AN IMPROVED POPOVICIU-TYPE INEQUALITY FOR A NEW BERNSTEIN-TYPE OPERATOR

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ABSTRACT. Recently we introduced a new Bernstein-type operator using Pólya's urn model with negative replacement, and we showed that it satisfies a Popoviciu-type inequality with a constant slightly larger than that of the corresponding inequality for the classical Bernstein operator.

In the present paper we prove an inequality for the rising factorial (of independent interest), and we use it in order to show that the constant in the Popoviciu inequality for the new operator is in fact smaller than the corresponding constant for the Bernstein operator.

1. INTRODUCTION

It is known that the classical Bernstein operator ([1]) defined by

$$B_n\left(f;x\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k \left(1-x\right)^{n-k} \tag{1.1}$$

satisfies the inequality

$$\left|B_n\left(f;x\right) - f\left(x\right)\right| \leq C\omega\left(n^{-1/2}\right), \qquad x \in [0,1], \ n = 1, 2, \ldots, \tag{1.2}$$

where $f : [0,1] \to \mathbb{R}$ is an arbitrary continuous function, $\omega(\cdot)$ denotes the modulus of continuity of f.

T. Popoviciu ([5]) proved the above inequality for the value of the constant $C = \frac{3}{2}$. Lorentz ([6], pp. 20 –21) improved the value of the constant to $C = \frac{5}{4}$, and also showed that the constant C cannot be less than one. The optimal value of the constant C for which the above inequality holds true for any continuous function was obtained by Sikkema ([9]), who obtained the value

$$C_{opt} = \frac{4306 + 837\sqrt{6}}{5932} \approx 1.0898873..., \tag{1.3}$$

attained in the case $n = 6$ for a particular function.

Recently ([8]), we introduced the Bernstein-type operator R_n defined by (2.5) and we showed that its also satisfies a Popoviciu-type inequality, with the constant $C = \frac{31}{27} \approx 1.14815$, smaller than Popoviciu's and Lorentz's constants, but slightly larger than Sikkema's optimal constant.

In the present paper we refine this result, by showing that the constant in the Popoviciu-type estimate is smaller than Sikkema's constant (Theorem 3.2). The proof is based on a certain inequality for the rising factorial (Lemma 3.1) which is of independent interest.

2. PRELIMINARIES

For $x, h \in \mathbb{R}$ and $n \in \mathbb{N}$ we set

$$x^{(n,h)} = x\left(x+h\right)\left(x+2h\right) \cdot \ldots \cdot \left(x+(n-1)h\right) \tag{2.1}$$

for the generalized (rising) factorial with increment h. When $n = 0$ we are using the convention $x^{(0,h)} = 1$ for any $x, h \in \mathbb{R}$.

A random variable $X_n^{a,b,c}$ has a Pólya's urn distribution (also known as Pólya-Eggenberger distribution, see [2], [4]) with parameters $n \geq 1$, $a, b \in \mathbb{R}_+$, and $c \in \mathbb{R}$ satisfying

$$a + (n-1)\,c \geq 0 \qquad \text{and} \qquad b + (n-1)\,c \geq 0, \tag{2.2}$$

if it is given by (see for example [7])

$$P\left(X_n^{a,b,c} = k\right) = p_{n,k}^{a,b,c} = C_n^k \frac{(a)^{(k,c)}\,(b)^{(n-k,c)}}{(a+b)^{(n,c)}}, \qquad k \in \{0, 1, \ldots, n\}. \tag{2.3}$$

In the case when $a, b \in \mathbb{N}$ and $c \in \mathbb{Z}$, the physical interpretation of the random variable $X_n^{a,b,c}$ is the total number of white balls obtained in n extractions from an urn containing initially a white balls and b black balls, when the extractions are made with c replacements (the extracted ball is returned to the urn together with c balls of the same color, a negative value of c being interpreted as removing $|c|$ balls from the urn).

It is known (e.g. [7]) that the mean and variance of $X_n^{a,b,c}$ are given by

$$E\left(X_n^{a,b,c}\right) = \frac{na}{a+b} \qquad \text{and} \qquad \sigma^2\left(X_n^{a,b,c}\right) = \frac{nab}{(a+b)^2}\left(1 + \frac{(n-1)\,c}{a+b+c}\right). \tag{2.4}$$

In [8] we considered the operator R_n defined on the space of real-valued functions on $[0,1]$ by

$$
\begin{aligned}
R_n\left(f;x\right) &= Ef\left(\frac{1}{n}X_n^{x,1-x,-\min\{x,1-x\}/(n-1)}\right) \tag{2.5}\\
&= \sum_{k=0}^{n} C_n^k \frac{x^{(k,-\min\{x,1-x\}/(n-1))}\,(1-x)^{(n-k,-\min\{x,1-x\}/(n-1))}}{1^{(n,-\min\{x,1-x\}/(n-1))}} f\left(\frac{k}{n}\right),
\end{aligned}
$$

and we showed that it satisfies pointwise estimates (in terms of the moduli of continuity of the function, of its first or second derivative) which improve the corresponding estimates for the classical Bernstein operator. We also showed that the operator R_n satisfies a global Popoviciu-type inequality of the form (1.2), with a constant slightly larger than the optimal constant found by Sikkema in the case of Bernstein operator.

In the next section we will show that the constant in the Popoviciu-type inequality for the operator R_n is in fact strictly smaller than Sikkema's optimal constant for the Bernstein operator B_n. This suggests (although we do not have a proof) that the operator R_n provides a better approximation than the Bernstein operator B_n, claim which is also supported by the numerical and theoretical results obtained in [8].

3. MAIN RESULTS

The proof of our main result rests on the following inequality for the rising factorial (or equivalently, Pochammer symbol), which may be of independent interest.

Lemma 3.1. *For any $x \in [0,1]$ and any non-negative integers $n > 1$ and $r \leq nx - \sqrt{n}$, we have*

$$\frac{x^{(r+1,c)}\,(1-x)^{(n-r,c)}}{1^{(n,c)}} \leq x^{r+1}\,(1-x)^{n-r} \tag{3.1}$$

for any $c \in \left[-\min\{x, 1-x\}/(n-1), 0\right]$.

Moreover, the above inequality is strict except for the case $c = 0$.

Proof. For $c = 0$ and the claim becomes an identity, so we may assume $c \neq 0$. The claim also holds true for $r = 0$, since

$$\frac{x\,(1-x)^{(n,c)}}{1^{(n,c)}} = x \prod_{i=0}^{n-1} \frac{1-x+ic}{1+ic} < x \prod_{i=0}^{n-1} (1-x) = x\,(1-x)^n.$$

Since $r \leq nx - \sqrt{n} < n - 1$, we may assume that $0 < r < n - 1$, which in particular shows that all the factors appearing in the rising factorials on the left of (3.1) are positive.

Taking logarithms and rearranging the terms, the claim (3.1) is equivalent to

$$\sum_{i=1}^{n-1} \ln \frac{1}{1+ic} \leq \sum_{i=1}^{r} \ln \frac{x}{x+ic} + \sum_{i=1}^{n-r-1} \ln \frac{1-x}{1-x+ic}. \tag{3.2}$$

It is easy to see that the function $\varphi : \left\{ (u,t) \in \mathbb{R}^2 : u, u + ct > 0 \right\} \to \mathbb{R}$ defined by

$$\varphi(u,t) = \ln \frac{u}{u+ct},$$

is convex and increasing in the variable t (holding $u > 0$ fixed), and it is also satisfies $\varphi(\alpha u, \alpha t) = \varphi(u,t)$ for any $\alpha > 0$. Using this and an area comparison, we obtain

$$\sum_{i=1}^{r} \ln \frac{x}{x+ic} = \sum_{i=1}^{r} \varphi(x,i) = r \sum_{i=1}^{r} \frac{1}{r} \varphi\left(\frac{x}{r}, \frac{i}{r}\right) > r \int_0^1 \varphi\left(\frac{x}{r}, t\right) dt = \int_0^1 r\varphi\left(1, \frac{rt}{x}\right) dt, \tag{3.3}$$

and similarly

$$\sum_{i=1}^{n-r-1} \ln \frac{1-x}{1-x+ic} > \int_0^1 (n-r-1) \varphi\left(1, \frac{(n-r-1)t}{1-x}\right) dt, \tag{3.4}$$

and

$$\sum_{i=1}^{n-1} \ln \frac{1}{1+ic} = n \sum_{i=1}^{n-1} \frac{1}{n} \varphi\left(\frac{1}{n}, \frac{i}{n}\right) < n \int_{1/n}^1 \varphi\left(\frac{1}{n}, t\right) dt. \tag{3.5}$$

Using a substitution, we can simplify the last term above as follows

$$n \int_{1/n}^1 \varphi\left(\frac{1}{n}, t\right) dt = n \left(\int_0^1 \varphi\left(\frac{1}{n}, t\right) dt - \int_0^{1/n} \varphi\left(\frac{1}{n}, t\right) dt \right)$$

$$= n \left(\int_0^1 \varphi\left(\frac{1}{n}, t\right) dt - \frac{1}{n} \int_0^1 \varphi\left(\frac{1}{n}, \frac{t}{n}\right) dt \right) = \int_0^1 n\varphi(1, nt) dt - \int_0^1 \varphi(1, t) dt.$$

Combining the above with (3.3) – (3.5), it follows that (3.2) holds if we prove the inequality

$$\int_0^1 \varphi(1, nt) dt \leq \int_0^1 \frac{1}{n} \varphi(1, t) dt + \frac{r}{n} \varphi\left(1, \frac{rt}{x}\right) + \frac{n-r-1}{n} \varphi\left(1, \frac{(n-r-1)t}{1-x}\right) dt. \tag{3.6}$$

The convexity of the function φ in the second variable shows that the right side of the above inequality is larger than

$$\int_0^1 \varphi\left(1, \frac{t}{n}\left(1 + \frac{r^2}{x} + \frac{(n-r-1)^2}{1-x}\right)\right) dt,$$

and using the monotonicity of the function φ in the second variable it follows that the inequality (3.6) holds provided we show that

$$nt \leq \frac{t}{n}\left(1 + \frac{r^2}{x} + \frac{(n-r-1)^2}{1-x}\right) \tag{3.7}$$

for any positive integer $r < nx - \sqrt{n}$.

Extending the above inequality to positive real values $r \leq nx - \sqrt{n}$, we have

$$\frac{d}{dr}\left(1 + \frac{r^2}{x} + \frac{(n-r-1)^2}{1-x}\right) = \frac{2r}{x} - \frac{2(n-r-1)}{1-x} = \frac{2(r-nx+x)}{x(1-x)} < \frac{2(r-nx+\sqrt{n})}{x(1-x)} \leq 0,$$

and therefore the right hand side of (3.7) is decreasing in $r \leq nx - \sqrt{n}$.

It follows that (3.7) holds, provided we show

$$n^2 \leq 1 + \frac{(nx-\sqrt{n})^2}{x} + \frac{(n(1-x)+\sqrt{n}-1)^2}{1-x},$$

or equivalent

$$n^2 \le 1 + n^2 x - 2n\sqrt{n} + \frac{n}{x} + n^2 (1-x) + 2n \left(\sqrt{n} - 1\right) + \frac{n - 2\sqrt{n} + 1}{1 - x}.$$

Rearranging terms, we have equivalent

$$\frac{(2n-1)x^2 - 2\left(n + \sqrt{n} - 1\right)x + n}{x(1-x)} \ge 0,$$

which holds true, since the numerator is a quadratic function of x with a negative discriminant $\Delta = -4(n-1)\left(\sqrt{n} - 1\right)^2 < 0$ for all $n > 1$, thus concluding the proof. $\qquad\square$

Using the above lemma, we can now prove the following.

Theorem 3.2. *There exists a constant $C \le 1.08970 < C_{opt} = 1.0898873...$ such that for any continuous function $f : [0,1] \to \mathbb{R}$ and any $n > 1$ we have*

$$|R_n(f;x) - f(x)| \le C\omega\left(n^{-1/2}\right), \qquad x \in [0,1], \tag{3.8}$$

where $\omega(\delta) = \omega^f(\delta)$ denotes the modulus of continuity of f.

Proof. For $a \in \mathbb{R}$ denote by $]a[$ the largest integer strictly smaller than a, that is $]a[= k \in \mathbb{Z}$ if $k < a \le k+1$.

Using the definition of the modulus of continuity of f, it can be seen that $\omega(\lambda\delta) \le (1+]\lambda[)\,\omega(\delta)$ for any $\lambda \ge 0$ and $\delta > 0$ (see [9]). Using this, with $\delta = n^{-1/2}$ and $\lambda = \left|x - \frac{k}{n}\right|$, $k \in \{0, 1, \ldots, n\}$, and $c \in \left[-\frac{\min\{x, 1-x\}}{n-1}, 0\right]$, we obtain

$$\left|Ef\left(\frac{1}{n}X_n^{x,1-x,c}\right) - f(x)\right| \le \left|\sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x)\right) p_{k,n}^{x,1-x,c}\right| \tag{3.9}$$

$$\le \omega\left(n^{-1/2}\right)\left(1 + \sum_{k=0}^{n}\left]\frac{\left|x - \frac{k}{n}\right|}{n^{-1/2}}\right[p_{n,k}^{x,1-x,c}\right)$$

$$\le \omega\left(n^{-1/2}\right)\left(1 + \sqrt{n}\sum_{\substack{k\in\{0,1,\ldots,n\}:\\ \left|x-\frac{k}{n}\right|>n^{-1/2}}} \left|x - \frac{k}{n}\right| p_{n,k}^{x,1-x,c}\right).$$

For $x \in (\frac{1}{\sqrt{n}}, 1]$, denoting by $r = r(x)$ the largest integer for which $x - \frac{r}{n} > n^{-1/2}$, or equivalent $r = r(x) =]nx - \sqrt{n}[\in \{0, 1, \ldots, n-1\}$, and using Lemma 3 in [3], we have

$$\sum_{\substack{k\in\{0,1,\ldots,n\}:\\ x-\frac{k}{n}>n^{-1/2}}} \left|x - \frac{k}{n}\right| p_{n,k}^{x,1-x,c} = \sum_{k=0}^{r}\left(x - \frac{k}{n}\right) p_{n,k}^{x,1-x,c} = \frac{1}{n}\sum_{k=0}^{r}(nx - k) p_{n,k}^{x,1-x,c} \tag{3.10}$$

$$= \frac{1}{n}(r+1)p_{n,r+1}^{x,1-x,c}\left(1 - x + (n-r-1)c\right) = C_{n-1}^{r}\frac{x^{(r+1,c)}(1-x)^{(n-r,c)}}{1^{(n,c)}}.$$

The above sum equal zero for $x \in \left[0, \frac{1}{\sqrt{n}}\right]$, and therefore we have

$$\sum_{\substack{k\in\{0,1,\ldots,n\}:\\ x-\frac{k}{n}>n^{-1/2}}}\left(x - \frac{k}{n}\right) p_{n,k}^{x,1-x,c} = F_n^c(x), \qquad x \in [0,1],$$

where $F_n^c : [0,1] \to \mathbb{R}$ is the function defined by

$$F_n^c(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{\sqrt{n}}\right] \\ C_{n-1}^r \frac{x^{(r+1,c)}(1-x)^{(n-r,c)}}{1^{(n,c)}}, & x \in (\frac{1}{\sqrt{n}}, 1] \end{cases}, \text{ where } r = r(x) =]nx - \sqrt{n}[. \tag{3.11}$$

For $x \in [0, 1 - \frac{1}{\sqrt{n}})$, denoting $s = s(x)$ the smallest integer for which $x - \frac{s}{n} < -\frac{1}{\sqrt{n}}$, or equivalent $s = [nx + \sqrt{n} + 1] \in \{1, \ldots, n\}$, and using again Lemma 3 in [3] and (2.4), we have

$$\sum_{\substack{k \in \{0,1,\ldots,n\}: \\ x - \frac{k}{n} < -n^{-1/2}}} \left| x - \frac{k}{n} \right| p_{n,k}^{x,1-x,c} = \frac{1}{n} \sum_{k=s}^{n} (k - nx) \, p_{n,k}^{x,1-x,c} = \frac{1}{n} \sum_{k=0}^{s-1} (nx - k) \, p_{n,k}^{x,1-x,c} \tag{3.12}$$

$$= C_{n-1}^{s-1} \frac{x^{(s,c)} (1-x)^{(n-s+1,c)}}{1^{(n,c)}}.$$

Denoting by $r' = n - s$ and $x' = 1 - x$, and using the property $-[a+1] =]-a[$, we have

$$r' = n - s = n - \left[n(1-x)' + \sqrt{n} + 1 \right] = -\left[-nx' + \sqrt{n} + 1 \right] =]nx' - \sqrt{n}[,$$

and using the definition (3.11) of F_n^c, we can rewrite the sum above as follows

$$\sum_{\substack{k \in \{0,1,\ldots,n\}: \\ x - \frac{k}{n} < -n^{-1/2}}} \left| x - \frac{k}{n} \right| p_{n,k}^{x,1-x,c} = C_{n-1}^{r'} \frac{(x')^{(r'+1,c)} (1-x')^{(n-r',c)}}{1^{(n,c)}} = F_n^c(x') = F_n^c(1-x), \tag{3.13}$$

for any $x \in [0, 1]$ (note that for $x \in \left[1 - \frac{1}{\sqrt{n}}, 1 \right]$ the left hand-side of the above equality also equals the right hand-side, both sides being equal to zero).

Combining (3.9) – (3.13) above, we obtain

$$\left| Ef \left(\frac{1}{n} X_n^{x,1-x,c} \right) - f(x) \right| \leq \omega \left(n^{-1/2} \right) \left(1 + \sqrt{n} \left(F_n^c(x) + F_n^c(1-x) \right) \right),$$

for any $x \in [0, 1]$ and $c \in \left[-\frac{\min\{x, 1-x\}}{n-1}, 0 \right]$. Considering in particular $c = -\frac{\min\{x, 1-x\}}{n-1}$, and using Lemma 3.1 (which shows that $F_n^c(x) \leq F_n^0(x)$ for any $x \in [0, 1]$), we obtain

$$|R_n(f;x) - f(x)| \leq \omega \left(n^{-1/2} \right) \left(1 + \sqrt{n} \left(F_n^0(x) + F_n^0(1-x) \right) \right), \qquad x \in [0, 1]. \tag{3.14}$$

In [9] (Section 4), the author obtained the following estimate

$$1 + \sqrt{n} \left(F_n^0(x) + F_n^0(1-x) \right) \leq 1.0897, \qquad x \in [0, 1], \tag{3.15}$$

valid any positive integer $n \neq 6$. Combining this with (3.14) proves the claim of the theorem for any positive integer $n \neq 6$.

To conclude the proof, we have left to consider the case $n = 6$. First note that $c = -\frac{\min\{x, 1-x\}}{5} = -\frac{x}{5}$ or $\frac{x-1}{5}$, depending whether $x \leq \frac{1}{2}$ or $x > \frac{1}{2}$, and the function $r = r(x) =]6x - \sqrt{6}[$ in the definition F_6^c takes the value $k \in \{0, 1, 2, 3\}$ for $x \in \left(\frac{1}{\sqrt{6}} + \frac{k}{6}, \frac{1}{\sqrt{6}} + \frac{k+1}{6} \right]$, thus in order to estimate F_6^c there are several cases to consider. For example, in the case $x \in \left(\frac{1}{\sqrt{6}}, \frac{1}{2} \right]$ we have $r = 0$ and $c = -\frac{x}{5}$, and from (3.11) we obtain

$$F_6^c(x) = \frac{x(1-x)^{(6,-x/5)}}{1^{(6,-x/5)}} < \frac{\frac{1}{2}(1 - \frac{1}{\sqrt{6}})^{(6,-\frac{1}{5\sqrt{6}})}}{1^{(6,-1/10)}} = \frac{193282 - 78887\sqrt{6}}{6804} \approx 0.00721673,$$

and for $x \in \left(\frac{1}{2}, \frac{1}{\sqrt{6}} + \frac{1}{6} \right]$ we have $r = 0$ and $c = \frac{x-1}{5}$, and we obtain

$$F_6^c(x) = \frac{x(1-x)^{(6,\frac{x-1}{5})}}{1^{(6,\frac{x-1}{5})}} \equiv 0.$$

In the remaining three cases we have $c = \frac{1-x}{5}$, and proceeding similarly we obtain

$$F_6^c(x) \leq \begin{cases} C_5^1 \dfrac{x^{\left(2, \frac{1-x}{5}\right)} (1-x)^{\left(5, \frac{1-x}{5}\right)}}{1^{\left(6, \frac{1-x}{5}\right)}}, & x \in \left(\frac{1}{\sqrt{6}} + \frac{1}{6}, \frac{1}{\sqrt{6}} + \frac{2}{6}\right] \\[3mm] C_5^2 \dfrac{x^{\left(3, \frac{1-x}{5}\right)} (1-x)^{4, \frac{1-x}{4}}}{1^{6, \frac{1-x}{5}}}, & x \in \left(\frac{1}{\sqrt{6}} + \frac{2}{6}, \frac{1}{\sqrt{6}} + \frac{3}{6}\right] \quad \leq 0.014271. \\[3mm] C_5^3 \dfrac{x^{\left(3, \frac{1-x}{5}\right)} (1-x)^{3, \frac{1-x}{5}}}{1^{6, \frac{1-x}{5}}}, & x \in \left(\frac{1}{\sqrt{6}} + \frac{3}{6}, 1\right] \end{cases}$$

In all cases above we obtained $F_6^c(x) \leq 0.014271$ for $x \in [0, 1]$, and therefore

$$1 + \sqrt{6} \left(F_6^c(x) + F_6^c(1-x)\right) \leq 1 + 2\sqrt{6} \cdot 0.014271 \approx 1.0699134 < 1.0897, \quad x \in [0, 1],$$

concluding the proof of the theorem.

$\square$

The above result leaves open the problem of finding the value of the optimal constant C in the above theorem. Although we do not have an answer here, we believe that the optimal constant is much smaller than the value hinted by the above theorem.

A second remark is that we believe that Lemma 3.1, on which the above proof rests, can be improved to show that the left hand-side of the inequality (3.1) is in fact an increasing function of $c \geq -\frac{\min\{x, 1-x\}}{n-1}$. If this conjecture is correct, an argument similar to the one used in the above proof would show that

$$\frac{\left| Ef\left(\frac{1}{n} X_n^{x, 1-x, c}\right) - f(x) \right|}{\omega\left(n^{-1/2}\right)}$$

is in fact a monotone increasing function of $c \geq -\min\{x, 1-x\}/(n-1)$, hence among all Pólya-Bernstein type operators of the form

$$P_n^{x, 1-x, c}(f; x) = Ef\left(\frac{1}{n} X_n^{x, 1-x, c}\right), \qquad x \in [0, 1],$$

the one that provides the best approximation in the class of continuous functions is the operator R_n given by (2.5), which corresponds to $c = -\min\{x, 1-x\}/(n-1)$.

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