)$. We also denote $\text{Top}_a^{(N)} = \text{Top}^{(N)} \cap \text{Top}_a$ and $\mathsf{Top}^{(N)}_b = \mathsf{Top}^{(N)} \cap \mathsf{Top}_b.$

Observe that, since $\text{Top}^{(N)} \subset \text{Top}$, we also have

$$
\sum_{R \in \mathsf{Top}^{(N)}} \chi_R(x) \leq \sum_{R \in \mathsf{Top}} \chi_R(x) \leq K \quad \text{ for all } x \in G_0^K.
$$

5.3. The Key Lemma. The main ingredient for the proof of the Main Lemma [2.13](#page-11-2) is the following result.

Lemma 5.3 (Key Lemma). *Given* $\eta \in (0,1)$ *and* $\lambda \in (0,c_3]$ *(with* c_3 *as in* [\(4.2\)](#page-16-1)*), there exists an exceptional family* $Ex(R)$ ⊂ $Stop(R)$ ∩ G *satisfying*

$$
\sum_{P \in \mathsf{Ex}(R)} \mu(P) \le \eta \,\mu(R)
$$

such that, for every Q ∈ Stop(R) ∩ G \ Ex(R)*, any* λ*-good corkscrew for* Q *can be joined to some* λ' -good corkscrew for R by a $C(\lambda, \eta)$ -good Harnack chain, with λ' depending on λ, η .

This lemma will be proved in the next Sections [6](#page-21-1) and [7.](#page-30-0) Using this result, in Section [8](#page-40-0) we will build the required carrot curves for the Main Lemma [2.13,](#page-11-2) which join the pole p to points from a suitable big piece of R_0 . If the reader prefers to see how this is applied before its long proof, they may go directly to Section [8.](#page-40-0) A key point in the Key Lemma is that the constant ε in the definition of the stopping cubes of the corona decomposition does not depend on the constants λ or η above.

To prove the Key Lemma [5.3](#page-20-3) we will need first to introduce the notion of "cubes with well separated big corkscrews" and we will split $Tree^{(N)}(R)$ into subtrees by introducing an additional stopping condition involving this type of cubes. Later on, in Section [6](#page-21-1) we will prove the "Geometric Lemma", which relies on a geometric construction which plays a fundamental role in the proof of the Key Lemma.

5.4. The cubes with well separated big corkscrews. Let $Q \in \mathcal{D}_{\mu}$ be a cube such that $b\beta(Q) \leq C_4 \varepsilon$. For example, Q might be a cube from $Q \in \text{Tree}^{(N)}(R) \cup \text{Stop}^{(N)}(R)$, with $R \in \text{Top}_{b}^{(N)}$ (which in particular implies that $b\beta(R) \leq \varepsilon$). We denote by L_Q a best approximating *n*-plane for $b\beta(Q)$, and we choose x_Q^1 and x_Q^2 to be two fixed points in B_Q such that $dist(x_Q^i, L_Q) = r(B_Q)/2$ and lie in different components of $\mathbb{R}^{n+1} \setminus L_Q$. So x_Q^1 and x_Q^2 are corkscrews for Q. We will call them "big corkscrews".

Since any corkscrew x for Q satisfies $\delta_{\Omega}(x) \geq \kappa_1 \ell(Q)$ and we have chosen $\varepsilon \ll \kappa_1$, it turns out that

$$
dist(x, L_Q) \ge \frac{1}{2} \kappa_1 \ell(Q) \gg \varepsilon \ell(Q).
$$

As a consequence, x can be joined either to x_Q^1 or to x_Q^2 by a C -good Harnack chain, with C depending only on n, C_0, κ_1 , and thus only on n, C_0 and the weak- A_{∞} constants in B_{R_0} . The following lemma follows by the same reasoning:

Lemma 5.4. Let $Q, Q' \in \mathcal{D}_{\mu}$ be cubes such that $b\beta(Q), b\beta(Q') \leq C_4\varepsilon$ and Q' is the parent of Q . Let x_Q^i , $x_{Q'}^i$, for $i = 1, 2$, be big corkscrews for Q and Q' respectively. Then, after relabelling the *corkscrews if necessary,* x i ^Q *can be joined to* x i ^Q′ *by a* C*-good Harnack chain, with* C *depending only on* n, C_0, κ_1 *.*

Given $\Gamma > 0$, we will write $Q \in WSBC(\Gamma)$ (or just $Q \in WSBC$, which stands for "well separated big corkscrews") if $b\beta(Q) \leq C_4\varepsilon$ and the big corkscrews x_Q^1 , x_Q^2 can *not* be joined by any Γ-good Harnack chain. The parameter Γ will be chosen below. For the moment, let us say that $\Gamma^{-1} \ll \varepsilon$. The reader should think that in spite of $b\beta(Q) \leq C_4\varepsilon$, the possible existence of "holes of size $C \varepsilon \ell(Q)$ in supp μ " makes possible the connection of the big corkscrews by means of Γ-Harnack chains passing through these holes. Note that if $Q \notin WSBC(\Gamma)$, then any pair of corkscrews for Q can be connected by a $C(\Gamma)$ -good Harnack chain, since any of these corkscrews can be joined by a good chain to one of the big corkscrews for Q, as mentioned above.

5.5. The tree of cubes of type WSBC and the subtrees. Given $R \in \text{Top}_{b}^{(N)}$, denote by $\text{Stop}_{\text{WSBC}}(R)$ the maximal subfamily of cubes from $Q \in \mathcal{D}_{\mu}(R)$ which satisfy that either

- $Q \notin \text{WSBC}(\Gamma)$, or
- $Q \notin \mathsf{Tree}^{(N)}(R)$.

Also, denote by Tree_{WSBC}(R) the cubes from $\mathcal{D}_{\mu}(R)$ which are not strictly contained in any cube from Stop_{WSBC}(R). So this tree is empty if $R \notin WSBC(\Gamma)$.

Observe that if $Q \in \mathsf{Stop}_{\mathsf{WSEG}}(R)$, it may happen that $Q \notin \mathsf{WSEC}(\Gamma)$. However, unless $Q = R$, it holds that $Q \in \mathsf{WSBC}(\Gamma')$, with $\Gamma' > \Gamma$ depending only on Γ and C_0 (because the parent of Q belongs to WSBC (Γ)).

For each $Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \setminus \mathsf{Stop}(R)$, we denote

$$
\mathsf{SubTree}(Q) = \mathcal{D}_\mu(Q) \cap \mathsf{Tree}^{(N)}(R), \qquad \mathsf{SubStop}(Q) = \mathsf{Stop}(R) \cap \mathcal{D}_\mu(Q).
$$

So we have

$$
\mathsf{Tree}^{(N)}(R) = \mathsf{Tree}_{\mathsf{WSEG}}(R) \cup \bigcup_{Q \in \mathsf{Stop}_{\mathsf{WSEG}}(R)} \mathsf{SubTree}(Q),
$$

and the union is disjoint. Observe also that we have the partition

(5.3)
$$
\mathsf{Stop}(R) = \left(\mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{Stop}(R) \right) \cup \bigcup_{Q \in \mathsf{Stop}_{\mathsf{WSE}}(R) \backslash \mathsf{Stop}(R)} \mathsf{SubStop}(Q).
$$

6. THE GEOMETRIC LEMMA

6.1. The geometric lemma for the tree of cubes of type WSBC. Let $R \in \text{Top}_{b}^{(N)}$ and suppose that Tree_{WSBC} $(R) \neq \emptyset$. We need now to define a family End (R) of cubes from \mathcal{D}_{μ} , which in a sense can be considered as a regularized version of $Stop(R)$. The first step consists of introducing the following auxiliary function:

$$
d_R(x) := \inf_{Q \in \mathsf{Tree}_{\mathsf{WSEC}}(R)} (\ell(Q) + \text{dist}(x, Q)), \quad \text{ for } x \in \mathbb{R}^{n+1}.
$$

Observe that d_R is 1-Lipschitz.

For each $x \in \partial\Omega$ we take the largest cube $Q_x \in \mathcal{D}_\mu$ such that $x \in Q_x$ and

(6.1)
$$
\ell(Q_x) \leq \frac{1}{300} \inf_{y \in Q_x} d_R(y).
$$

We consider the collection of the different cubes Q_x , $x \in \partial \Omega$, and we denote it by End(R).

Lemma 6.1. *Given* $R \in \text{Top}_{b}^{(N)}$, the cubes from $\text{End}(R)$ are pairwise disjoint and satisfy the following *properties:*

- *(a) If* $P \in \text{End}(R)$ *and* $x \in 50B_P$ *, then* $100 \ell(P) \le d_R(x) \le 900 \ell(P)$ *.*
- *(b)* There exists some absolute constant C such that if $P, P' \in End(R)$ and $50B_P \cap 50B_{P'} \neq \emptyset$, *then* $C^{-1}\ell(P) \leq \ell(P') \leq C \ell(P)$ *.*
- *(c)* For each $P \in \text{End}(R)$, there at most N cubes $P' \in \text{End}(R)$ such that $50B_P ∩ 50B_{P'} \neq ∅$, *where* N *is some absolute constant.*
- *(d)* If $P \in \text{End}(R)$ and $dist(P, R) \leq 20 \ell(R)$, then there exists some $Q \in \text{Tree}_{\text{WSRC}}(R)$ such that $P \subset 22Q$ and $\ell(Q) \leq 2000 \, \ell(P)$.

Proof. The proof is a routine task. For the reader's convenience we show the details.. To show (a), consider $x \in 50B_P$. Since $d_R(\cdot)$ is 1-Lipschitz and, by definition, $d_R(z_P) \geq 300 \ell(P)$, we have

$$
d_R(x) \ge d_R(z_P) - |x - z_P| \ge d_R(z_P) - 50 r(B_P) \ge 300 \ell(P) - 200 \ell(P) = 100 \ell(P).
$$

To prove the converse inequality, by the definition of $\text{End}(R)$, there exists some $z' \in \widehat{P}$, the parent of P , such that

$$
d_R(z') \leq 300 \ell(\widehat{P}) = 600 \ell(P).
$$

Also, we have

$$
|x - z'| \le |x - z_P| + |z_P - z'| \le 50 r(B_P) + 2\ell(P) \le 300 \ell(P).
$$

Thus,

$$
d_R(x) \le d_R(z') + |x - z'| \le (600 + 300) \,\ell(P).
$$

The statement (b) is an immediate consequence of (a), and (c) follows easily from (b). To show (d), observe that, for any $S \in \text{Tree}_{\text{WSBC}}(R)$,

$$
\ell(P) \le \frac{d(z_P)}{300} \le \frac{\ell(S) + \text{dist}(z_P, S)}{300} \le \frac{\ell(P) + \ell(S) + \text{dist}(P, S)}{300}.
$$

Thus,

$$
\ell(P) \le \frac{\ell(S) + \text{dist}(P, S)}{299}.
$$

In particular, choosing $S = R$, we deduce

$$
\ell(P) \le \frac{\ell(R) + \operatorname{dist}(P, R)}{299} \le \frac{21}{299} \ell(R) \le \ell(R),
$$

and thus, using again that $dist(P, R) \leq 20\ell(R)$, it follows that $P \subset 22R$. Let $S_0 \in \text{Tree}_{\text{WSBC}}(R)$ be such that $d(z_P) = \ell(S_0) + \text{dist}(z_P, S_0)$, and let $Q \in \mathcal{D}_{\mu}$ be be the smallest cube such that $S_0 \subset Q$ and $P \subset 22Q$. Since $S_0 \subset R$ and $P \subset 22R$, we deduce that $S_0 \subset Q \subset R$, implying that $Q \in$ Tree $_{\text{WSBC}}(R)$.

So it just remains to check that $\ell(Q) \le 2000 \ell(P)$. To this end, consider a cube $\widetilde{Q} \supset S_0$ such that

$$
\ell(P) + \ell(S_0) + \text{dist}(P, S_0) \leq \ell(\widetilde{Q}) \leq 2(\ell(P) + \ell(S_0) + \text{dist}(P, S_0)).
$$

From the first inequality, it is clear that $P \subset 2\tilde{Q}$ and then, by the definition of Q, we infer that $Q \subset \tilde{Q}$. This inclusion and the second inequality above imply that

$$
\ell(Q) \leq \ell(\widetilde{Q}) \leq 2(2\ell(P) + \ell(S_0) + \text{dist}(z_P, S_0)) = 4\ell(P) + 2 d_R(z_P).
$$

By (a) we know that $d_R(z_P) \le 900 \ell(P)$, and so we derive $\ell(Q) \le 2000 \ell(P)$.

Lemma 6.2. *Given* $R \in \text{Top}_{b}^{(N)}$ *, if* $Q \in \text{End}(R)$ *and* $\text{dist}(P, R) \leq 20 \ell(R)$ *, then* $b\beta(Q) \leq C \varepsilon$ *and* $Q \in \text{WSBC}(\Gamma')$, with $\Gamma' = c_6 \Gamma$, for some absolute constants $C, c_6 > 0$.

Proof. This immediate from the fact that, by (d) in the previous lemma, there exists some cube $Q' \in$ Tree_{WSBC}(*R*) such that $Q \subset 22Q'$ and $\ell(Q') \le 2000 \ell(Q)$, so that $b\beta(Q') \le \varepsilon$ and $Q' \in \text{WSBC}(\Gamma)$. \Box

Next we consider the following Whitney decomposition of Ω : we let W be a family of dyadic cubes from \mathbb{R}^{n+1} , contained in Ω , with disjoint interiors, such that

$$
\bigcup_{I \in \mathcal{W}} I = \Omega,
$$

and such that moreover there are some constants $M_0 > 20$ and $D_0 \ge 1$ satisfying the following for every $I \in \mathcal{W}$:

(i) $10I \subset \Omega$;

- (ii) $M_0I \cap \partial\Omega \neq \emptyset$;
- (iii) there are at most D_0 cubes $I' \in W$ such that $10I \cap 10I' \neq \emptyset$. Further, for such cubes I' , we have $\ell(I') \approx \ell(I)$, where $\ell(I')$ stands for the side length of I' .

From the properties (i) and (ii) it is clear that $dist(I, \partial \Omega) \approx \ell(I)$. We assume that the Whitney cubes are small enough so that

(6.2)
$$
\text{diam}(I) < \frac{1}{100} \text{dist}(I, \partial \Omega).
$$

This can be achieved by replacing each cube $I \in \mathcal{W}$ by its descendants $I' \in \mathcal{D}_k(I)$, for some fixed $k > 1$, if necessary.

For each $I \in \mathcal{W}$, we denote by B^I a ball concentric with I and radius $C_5\ell(I)$, where C_5 is a universal constant big enough so that

$$
g(p,x) \lesssim \frac{\omega^p(B^I)}{\ell(I)^{n-1}} \quad \text{ for all } x \in 4I.
$$

Obviously, the ball B^I intersects $\partial\Omega$, and the family $\{B^I\}_{I \in \mathcal{W}}$ does not have finite overlapping.

To state the Geometric Lemma we need some additional notation. Given a cube $R' \in \text{Tree}_{\text{WSRC}}(R)$, we denote by $Tree_{WSBC}(R')$ the family of cubes from \mathcal{D}_{μ} with side length at most $\ell(R')$ which are contained in $100B_{R'}$ and are not contained in any cube from End (R) . We also denote by End (R') the subfamily of the cubes from $\text{End}(R)$ which are contained in some cube from $\text{Tree}_{\text{WSBC}}(R')$. Note that Tree_{WSBC} (R') is not a tree, in general, but a union of trees.

Lemma 6.3 (Geometric Lemma). Let $0 < \gamma < 1$, and assume that the constant $\Gamma = \Gamma(\gamma)$ in the $definition$ of WSBC is big enough. Let $R \in \mathsf{Top}^{(N)}_b \cap \mathsf{WSBC}(\Gamma)$ and let $R' \in \mathsf{Tree}_{\mathsf{WSBC}}(R)$ be such

that $\ell(R') = 2^{-k_0} \ell(R)$, with $k_0 = k_0(\gamma) \ge 1$ big enough. Then there are two connected open sets $V_1, V_2 \subset CB_{R'} \cap \Omega$ *with disjoint closures which satisfy the following properties:*

- (a) There are subfamilies $\mathcal{W}_i \subset \mathcal{W}$ such that $V_i = \bigcup_{I \in \mathcal{W}_i} 1.1 \mathring{I}$.
- *(b)* Each V_i contains a ball B_i with $r(B_i) \approx \ell(R')$, and each corkscrew point for R' contained in $2B_{R'}\cap V_i$ can be joined to the center z_i of B_i by a good Harnack chain contained in V_i . Further, *any point* $x \in V_i$ *can be joined to* z_i *by a good Harnack chain (not necessarily contained in* Vi*).*
- *(c) For each* Q ∈ Tree_{WSBC} $(R) \cap \mathcal{D}_{\mu}(R')$ *there are big corkscrews* x_Q^1 ∈ $V_1 \cap 2B_Q$ *and* x_Q^2 ∈ $V_2 \cap 2B_Q$ *, and if* \widehat{Q} *is an ancestor of* Q *which also belongs to* $\mathsf{Tree}_{\mathsf{WSBC}}(R) \cap \mathcal{D}_\mu(R')$ *<i>, then* x_Q^i can be joined to $x_{\widehat{Q}}^i$ by a good Harnack chain, for each $i = 1, 2$.
- *(d)* $(\partial V_1 \cup \partial V_2) \cap 10B_{R'} \subset \bigcup_{P \in \widetilde{\text{End}}(R')} 2B_P$.
- (e) *If* $P \in End(R')$ *is such that* $2B_P ∩ 10B_{R'} ≠ ∅$, then $\partial V_i ∩ 2B_P$ *is contained in the union of cubes of a subfamily* $W_P \subset W$ *such that (i)*

$$
m_{4I}g(p,\cdot) \leq \gamma \frac{\ell(P)}{\mu(R_0)}
$$
 for each $I \in \mathcal{W}_P$,

and (ii)

$$
\sum_{I \in \mathcal{W}_P} \ell(I)^n \lesssim \ell(P)^n \quad \text{and} \quad \sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P),
$$

for some universal constant $C > 1$ *.*

The constants involved in the Harnack chain and corkscrew conditions may depend on ε , Γ *, and* γ ^{[2](#page-24-1)}

6.2. **Proof of the Geometric Lemma [6.3.](#page-23-0)** In this whole subsection we fix $R \in \text{Top}_{b}^{(N)}$ and we assume Tree_{WSBC} $(R) \neq \emptyset$, as in Lemma [6.3.](#page-23-0) We let $R' \in \text{Tree}_{\text{WSRC}}(R)$ be such that $\ell(R') = 2^{-k_0} \ell(R)$, with $k_0 = k_0(\gamma) \ge 1$ big enough, as in Lemma [6.3,](#page-23-0) and we consider the associated families $\text{Tree}_{\text{WSBC}}(R')$ and $\widetilde{\mathsf{End}}(R')$.

Remark 6.4. By arguments analogous to the ones in Lemma [6.2,](#page-23-1) it follows easily that if $Q \in \text{Tree}_{\text{WSBC}}(R')$, for $R' \in \mathsf{Tree}_{\mathsf{WSBC}}(R)$ such that $\ell(R') = 2^{-k_0} \ell(R)$, then there exists some cube $S \in \mathsf{Tree}_{\mathsf{WSBC}}(R)$ such that $Q \subset 22S$ and $\ell(S) \leq 2000\ell(Q)$. This implies that $b\beta(Q) \leq C \varepsilon$ and $Q \in \mathsf{WSBC}(c_6\Gamma)$ too.

In order to define the open sets V_1 , V_2 described in the lemma, first we need to associate some open sets $U_1(Q)$, $U_2(Q)$ to each $Q \in \text{Tree}_{\text{WSBC}}(R') \cup \text{End}(R')$. We distinguish two cases:

• For $Q \in \text{Tree}_{\text{WSBC}}(R')$, we let $\mathcal{J}_i(Q)$ be the family of Whitney cubes $I \in \mathcal{W}$ which intersect

$$
\{y \in 20B_Q : \text{dist}(y, L_Q) > \varepsilon^{1/4} \,\ell(Q)\}
$$

and are contained in the same connected component of $\mathbb{R}^{n+1} \setminus L_Q$ as x_Q^i , and then we set

$$
U_i(Q) = \bigcup_{I \in \mathcal{J}_i(Q)} 1.1 \mathring{I}.
$$

²To guarantee the existence of the sets V_i and the fact that they are contained in Ω we use the assumption that $\Omega = (\partial \Omega)^c$.

• For $Q \in End(R')$ the definition of $U_i(Q)$ is more elaborated. First we consider an auxiliary ball \widetilde{B}_Q , concentric with B_Q , such that $19B_Q \subset \widetilde{B}_Q \subset 20B_Q$ and having thin boundaries for ω^p . This means that, for some absolute constant C,

(6.3)
$$
\omega^p(\left\{x \in 2\widetilde{B}_Q : \text{dist}(x, \partial \widetilde{B}_Q) \le tr(\widetilde{B}_Q)\right\}) \le C t \omega^p(2\widetilde{B}_Q) \quad \text{for all } t > 0.
$$

The existence of such ball \widetilde{B}_Q follows by well known arguments (see for example [\[To,](#page-44-15) p.370]). Next we denote by $\mathcal{J}(Q)$ the family of Whitney cubes $I \in \mathcal{W}$ which intersect B_Q and satisfy $\ell(I) \ge \theta \ell(Q)$ for $\theta \in (0,1)$ depending on γ (the reader should think that $\theta \ll \varepsilon$ and that $\theta = 2^{-j_1}$ for some $j_1 \gg 1$), and we set

(6.4)
$$
U(Q) = \bigcup_{I \in \mathcal{J}(Q)} 1.1\mathring{I}.
$$

For a fixed $i = 1$ or 2, let $\{D_j^i(Q)\}_{j \geq 0}$ be the connected components of $U(Q)$ which satisfy one of the following properties:

- – either $x_Q^i \in D_j^i(Q)$ (recall that x_Q^i is a big corkscrew for Q), or
- there exists some $y \in D_j^i(Q)$ such that $g(p, y) > \gamma \ell(Q) \mu(R_0)^{-1}$ and there is a $C_6(\gamma, \theta)$ good Harnack chain that joins y to x_Q^i , for some constant $C_6(\gamma, \theta)$ to be chosen below.

Then we let $U_i(Q) = \bigcup_j D_j^i(Q)$. After reordering the sequence, we assume that $x_Q^i \in D_0^i(Q)$.

In the case $Q \in \text{Tree}_{\text{WSBC}}(R')$, from the definitions, it is clear that the sets $U_i(Q)$ are open and connected and

$$
(6.5) \t\t\t \overline{U_1(Q)} \cap \overline{U_2(Q)} = \varnothing.
$$

In the case $Q \in \text{End}(R')$, the sets $U_i(Q)$ may fail to be connected. However, [\(6.5\)](#page-25-0) still holds if Γ is chosen big enough (which will be the case). Indeed, if some component D_j^i can be joined by $C_6(\gamma, \theta)$ good Harnack chains both to x_Q^1 and x_Q^2 , then there is a $C(\gamma, \theta)$ -good Harnack chain that joins x_Q^1 to x_Q^2 , and thus Q does not belong to WSBC($c_6\Gamma$) if Γ is taken big enough, which cannot happen by Lemma [6.2.](#page-23-1) Note also that the two components of

$$
\{y \in \widetilde{B}_Q : \text{dist}(y, L_Q) > \varepsilon^{1/2} \ell(Q)\}
$$

are contained in $D_0^1(Q) \cup D_0^2(Q)$, because $b\beta(Q) \leq C\varepsilon$ and we assume $\theta \ll \varepsilon$.

The following is immediate:

Lemma 6.5. Assume that we relabel appropriately the sets $U_i(P)$ and corkscrews x^i_P for $P \in \text{Tree}_{\text{WSBC}}(R') \cup$ $\text{End}(R')$. Then for all Q , $\widehat{Q} \in \text{Tree}_{\text{WSBC}}(R') \cup \text{End}(R')$ *such that* \widehat{Q} *is the parent of* Q *we have*

 (6.6) $[x_Q^1, x_{\widehat{Q}}^1] \subset U_1(Q) \cap U_1(\widehat{Q})$ and $[x_Q^2, x_{\widehat{Q}}^2] \subset U_2(Q) \cap U_2(\widehat{Q})$.

Further,

$$
dist([x_Q^i, x_{\widehat{Q}}^i], \partial \Omega) \ge c \,\ell(Q) \quad \text{for } i = 1, 2,
$$

where c depends at most on *n* on C_0 *.*

The labelling above can be chosen inductively. First we fix the sets $U_i(T)$ and corkscrews x_T^i for every maximal cube T from $\text{Tree}_{\text{WSBC}}(R')$ (contained in $100B_{R'}$ and with side length equal to $\ell(R')$). Further we assume that, for any maximal cube T, the corkscrew x_T^i is at the same side of $L_{R'}$ as $x_{R'}^i$, for each $i = 1, 2$ (this property will be used below). Later we label the sons of each T so that [\(6.6\)](#page-25-1)

holds for any son Q of T . Then we proceed with the grandsons of T , and so on. We leave the details for the reader.

The following result will be used later to prove the property (e)(i).

Lemma 6.6. *Suppose that the constant* $k_0(\gamma)$ *in Lemma [6.3](#page-23-0) is big enough. Let* $Q \in \text{End}(R')$ *and assume* θ *small enough and* $C_6(\gamma, \theta)$ *big enough in the definition of* $U_i(Q)$ *. If* $y \in \widetilde{B}_Q$ *satisfies* $g(p, y)$ > $\gamma \,\ell(Q) \,\mu(R_0)^{-1}$, then $y \in U_1(Q) \cup U_2(Q)$.

Proof. By the definition of $U_i(Q)$, it suffices to show that y belongs to some component $D_j^i(Q)$ and that there is a $C_6(\gamma, \theta)$ -good Harnack chain that joins y to x_Q^i . To this end, observe that by the boundary Hölder continuity of $g(p, \cdot)$,

$$
\gamma\,\frac{\ell(Q)}{\mu(R_0)}\le g(p,y)\le C\,\left(\frac{\delta_\Omega(y)}{\ell(Q)}\right)^\alpha\, m_{30B_Q}g(p,\cdot)\le C\,\left(\frac{\delta_\Omega(y)}{\ell(Q)}\right)^\alpha\,\frac{\ell(Q)}{\mu(R_0)},
$$

where in the last inequality we used Lemma [2.5.](#page-6-1) Thus,

$$
\delta_{\Omega}(y) \ge c \,\gamma^{1/\alpha} \,\ell(Q),
$$

and if θ is small enough, then y belongs to some connected component of the set $U(Q)$ in [\(6.4\)](#page-25-2). By Lemma [6.1\(](#page-22-0)d) there is a cube $Q' \in \text{Tree}_{\text{WSBC}}(R)$ such that $Q \subset 22Q'$ and $\ell(Q') \approx \ell(Q)$. In particular, $WA(p, \Lambda) \cap Q' \supseteq G_0 \cap Q' \neq \emptyset$ and thus, by applying Lemma [3.2](#page-12-0) with $q = y$ and $r_0 = Cr(B_Q)$ (for a suitable $C > 1$), it follows that there exists a κ_1 -corkscrew $y' \in C(\gamma) B_Q$, with $C(\gamma) > 20$ say, such that y can be joined to y' by a $C'(\gamma)$ -good Harnack chain. Assuming that the constant $k_0(\gamma)$ in Lemma [6.3](#page-23-0) is big enough, it turns out that $y' \in CB_{Q''}$ for some $Q'' \in Tree_{\text{WSBC}}(R)$ such that $22Q'' \supset Q$. Since all the cubes S such that $Q \subset S \subset 22Q''$ satisfy $b\beta(S) \leq C \varepsilon$, by applying Lemma [5.4](#page-21-3) repeatedly, it follows that y' can be joined either to x_Q^1 or x_Q^2 by a $C''(\gamma)$ -good Harnack chain. Then, joining both Harnack chains, it follows that y can be joined either to x_Q^1 or x_Q^2 by a $C'''(\gamma)$ -good Harnack chain. So y belongs to one of the components D_j^i , assuming $C_6(\gamma, \theta)$ big enough.

From now on we assume θ small enough and $C_6(\gamma, \theta)$ big enough so that the preceding lemma holds. Also, we assume $\theta \ll \varepsilon^4$. We define

$$
V_1=\bigcup_{Q\in \widetilde{\operatorname{Tree}}_{\mathsf{WSEC}}(R')\cup \widetilde{\operatorname{End}}(R')} U_1(Q),\qquad V_2=\bigcup_{Q\in \widetilde{\operatorname{Tree}}_{\mathsf{WSEC}}(R')\cup \widetilde{\operatorname{End}}(R')} U_2(Q).
$$

Next we will show that

$$
\overline{V_1} \cap \overline{V_2} = \varnothing.
$$

Since the number of cubes $Q \in \text{Tree}_{\text{WSEC}}(R') \cup \text{End}(R')$ is finite (because of the truncation in the corona decomposition), this is a consequence of the following:

Lemma 6.7. *Suppose* Γ *is big enough in the definition of* WSBC *(depending on* θ *). For all* $P, Q \in$ $Tree_{\text{WSBC}}(R') \cup \text{End}(R')$ *, we have*

$$
\overline{U_1(P)} \cap \overline{U_2(Q)} = \varnothing.
$$

Proof. We suppose that $\ell(Q) \geq \ell(P)$ We also assume that $\overline{U_1(P)} \cap \overline{U_2(Q)} \neq \emptyset$ and then we will get a contradiction. Notice first that if $\ell(P) = \ell(Q) = 2^{-j}\ell(R')$ for some $j \ge 0$, then the corkscrews x_P^i and x_Q^i are at the same side of L_Q for each $i = 1, 2$. This follows easily by induction on j.

28 AZZAM, MOURGOGLOU, AND TOLSA

1. Suppose first that $P, Q \in \text{Tree}_{\text{WSBC}}(R')$. Since the cubes from $\mathcal{J}_2(Q)$ have side length at least $c \epsilon^{1/4} \ell(Q)$, it follows that at least one of the cubes from $\mathcal{J}_1(P)$ has side length at least $c' \epsilon^{1/4} \ell(Q)$, which implies that $\ell(P) \geq c'' \varepsilon^{1/4} \ell(Q)$, by the construction of $U_1(P)$.

Since $U_1(P) \cap U_2(Q) \neq \emptyset$, there exists some curve $\gamma = \gamma(x_P^1, x_Q^2)$ that joins x_P^1 and x_Q^2 such that $dist(\gamma, \partial \Omega) \geq c \epsilon^{1/2} \ell(Q)$ because all the cubes from $\mathcal{J}_2(Q)$ have side length at least $c \epsilon^{1/4} \ell(Q)$, and the ones from $\mathcal{J}_1(P)$ have side length $\geq c \varepsilon^{1/4} \ell(P) \geq c \varepsilon^{1/2} \ell(Q)$.

Let \widehat{P} be the ancestor of P such that $\ell(\widehat{P}) = \ell(Q)$. From the fact that $\overline{U_1(P)} \cap \overline{U_2(Q)} \neq \emptyset$, we deduce that $20B_P \cap 20B_Q \neq \emptyset$ and thus $20B_{\widehat{P}} \cap 20B_Q \neq \emptyset$, and so $20B_{\widehat{P}} \subset 60B_Q$. This implies that $x_{\hat{P}}^1$ is in the same connected component as x_Q^1 and also that $dist([x_Q^1, x_{\hat{P}}^1], \partial \Omega) \gtrsim \ell(Q)$, because $b\beta(100B_Q) \leq \varepsilon \ll 1$ and they are at the same side of L_Q .

Consider now the chain $P = P_1 \subset P_2 \subset \ldots \subset P_m = P$, so that P_{i+1} is the parent of P_i . Form the curve $\gamma' = \gamma'(x_{\hat{P}}^1, x_P^1)$ with endpoints $x_{\hat{P}}^1$ and x_P^1 by joining the segments $[x_{P_i}^1, x_{P_{i+1}}^1]$. Since these segments satisfy

$$
\text{dist}\big([x^1_{P_i}, x^1_{P_{i+1}}], \partial\Omega\big) \ge c\,\ell(P_i) \ge c\,\ell(P) \ge c\,\varepsilon^{1/4}\,\ell(Q),
$$

it is clear that $dist(\gamma', \partial \Omega) \geq c \, \varepsilon^{1/4} \, \ell(Q)$.

Next we form a curve $\gamma'' = \gamma''(x_Q^1, x_Q^2)$ which joins x_Q^1 to x_Q^2 by joining $[x_Q^1, x_{\hat{P}}^1]$, $\gamma'(x_{\hat{P}}^1, x_P^1)$, and $\gamma(x_P^1, x_Q^2)$. It follows easily that this is contained in $90B_Q$ and that $dist(\gamma'', \partial \Omega) \geq c \epsilon^{1/2} \ell(Q)$. However, this is not possible because x_Q^1 and x_Q^2 are in different connected components of $\mathbb{R}^{n+1} \setminus L_Q$ and $b\beta(Q) \leq \varepsilon \ll \varepsilon^{1/2}$ (since we assume $\varepsilon \ll 1$).

2. Suppose now that $Q \in \text{End}(R')$. The arguments are quite similar to the ones above. In this case, the cubes from $\mathcal{J}_2(Q)$ have side length at least $\theta \ell(Q)$ and thus at least one of the cubes from $\mathcal{J}_1(P)$ has side length at least $c \theta \ell(Q)$, which implies that $\ell(P) \geq c' \theta \ell(Q)$.

Now there exists a curve $\gamma = \gamma(x_P^1, x_Q^2)$ that joints x_P^1 and x_Q^2 such that $dist(\gamma, \partial \Omega) \geq c \theta^2 \ell(Q)$ because all the cubes from $\mathcal{J}_2(Q)$ have side length at least $\theta \ell(Q)$, and the ones from $\mathcal{J}_1(P)$ have side length $\theta \ell(P) \geq c \theta^2 \ell(Q)$.

We consider again cubes \widehat{P} and P_1, \ldots, P_m defined exactly as above. By the same reasoning as above, dist $([x_Q^1, x_{\hat{P}}^1], \partial \Omega) \gtrsim \ell(Q)$. We also define the curve $\gamma' = \gamma'(x_{\hat{P}}^1, x_P^1)$ which joins $x_{\hat{P}}^1$ to x_P^1 in the same way. In the present case we have

$$
dist(\gamma', \partial \Omega) \gtrsim \ell(P) \ge c \theta \ell(Q).
$$

Again construct a curve $\gamma'' = \gamma''(x_Q^1, x_Q^2)$ which joins x_Q^1 to x_Q^2 by gathering $[x_Q^1, x_{\hat{P}}^1], \gamma'(x_{\hat{P}}^1, x_P^1)$, and $\gamma(x_P^1, x_Q^2)$. This is contained in CB_Q (for some $C > 1$ possibly depending on γ) and satisfies $dist(\gamma'', \partial \Omega) \geq c \theta^2 \ell(Q)$. From this fact we deduce that x_Q^1 and x_Q^2 can be joined by $C(\theta)$ -good Harnack chain. Taking Γ big enough (depending on $C(\theta)$), this implies that the big corkscrews for Q can be joined by a $(c_6\Gamma)$ -good Harnack chain, which contradicts Lemma [6.2.](#page-23-1)

3. Finally suppose that $P \in \text{End}(R')$. We consider the same auxiliary cube \widehat{P} and the same curve $\gamma = \gamma(x_P^1, x_Q^2)$ satisfying $dist(\gamma, \partial \Omega) \ge c \theta \ell(P)$. By joining the segments $[x_{P_i}^2, x_{P_{i+1}}^2]$, we construct a curve $\gamma'_2 = \gamma'_2(x_{\hat{P}}^2, x_P^2)$ analogous to $\gamma' = \gamma'(x_{\hat{P}}^1, x_P^1)$ from the case 2, so that this joins $x_{\hat{P}}^2$ to x_P^2 and satisfies dist $(\gamma_2', \partial \Omega) \gtrsim \ell(P)$.

We construct a curve γ''' that joins x_P^1 to x_P^2 by joining $\gamma(x_P^1, x_Q^2)$, $[x_Q^2, x_P^2]$, and $\gamma_2'(x_P^2, x_P^2)$. Again this is contained in CB_Q and it holds $dist(\gamma''', \partial \Omega) \geq c \theta \ell(P)$. This implies that x_P^1 and x_P^2 can be

joined by $C(\theta)$ -good Harnack chain. Taking Γ big enough, we deduce the big corkscrews for P can be joined by a $(c_6\Gamma)$ -good Harnack chain, which is a contradiction.

By the definition of V_1 and V_2 it is clear that the properties (a), (b) and (c) in Lemma [6.3](#page-23-0) hold. So to complete the proof of the lemma it just remains to prove (d) and (e).

Proof of Lemma [6.3\(](#page-23-0)d). Let $x \in (\partial V_1 \cup \partial V_2) \cap 10B_{R'}$. We have to show that there exists some $S \in End(R')$ such that $x \in 2B_S$. To this end we consider $y \in \partial \Omega$ such that $|x - y| = \delta_{\Omega}(x)$. Since $z_{R'} \in \partial \Omega$, it follows that $y \in 20B_{R'}$. Let $S \in \text{End}(R')$ be such that $y \in S$. Observe that

(6.7)
$$
\ell(S) \le \frac{1}{300} d_R(y) \le \frac{1}{300} \left(\ell(R') + 20 r(B_{R'}) \right) = \frac{81}{300} \ell(R') \le \frac{1}{3} \ell(R').
$$

We claim that $x \in 2B_S$. Indeed, if $x \notin 2B_S$, taking also into account [\(6.7\)](#page-28-0), there exists some ancestor Q of S contained in $100B_{R'}$ such that $x \in 2B_Q$ and $\delta_{\Omega}(x) = |x - y| \approx \ell(Q)$. From the fact that $S \subsetneq Q \subset 100B_{R'}$ we deduce that $Q \in \text{Tree}_{\text{WSBC}}(R')$. By the construction of the sets $U_i(Q)$, it is immediate to check that the condition that $\delta_{\Omega}(x) \approx \ell(Q)$ implies that $x \in U_1(Q) \cup U_2(Q)$. Thus $x \in V_1 \cup V_2$ and so $x \notin \partial (V_1 \cup V_2) = \partial V_1 \cup \partial V_2$ (for this identity we use that $dist(V_1, V_2) > 0$), which is a contradiction. \Box

To show (e), first we need to prove the next result:

Lemma 6.8. *For each* $i = 1, 2$ *, we have*

$$
\partial V_i \cap 10B_{R'} \subset \bigcup_{Q \in \overline{\text{End}}(R')} \partial U_i(Q).
$$

Proof. Clearly, we have

$$
\partial V_i \cap 10B_{R'} \subset \bigcup_{\substack{P \in \widetilde{\operatorname{Tree}}_{\operatorname{WSE}}(R') : \\P \cap 10B_{R'} \neq \varnothing}} \partial U_i(P) \cup \bigcup_{\substack{Q \in \widetilde{\operatorname{End}}(R') : \\Q \cap 10B_{R'} \neq \varnothing}} \partial U_i(Q).
$$

So it suffices to show that

(6.8)
$$
\bigcup_{\substack{P \in \widetilde{\text{Tree}}_{\text{WSE}}(R') : \\ P \cap 10B_{R'} \neq \varnothing}} \partial U_i(P) \cap \partial V_i \cap 10B_{R'} = \varnothing.
$$

Let $x \in \partial U_i(P) \cap \partial V_i \cap 10B_{R'}$, with $P \in \text{Tree}_{\text{WSBC}}(R')$, $P \cap 10B_{R'} \neq \emptyset$. From the definition of $U_i(P)$, it follows easily that

$$
\delta_{\Omega}(x) \gtrsim \varepsilon^{1/4} \ell(P).
$$

On the other hand, by Lemma [6.3\(](#page-23-0)d), there exists some $Q \in \text{End}(R')$ such that $x \in 2B_Q$. By the definition of $U_i(Q)$, since $\theta \ll \varepsilon$, it also follows easily that

$$
\left\{y \in 2B_Q : \delta_{\Omega}(y) > \varepsilon^{1/2} \ell(Q)\right\} \subset V_1 \cup V_2.
$$

Hence, $dist(\partial V_i \cap 2B_Q, \partial \Omega) \leq \varepsilon^{1/2} \ell(Q)$, and so

$$
(6.10) \t\t \delta_{\Omega}(x) \le \varepsilon^{1/2} \,\ell(Q).
$$

We claim that $\ell(Q) \leq \ell(P)$. Indeed, from the fact that $x \in \partial U_i(P) \subset 30B_P$, we infer that

$$
30B_P \cap 2B_Q \neq \varnothing.
$$

Suppose that $\ell(Q) \geq \ell(P)$. This implies that $B_P \subset 33B_Q$. Consider now a cube $S \subset P$ belonging to $\text{End}(R')$. Since $B_S \cap 33B_Q \neq \emptyset$, by Lemma [6.1](#page-22-0) (b) we have

$$
\ell(Q) \approx \ell(S) \le \ell(P),
$$

which proves our claim. Together with (6.9) and (6.10) , this yields

$$
\varepsilon^{1/4}\ell(P) \lesssim \delta_{\Omega}(x) \lesssim \varepsilon^{1/2}\,\ell(Q) \lesssim \varepsilon^{1/2}\,\ell(P),
$$

which is a contradiction for ε small enough. So there does not exist any $x \in \partial U_i(P) \cap \partial V_i \cap 10B_{R'}$, which proves (6.8) .

Proof of Lemma [6.3\(](#page-23-0)e). Let $P \in \text{End}(R')$ be such that $2B_P \cap 10B_{R'} \neq \emptyset$. The statement (i) is an immediate consequence of Lemma [6.6.](#page-26-0) In fact, this lemma implies that any $y \in 2B_P$ such that $g(p, y) > \gamma \ell(P) \mu(R_0)^{-1}$ is contained in $U_1(P) \cup U_2(P)$ and thus in $V_1 \cup V_2$. In particular, $y \notin$ $\partial(V_1 \cup V_2) = \partial V_1 \cup \partial V_2$. Thus, if $y \in 2B_P \cap \partial V_i$, then

$$
g(p, y) \le \gamma \frac{\ell(P)}{\mu(R_0)}.
$$

It is easy to check that this implies the statement (i) in Lemma [6.3\(](#page-23-0)e) (possibly after replacing γ by C_{γ}).

Next we turn our attention to (ii). To this end, denote by J_P the subfamily of the cubes $Q \in \text{End}(R')$ such that $30B_Q \cap 2B_P \neq \emptyset$. By Lemma [6.8,](#page-28-4)

(6.11)
$$
\partial V_i \cap 2B_P \subset \bigcup_{Q \in J_P} \partial U_i(Q) \cap 2B_P.
$$

We will show that

(6.12)
$$
\sum_{I \in \mathcal{W}_P} \ell(I)^n \lesssim \ell(P)^n \quad \text{and} \quad \sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P),
$$

where W_P the family of Whitney cubes $I \subset V_1 \cup V_2$ such that $1.1\overline{I} \cap \partial(V_1 \cup V_2) \cap 2B_P \neq \emptyset$. To this end, observe that, by [\(6.11\)](#page-29-0) and the construction of $U_i(Q)$, for each $I \in \mathcal{W}_P$ there exists some $Q \in J_P$ such that $I \subset 30B_Q$ and either $\ell(I) = \theta \ell(Q)$ or $1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \emptyset$. Using the *n*-AD-regularity of μ , it is immediate to check that for each $Q \in J_P$,

$$
\sum_{\substack{I \subset 30B_Q:\\\ell(I) = \theta\ell(Q)}} \ell(I)^n \lesssim \ell(Q)^n.
$$

Also,

$$
\sum_{\substack{I \in \mathcal{W}: \\ 1.1 \overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \ell(I)^n \lesssim \sum_{\substack{I \in \mathcal{W} \\ 1.1 \overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \mathcal{H}^n(2I \cap \partial \widetilde{B}_Q) \lesssim \mathcal{H}^n(\partial \widetilde{B}_Q) \lesssim \ell(Q)^n.
$$

Since the number of cubes $Q \in J_P$ is uniformly bounded (by Lemma [6.1\(](#page-22-0)b)) and $\ell(Q) \approx \ell(P)$, the above inequalities yield the first estimate in [\(6.12\)](#page-29-1).

To prove the second one we also distinguish among the two types of cubes $I \in J_P$ above. First, by the bounded overlap of the balls B^I such that $\ell(I) = \theta \ell(Q)$, we get

(6.13)
$$
\sum_{\substack{I \subset 30B_Q \\ \ell(I) = \theta \ell(Q)}} \omega^p(B^I) \lesssim \omega^p(CB_P),
$$

since the balls B^I in the sum are contained CB_P for a suitable universal constant $C > 1$. To deal with the cubes $I \in W$ such that $1.\overline{1I} \cap \partial \widetilde{B}_Q \neq \emptyset$ we intend to use the thin boundary property of \widetilde{B}_Q in [\(6.3\)](#page-25-3). To this end, we write

$$
\sum_{\substack{I \in \mathcal{W}: \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \omega^p(B^I) = \sum_{k \ge 0} \sum_{\substack{I \in \mathcal{W}: \\ 1.1\overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \omega^p(B^I) \lesssim \sum_{k \ge 0} \omega^p(\mathcal{U}_{2^{-k+1} \operatorname{diam}(Q)}(\partial \widetilde{B}_Q)),
$$

where $\mathcal{U}_d(A)$ stands for the d-neighborhood of A. By [\(6.3\)](#page-25-3) it follows that

$$
\omega^p(\mathcal{U}_{2^{-k}\ell(Q)}(\partial \widetilde{B}_Q)) \lesssim 2^{-k}\omega^p(C'B_Q),
$$

and thus

$$
\sum_{\substack{I \in \mathcal{W}: \\ 1.1 \overline{I} \cap \partial \widetilde{B}_Q \neq \varnothing}} \omega^p(B^I) \lesssim \omega^p(C' B_Q) \lesssim \omega^p(C B_P),
$$

for a suitable $C > 1$. Together with [\(6.13\)](#page-30-2), this yields the second inequality in [\(6.12\)](#page-29-1), which completes the proof of Lemma [6.3\(](#page-23-0)e). \Box

7. PROOF OF THE KEY LEMMA

We fix $R_0 \in \mathcal{D}_{\mu}$ and a corkscrew point $p \in \Omega$ as in the preceding sections. We consider $R \in \text{Top}_{b}^{(N)}$ and we assume $Tree_{\text{WSBC}}(R) \neq \emptyset$, as in Lemma [6.3.](#page-23-0) We let $R' \in Tree_{\text{WSBC}}(R)$ be such that $\ell(R') =$ $2^{-k_0}\ell(R)$, with $k_0 = k_0(\gamma) \ge 1$ big enough. Given $\lambda > 0$ and $i = 1, 2$, we set

$$
(7.1) \qquad \mathsf{H}_i(R') = \big\{ Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathcal{D}_{\mu}(R') \cap \mathsf{G} : g(p, x_Q^i) > \lambda \,\ell(Q)\,\mu(R_0)^{-1} \big\},
$$

so that $\text{Stop}_{\text{WSEG}}(R) \cap \mathcal{D}_{\mu}(R') \cap G = H_1(R') \cup H_2(R')$. Here we are assuming that the corkscrews x_Q^i belong to the set V_i from Lemma [6.3](#page-23-0) and that λ is small enough.

Lemma 7.1 (Baby Key Lemma). Let p, R_0, R, R' be as above. Given $\lambda > 0$, define also $H_i(R')$ as *above. For a given* $\tau > 0$ *, suppose that*

$$
\mu\bigg(\bigcup_{Q\in\mathsf{H}_i(R')}Q\bigg)\geq\tau\,\mu(R').
$$

If γ is small enough in the definition of V_i in Lemma [6.3](#page-23-0) (depending on τ and λ), then

$$
g(p, x_{R'}^{i}) \ge c(\lambda, \tau) \frac{\ell(R')}{\mu(R_0)}.
$$

Remark that Γ depends on γ (see Lemma [6.3\)](#page-23-0), and thus the families WSBC(Γ), Stop_{WSBC}(R), $H_i(R')$ also depend on γ . The reader should thing that $\Gamma \to \infty$ as $\gamma \to 0$.

A key fact in this lemma is that the constants λ , τ can be taken arbitrarily small, without requiring $\varepsilon \to 0$ as $\lambda \tau \to 0$. Instead, the lemma requires $\gamma \to 0$, which does not affect the packing condition in Lemma [5.2.](#page-19-1)

We denote

$$
\mathsf{Bdy}(R') = \bigcup_{P \in \widetilde{\mathsf{End}}(R'): 2B_P \cap 10 B_{R'} \neq \varnothing} \mathcal{W}_P,
$$

with W_P as in the Lemma [6.3.](#page-23-0) That is, W_P the family of Whitney cubes $I \subset V_1 \cup V_2$ such that $1.1\overline{I} \cap \partial(V_1 \cup V_2) \cap 2B_P \neq \emptyset$. So the family Bdy(R') contains Whitney cubes which intersect the boundaries of V_1 or V_2 and are close to $10B_{R'}$.

To prove Lemma [7.1,](#page-30-1) first we need the following auxiliary result.

Lemma 7.2. Let p, R_0, R, R' be as above and, for $i = 1$ or 2, let $Q \in H_i(R')$. Let V_i be as in *Lemma* [6.3](#page-23-0) and let $q \in \Omega$ be a corkscrew point for Q which belongs to V_i . Denote $r = 2\ell(R')$ and for $\delta \in (0, 1/100)$ *set*

$$
A_r^{\delta} = \{ x \in A(q, r, 2r) \cap \Omega : \delta_{\Omega}(x) > \delta r \}.
$$

Then we have

$$
g(p,q) \lesssim \frac{1}{r} \sup_{y \in A_r^{\delta} \cap V_i} \frac{g(p,y)}{\delta_{\Omega}(y)} \int_{A_r^{\delta}} g(q,x) dx + \frac{\delta^{\alpha/2}}{r^{n+3}} \int_{A(q,r,2r)} g(p,x) dx \int_{A(q,r,2r)} g(q,x) dx + \sum_{I \in \text{Bdy}(R')} \frac{1}{\ell(I)} \int_{2I} \left| g(p,x) \nabla g(q,x) - \nabla g(p,x) g(q,x) \right| dx.
$$

Note that the fact that q is a corkscrew for Q contained in V_i implies that $dist(q, \partial V_i) \approx \ell(Q)$, by the construction of the sets V_i in Lemma [6.3.](#page-23-0)

Proof. We fix $i = 1$, for definiteness. Recall that $V_1 = \bigcup_{I \in \mathcal{W}_1} 1.1 \hat{I}$. For each $I \in \mathcal{W}_1$, consider a smooth function η_I such that $\chi_{0.9I} \leq \eta_I \leq \chi_{1.09I}$ with $\|\nabla \eta_I\|_{\infty} \lesssim \ell(I)^{-1}$ and

$$
\eta := \sum_{I \in \mathcal{W}_1} \eta_I \equiv 1 \quad \text{ on } V_1 \cap 10B_{R'} \setminus \bigcup_{I \in \mathsf{Bdy}(R')} 2I.
$$

It follows that supp $\eta \subset V_1$ and so supp $\eta \cap V_2 = \emptyset$, and also

$$
\mathrm{supp}(\nabla \eta) \cap 10 B_{R'} \subset \bigcup_{I \in \mathsf{Bdy}(R')} 2I.
$$

Let φ_0 be a smooth function such that $\chi_{B(q,1.2r)} \leq \varphi_0 \leq \chi_{B(q,1.8r)}$, with $\|\nabla \varphi_0\|_{\infty} \lesssim 1/r$. Then we set

$$
\varphi = \eta \, \varphi_0.
$$

So φ is smooth, and it satisfies

$$
\operatorname{supp} \nabla \varphi \subset (A(q,r,2r) \cap V_1) \cup \bigcup_{I \in \mathsf{Bdy}(R')} 2I.
$$

Observe that, in a sense, φ is a smooth version of the function $\chi_{B(q,r) \cap V_1}$.

Since $g(p,q) = g(p,q) \varphi(q)$ and $g(p, \cdot) \varphi$ is a continuous function from $W_0^{1,2}$ $\mathcal{O}^{1,2}(\Omega)$, we have

$$
g(p,q) = \int_{\Omega} \nabla(g(p,\cdot)\varphi)(x) \nabla g(q,x) dx
$$

=
$$
\int_{\Omega} g(p,x) \nabla \varphi(x) \nabla g(q,x) dx + \int_{\Omega} \varphi(x) \nabla g(p,x) \nabla g(q,x) dx
$$

=: $I_1 + I_2$.

First we estimate I₂. For ε with $0 < \varepsilon < 1/10$, we consider a smooth function φ_{ε} such that $\chi_{B(q,\varepsilon\delta_{\Omega}(q))} \leq \varphi_{\varepsilon} \leq \chi_{B(q,2\varepsilon\delta_{\Omega}(q))}$, with $\|\nabla \varphi_{\varepsilon}\|_{\infty} \lesssim 1/(\varepsilon\delta_{\Omega}(q))$. Since $\varphi_{\varepsilon} \varphi = \varphi_{\varepsilon}$, we have

$$
I_2 = \int_{\Omega} \varphi_{\varepsilon}(x) \nabla g(p,x) \nabla g(q,x) dx + \int_{\Omega} \varphi(x) (1 - \varphi_{\varepsilon}(x)) \nabla g(p,x) \nabla g(q,x) dx =: I_{2,a} + I_{2,b}.
$$

To deal with $I_{2,a}$ we use the fact that for $x \in B(q, 2\varepsilon \delta_{\Omega}(q))$ we have

$$
|\nabla g(q,x)| \lesssim \frac{1}{|x-q|^n}
$$
 and $|\nabla g(p,x)| \lesssim \frac{g(p,q)}{\delta_{\Omega}(q)}$.

Then we get

$$
|I_{2,a}| \lesssim \frac{g(p,q)}{\delta_{\Omega}(q)} \int_{B(q,2\varepsilon\delta_{\Omega}(q))} \frac{1}{|x-q|^n} dx \lesssim \frac{g(p,q)}{\delta_{\Omega}(q)} \varepsilon \delta_{\Omega}(q) = \varepsilon g(p,q).
$$

Let us turn our attention to $I_{2,b}$. We denote $\psi = \varphi(1 - \varphi_{\varepsilon})$. Integrating by parts, we get

$$
I_{2,b} = \int \nabla g(p,x) \, \nabla(\psi \, g(q,\cdot))(x) \, dx - \int \nabla g(p,x) \, \nabla \psi(x) \, g(q,x) \, dx.
$$

Observe now that the first integral vanishes because $\psi g(q, \cdot) \in W_0^{1,2}$ $O_0^{1,2}(\Omega) \cap C(\Omega)$ and vanishes at $\partial \Omega$ and at p. Hence, since $\nabla \psi = \nabla \varphi - \nabla \varphi_{\varepsilon}$, we derive

$$
I_{2,b} = -\int \nabla g(p,x) \nabla \varphi(x) g(q,x) dx + \int \nabla g(p,x) \nabla \varphi_{\varepsilon}(x) g(q,x) dx = I_3 + I_4.
$$

Since I are the into account that $|\nabla \varphi| \leq \varepsilon$.

To estimate I_4 we take into account that $|\nabla \varphi_{\varepsilon}| \lesssim \chi_{A(q,\varepsilon \delta_{\Omega}(q),2\varepsilon \delta_{\Omega}(q))}/(\varepsilon \delta_{\Omega}(q))$, and then we derive

$$
|I_4| \lesssim \frac{1}{\varepsilon \,\delta_{\Omega}(q)} \int_{A(q,\varepsilon \delta_{\Omega}(q),2\varepsilon \delta_{\Omega}(q))} |\nabla g(p,x)| \, g(q,x) \, dx.
$$

Using now that, for x in the domain of integration,

$$
g(q,x) \lesssim \frac{1}{(\varepsilon \delta_{\Omega}(q))^{n-1}}
$$
 and $|\nabla g(p,x)| \lesssim \frac{g(p,q)}{\delta_{\Omega}(q)},$

we obtain

$$
|I_4| \lesssim \frac{1}{\varepsilon \, \delta_{\Omega}(q)} \frac{1}{(\varepsilon \, \delta_{\Omega}(q))^{n-1}} \, \frac{g(p,q)}{\delta_{\Omega}(q)} \, (\varepsilon \, \delta_{\Omega}(q))^{n+1} \lesssim \varepsilon \, g(p,q).
$$

From the above estimates we infer that

$$
g(p,q) \leq |I_1 + I_3| + c \varepsilon g(p,q).
$$

Since neither I_1 nor I_3 depend on ε , letting $\varepsilon \to 0$ we get

$$
g(p,q) \leq |I_1 + I_3|
$$

\$\leq \left| \int g(p,x) \nabla \varphi(x) \nabla g(q,x) dx - \int \nabla g(p,x) \nabla \varphi(x) g(q,x) dx \right|\$

34 AZZAM, MOURGOGLOU, AND TOLSA

$$
\leq \int |\nabla \varphi(x)| |g(p,x) \nabla g(q,x) - \nabla g(p,x) g(q,x)| dx.
$$

We denote

$$
\widetilde{F}=\bigcup_{I\in{\sf{Bdy}}(R')}2I,
$$

$$
\widetilde{A}_r^{\delta} = \{ x \in A(q, 1.2r, 1.8r) \cap V_1 \setminus \widetilde{F} : \delta_{\Omega}(x) > \delta r \},
$$

and

$$
\widetilde{A}_{r,\delta} = \{ x \in A(q, 1.2, 1.8r) \cap V_1 \setminus \widetilde{F} : \delta_{\Omega}(x) \le \delta r \}.
$$

Next we split the last integral as follows:

(7.2)
$$
g(p,q) \leq \int_{\widetilde{A}_r^{\delta}} |\nabla \varphi(x)| |g(p,x) \nabla g(q,x) - \nabla g(p,x) g(q,x)| dx
$$

$$
+ \int_{\widetilde{A}_{r,\delta}} |\nabla \varphi(x)| |g(p,x) \nabla g(q,x) - \nabla g(p,x) g(q,x)| dx
$$

$$
+ \int_{\widetilde{F}} |\nabla \varphi(x)| |g(p,x) \nabla g(q,x) - \nabla g(p,x) g(q,x)| dx
$$

$$
=: J_1 + J_2 + J_3.
$$

Concerning J_1 , we have

$$
|\nabla g(p,x)| \lesssim \frac{g(p,x)}{\delta_\Omega(x)} \quad \text{ and } \quad |\nabla g(q,x)| \lesssim \frac{g(q,x)}{\delta_\Omega(x)} \quad \text{ for all } x \in \widetilde{A}_r^\delta.
$$

Thus, using also that $|\nabla \varphi| \lesssim 1/r$ outside $\widetilde{F},$

(7.3)
$$
J_1 \lesssim \frac{1}{r} \sup_{x \in A_r^{\delta} \cap V_1} \frac{g(p,x)}{\delta_{\Omega}(x)} \int_{A_r^{\delta}} g(q,x) dx.
$$

Regarding J_2 , using Cauchy-Schwarz, we get

(7.4)
\n
$$
J_2 \lesssim \frac{1}{r} \int_{\widetilde{A}_{r,\delta}} |g(p,x) \nabla g(q,x) - \nabla g(p,x) g(q,x)| dx
$$
\n
$$
\leq \frac{1}{r} \left(\int_{\widetilde{A}_{r,\delta}} g(p,x)^2 dx \right)^{1/2} \left(\int_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 dx \right)^{1/2}
$$
\n
$$
+ \frac{1}{r} \left(\int_{\widetilde{A}_{r,\delta}} |\nabla g(p,x)|^2 dx \right)^{1/2} \left(\int_{\widetilde{A}_{r,\delta}} g(q,x)^2 dx \right)^{1/2}.
$$

To estimate the integral $\int_{\tilde{A}_{r,\delta}} g(p,x)^2 dx$, we take into account that, for all $x \in \tilde{A}_{r,\delta}$,

$$
g(p,x) \lesssim \delta^{\alpha} \int_{A(q,r,2r)} g(p,y) \, dy.
$$

Then we deduce

$$
\int_{\widetilde{A}_{r,\delta}} g(p,x)^2 dx \lesssim \frac{\delta^{\alpha}}{r^{n+1}} \left(\int_{A(q,r,2r)} g(p,x) dx \right)^2.
$$

Next we estimate the integral $\int_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 dx$. By covering $\widetilde{A}_{r,\delta}$ by a finite family of balls of radius $r/100$ and applying Cacciopoli's inequality to each one, it follows that

$$
\int_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2 dx \lesssim \frac{1}{r^2} \int_{A(q,1.1r,1.9r)} g(q,x)^2 dx.
$$

Since

$$
g(q,x) \lesssim \int_{A(q,r,2r)} g(q,y) dy \quad \text{ for all } x \in A(q,1.1r,1.9r),
$$

we get

$$
\int_{\widetilde{A}_{r,\delta}} |\nabla g(q,x)|^2\,dx \lesssim \frac{1}{r^2} \int_{A(q,1.1r,1.9r)} g(q,x)^2\,dx \lesssim \frac{1}{r^{n+3}} \left(\int_{A(q,r,2r)} g(q,x)\,dx \right)^2.
$$

So we obtain

$$
\begin{array}{l} \displaystyle \left(\int_{\widetilde{A}_{r,\delta}}\! g(p,x)^2\,dx \right)^{1/2}\, \left(\int_{\widetilde{A}_{r,\delta}}\!|\nabla g(q,x)|^2\,dx \right)^{1/2} \\\displaystyle\qquad\qquad \lesssim \frac{\delta^{\alpha/2}}{r^{n+2}}\int_{A(q,r,2r)} g(p,x)\,dx \, \int_{A(q,r,2r)} g(q,x)\,dx. \end{array}
$$

By interchanging, p and q , it is immediate to check that an analogous estimate holds for the second summand on the right hand side of [\(7.4\)](#page-33-0). Thus we get

(7.5)
$$
J_2 \lesssim \frac{\delta^{\alpha/2}}{r^{n+3}} \int_{A(q,r,2r)} g(p,x) dx \int_{A(q,r,2r)} g(q,x) dx.
$$

Concerning J_3 , we just take into account that $|\nabla \varphi| \lesssim 1/\ell(I)$ in 2*I*, and then we obtain

$$
J_3 \lesssim \sum_{I \in {\text {Bdy}}(R')} \frac{1}{\ell(I)} \int_{2I} \left| g(p,x) \nabla g(q,x) - \nabla g(p,x) \, g(q,x) \right| dx.
$$

Together with (7.2) , (7.3) , and (7.5) , this yields the lemma.

Proof of Lemma [7.1.](#page-30-1) We fix $i = 1$, for definiteness. By a Vitali type covering theorem, there exists a subfamily $\check{H}_1(R') \subset H_1(R')$ such that the balls $\{8B_Q\}_{Q \in \widetilde{H}_1(R')}$ are disjoint and

$$
\sum_{Q \in \mathsf{H}_1(R')} \mu(Q) \lesssim \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \mu(Q).
$$

By Lemma [7.2,](#page-31-0) for each $Q \in \mathsf{H}_1(R')$ we have

$$
g(p, x_Q^1) \lesssim \frac{1}{r} \sup_{y \in 2B_{R'} \cap V_1 : \delta_{\Omega}(y) \ge \delta\ell(R')} \frac{g(p, y)}{\delta_{\Omega}(y)} \int_{A(x_Q^1, r, 2r)} g(x_Q^1, x) dx
$$

+
$$
\frac{\delta^{\alpha/2}}{r^{n+3}} \int_{A(x_Q^1, r, 2r)} g(p, x) dx \int_{A(x_Q^1, r, 2r)} g(x_Q^1, x) dx
$$

+
$$
\sum_{I \in \text{Bdy}(R')} \frac{1}{\ell(I)} \int_{2I} \left| g(p, x) \nabla g(x_Q^1, x) - \nabla g(p, x) g(x_Q^1, x) \right| dx
$$

$$
=: I_1(Q) + I_2(Q) + I_3(Q),
$$

with $r = 2\ell(R')$. Since $g(p, x_Q^1) > \lambda \ell(Q)/\mu(R_0)$, we derive (7.6)

$$
\lambda \tau \mu(R') \lesssim \lambda \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \mu(Q) \lesssim \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} g(p, x^1_Q) \,\ell(Q)^{n-1} \,\mu(R_0) \lesssim \sum_{j=1}^3 \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_j(Q) \,\ell(Q)^{n-1} \,\mu(R_0).
$$

Estimate of $\sum_{Q \in \widetilde{H}_1(R')} I_1(Q) \ell(Q)^{n-1}$. We have

$$
\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_1(Q) \,\ell(Q)^{n-1} \le \frac{1}{r} \sup_{y \in 2B_{R'} \cap V_1: \delta_{\Omega}(y) \ge \delta\ell(R')} \frac{g(p,y)}{\delta_{\Omega}(y)} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \int_{A(x^1_Q, r, 2r)} g(x^1_Q, x) \, dx \,\ell(Q)^{n-1}.
$$

Note now that

$$
\sum_{Q \in \widetilde{{\sf H}}_1(R')} \int_{A(x^1_Q,r,2r)} g(x^1_Q,x) \, dx \, \ell(Q)^{n-1} \lesssim \int_{2B_{R'}} \sum_{Q \in \widetilde{{\sf H}}_1(R')} \omega^x(4Q) \, dx \le \int_{2B_{R'}} 1 \, dx \lesssim \ell(R')^{n+1}.
$$

Since $r \approx \ell(R')$, we derive

$$
\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_1(Q) \, \ell(Q)^{n-1} \lesssim \sup_{y \in 2B_{R'} \cap V_1: \delta_\Omega(y) \ge \delta \ell(R')} \frac{g(p,y)}{\delta_\Omega(y)} \, \mu(R').
$$

Estimate of $\sum_{Q \in \tilde{H}_1(R')} I_2(Q) \ell(Q)^{n-1}$. First we estimate $\int_{A(x_Q^1,r,2r)} g(p,x) dx$ by applying Lemma $2.5:$

$$
\int_{A(x_{Q}^1,r,2r)} g(p,x) dx \leq \int_{2B_{R'}} g(p,x) dx \lesssim \ell(R')^{n+1} \frac{\omega^p (8B_{R'})}{\ell(R')^{n-1}} \lesssim \ell(R')^2 \frac{\mu(R')}{\mu(R_0)} \approx \frac{r^{n+2}}{\mu(R_0)}.
$$

So we have

$$
\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} I_2(Q) \,\ell(Q)^{n-1} \lesssim \frac{\delta^{\alpha/2}}{r \,\mu(R_0)} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \int_{A(x^1_Q, r, 2r)} g(x^1_Q, x) \, dx \,\ell(Q)^{n-1}
$$
\n
$$
\lesssim \frac{\delta^{\alpha/2}}{r \,\mu(R_0)} \int_{2B_{R'}} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \omega^x(4Q) \, dx
$$
\n
$$
\lesssim \frac{\delta^{\alpha/2}}{r \,\mu(R_0)} \int_{2B_{R'}} 1 \, dx \lesssim \frac{\delta^{\alpha/2} \,\mu(R')}{\mu(R_0)}.
$$

Estimate of $\sum_{Q \in \tilde{H}_1(R')} I_3(Q) \ell(Q)^{n-1}$. Note first that, for each $I \in \text{Bdy}(R')$, since $x_Q^1 \notin 4I$, using the subharmonicity of $g(p, \cdot)$ and $g(x_Q^1, \cdot)$ in 4*I*, and Caccioppoli's inequality,

$$
\frac{1}{\ell(I)}\int_{2I} \left|g(p,x)\nabla g(x_Q^1, x)\right| dx \lesssim \frac{1}{\ell(I)} \sup_{x \in 2I} g(p,x) \int_{2I} \left|\nabla g(x_Q^1, x)\right| dx
$$

$$
\lesssim \ell(I)^{n-1} m_{4I} g(p, \cdot) m_{4I} g(x_Q^1, \cdot).
$$

By very similar estimates, we also get

$$
\frac{1}{\ell(I)}\int_{2I} \left|\nabla g(p,x)\,g(x_Q^1,x)\right|dx \lesssim \ell(I)^{n-1} \, m_{4I}g(p,\cdot) \, m_{4I}g(x_Q^1,\cdot).
$$

Recall now that, by Lemma [6.3\(](#page-23-0)e)(i),

$$
m_{4I}g(p,\cdot) \leq \gamma \frac{\ell(P)}{\mu(R_0)}
$$
 for each $I \in \mathcal{W}_P$, with $P \in \widetilde{\text{End}}(R')$ such that $2B_P \cap 10B_{R'} \neq \emptyset$.

We distinguish two types of Whitney cubes $I \in Bdy(R')$. We write $I \in T_1$ if $\ell(I) \geq \gamma^{1/2} \ell(P)$ for some P such that $I \in W_P$ and $2B_P \cap 10B_{R'} \neq \emptyset$, and we write $I \in T_2$ otherwise (there may exist more than one P such that $I \in W_P$, but if $W_P \cap W_{P'} \neq \emptyset$, then $\ell(P) \approx \ell(P')$). So we split

$$
\sum_{Q \in \widetilde{H}_1(R')} I_3(Q) \ell(Q)^{n-1} \leq \sum_{Q \in \widetilde{H}_1(R')} \sum_{I \in \text{Bdy}(R')} \ell(I)^{n-1} m_{4I} g(p, \cdot) m_{4I} g(x^1_Q, \cdot) \ell(Q)^{n-1}
$$
\n
$$
= \sum_{Q \in \widetilde{H}_1(R')} \sum_{I \in T_1} \dots + \sum_{Q \in \widetilde{H}_1(R')} \sum_{I \in T_2} \dots =: S_1 + S_2.
$$

Concerning the sum S_1 we have

$$
S_1 \lesssim \gamma \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'):\\2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P \cap T_1} \frac{\ell(P)}{\mu(R_0)} \ell(I)^{n-1} m_{4I} g(x_Q^1, \cdot) \ell(Q)^{n-1}
$$

$$
\lesssim \gamma^{1/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathsf{End}}(R'):\\2B_P \cap 10B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P} \frac{\ell(I)^n}{\mu(R_0)} m_{4I} g(x_Q^1, \cdot) \ell(Q)^{n-1}
$$

Next we take into account that

$$
\ell(Q)^{n-1} m_{4I} g(x_Q^1, \cdot) \lesssim \omega^{x_I}(4Q),
$$

where x_I stands for the center of I . Then we derive

$$
S_1 \lesssim \gamma^{1/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \widetilde{\mathrm{End}}(R') : \\ 2B_P \cap 10 B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P} \omega^{x_I}(4Q) \, \frac{\ell(I)^n}{\mu(R_0)}.
$$

Since $\sum_{Q \in \widetilde{H}_1(R')} \omega^{x_I}(4Q) \lesssim 1$ for each *I*, we get

$$
S_1 \lesssim \gamma^{1/2} \sum_{\substack{P \in \widetilde{\operatorname{End}}(R') : \\ 2B_P \cap 10 B_{R'} \neq \varnothing}} \sum_{I \in \mathcal{W}_P} \frac{\ell(I)^n}{\mu(R_0)}.
$$

By Lemma [6.3\(](#page-23-0)e)(ii), we have $\sum_{I \in \mathcal{W}_P} \ell(I)^n \lesssim \ell(P)^n$, and so we deduce

$$
S_1 \lesssim \gamma^{1/2} \sum_{\substack{P \in \widetilde{\operatorname{End}}(R') : \\ 2B_P \cap 10 B_{R'} \neq \varnothing}} \frac{\mu(P)}{\mu(R_0)} \lesssim \gamma^{1/2} \frac{\mu(R')}{\mu(R_0)}.
$$

Next we turn our attention to the sum S_2 in [\(7.7\)](#page-36-0). Recall that

$$
S_2 = \sum_{Q \in \widetilde{H}_1(R')} \sum_{I \in T_2} \ell(I)^{n-1} m_{4I} g(p,\cdot) m_{4I} g(x_Q^1,\cdot) \ell(Q)^{n-1}.
$$

Let us remark that we assume the condition that $I \in W_P$ for some $2P \in \text{End}(R')$ such that $2B_P \cap$ $10B_{R'} \neq \emptyset$ to be part of the definition of $I \in T_2$. Using the estimate $m_{4I}g(p, \cdot) \lesssim \omega^p(B^I)\ell(I)^{1-n}$, we derive

$$
S_2 \lesssim \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2} \omega^p(B^I) m_{4I} g(x_Q^1, \cdot) \ell(Q)^{n-1}
$$

=
$$
\sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2: 20I \cap 20B_Q \neq \varnothing} \dots + \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{I \in T_2: 20I \cap 20B_Q = \varnothing} \dots =: A + B.
$$

To estimate the term A we take into account that if $20I \cap 20B_Q \neq \emptyset$ and $I \in \mathcal{W}_P$, then $\ell(P) \leq \ell(Q)$ and thus $\ell(I) \lesssim \gamma^{1/2} \ell(Q)$ because $I \in T_2$. As a consequence, $I \subset 21B_Q$ and also, by the Hölder continuity of $g(x_Q^1, \cdot)$, if we let B be a ball concentric with B^I with radius comparable to $\ell(Q)$ and such that $dist(x_Q^1, B) \approx \ell(Q)$, we obtain

$$
m_{2B^I}g(x_Q^1,\cdot)\lesssim \left(\frac{r(B^I)}{r(B)}\right)^{\alpha}m_Bg(x_Q^1,\cdot)\lesssim \gamma^{\alpha/2}\,\frac{1}{\ell(Q)^{n-1}},
$$

where $\alpha > 0$ is the exponent of Hölder continuity. Hence,

$$
A \lesssim \gamma^{\alpha/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \sum_{\substack{P \in \text{End}(R') : \\ 2B_P \cap 10B_{R'} \neq \varnothing \\ 20B_P \cap 20B_Q \neq \varnothing}} \sum_{I \in \mathcal{W}_P \cap T_2} \omega^p(B^I).
$$

By Lemma [6.3\(](#page-23-0)e)(ii), we have $\sum_{I \in W_P} \omega^p(B^I) \leq \omega^p(CB_P)$, and using also that, for P as above, $CB_P \subset C'B_Q$ for some absolute constant C' , we obtain

$$
A \lesssim \gamma^{\alpha/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \omega^p(C'B_Q) \lesssim \gamma^{\alpha/2} \sum_{Q \in \widetilde{\mathsf{H}}_1(R')} \frac{\mu(Q)}{\mu(R_0)} \lesssim \gamma^{\alpha/2} \frac{\mu(R')}{\mu(R_0)}.
$$

Finally, we turn our attention to the term B . We have

$$
B = \sum_{Q \in \widetilde{H}_1(R')} \sum_{I \in T_2: 20I \cap 20B_Q = \varnothing} \omega^p(B^I) \, m_{4I} g(x_Q^1, \cdot) \, \ell(Q)^{n-1}
$$

=
$$
\sum_{I \in T_2} \omega^p(B^I) \int_{4I} \sum_{Q \in \widetilde{H}_1(R'): 20I \cap 20B_Q = \varnothing} g(x_Q^1, x) \, \ell(Q)^{n-1} \, dx
$$

$$
\lesssim \sum_{I \in T_2} \omega^p(B^I) \int_{4I} \sum_{Q \in \widetilde{H}_1(R'): 20I \cap 20B_Q = \varnothing} \omega^x(8B_Q) \, dx.
$$

We claim now that, in the last sum, if $20I \cap 20B_Q = \emptyset$, then $dist(I, 8B_Q) \geq c \gamma^{-1/2} \ell(I)$. To check this, take $P \in \text{End}(R')$ such that $I \in \mathcal{W}_P$. Then note that

$$
\ell(P) \le \frac{1}{300} d_R(z_P) \le \frac{1}{300} \left(\text{dist}(z_P, Q) + \ell(Q) \right)
$$

$$
\leq \frac{1}{300} \left(\text{dist}(z_P, I) + \text{diam}(I) + \text{dist}(I, 8B_Q) + C\ell(Q) \right).
$$

Using that $I \cap 2B_P \neq \emptyset$, $\text{diam}(I) \leq C \gamma^{1/2} \ell(P) \ll \ell(P)$, and $\ell(Q) \leq \text{dist}(I, 8B_Q)$, we get

$$
\ell(P) \le \frac{1}{300} \left(\text{dist}(I, 8B_Q) + 3r(B_P) + C \,\ell(Q) \right) \le C \,\text{dist}(I, 8B_Q) + \frac{12}{300} \,\ell(P),
$$

which implies that

$$
\ell(I) \le C\gamma^{1/2} \,\ell(P) \le C\,\gamma^{1/2}\,\text{dist}(I, 8B_Q),
$$

and yields our claim.

Taking into account that the balls $\{8B_Q\}_{Q\in \widetilde{\mathsf{H}}_1(R')}$ are disjoint and the Hölder continuity of $\omega^{(\cdot)}(\partial\Omega\setminus$ $c\gamma^{-1/2}I$), for all $x \in 4I$ we get

$$
\sum_{Q\in \widetilde{\mathsf{H}}_1(R'): 20I\cap 20B_Q=\varnothing}\omega^x(8B_Q)\lesssim \omega^x(\partial\Omega\setminus c\gamma^{-1/2}I)\lesssim \gamma^{\alpha/2}.
$$

Thus,

$$
B \lesssim \gamma^{\alpha/2} \sum_{I \in T_2} \omega^p(B^I) \le \gamma^{\alpha/2} \sum_{\substack{P \in \overline{\text{End}}(R') : \\ 2B_P \cap 10B_{R'} \ne \varnothing}} \sum_{I \in \mathcal{W}_P \cap T_2} \omega^p(B^I).
$$

Recalling again that $\sum_{I \in \mathcal{W}_P} \omega^p(B^I) \lesssim \omega^p(CB_P)$, we deduce

$$
B \lesssim \gamma^{\alpha/2} \sum_{\substack{P \in \widetilde{\operatorname{End}}(R') : \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \omega^p(CB_P) \lesssim \gamma^{\alpha/2} \sum_{\substack{P \in \widetilde{\operatorname{End}}(R') : \\ 2B_P \cap 10B_{R'} \neq \varnothing}} \frac{\mu(P)}{\mu(R_0)} \lesssim \gamma^{\alpha/2} \frac{\mu(R')}{\mu(R_0)}.
$$

Remark that for the second inequality we took into account that P is contained in a cube of the form 22P' with $P' \in \text{Tree}_{\text{WSBC}}(R)$ and $\ell(P') \approx \ell(P)$, by Lemma [6.1.](#page-22-0) This implies that $\omega^p(CB_P) \leq$ $\omega^p(C'B_{P'}) \lesssim \mu(P')\,\mu(R_0)^{-1} \lesssim \mu(P)\,\mu(R_0)^{-1}.$

Gathering the estimates above and recalling [\(7.6\)](#page-35-3), we deduce

$$
\lambda \tau \mu(R') \lesssim \sup_{y \in 2B_{R'} \cap V_1 : \delta_{\Omega}(y) \ge \delta \ell(R')} \frac{g(p, y)}{\delta_{\Omega}(y)} \mu(R') \mu(R_0) + \delta^{\alpha/2} \mu(R') + \gamma^{\alpha/2} \mu(R').
$$

So, if δ and γ are small enough (depending on λ , τ), we infer that

$$
\lambda \tau \mu(R') \lesssim \sup_{y \in 2B_{R'} \cap V_1 : \delta_{\Omega}(y) \ge \delta \ell(R')} \frac{g(p, y)}{\delta_{\Omega}(y)} \mu(R') \mu(R_0).
$$

That is, there exists some $y_0 \in 2B_{R'} \cap V_1$ with $\delta_{\Omega}(y_0) \ge \delta \ell(R')$ such that

$$
\frac{g(p, y_0)}{\delta_{\Omega}(y)} \gtrsim \frac{\lambda \tau}{\mu(R_0)},
$$

with δ depending on λ, τ . Since $x_{R'}^1$ and y_0 can be joined by a C-good Harnack chain (for some C depending on δ and γ , and thus on λ , τ), we deduce that

$$
\frac{g(p, x^1_{R'})}{\ell(R')} \gtrsim \frac{c(\lambda, \tau)}{\mu(R_0)},
$$

as wished. \square

Lemma 7.3. Let $\eta \in (0,1)$ and $\lambda > 0$. Choose $\gamma = \gamma(\lambda, \tau)$ small enough as in Lemma [7.1](#page-30-1) with τ = η/2*. Assume that the family* WSBC(Γ) *is defined by choosing* Γ *big enough depending on* γ *(and thus on* λ *and* η *) as in Lemma* [6.3.](#page-23-0) Let $R \in Top_b^{(N)}$ *and suppose that* $Tree_{\text{WSEC}}(R) \neq \emptyset$. Then, there *exists an exceptional family* $Ex_{\text{WSBC}}(R) \subset \text{Stop}_{\text{WSBC}}(R) \cap G$ *satisfying*

$$
\sum_{P \in \mathsf{Ex}_{\mathsf{WSEC}}(R)} \mu(P) \le \eta \,\mu(R)
$$

such that, for every $Q \in \text{Stop}_{\text{WSRC}}(R) \cap G \setminus \text{Ex}_{\text{WSRC}}(R)$ *, any* λ -good corkscrew for Q can be joined *to some* λ' -good corkscrew for R by a $C(\lambda, \eta)$ -good Harnack chain, with λ' depending on λ, η .

Proof. For any $R' \in \mathcal{D}_{\mu,k_0} \cap \text{Tree}_{\text{WSBC}}(R)$, with $k_0 = k_0(\gamma)$, we define $H_i(R')$ as in [\(7.1\)](#page-30-3), so that

$$
\mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{G} \cap \mathcal{D}_{\mu}(R') = \mathsf{H}_1(R') \cup \mathsf{H}_2(R').
$$

For each R' , we set

$$
\mathsf{Ex}_{\mathsf{WSBC}}(R') = \bigcup_{i=1}^2 \Big\{ Q \in \mathsf{H}_i(R') : \sum_{P \in \mathsf{H}_i(R')} \mu(P) \leq \tau \mu(R') \Big\}.
$$

That is, for fixed $i = 1$ or 2, if $\sum_{P \in H_i(R')} \mu(P) \leq \tau \mu(R')$, then all the cubes from $H_i(R')$ belong to $Ex_{WSBC}(R')$. In this way, it is clear that

(7.8)
$$
\sum_{P \in \mathsf{Ex}_{\mathsf{WSEC}}(R')} \mu(P) \leq 2\tau \mu(R) = \eta \mu(R').
$$

We claim that the λ -good corkscrews of cubes from $\text{Stop}_{\text{WSBC}}(R) \cap \text{G} \cap \mathcal{D}_{\mu}(R') \setminus \text{Ex}_{\text{WSBC}}(R')$ can be joined to some $\tilde{\lambda}$ -good corkscrew for R' by a \tilde{C} -good Harnack chain, with $\tilde{\lambda}$ depending on λ , η , and \tilde{C} depending on Γ and thus on λ , η too. Indeed, if $Q \in H_i(R') \setminus E \times_{WSE}(R')$ and x_Q^i is λ -good corkscrew belonging to V_i (we use the notation of Lemma [7.1](#page-30-1) and [6.3\)](#page-23-0), then $\sum_{P \in H_i(R')} \mu(P) > \tau \mu(R')$ by the definition above and thus Lemma [7.1](#page-30-1) ensures that $g(p, x_{R'}^i) \ge c(\lambda, \tau) \frac{\ell(R')}{\mu(R_0)}$ $\frac{\ell(R')}{\mu(R_0)}$. So $x_{R'}^i$ is a λ -good corkscrew, which by Lemma [6.3\(](#page-23-0)c) can be joined to x_Q^i by a \tilde{C} -good Harnack chain. In turn, this λ good corkscrew for R' can be joined to some λ' -good corkscrew for R by a C'-good Harnack chain, by applying Lemma [5.4](#page-21-3) k_0 times, with C' depending on k_0 and thus on λ and η .

On the other hand, the cubes $Q \in \mathsf{Stop}_{\mathsf{WSBC}}(R) \cap \mathsf{G}$ which are not contained in any cube $R' \in \mathsf{C}$ $\mathcal{D}_{\mu,k_0} \cap \mathsf{Tree}_{\mathsf{WSEG}}(R)$ satisfy $\ell(Q) \geq 2^{-k_0}$, and then, arguing as above, their associated λ -good corkscrews can be joined to some λ' -good corkscrew for R by a C' -good Harnack chain, by apply-ing Lemma [5.4](#page-21-3) at most k_0 times. Hence, if we define

$$
\mathsf{Ex}_{\mathsf{WSBC}}(R) = \bigcup_{R' \in \mathcal{D}_{\mu,k_0}(R)} \mathsf{Ex}_{\mathsf{WSBC}}(R'),
$$

taking into account [\(7.8\)](#page-39-0), the lemma follows. \square

Proof of the Key Lemma [5.3.](#page-20-3) We choose $\Gamma = \Gamma(\lambda, \eta)$ as in Lemma [7.3](#page-39-1) and we consider the associated family WSBC(Γ). In case that Tree_{WSBC}(*R*) = \varnothing , we set Ex(*R*) = \varnothing . Otherwise, we consider the family $Ex_{WSBC}(R)$ from Lemma [7.3,](#page-39-1) and we define

$$
\text{Ex}(R) = \big(\text{Ex}_{\text{WSBC}}(R) \cap \text{Stop}(R)\big) \cup \bigcup_{Q \in \text{Ex}_{\text{WSBC}}(R) \backslash \text{Stop}(R)} (\text{SubStop}(Q) \cap \text{G}).
$$

It may be useful for the reader to compare the definition above with the partition of $\text{Stop}(R)$ in [\(5.3\)](#page-21-4). By Lemma [7.3](#page-39-1) we have

$$
\sum_{P \in \text{Ex}(R)} \mu(P) \le \sum_{Q \in \text{Ex}_{\text{WSEC}}(R)} \mu(P) \le \eta \mu(R).
$$

Next we show that for every $P \in \text{Stop}(R) \cap G \setminus \text{Ex}(R)$, any λ -good corkscrew for P can be joined to some λ' -good corkscrew for R by a $C(\lambda, \eta)$ -good Harnack chain. In fact, if $P \in \text{Stop}_{\text{WSBC}}(R)$, then $P \in \text{Stop}_{\text{WSBC}}(R) \cap G \setminus \text{Ex}_{\text{WSBC}}(R)$ since such cube P cannot belong to SubStop(Q) for any $Q \in \text{StopWSR}(R) \setminus \text{Stop}(R)$ (recall the partition [\(5.3\)](#page-21-4)), and thus the existence of such Harnack chain is ensured by Lemma [7.3.](#page-39-1) On the other hand, if $P \notin \text{Stop}_{\text{WSEG}}(R)$, then P is contained in some cube $Q(P) \in Stop_{WSBC}(R) \setminus WSBC(\Gamma)$. Consider the chain $P = S_1 \subset S_2 \subset \cdots \subset S_m = Q(P)$, so that each S_i is the parent of S_{i-1} . For $1 \leq i \leq m$, choose inductively a big corkscrew x_i for S_i in such a way that x_1 is at the same side of L_P as the good λ corkscrew x_P for P, and x_{i+1} is at the same side of L_{S_i} as x_i for each i. Using that $b\beta(S_i) \leq C\varepsilon \ll 1$ for all i, it easy to check that the line obtained by joining the segments $[x_P, x_1], [x_1, x_2], \ldots, [x_{m-1}, x_m]$ is a good carrot curve and so gives rise to a good Harnack chain that joins x_P to x_m . It may happen that x_m is not a λ -good corkscrew. However, since $Q(P) \notin WSBC(\Gamma)$, it turns out that x_m can be joined to some c_3 -good corkscrew $x_{Q(P)}$ for $Q(P)$ by some $C(\Gamma)$ -good Harnack chain, with c_3 given by [\(4.2\)](#page-16-1) (and thus independent of λ and η), because $Q(P) \in G$. Note that since $\lambda \leq c_3$, $x_{Q(P)}$ is also a λ -good corkscrew. In turn, since $Q(P) \notin \mathsf{Ex}_{\mathsf{WSBC}}(R)$, $x_{Q(P)}$ can be joined to some λ' -good corkscrew for R by another $C'(\lambda, \eta)$ -good Harnack chain. Altogether, this shows that x_P can be connected to some λ' -good corkscrew for R by a $C''(\lambda, \eta)$ -good Harnack chain, which completes the proof of the lemma.

Below we will write $Ex(R, \lambda, \eta)$ instead of $Ex(R)$ to keep track of the dependence of this family on the parameters λ and η .

8. PROOF OF THE MAIN LEMMA [2.13](#page-11-2)

8.1. Notation. Recall that by the definition of G_0^K in [\(5.2\)](#page-19-2), $\sum_{R \in \text{Top}} \chi_R(x) \leq K$ for all $x \in G_0^K$. For such x, let Q be the smallest cube from Top that contains x, and denote $n_0(x) = -\log_2 \ell(Q)$, so that $Q \in \mathcal{D}_{\mu,n_0(x)}$. Next let $N_0 \in \mathbb{Z}$ be such that

$$
\mu\big(\big\{x \in G_0^K : n_0(x) \le N_0 - 1\big\}\big) \ge \frac{1}{2} \,\mu(G_0^K),
$$

and denote

$$
\widetilde{G}_0^K = \{ x \in G_0^K : n_0(x) \le N_0 - 1 \}.
$$

Fix

and set

$$
\mathsf{T}'_a=\mathcal{D}_{\mu,N}(R_0)\cup\mathsf{Top}_a^{(N)},
$$

 $N = N_0 - 1$,

and also

$$
\mathsf{T}'_b = \mathsf{Top}^{(N)}_b \setminus \mathcal{D}_{\mu,N}(R_0)
$$

(recall that $\text{Top}_a^{(N)}$ and $\text{Top}_b^{(N)}$ were defined in Section [5.2\)](#page-20-0). So if $R \in \textsf{T}'_a \setminus \mathcal{D}_{\mu,N}(R_0)$, then $\textsf{Stop}^N(R)$ coincides the family of sons of R, and it $R \in \mathsf{T}'_b$ this will not be the case, in general. Next we denote by T_a and T_b the respective subfamilies of cubes from T'_a and T'_b which intersect \tilde{G}_0^K .

For $j \geq 0$, we set

$$
\mathsf{T}_{b}^{j} = \Big\{ R \in \mathsf{T}_{b} : \sum_{Q \in \mathsf{T}_{b}: Q \supset R} \chi_{Q} = j \text{ on } R \Big\}.
$$

We also denote

$$
\mathsf{S}_b^j = \big\{ Q \in \mathcal{D}_\mu : Q \in \mathsf{Stop}^N(R) \text{ for some } R \in \mathsf{T}_b^j \big\}, \qquad \mathsf{S}_b = \bigcup_j \mathsf{S}_b^j,
$$

and we let T_a^j be the subfamily of cubes $R \in \mathsf{T}_a$ such that there exists some $Q \in \mathsf{S}_b^{j-1}$ b^{j-1} such that $Q \supset R$ and R is not contained in any cube from S_b^k with $k \geq j$.

8.2. Two auxiliary lemmas.

Lemma 8.1. *The following properties hold for the family* T_b^1 *:*

- (a) The cubes from T^1_b are pairwise disjoint and cover \widetilde{G}_0^K , assuming N_0 big enough.
- *(b)* If $R \in \mathsf{T}^1_b$, then $\ell(R) \approx_K \ell(R_0)$.
- *(c) Given* $R \in \mathcal{D}_{\mu}(R_0)$ *with* $\ell(R) \ge c \ell(R_0)$ *(for example,* $R \in \mathsf{T}^1_b$ *)* and $\lambda > 0$ *, if* x_R *is a* λ -good *corkscrew point for* R, then there is a $C(\lambda, c)$ -good Harnack chain that joins x_R to p.

Proof. Concerning the statement (a), the cubes from T_b^1 are pairwise disjoint by construction. Suppose that $x \in \widetilde{G}_0^K$ is not contained in any cube from T_b^1 . By the definition of the family Top^N , this implies that all the cubes $Q \subset R_0$ with $2^{-N} \ell(R_0) \leq \ell(Q) \leq 2^{-10} \ell(R_0)$ containing x belong to T_a . However, there are most K cubes Q of this type, which is not possible if N is taken big enough. So the cubes from T^1_b cover \widetilde{G}_0^K .

The proof of (b) is analogous. Given $R \in \mathsf{T}^1_b$, all the cubes Q which contain R and have side length smaller or equal that $2^{-10}\ell(R_0)$ belong to T_a . Hence there at most $K-1$ cubes Q of this type, because $\widetilde{G}_0^K \cap R \neq \emptyset$. Thus, $\ell(R) \geq 2^{-K-10} \ell(R_0)$.

The statement (c) is an immediate consequence of (b) and Lemma [4.4.](#page-16-3)

Lemma 8.2. Let $Q \in \mathsf{T}^j_a \cup \mathsf{T}^j_b$ $\frac{d}{b}$ for some $j \geq 2$ and let x_Q be a λ -good corkscrew for Q , with $\lambda > 0$. *There exists some constant* $\gamma(\lambda, K) > 0$ *such if* $\ell(Q) \leq \gamma(\lambda, K) \ell(R_0)$ *, then there exists some cube* $R \in \mathsf{S}_b$ such that $R \supset Q$ with a λ' -good corkscrew x_R for R such that x_R can be joined to x_Q by a $C(\lambda, K)$ -good Harnack chain, with λ' depending on λ and K .

Proof. We assume $\gamma(\lambda, K) > 0$ small enough. Then we can apply Lemma [4.5](#page-17-2) $K + 1$ times to get cubes R_1, \ldots, R_{K+1} satisfying:

- $Q \subsetneq R_1 \subsetneq R_2 \subsetneq \ldots \subsetneq R_{K+1}$ and $\ell(R_{K+1}) \leq 2^{-10} \ell(R_0)$,
- each R_j has an associated λ' -good corkscrew x_{R_i} (with λ' depending on λ, K) and there exists a $C(\lambda, K)$ -good Harnack chain joining x_Q and $x_{R_1}, \ldots, x_{R_{K+1}}$.

Since $Q \cap \widetilde{G}_0^K \neq \emptyset$, at least one of the cubes R_1, \ldots, R_{K+1} , say R_j , does not belong to Top. This implies that $R_j \in \text{Tree}^{(N)}(\widetilde{R})$ for some $\widetilde{R} \in \text{Top}(K)$. Let $R \in \text{Stop}^{(N)}(\widetilde{R})$ be the stopping cube that contains Q. Then Lemma [6.3](#page-23-0) ensures that there is a good Harnack chain that connects x_{R_j} to some corkscrew x_R for R. Notice that $\ell(R_i) \approx_{\lambda,K} \ell(Q) \approx_{\lambda,K} \ell(R)$ because $Q \subset R \subset R_i$. This implies that $g(p, x_R) \approx_{K,\lambda} g(p, x_{R_j}) \approx_{K,\lambda} g(p, x_Q)$. Further, gathering the Harnack chain that joins x_Q to $x_{\tilde{R}}$ and the one that joins x_{R_j} to x_R , we obtain the good Harnack chain required by the lemma.

8.3. The algorithm to construct good Harnack chains. We will construct good Harnack chains that join good corkscrews from "most" cubes from $\mathcal{D}_{\mu,N}$ that intersect \tilde{G}_0^K to good corkscrews from cubes belonging to $R \in \mathsf{T}^1_b$, and then we will join these latter good corkscrews to p using the fact that $\ell(R) \approx \ell(R_0)$. To this end we choose $\eta > 0$ such that

$$
\eta \le \frac{1}{2K} \, \frac{\mu(\widetilde{G}_0^K)}{\mu(R_0)},
$$

and we denote

$$
m = \max_{x \in \widetilde{G}_0^K} \sum_{R \in \mathsf{T}_b} \chi_R(x)
$$

(so that $m \leq K$) and we apply the following algorithm: we set $a_{m+1} = c_3$, so that [\(4.2\)](#page-16-1) ensures that for each $Q \in T_a \cup T_b$ there exists some good a_{m+1} -good corkscrew x_Q . For $j = m, m-1, \ldots, 1$, we perform the following procedure:

(1) Join a_{j+1} -good corkscrews of cubes Q from $\mathsf{T}_a^{j+1} \cup \mathsf{T}_b^{j+1}$ b^{j+1} such that $\ell(Q) \leq c'_j \, \ell(R_0)$ to a'_j -good corkscrews of cubes $R(Q)$ from $\mathsf{S}^1_b\cup\ldots\cup\mathsf{S}^j_b$ ∂_b^j by C'_j -good Harnack chains, with $a'_j \le a_{j+1}$, so that $R(Q)$ is an ancestor of Q. This step can be performed because of Lemma [8.2,](#page-41-1) with $c'_j = \gamma(a_{j+1}, K)$ in the lemma. The constants a'_j, c'_j , and C'_j depend on a_{j+1} and K .

(2) Set

$$
NC_j = \bigcup_{R \in T_b^j} \mathsf{Ex}(R, a'_j, \eta),
$$

and join a'_j -good corkscrews for all cubes $Q \in \mathsf{S}^j_b$ $\mathcal{B}_b^{\jmath} \setminus {\sf NC}_j$ to a_j -good corkscrews for cubes $R(Q) \in \mathsf{T}_{h}^{j}$ b^j by C_j -good Harnack chains, with $a_j \le a'_j$, so that $R(Q)$ is an ancestor of Q. To this end, one applies Lemma [5.3,](#page-20-3) which ensures the existence of such Harnack chains connecting a'_j -good corkscrew points for cubes from S_b^j $\frac{\jmath}{b}\setminus \mathsf{NC}_j$ to a_j -good corkscrew points for cubes from T^j_b b_b . The constants a_j and C_j depend on a'_j and K.

After iterating the procedure above for $j = m, m - 1, \ldots, 1$ and joining some Harnack chains arisen in the different iterations, we will have constructed C-good Harnack chains that join a_{m+1} good corkscrew points for all cubes $Q \in \mathsf{T}_a$ not contained in $\bigcup_{j=1}^m \bigcup_{P \in \mathsf{NC}_j} P$ to a_1 -good corkscrews of some ancestors $R(Q)$ belonging either T^1_b or, more generally, such that $\ell(R(Q)) \gtrsim \ell(R_0)$. The constants c'_j , a'_j , a_j , C_j worsen at each step j. However, this is not harmful because the number of iterations of the procedure is at most m, and $m \leq K$.

Denote by I_N the cubes from $\mathcal{D}_{\mu,N}$ which intersect \tilde{G}_0^K and are not contained in any cube from ${P \in NC_j : j = 1,...,m}$. By the algorithm above we have constructed good Harnack chains that join a_{m+1} -good corkscrew points for all cubes $Q \in I_N$ to some to some a_1 -good corkscrew for cubes $R(Q) \in \mathcal{D}_{\mu}(R_0)$ with $\ell(R(Q)) \approx \ell(R_0)$. Also, by applying Lemma [8.1](#page-41-2) (c) we can connect the a_1 -good corkscrew for $R(Q)$ to p by a good Harnack chain.

Consider now an arbitrary point $x \in \widetilde{G}_0^K \cap Q$, with $Q \in I_N$. By the definition of \widetilde{G}_0^K and the choice $N = N_0$, all the cubes $P \in \mathcal{D}_{\mu}$ containing x with side length smaller or equal than $\ell(Q)$ satisfy $b\beta(P) \leq \varepsilon$. Then, by an easy geometric argument (see the proof of Lemma [5.3](#page-20-3) for a related argument) it is easy to check that there is a good Harnack chain joining any good corkscrew for Q to x . Hence, for all the points $x \in \bigcup_{Q \in I_N} Q \cap \widetilde{G}_0^K$ there is a good Harnack chain that joins x to p.

Finally, observe that, for each j , by Lemma [5.3,](#page-20-3)

$$
\sum_{P\in {\rm NC}_j}\mu(P)=\sum_{R\in \mathsf{T}^j_b}\sum_{P\in {\rm Ex}(R,a'_j,\eta)}\mu(P)\leq \eta\sum_{R\in \mathsf{T}^j_b}\mu(R)\leq \eta\,\mu(R_0)\leq \frac{1}{2K}\,\mu(\widetilde G_0^K).
$$

Therefore,

$$
\sum_{j=1}^m \sum_{P\in {\rm NC}_j} \mu(P) \leq \frac{m}{2K}\,\mu(\widetilde{G}_0^K) \leq \frac{1}{2}\,\mu(\widetilde{G}_0^K),
$$

and thus

$$
\sum_{Q \in I_N} \mu(Q) \ge \mu(\widetilde{G}_0^K) - \sum_{j=1}^m \sum_{P \in \mathsf{NC}_j} \mu(P) \ge \frac{1}{2} \mu(\widetilde{G}_0^K) \approx \mu(R_0).
$$

This finishes the proof of the Main Lemma [2.13.](#page-11-2)

Remark 8.3. Recall that in the arguments above we assumed that $\Omega = \mathbb{R}^{n+1} \setminus \partial\Omega$. For the general case, we define the auxiliary open set $\widetilde{Q} = \mathbb{R}^{n+1} \setminus \partial\Omega$, and we apply the arguments above to $\widetilde{\Omega}$. Then we will get carrot curves contained in Ω that join points from a big piece of \tilde{G}_0^K to p. A quick inspection of the construction above shows that these carrot curves are contained in the set $\{x \in \Omega : g(p, x) > 0\},\$ which is a subset of Ω , which implies the required connectivity condition to conclude the proof of the Main Lemma [2.13.](#page-11-2)

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