CHARACTER VALUES AND HOCHSCHILD HOMOLOGY

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Abstract. We present a conjecture (and a proof for $G = SL(2)$) generalizing a result of J. Arthur which expresses a character value of a cuspidal representation of a p-adic group as a weighted orbital integral of its matrix coefficient. It also generalizes a conjecture by the second author proved by Schneider-Stuhler and (independently) the first author. The latter statement expresses an elliptic character value as an orbital integral of a pseudo-matrix coefficient defined via the Chern character map taking value in zeroth Hochschild homology of the Hecke algebra. The present conjecture generalizes the construction of pseudo-matrix coefficient using compactly supported Hochschild homology, as well as a modification of the category of smooth representations, the so called compactified category of smooth G-modules. This newly defined "compactified pseudo-matrix coefficient" lies in a certain space $\mathcal{K}$ on which the weighted orbital integral is a conjugation invariant linear functional, our conjecture states that evaluating a weighted orbital integral on the compactified pseudo-matrix coefficient one recovers the corresponding character value of the representation.

We also discuss the properties of the averaging map from $\mathcal{K}$ to the space of invariant distributions, partly building on works of Waldspurger and Beuzart-Plessis.

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1. Introduction

Let G be a reductive group over a local nonArchimedean field F.

The goal of the article is to present an algebraic expression for a character of an admissible representation of G on a compact element.

The statement is presented as a conjecture (see Conjecture 4.3) for a general reductive group, it is proved in the paper for $G = SL(2)$. We also describe a modification of the category $Sm = Sm(G)$ of finitely generated smooth representations, the so called *compactified category* of smooth G-modules, which plays a key role in our algebraic description of character values and may have an independent interest.

To describe the context for these constructions recall a conjecture of [14] proved in [19] and [7].

Let $\mathcal{H} = \cup_K \mathcal{H}_K$ be the Hecke algebra of locally constant compactly supported $\mathbb{C}$-valued measures. Thus $Sm(G)$ is identified with the category of finitely generated nondegenerate $\mathcal{H}$ modules [10].

Let $C(\mathcal{H}) = \mathcal{H}/[\mathcal{H},\mathcal{H}] = \mathcal{H}_G = HH_0(\mathcal{H}) = HH_0(Sm)$ be the cocenter of $\mathcal{H}$. Here $\mathcal{H}_G$ denotes coinvariants with respect to the conjugation action, while HH_* stands for Hochschild homology, and its second appearance refers to the notion of Hochschild homology of an abelian category.

Since $\mathcal{H}$ is Noetherian and has finite homological dimension, there is a well defined Chern character (also called the Hattori-Stallings or Dennis trace) map $ch : K^0(Sm) \to C(\mathcal{H})$ (we will abbreviate $ch([M])$ to $ch(M)$). It has been conjectured in [14] and proven in [19], [7] that for an *elliptic* regular semisimple element $g \in G$ and an admissible representation ρ we have

$$\chi_\rho(g) = O_g(ch(\rho)), \tag{1}$$

where O_g denotes the orbital integral. Here we use that $O_g : \mathcal{H} \to \mathbb{C}$ being conjugation invariant factors through $C(\mathcal{H})$.

If ρ is a cuspidal irreducible representation then (assuming that G has a compact center) a matrix coefficient $m_\rho \in \mathcal{H}$ is a representative of the class $ch(\rho) \in C(\mathcal{H})$. Thus in this case (1) reduces to an earlier result of Arthur [2]. However, the latter applies also to nonelliptic regular semisimple elements: for such an element g and a cuspidal irreducible representation ρ Arthur has proved that

$$\chi_\rho(g) = WO_g(m_\rho), \tag{2}$$

where WO_g denotes the *weighted orbital integral*. Our Conjecture 4.3 provides a generalization of (1) to all regular semisimple compact elements g, which for a cuspidal representation ρ reduces to (2).

The first step in this direction is a generalization of the map $ch : K^0(Sm) \to C(\mathcal{H})$. For our present purposes we need to modify both the source and the target of this map. We replace the target $C(\mathcal{H}) = \mathcal{H}_G$ by $\mathcal{K}_G$ where $\mathcal{K} \subset \mathcal{H}$ is a subspace invariant under the conjugation action of G, the so called space of "weightless" functions.[1] Definition and some properties of $\mathcal{K}$ are discussed in section 2. The key property is that $WO_g|_{\mathcal{K}}$ is a G-invariant functional for any regular semisimple element $g \in G$; furthermore, there is a well defined averaging map Av from $\mathcal{K}$ to the space of invariant generalized functions on G and for $\phi \in \mathcal{K}$ the value of $WO_g(\phi)$ coincides with $Av(\phi)(g)$ (the latter is well defined since $Av(\phi)$ is in fact a locally constant function on the set of regular elements in G). We also provide a conjecture with a proof for $PGL(2, F)$, $char(F) = 0$ describing the image of the averaging map from $\mathcal{K}$ to the space of invariant distributions.

[1]This adjective reflects the fact that weighted orbital integrals restricted to this space are independent of the choices involved in choosing the weight function on an orbit. This space has appeared in the literature (see [6], [24] and references therein) where it was called the space of *strongly* cuspidal function. We refrain from using this terminology since we use the term "cuspidal function" in the sense of [14] where it refers to a function acting by zero in any parabolically induced representation, thus in our terminology $\mathcal{K}$ contains the space of cuspidal functions. The term "cuspidal function" was used in a different sense in [6], [24] etc., so that $\mathcal{K}$ is contained in the set of cuspidal functions in the sense of *loc. cit.*

To describe the source of the map generalizing ch we need some new ingredients. One of them is the so called *compactified category* of smooth (finitely generated) representations $\overline{Sm}$.

The abelian category $\overline{Sm}$ is defined in section 3. Recall that according to Bernstein [9], Sm can be identified with the category of coherent sheaves of modules over a certain sheaf of algebras over a scheme which is an infinite union of affine algebraic varieties, the spectrum $\mathfrak{Z}$ of the Bernstein center of G. The category $\overline{Sm}$ can be described as the category of coherent sheaves of modules over a certain coherent sheaf of algebras over a (componentwise) compactification of $\mathfrak{Z}$. An admissible module ρ can also be viewed as an object in $\overline{Sm}$, so we can apply the Chern character to the class of ρ obtaining $\bar{ch}(\rho) \in HH_0(\overline{Sm})$.

We also need another invariant of the category Sm, namely the *compactly supported* Hochschild homology $HH^c_*(Sm)$ which is the derived global sections with compact support in the sense of [12] of localized Hochschild homology $R\underline{Hom}_{\mathcal{H}\otimes\mathcal{H}^{op}}(\mathcal{H},\mathcal{H})$.

We have natural maps $HH^c_*(Sm) \to HH_*(\overline{Sm}) \to HH_*(Sm)$; for an admissible module ρ we have its compactly supported Chern character $ch^c(\rho) \in HH^c_0(Sm)$, so that $\bar{ch}(\rho)$ and $ch(\rho)$ equal the images of $ch^c(\rho)$ under the corresponding maps.

The first statement in the main conjecture (a theorem for $SL(2)$) provides a natural isomorphism

$$\mathcal{K}^c_G \cong Im(HH^c_0(Sm) \to HH_0(\overline{Sm})),$$

where $\mathcal{K}^c \subset \mathcal{K}$ is the subspace of measures supported on compact elements. By the previous paragraph, $\bar{ch}(\rho)$ belongs to that image, thus we obtain a homological construction of an element in $\mathcal{K}^c_G$ from an admissible representation, the so called "compactified pseudo-matrix coefficient" of the representation.

The second main statement (proved for $SL(2)$) asserts that for a compact regular element g and an admissible representation ρ we have $WO_g(\bar{ch}(\rho)) = \chi_\rho(g)$. Notice that for a noncompact regular element g the value of $\chi_\rho(g)$ coincides with a character value of the Jacquet functor applied to ρ [11], in particular it vanishes for a cuspidal module.

We view Conjecture 4.3 as an algebraic statement underlying some aspects of Arthur's local trace formula [1], while equality (1) underlies the elliptic part of the local trace formula (see also Remark 3.16 below). We plan to develop this theme in a future publication.

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2. Weightless functions and invariant distributions

Let $\mathcal{H} = \mathcal{H}(G)$ be the Hecke algebra of compactly supported locally constant measures on G, the convolution product on $\mathcal{H}(G)$ will be denoted by $*$. For an open subsemigroup $S \subset G$ we let $\mathcal{H}(S) \subset \mathcal{H}(G)$ denote the subalgebra of measures supported on S. We denote by $\mathcal{D}$ the space of generalized functions on G (that is the space of linear functionals on $\mathcal{H}(G)$) and by $\mathcal{D}^G \subset \mathcal{D}$ the subspace of invariant generalized functions. Let $\mathcal{H}_{cusp}$, $\mathcal{D}^G_{cusp}$ be the cuspidal part of $\mathcal{H}$, $\mathcal{D}^G$, i.e. $\mathcal{H}_{cusp}$ consists of functions acting by zero in any parabolically induced representation and

$\mathcal{D}^G_{cusp}$ consists of distributions vanishing on the orthogonal complement of $\mathcal{H}_{cusp}$ (cf. footnote 1 above). Until the end of the section we assume for simplicity of notation that G has compact center.

Then *averaging* with respect to conjugations yields a well defined map $H_0(G, \mathcal{H}_{cusp}) \to \mathcal{D}^G_{cusp}$ given by $f \mapsto \int_G \frac{{}^g f}{dg} dg$ for a Haar measure dg on G.

In this section we define a larger subspace $\mathcal{K}$ in $\mathcal{H}$ on which the averaging map is still well defined and conjecture that the map τ defines an embedding $H_0(G, \mathcal{K}) \hookrightarrow \mathcal{D}^G$. Moreover, we propose a conjectural description of the image of τ. We prove this conjecture in the case when $G = PGL(2)$.

2.1. **The conjecture.** We start with some notation. Let $\mathcal{O} \subset F$ be the ring of integers and π be a generator of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$. Let $val : F^\star \to \mathbb{Z}$ be the valuation such that $val(\pi) = 1$. We define $\|x\| = q^{-val(x)}, x \in F^\star$ where $q = |\mathcal{O}/\mathfrak{m}|$.

For any smooth F-variety X we denote by $\mathcal{S}(X)$ the space of locally constant measures on X with compact support. In the case when X is a homogeneous G-variety with a G-invariant measure dx the map $f \to f/dx$ identifies $\mathcal{S}(X)$ with the space of compactly supported locally constant functions on X. In this case we will not distinguish between functions and measures on X. Also, we will work with spaces $Y_P = (G/U_P \times G/U_P)/L$ for a parabolic subgroup $P = LU_P \subset G$. In this case $\mathcal{S}(Y_P)$ will denote the space of integral kernels of operators $\mathcal{S}(G/U_P) \to \mathcal{S}(G/U_P)$, i.e. locally constant compactly supported sections of the G-equivariant locally constant sheaf $pr_1^*(\mu)$, where $pr_1 : Y_P \to G/P$ is the first projection.

In particular, we fix a Haar measure dg on G and identify the space $\mathcal{S}(G) = \mathcal{H}(G)$ with the space of compactly supported locally constant functions on G. Then Δ is identified with the space of distributions, we also get the L^2 pairing $\langle \ , \ \rangle$ on $\mathcal{H}(G)$.

Let $\mathfrak{g}$ be the Lie algebra of G. We will assume that $\mathfrak{g}$ has a finite number of nilpotent conjugacy classes and that there exists a G-equivariant F-analytic bijection ϕ between a neighbourhood of 0 in $\mathfrak{g}$ and a neighbourhood of e in G. We also assume that for a semisimple element $s \in G$ the Lie algebra $\mathfrak{z}$ of its centralizer $Z_G(s)$ admits a $Z_G(s)$-invariant complement. These assumptions are well known to hold if $char(F) = 0$ or if $char(F) > N$ for some N depending on the rank of G, see [15, §1.8] for more precise information.

Definition 2.1.

- We denote by $\mathcal{G}_{G,e}$ the space of germs of Ad-invariant distributions near 0 on $\mathfrak{g}$ which are restrictions of linear combinations of the Fourier transforms of invariant measures on a nilpotent orbits. Using the bijection ϕ we consider $\mathcal{G}_{G,e}$ as a space of germs of Ad-invariant distributions on G near the identity.
- For a semisimple element $s \in G$ we denote by $\mathcal{G}_{Z_G(s),s}$ the space of germs of distributions on $Z_G(s)$ at s obtained from the space $\mathcal{G}_{Z_G(s),e}$ by the shift by s.
- Let $s \in G$ be a semisimple element, $X_s = G/Z_G(s), r : G \to X_s$ be the natural projection and $\gamma : X_s \to G$ be a continuous section. We denote by dz a G-invariant measure on X_s.
- We denote by $\tilde{\kappa} : Z_G(s) \times X_s \to G$ the map given by $(z, x) \to \gamma(x) s z (\gamma(x))^{-1}$.

Let $\mathfrak{z} \subset \mathfrak{g}$ be the Lie algebra of $Z_G(s)$. By assumption there exists a complementary $\mathfrak{z}$-invariant subspace $W \subset \mathfrak{g}$ and the map $\kappa_0 : \mathfrak{z} \oplus W \to \mathfrak{g}, (z, w) \to z + Ad(s)w$ is a bijection. Therefore there exists an open neighborhood $R \subset Z_G(s)$ of e such that the restriction κ of $\tilde{\kappa}$ on $R \times X_s$ is an open embedding.

Definition 2.2. For any $\bar{\psi} \in \mathcal{G}_{Z_G(s),s}$ we choose a representative $\tilde{\psi} \in \mathcal{D}^{Z_G(s)}(Z_G(s))$ of $\bar{\psi}$.

- For a function $f \in \mathcal{S}(G)$ and $z \in R \subset X_s$ we define $f_z \in \mathcal{S}(Z_G(s))$ by $f_z := f(\kappa(z, x))$.

- We define a function $\bar{f}$ on R by $\bar{f}(z) := \tilde{\psi}(f_z)$.
- We define a distribution $\psi(\tilde{\psi})$ on G by $\psi(f) := \int_{X_s} \bar{f}(z)$.
- We denote by $[\psi(\tilde{\psi})]$ the germ of the distribution ψ at s.

It is clear that for any two choices $\tilde{\psi}, \tilde{\psi}'$ of representatives of $\bar{\psi}$ the difference $\psi(\tilde{\psi}) - \psi(\tilde{\psi}')$ vanishes on a G-invariant open neighborhood of s. Therefore the germ $[\bar{\psi}]$ does not depend on a choice of a representative of $\bar{\psi}$.

Definition 2.3.
- We denote by $\mathcal{G}_s$ the space of germs at s of Ad-invariant distributions of the form $[\psi], \psi \in \mathcal{G}_{Z_G(s),s}$.
- We denote by $\mathcal{E} \subset \mathcal{D}^G$ the subspace of distributions α such that
  a) there exists a compact subset C in G such that $supp(\alpha) \subset G(C)$ and
  b) for any semisimple $s \in G$ the germ of α at s belongs to $\mathcal{G}_s$.

Remark 2.4. If $char(F) = 0$ then $\mathcal{E}$ admits an equivalent description as the space of invariant distributions α satisfying the following requirements:

a) there exists a compact subset C in G such that $supp(\alpha) \subset G(C)$;

b) there exists a compact open subgroup $K \subset G$ such that for every element z in the Bernstein center satisfying $\delta_K * z = 0$ we have $\alpha * z = 0$.

Equivalence of the two definitions of $\mathcal{E}$ follows from[2] [13, Theorem 16.2].

Definition 2.5.
- We define the space $\mathcal{K}(G)$ of *weightless* functions as the subspace in $\mathcal{S}(G)$ of functions f such that $\int_{u \in U_Q} f(lu)du = 0, l \in L$ for all proper parabolic subgoups $Q = LU_Q \subset G$.
- For a closed conjugation invariant subset X of G we define the space $\mathcal{K}(X) = \mathcal{K}_X \subset \mathcal{S}(X)$ as the subspace of functions f such that
$$\int_{u \in U_Q} f(lu)du = 0,$$
for all proper parabolic subgroups $Q = LU_Q \subset G$ and $l \in L$ such that $lU_Q \subset X$.

Remark 2.6. For $f \in \mathcal{H}$ and a parabolic $P = LU_P \subset G$ let $A_P(f) \in \mathcal{S}(Y_P)$ denote its orishperic transform, i.e. the integral kernel of the action of f on $\mathcal{S}(G/U_P)$. Let $\Delta_{Y_P} \subset Y_P$ be the preimage of diagonal under the projection $Y_P \to (G/P)^2$. Then $\Delta_{Y_P} \cong (G/U_P \times L)/L$, where L acts on the first factor by right translations and on the second one by conjugation. It is easy to see that for $f \in \mathcal{H}$ we have $f \in \mathcal{K}$ iff for any parabolic subgroup $P \subsetneq G$ we have $A_P(f)|_{\Delta_{Y_P}} = 0$.

The following result is due to J.-L. Waldspurger (see [24, Lemma 9]).

Proposition 2.7. For $f \in \mathcal{H}(G)$ the following are equivalent:

a) $f \in \mathcal{K}$.

b) For any $h \in \mathcal{H}(G)$ the function $g \mapsto \langle {}^g f, h \rangle$ has compact support.

For any $f \in \mathcal{S}(G)$ we define distribution $\hat{f}$ by:
$$\langle \hat{f}, h \rangle := \int_{g \in G} \langle {}^g f, h \rangle dg.$$

For future reference we mention the following.

Lemma 2.8. For $f \in \mathcal{K}$ the distribution $\hat{f}|_{G^{rs}}$ (where G^{rs} is the open set of regular semisimple elements) is a locally constant function. For $g \in G^{rs}$ we have
$$\hat{f}(g) = WO_g(f),$$
where WO_g denotes the *weighted orbital integral*.

[2]We thank Raphaël Beuzart-Plessis for pointing this out to us.

Proof follows from the definition and basic properties of the weighted orbital integral, see e.g. [3, §I.11]. □

The group G acts on $\mathcal{K}$ by conjugation. It is clear that the map $f \to \hat{f}$ factors through a map $\tau : H_0(G, \mathcal{K}) \to \mathcal{D}^G$. For any $f \in \mathcal{K}$ we denote by $[f]$ it image in $H_0(G, \mathcal{K})$.

Conjecture 2.9. a) $\hat{f} \in \mathcal{E}$ for $f \in \mathcal{K}$.

b) The map τ defines an isomorphism between $H_0(G, \mathcal{K})$ and $\mathcal{E}$.

c) $\dim(H_0(G, \mathcal{K}(\Omega))) = 1$ for any regular semisimple conjugacy class $\Omega \subset G$.

Remark 2.10. One can check that $\hat{f}$ satisfies the conditions of Remark 2.4, thus if $char(F) = 0$ then part (a) of the conjecture follows from Harish-Chandra's Theorem [13, Theorem 16.2]; see [23, Corollary 5.9], [6, Proposition 5.6.1] for details.

Remark 2.11. It is clear that part *c*) follows from *a*) and *b*).

Remark 2.12. If the centralizer of an element $g \in \Omega$ is an anisotropic (compact) torus then statement (c) clearly follows from uniqueness (up to scaling) of a Haar measure on G. In the case when that centralizer has split rank one the statement is checked in the next subsection.

2.2. **Almost elliptic orbits.** To simplify the wording we assume in this subsection that the center of G is compact. A regular semisimple element $g \in G$ will be called almost elliptic if the split rank of its centralizer is at most one. We now prove Conjecture 2.9(c) in the case when Ω consists of almost elliptic elements.

Fix $g \in \Omega$ and let T be the centralizer of G, thus $\Omega \cong G/T$. In view of Remark 2.12 it suffices to consider the case when the split rank of T equals one; we also assume without loss of generality that G is almost simple.

There are exactly two parabolic subgroups P, $P' \subsetneq G$ containing T. Let U, U' be their unipotent radicals.

Consider the complex

$$0 \to \mathcal{K}_\Omega \to \mathcal{S}(G/T) \to \mathcal{S}(G/TU) \oplus \mathcal{S}(G/TU') \to \mathbb{C} \to 0; \tag{3}$$

here $\mathcal{S}$ stands for the space of locally constant compactly supported measures as before, the third arrow send ϕ to $(pr_*(\phi), pr'_*(\phi))$, where $pr : G/T \to G/TU$, $pr' : G/T \to G/TU'$ are the projections and the fourth arrow sends (ϕ, ϕ') to $\int \phi - \int \phi'$.

Lemma 2.13. The complex (3) is exact.

Proof. Exactness at all the terms except for $\mathcal{S}(G/TU) \oplus \mathcal{S}(G/TU')$ is clear. Suppose that $(\psi, \psi') : \mathcal{S}(G/TU) \oplus \mathcal{S}(G/TU') \to \mathbb{C}$ is a linear functional vanishing on the image of $\mathcal{S}(G/T)$. Let $\tilde{\psi} : \mathcal{S}(G) \to \mathbb{C}$ be the composition of the direct image map $\mathcal{S}(G) \to \mathcal{S}(G/TU)$ with ψ. Then ψ is a right TU invariant generalized function on G. On the other hand, $-\psi$ is equal to the composition of the direct image map $\mathcal{S}(G) \to \mathcal{S}(G/TU')$ with ψ', which shows that $\tilde{\psi}$ is also right TU' invariant. Since G is assumed to be almost simple, U and U' together generate G, thus we see that $\tilde{\psi}$ is right G invariant. It follows that $\tilde{\psi}$ is proportional to the functional $\phi \mapsto \int \phi$, hence the functional (ψ, ψ') factors through the differential in (3), which yields exactness of (3). □

We can now finish the proof of Conjecture 2.9(c) in the present case. Breaking (3) into short exact sequences we get

$$0 \to \mathcal{K}_\Omega \to \mathcal{S}(G/T) \to M \to 0,$$

$$0 \to M \to \mathcal{S}(G/TU) \oplus \mathcal{S}(G/TU') \to \mathbb{C} \to 0.$$

Considering the corresponding long exact sequences on homology we see that it suffices to check that the map $\mathbb{C} = H_0(G, \mathcal{S}(G/T)) \to H_0(G, M)$ is nonzero while the map $H_1(G, \mathcal{S}(G/T)) \to H_1(G, M)$ has one-dimensional cokernel. The former statement is clear since the composition $H_0(G, \mathcal{S}(G/T)) \to H_0(G, M) \to H_0(G, \mathcal{S}(G/TU))$ is nonzero. To check the latter recall that $H_i(G, \mathbb{C}) = 0$ for $i > 0$ since the resolution of $\mathbb{C}$ provided by the simplicial complex for computation of homology of the Bruhat-Tits building $\mathfrak{B}$ shows that $H_i(G, \mathbb{C}) \cong H_i(\mathfrak{B}/G)$, while $\mathfrak{B}/G$ is a product of simplices. Thus

$$H_1(G, M) \cong H_1(G, \mathcal{S}(G/TU) \oplus \mathcal{S}(G/TU')) \cong H_1(T, \mathbb{C}) \oplus H_1(T, \mathbb{C}).$$

Since $\mathcal{S}(G/TU) \cong H_1(T, \mathbb{C})$ we see that

$$CoKer\,(H_1(G, \mathcal{S}(G/T)) \to H_1(G, M)) \cong H_1(T, \mathbb{C}),$$

which is one-dimensional. □

2.3. **The case of $PGL(2)$.** To simplify the argument we assume in this subsection that $char(F) = 0$.

Theorem 2.14. Conjecture 2.9 is true for $G = PGL(2)$.

The rest of the subsection is devoted to the proof of the Theorem.

Claim 2.15. We have $\mathcal{K}_{cusp} \subset \mathcal{K}$.

Proposition 2.16. The map $\tau : H_0(G, \mathcal{K}) \to \mathcal{D}^G$ is an embedding.

Proof. We need more notation.

Definition 2.17.
- For $\epsilon \geq 0$ we define $G^\epsilon = \{g \in G \| p(g)| \leq \epsilon\}$, where $p(g) = \frac{tr^2(\tilde{g})}{det(\tilde{g})} - 4$; here $\tilde{g} \in GL(2, F)$ is a representative of g.
- We let G_s, G_e, N, $\bar{N}$ denote, respectively, the sets of regular semisimple split, regular semisimple elliptic, regular unipotent and all unipotent elements.
- We set
$$\mathcal{K}_u = \mathcal{K}(\bar{N}) := \{f \in \mathcal{S}(\bar{N}) \mid \int\limits_{u \in U} f(u)du = 0 \;\forall B = TU \subset G\},$$
$$\mathcal{K}_0 := \{f \in \mathcal{K}_u | f(e) = 0\},$$
where B runs over the set of Borel subgroups in G.
- For $f \in \mathcal{K}$ we denote by $\kappa(f) \in \mathcal{K}_u$ the restriction of f to $\bar{N}$ and by $[\kappa(f)]$ the image of $\kappa(f)$ in $H_0(G, \mathcal{K}_u)$.

We start with the following geometric statement.

Lemma 2.18. Let $f \in \mathcal{S}(G)$ be such that $f|_{\bar{N}} \in \mathcal{K}_u$. Then there exists $\epsilon > 0$ such that $f|_{G^\epsilon} \in \mathcal{K}$.

Proof. Recall notations of Remark 2.6. We have $\Delta_{Y_B} = G/B \times T$, where T is the (abstract) Cartan subgroup of G. It is easy to see that condition $f|_{\bar{N}} \in \mathcal{K}_u$ is equivalent to vanishing of the restriction of $A_B(f)$ to $G/B \times \{1\} \subset \Delta_{Y_B}$. Also, condition $f|_{G^\epsilon} \in \mathcal{K}$ is equivalent to vanishing of $A_B(f)$ on $G/B \times T_\epsilon \subset \Delta_{Y_B}$, where $T_\epsilon = G_\epsilon \cap T$ (here we abuse notation by identifying the abstract Cartan subgroup T with an arbitrarily chosen Cartan subgroup). Since $A_B(f)$ is locally constant for $f \in \mathcal{H}$, the statement follows from compactness of G/B. □

Corollary 2.19. For any $f \in \mathcal{K}$ such that $[\kappa(f)] = 0$ there exists $f' \in \mathcal{K}$ with the same image in $H_0(G, \mathcal{K})$ and such that $f'|_{\bar{N}} = 0$.

Proof. Since $[f|_{\bar{N}}] = 0$ we can write the restriction of f to $\bar{N}$ as a finite sum $\sum_i (\tilde{f}_i^{g_i} - \tilde{f}_i), \tilde{f}_i \in \mathcal{K}_0', g_i \in G$. As follows from the Lemma 2.18 we can choose $f_i \in \mathcal{K}$ such that $\tilde{f}_i = f_i|_{\bar{N}}$. Then the function $f' := f - \sum_i (f_i^{g_i} - f_i)$ satisfies the conditions of Corollary. □

Proposition 2.20. The space $H_0(G,\mathcal{K}_u)$ is two dimensional.

Proof. We first show that $\dim(H_0(G,\mathcal{K}_0))=1$.

Let $\mathcal{B}$ be the variety of Borel subgroups, $p:N\to\mathcal{B}$ the map which associates to $u\in N$ the Borel subgroup containing u. By definition we have an exact sequence

$$0\to\mathcal{K}_0\to\mathcal{S}(N)\to\mathcal{S}(\mathcal{B})\to 0$$

and therefore an exact sequence

$$H_1(G,\mathcal{S}(N))\to H_1(G,\mathcal{S}(\mathcal{B}))\to H_0(G,\mathcal{K}_0)\to H_0(G,\mathcal{S}(N))\to H_0(G,\mathcal{S}(\mathcal{B})).$$

Lemma 2.21. $H_1(G,\mathcal{S}(N))=0$.

Proof. Fix a Borel subgroup $B=TU$. We can write U as a union of open compact subgroup $U_1\subset ...U_n\subset ....$ Therefore $\mathcal{S}(N)=\mathcal{S}(G/U)$ is the direct limit of $\mathcal{S}(G/U_n)$. Since the functor $M\mapsto H_1(G,M)$ commutes with direct limits it is sufficient to show that $H_1(G,\mathcal{S}(G/U_n))=\{0\}$. Since $U_n\subset G$ is a compact subgroup the space $H_1(G,\mathcal{S}(G/U_n))$ is a direct summand of $\mathcal{S}(G)$. Since $H_1(G,\mathcal{S}(G))=0$ the Lemma is proven. $\square$

Since G acts transitively on N and on $\mathcal{B}$ we have $H_0(G,\mathcal{S}(N))\widetilde{\longrightarrow}H_0(G,\mathcal{S}(\mathcal{B}))=\mathbb{C}$. Since $H_1(G,\mathcal{S}(N))=0$ we see that the map $H_1(G,\mathcal{S}(\mathcal{B}))\to H_0(G,\mathcal{K}_0)$ is an isomorphism. Since $\mathcal{B}=G/B$ we have:

$$H_1(G,\mathcal{S}(\mathcal{B}))=H_1(B,\mathbb{C})=H_1(T,\mathbb{C})=\mathbb{C}.$$

So $\dim(H_0(G,\mathcal{K}_0))=1$.

To conclude the argument, recall the short exact sequence $0\to\mathcal{K}_0\to\mathcal{K}_u\to\mathbb{C}\to 0$.

Since $H_1(G,\mathbb{C})=0$ we have an exact sequence:

$$0\to H_0(G,\mathcal{K}_0)\to H_0(G,\mathcal{K}_u)\to\mathbb{C}\to 0$$

So $\dim(H_0(G,\mathcal{K}_u)=2$. $\square$

Let $r:\mathcal{K}_{cusp}\to\mathcal{K}_u$ be the restriction and $[r]:\mathcal{K}_{cusp}\to H_0(G,\mathcal{K}_u)$ be the composition of r and projection $\mathcal{K}_u\to H_0(G,\mathcal{K}_u)$.

Lemma 2.22. The map $[r]$ is onto.

Proof. Recall the map $\tau:H_0(G,\mathcal{K})\to\mathcal{D}^G$, $[f]\to\hat{f}$. Let $\bar{\mathcal{D}}^G$ be the space of germs of invariant distributions at e and $\bar{\tau}:H_0(G,\mathcal{K})\to\bar{\mathcal{D}}^G$ be the composition of τ with the restriction map.

Corollary 2.19 implies that the map $\bar{\tau}$ vanishes on the kernel of the map $H_0(G,\mathcal{K})\to H_0(G,\mathcal{K}_u)$. Thus it suffices to show that $\bar{\tau}|_{\mathcal{K}_{cusp}}$ has rank at least two, i.e. that there exist irreducible cuspidal representations ρ_1, ρ_2, such that their characters restricted to any G-invariant open neighborhood of identity are not proportional. This is easily done by inspecting the character tables, see e.g. [20, §2.6]. $\square$

Corollary 2.23. a) For any $f\in\mathcal{K}$ there exists $f_{cusp}\in\mathcal{K}_{cusp}$ such that $[\kappa(f)]=[\kappa(f_{cusp'})]$.

b) For any $f_0\in\mathcal{K}_u$ there exists $f\in\mathcal{K}_{cusp}$ such that $[f_0]=[\kappa(f)]$. $\square$

Let $s\in G$ be a regular split semisimple element and $\Omega\subset G$ be the conjugacy class of s.

Proposition 2.24. $\dim(H_0(G,\mathcal{K}_\Omega))=1$.

Proof. Let $T=Z_G(t)$ be the split torus and $B,B'\subset G$ be Borel subgroups containing T. Since $\Omega=G/T$ we have maps $r:\Omega\to G/B$ and $r':\Omega\to G/B'$ and therefore morphisms $r_\star:\mathcal{S}(\Omega)\to\mathcal{S}(G/B)$ and $r'_\star:\mathcal{S}(\Omega)\to\mathcal{S}(G/B')$.

As a special case of Lemma 2.13 we get:

Lemma 2.25. The sequence

$$0 \to \mathcal{K}_\Omega \to \mathcal{S}(\Omega) \to \mathcal{S}(G/B) \oplus \mathcal{S}(G/B') \to \mathbb{C} \to 0, \tag{4}$$

where the last map l is given by $(\nu, \nu') \mapsto \int \nu - \int \nu'$, is exact. □

Let $L := ker(l)$. We have an exact sequence

$$0 \to L \to \mathcal{S}(G/B) \oplus \mathcal{S}(G/B') \to \mathbb{C} \to 0.$$

Using that G has homological dimension one, we get that the corresponding long exact sequence of homology contains the following fragment:

$$0 \to H_1(G, L) \to H_1(B, \mathbb{C}) \oplus H_1(B', \mathbb{C}) \to H_1(G, \mathbb{C}).$$

It is easy to see that

$$\dim(H_1(B, \mathbb{C})) = \dim(H_1(B', \mathbb{C})) = \dim(H_1(T, \mathbb{C})) = 1$$

Since the quotient $G/[G, G]$ is finite we see that $H_1(G, \mathbb{C}) = 0$. Therefore $\dim(H_1(G, L)) = 2$.

On the other hand we have an exact sequence

$$0 \to \mathcal{K}_\Omega \to \mathcal{S}(\Omega) \to L \to 0$$

and therefore an exact sequence

$$H_1(G, \mathcal{S}(\Omega)) \to H_1(G, L) \to H_0(G, \mathcal{K}_\Omega) \to H_0(G, \mathcal{S}(\Omega)).$$

Since $H_1(G, \mathcal{S}(\Omega)) = H_1(T, \mathbb{C})$ we see $\dim(H_1(G, \mathcal{S}(\Omega)) = 1$.

Lemma 2.26. The map $a : H_1(G, \mathcal{S}(\Omega)) \to H_1(G, L)$ is an embedding.

Proof. It is sufficient to show that the map $H_1(G, \mathcal{S}(\Omega)) \to H_1(G, \mathcal{S}(G/B))$ induced by the composition $p_\star \circ a : L \to \mathcal{S}(G/B)$ is an embedding.

Since $H_1(G, \mathcal{S}(\Omega)) = H_1(T, \mathbb{C})$, $H_1(G, \mathcal{S}(G/B)) = H_1(B, \mathbb{C})$ and $H_{>0}(U, \mathbb{C}) = 0$, we see that this map is an isomorphism. □

We can now finish the proof of Proposition 2.24. Since G acts transitively on G/B the map $\mu \to \int\limits_{G/B} \mu$ defines an isomorphism $H_0(G, \mathcal{S}(G/B)) \to \mathbb{C}$. On the other hand, since $\int\limits_\Omega \nu = 0$ for any $\nu \in \mathcal{K}_\Omega$ the map $H_0(G, \mathcal{K}_\Omega) \to H_0(G, \mathcal{S}(\Omega))$ equals zero. So $\dim(H_0(G, \mathcal{K}_\Omega)) = 1$. □

Recall that $G_s \subset G$ is the subset of regular split semisimple elements, let $\mathcal{K}_s \subset \mathcal{K}$ be the subspace of functions in $\mathcal{K}$ supported on G_s. We fix a Cartan subgroup T, the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$ acts on T and on G/T in the usual way. Then the map

$$(T - \{e\}) \times G/T \to G_s, (s, g) \to gsg^{-1}$$

induces an isomorphism

$$\mathcal{S}(G_s) \to (\mathcal{S}(T - \{e\}) \otimes \mathcal{S}(G/T))^W. \tag{5}$$

Corollary 2.27. a) For $f \in \mathcal{K}_s$ the distribution $\hat{f}$ is locally constant on G_s.

b) The map $f \mapsto (t \mapsto \hat{f}(t))$ induces an isomorphism

$$H_0(G, \mathcal{K}_s) \to \mathcal{S}(T - \{e\})^W.$$

Proof. The isomorphism (5) is clearly compatible with the averaging map $f \mapsto \hat{f}$, which implies a). Likewise, restriction to an orbit is compatible with averaging, thus in view of (5) it suffices to show that for a fixed orbit $\Omega \subset G_s$ the map $f \mapsto \hat{f}(t)$, $t \in \Omega$ induces an isomorphism $H_0(G, \mathcal{K}(\Omega)) \to \mathbb{C}$. By Proposition 2.24 it suffices to see that this map is nonzero. This follows, for example, from the fact that the character of a cuspidal representation does not necessarily vanish on G_s, while the character of an irreducible cuspidal representation is obtained by averaging from its matrix coefficient. □

Now we can finish the proof of Proposition 2.16. Let $f \in \mathcal{K}$ be such that $\hat{f} = 0$. It follows from Corollary 2.19 and Lemma 2.18 that we can can find f' with the same image in $H_0(G, \mathcal{K})$ such that $f' = f_s + f_e$ where f_s is supported on regular split semisiple elements and f_e on regular elliptic elements. The condition $\hat{f} = 0$ implies that $\hat{f}' = 0$, hence $\hat{f}_s = 0$ and $\hat{f}_e = 0$. It is easy to see that the condition $\hat{f}_e = 0$ implies that $[f_e] = 0$. So we may assume that the support of f is contained in the subset $G_s \subset G$ of regular split semisimple elements. Now Proposition 2.16 follows from Corollary 2.27. □

Let $\tilde{\mathcal{D}}_e$ be the space of germs of distributions at e and $\mathcal{D}_e \subset \tilde{\mathcal{D}}_e$ be the subspaces spanned by germs of characters of irreducible representations.

Lemma 2.28. The space $\mathcal{D}_e$ is 2-dimensional. It is spanned by germs of characters of irreducible cuspidal representations.

Proof. The second statement is a special case of a theorem of Harish-Chandra [13]. The first one also follows from *loc. cit.*, as it shown there that more generally the space $\mathcal{D}_e$ has a basis indexed by unipotent orbits. □

Recall that $\mathcal{E} \subset \mathcal{D}$ is the subspace of distributions α satisfying the following three conditions:

a) There exists a compact subset C in G such that $supp(\alpha) \subset C^G$.

b) The restriction of α on $G - \{e\}$ is given by a locally constant function.

c) The germ of α at e belongs to $\mathcal{D}_e$.

Lemma 2.29. $\tau(\mathcal{K}) \subset \mathcal{E}$.

Proof. Fix $f \in \mathcal{K}$. It is clear that the distribution $\hat{f}$ satisfies condition a).

To prove that $\hat{f}$ satisfies condition b) we have to show that for any semisimple element $s \in G - \{e\}$ there exists an open neighborhood $R \subset G$ of s such that the restriction $\hat{f}|_R$ is a constant. If s is split then this follows from Corollary 2.27(a), if s is elliptic the proof is similar.

To prove that $\hat{f}$ satisfies condition c) we observe that Corollary 2.23 implies existence of $f_{cusp} \in \mathcal{K}_{cusp}$ such that $[\kappa(f)] = [\kappa(f_{cusp})]$.

It is easy to see that when f is a matrix coefficient of an irreducible cuspidal representation ρ then $\hat{f}$ is proportional to the character of ρ. Thus condition c) is satisfied by $\alpha_{cusp} = \widehat{f_{cusp}}$. However, by Lemma 2.18 and Corollary 2.19 the germs of α and α_{cusp} at e coincide. □

Proposition 2.30. $\tau(\mathcal{K}) = \mathcal{E}$.

Proof. It remains to show that every $\alpha \in \mathcal{E}$ is in the image of τ. Lemma 2.28 shows that there exists $\beta \in \mathcal{E}$ which is a linear combination of characters of cuspidal representations such $\alpha - \beta$ vanishes on an open neighborhood of e. Thus we have $\alpha - \beta = \alpha_s + \alpha_e$, where α_s is supported on G_s, while α_e is supported on G_e.

Now α_s is in the image of τ by Corollary 2.27, while α_e is in the image of τ by a similar argument. Also, β is in the image of τ since the character of an irreducible cuspidal representation ρ equals $\hat{f}$ where f is a matrix coefficient of ρ.

Proposition 2.30 and therefore Theorem 2.14 are proven. □

3. The compactified category of smooth modules

3.1. **Definition of the compactified category.** For a parabolic $P = LU$ let $L^0 \subset L$ be the subgroup generated by compact subgroups; thus L^0 is the kernel of the unramified characters of L. Set $\check{\Lambda}_P = L/L^0$.

Let Λ_P be the group of F-rational characters of L and Λ_P^+ be the subset of P-dominant weights, i.e. weights which are (non-strictly) dominant with respect to any (not necessarily F-rational) Borel subgroup $B \subset P$. We have a nondegenerate pairing between the lattices $\check{\Lambda}_P$ and Λ_P given by:

$$\langle xL^0, \lambda\rangle = val_F(\lambda(x)).$$

Let $\check{\Lambda}_P^+$ be the subsemigroup defined by:

$$\check{\Lambda}_P^+ = \{x \in \check{\Lambda}_P \mid \langle x, \lambda\rangle \geq 0 \ \ \forall \lambda \in \Lambda_P^+\},$$

and let $L_P^+ \subset L$ be the preimage of $\check{\Lambda}_P^+$ under the projection $L \to \check{\Lambda}_P$.

For a pair of parabolics $P \supset Q$ let $L_Q^{P+} \subset L_Q$ denote the image of $L_P^+ \cap Q$ in $L_Q = Q/U_Q$. It is easy to see that $L_Q^{P+} \supset L_Q^+$.

For an open submonoid $M \subset G$ we let $Sm(M)$ denote the category of nondegenerate finitely generated $\mathcal{H}(M)$-modules; this is easily seen to be equivalent to the category of finitely generated smooth M-modules.

For parabolic subgroups $P \supset Q$ we have the "Jacquet" functor $J_Q^P : Sm(L_P^+) \to Sm(L_Q^{P+})$, $M \mapsto M_{\overline{U_Q}}$, where $\overline{U_Q}$ is image of U_Q in $L_P = P/U_P$.

To simplify the wording in the following definition we fix a minimal parabolic P_0, then by a standard parabolic we mean a subgroup P containing P_0.

Definition 3.1. The *compactified category* of smooth G-modules $\overline{Sm}^{qc} = \overline{Sm}^{qc}(G)$ is the category whose object is a collection (M_P) indexed by standard parabolic subgroups $P = L_P U_P$, where M_P is a smooth module over L_P^+, together with isomorphisms

$$J_Q^P(M_P) \cong \mathcal{H}(L_Q^{P+}) \otimes_{\mathcal{H}(L_Q^+)} M_Q \tag{6}$$

fixed for every pair of standard parabolic subgroups $P = L_P U_P \supset Q = L_Q U_Q$; here $\mathcal{H}$ denotes the algebra of locally constant compactly supported distributions. The isomorphisms are required to satisfy the associativity identity for each triple of parabolics $P_1 \supset P_2 \supset P_3$.

An object in the compactified category is called coherent if the module M_P is finitely generated for all P.

We let $\overline{Sm} = \overline{Sm}(G) \subset \overline{Sm}^{qc}$ denote the full subcategory of coherent objects.

It is easy to see that $\overline{Sm}(G)$ is an abelian category, the functor sending $(M_P) \in \overline{Sm}$ to M_G identifies $Sm(G)$ with a Serre quotient of $\overline{Sm}$.

We also have an adjoint functor $Sm(G) \to \overline{Sm}^{qc}$. This functor sends admissible modules but not general finitely generated modules to $\overline{Sm}$.

Example 3.2. Let $G = SL(2)$. In this case the category $\overline{Sm}$ admits the following more direct description. A component of the spectrum $\mathfrak{Z}$ of Bernstein center is in this case either a point or an affine curve, thus $\mathfrak{Z}$ admits a canonical (componentwise) compactification $\overline{\mathfrak{Z}}$. Notice that $\partial\mathfrak{Z} = \overline{\mathfrak{Z}} \setminus \mathfrak{Z}$ is identified with the set $(O^\times)^*$ of characters of $T_0 = O^\times$. Let $T^+ = O \setminus \{0\} \subset F^\times = T$, thus $T^+ \cong O^\times \times \mathbb{Z}_{\geq 0}$. Set $\tilde{\mathfrak{Z}}^+ = Spec(\mathcal{H}(T^+)) \cong (O^\times)^* \times \mathbb{A}^1$. Notice that we have a natural map $\pi : \tilde{\mathfrak{Z}}^+ \to \overline{\mathfrak{Z}}$ inducing an isomorphism $(O^\times)^* \times \{0\} \to \partial\mathfrak{Z}$. Moreover, π is etale at $(O^\times)^* \times \{0\}$.

The full Hecke algebra $\mathcal{H}$ defines a quasicoherent sheaf of algebras on $\mathfrak{Z}$, we now describe its extension to a quasicoherent sheaf of algebras $\overline{\mathcal{H}}$ on $\overline{\mathfrak{Z}}$. The latter depends on the choice of a maximal open compact subgroup $K_0 = SL(2, O)$. Fixing this choice we set $\tilde{\mathcal{H}}^+ = End_{T^+}(\mathcal{S}(G/U^+))$, where $(G/U)^+ = O^2 \setminus \{0\} \subset F^2 \setminus \{0\} = G/U$. We also let $\tilde{\mathcal{H}} = End_T(\mathcal{S}(G/U))$. It is clear that $\tilde{\mathcal{H}}^+$ defines a quasicoherent sheaf on $\tilde{\mathfrak{Z}}^+$ whose restriction to the open subset $\tilde{\mathfrak{Z}} := \tilde{\mathfrak{Z}}^+ \setminus (O^\times)^* \times \{0\}$ is the quasicoherent sheaf defined by $\tilde{\mathcal{H}}$.

The action of $\mathcal{H}$ on $\mathcal{S}(G/U)$ defines a homomorphism $\pi^*(\mathcal{H}) \to \tilde{\mathcal{H}}$ which is an isomorphism on a Zariski neighborhood of $\partial\tilde{\mathfrak{Z}}^+ = \tilde{\mathfrak{Z}}^+ \setminus \tilde{\mathfrak{Z}}$. Thus we get a well defined quasicoherent sheaf of algebras $\overline{\mathcal{H}}$ on $\overline{\mathfrak{Z}}$ such that $\overline{\mathcal{H}}|_{\mathfrak{Z}} = \mathcal{H}$ and the induced map $\pi^*(\overline{\mathcal{H}})|_{\tilde{\mathfrak{Z}}} \to \tilde{\mathcal{H}}$ extends to a map $\pi^*(\overline{\mathcal{H}}) \to \tilde{\mathcal{H}}^+$ which is an isomorphism on a neighborhood of $\partial\tilde{\mathfrak{Z}}^+$.

It is clear that K_0^2 acts on $\overline{\mathcal{H}}$ and for an open subgroup $K \subset K_0$ the subsheaf $\overline{\mathcal{H}}_K$ of K^2 invariants is a coherent sheaf of algebras.

We leave it to the reader to show that although $\overline{\mathcal{H}}$ depends on an auxiliary choice, different choices lead to algebras which are canonically Morita equivalent. Thus we can consider the category of sheaves of nondegenerate $\overline{\mathcal{H}}$-modules which can be checked to be canonically equivalent to $\overline{Sm}$. If the subgroup $K \subset K_0$ is nice in the sense of [9] then for every component X of $\overline{\mathfrak{Z}}$ either the coherent sheaf of algebras $\overline{\mathcal{H}}_K|_X$ is zero or the corresponding summand in $\overline{Sm}$ (respectively, $\overline{Sm}^{qc}$) is canonically equivalent to the category of coherent (respectively, quasicoherent) sheaves of $\overline{\mathcal{H}}_K|_X$-modules.

3.2. **Compactified center and a spectral description of the compactifed category.** Let $Z = Z(G)$ be the Bernstein center of G and $\mathfrak{Z} = Spec(Z)$ be its spectrum. By the main result of [9] (the set of closed points of) $\mathfrak{Z}$ is in bijection with the set $Cusp(G)$ of *cuspidal data*, i.e. the set of G-conjugacy classes of pairs (L, ρ), where $L \subset G$ is a Levi subgroup and ρ is a cuspidal irreducible representation of L.

3.2.1. *Compactified center.* Let $\overline{\mathfrak{Z}}$ denote its compactification described as follows. We have a canonical isomorphism $\mathfrak{Z} = \tilde{\mathfrak{Z}}/W$ where W is the Weyl group, and $\tilde{\mathfrak{Z}}$ parametrizes pairs (L, ρ) where L is a Levi subgroup containing a fixed maximally split Cartan T and ρ is a cuspidal representation of L. The complex torus ${}^LT = \mathcal{X}(L)$ acts nn the union $\tilde{\mathfrak{Z}}_L$ of components corresponding to a given Levi subgroup $L \supset T$; here $\mathcal{X}(L)$ stands for the group of unramified characters of L acting on the set of representations by twisting. Notice that ${}^LT$ is a torus with $X^*({}^LT) = L/L^0$. The action is transitive on each component and the stabilizer of each point is finite. The space $X^*({}^LT)_{\mathbb{R}} = X_*(Z(L))_{\mathbb{R}}$ (where X_* stands for the lattice of F-rational cocharacters) contains hyperplanes corresponding to the roots of $Z(L)$ in $\mathfrak{g}$; the fan formed by these hyperplanes defines an equivariant compactification $\overline{{}^LT}$ of ${}^LT$. We set

$$\overline{\mathfrak{Z}} = \tilde{\overline{\mathfrak{Z}}}/W, \qquad \tilde{\overline{\mathfrak{Z}}} = \bigcup_L \overline{{}^LT} \times^{{}^LT} \tilde{\mathfrak{Z}}_L,$$

where the right hand side makes sense because the action of W on $\tilde{\mathfrak{Z}}$ extends to the compactification, here we use the notation $X \times^H Y = (X \times Y)/H$. Notice that every component of $\tilde{\mathfrak{Z}}_L$ is of the form ${}^LT/A$ for a finite subgroup $A \subset {}^LT$, thus the corresponding component of $\overline{{}^LT} \times^{{}^LT} \tilde{\mathfrak{Z}}_L$ is identified with $\overline{{}^LT}/A$.

For a parabolic $P = LU$ let $Z^0(L) \subset Z(L)$, $Z^+(L) \subset Z(L)$ be the subalgebras consisting of distributions supported on L^0 and L_P^+ respectively, set also $\mathfrak{Z}^0(L) = Spec(Z^0(L))$, $\mathfrak{Z}^+(L) = Spec(Z^+(L))$.

It is clear that

$$\mathfrak{Z}^0(L) = \mathfrak{Z}(L)/\mathcal{X}(L), \tag{7}$$

where $\mathcal{X}(L)$ is the group of unramified characters of L.

Proposition 3.3. a) $\overline{\mathfrak{Z}}$ admits a canonical stratification indexed by conjugacy classes of parabolic subgroups, where the stratum $\overline{\mathfrak{Z}}_P$ corresponding to the class of a parabolic P is identified with $\mathfrak{Z}^0(L)$.

b) The embedding $\mathfrak{Z}^0(L) \to \overline{\mathfrak{Z}}$ canonically extends to a map $\mathfrak{Z}^+(L) \to \overline{\mathfrak{Z}}$ which is etale on a Zariski neighborhood of $\overline{\mathfrak{Z}}_P \cong \mathfrak{Z}^0(L)$.

Given two parabolics $P \subset Q$ we have a canonical map $c_P^Q : \mathfrak{Z}^+(L_P) \to \mathfrak{Z}^+(L_Q)$ which is compatible with maps to $\overline{\mathfrak{Z}}$.

Moreover, for three parabolics $P_1 \subset P_2 \subset P_3$ we have

$$c_{P_1}^{P_3} = c_{P_2}^{P_3} c_{P_1}^{P_2}.$$

Proof. Let $\tilde{\tilde{\overline{\mathfrak{z}}}}_L = \overline{{}^LT} \times^{{}^LT} \tilde{\mathfrak{z}}_L$.

It is a standard fact that ${}^LT$-orbits in $\overline{{}^LT}$ are in bijection with parabolic subgroups containing L, so that the orbit $\overline{{}^LT}_Q$ corresponding to a parabolic $Q = MU_Q$ is identified with $\mathcal{X}(L)/\mathcal{X}(M)$. The stratification of $\overline{{}^LT}$ by ${}^LT$-orbits induces a stratification on $\tilde{\tilde{\overline{\mathfrak{z}}}}_L$, the stratum corresponding to a parabolic Q will be denoted by $\tilde{\tilde{\overline{\mathfrak{z}}}}_L\{Q\}$.

Fix a conjugacy class $\mathbf{P}$ of parabolic subgroups and set $\tilde{\tilde{\overline{\mathfrak{z}}}}_L(\mathbf{P}) = \bigcup\limits_{Q\in\mathbf{P}} \tilde{\tilde{\overline{\mathfrak{z}}}}_L\{Q\}$. Let $\tilde{\tilde{\overline{\mathfrak{z}}}}_{\mathbf{P}} = \bigcup\limits_L \tilde{\tilde{\overline{\mathfrak{z}}}}_L(\mathbf{P})$.

It is clear that $(\tilde{\tilde{\overline{\mathfrak{z}}}}_{\mathbf{P}})$ is a stratification of $\tilde{\tilde{\overline{\mathfrak{z}}}}$ and each stratum is W-invariant. Thus $\overline{\mathfrak{z}}_{\mathbf{P}} := \tilde{\tilde{\overline{\mathfrak{z}}}}_{\mathbf{P}}/W$ are strata of a stratification of $\overline{\mathfrak{z}}$.

The map $Q \mapsto \tilde{\tilde{\overline{\mathfrak{z}}}}_L\{Q\}$ is easily seen to be W-equivariant, it follows that for a parabolic $P = LU \in \mathbf{P}$ we have

$$\overline{\mathfrak{z}}_{\mathbf{P}} \cong \bigcup_{M,T\subset M\subset L} \tilde{\tilde{\overline{\mathfrak{z}}}}_M\{P\}/W_L.$$

The above isomorphism $\overline{{}^LT}_Q \cong \mathcal{X}(L)/\mathcal{X}(M)$ shows that $\tilde{\tilde{\overline{\mathfrak{z}}}}_M\{P\} \cong \overline{\mathfrak{z}}_M(L)/\mathcal{X}(L)$. Passing to the union over M and taking quotient by the action of W_L (which commutes with the action of $\mathcal{X}(L)$) we get $\overline{\mathfrak{z}}_{\mathbf{P}} \cong \mathfrak{z}(L)/\mathcal{X}(L)$ which yields (a) in view of (7).

To check (b) observe that for parabolic subgroups $Q = MU_Q \supset P = LU_P \supset T$ the cone $\mathbb{R}^{\geq 0}\Lambda_Q^+$ belongs to the fan defining the toric variety $\overline{{}^LT}$. Let $V_L\{Q\}$ be the corresponding affine open subset in $\overline{{}^LT}$ and $\mathcal{V}_L\{Q\}$ be the corresponding open affine in $\tilde{\tilde{\overline{\mathfrak{z}}}}_L$. Thus $\mathcal{V}_L\{Q\}$ is a Zariski open neighborhood of $\tilde{\tilde{\overline{\mathfrak{z}}}}_L\{Q\}$.

It is easy to see that $\mathcal{V}_L\{Q\}$ is W_M invariant and $\mathcal{V}_L\{Q\}/W_M \cong \mathfrak{z}^+(L)$. Since $\overline{\mathfrak{z}} = \tilde{\tilde{\overline{\mathfrak{z}}}}/W$, claim (b) follows from the fact that the stabilizer of any point $x \in \tilde{\tilde{\overline{\mathfrak{z}}}}\{Q\}$ is contained in W_M.

c) follows by inspection. □

In order to relate $\overline{Sm}$ to $\overline{\mathfrak{z}}$ we will need the following general concept. Let X be an algebraic variety. By a *quasicoherent enrichment* of a category $\mathcal{C}$ over X we will mean assigning to objects $M,\ N \in \mathcal{C}$ an object $\underline{\mathrm{Hom}}(M,N) \in QCoh(X)$ together with an isomorphism $Hom(M,N) = \Gamma(\underline{\mathrm{Hom}}(M,N))$ and maps $\underline{\mathrm{Hom}}(M_1,M_2)\otimes_{\mathcal{O}_X} \underline{\mathrm{Hom}}(M_2,M_3) \to \underline{\mathrm{Hom}}(M_1,M_3)$ satisfying the associativity constraint and compatible with the composition of morphisms in $\mathcal{C}$. If the quasicoherent sheaf $\underline{\mathrm{Hom}}(M,N)$ is actually coherent for all $M,N\in\mathcal{C}$ we say that the enrichment is coherent.

Proposition 3.4. The category $\overline{Sm}$ (respectively, $\overline{Sm}^{qc}(G)$) admits a natural lifts to a category coherent (respectively, quasicoherent) enrichment over $\overline{\mathfrak{z}}$.

Corollary 3.5. The categories $\overline{Sm}$, $\overline{Sm}^{qc}$ split as a direct sum indexed by components of $\mathfrak{z}$. □

Before proceeding to prove the Proposition we state a general elementary Lemma.

Lemma 3.6. Let $X = \coprod X_i$ be a scheme with a fixed stratification (i.e. the closure of X_i coincides with $\coprod_{j\leq i} X_j$ for some partial order $\leq$ on the set I of strata). Set $U_i = \coprod_{j\geq i} X_j$, this is an open subset of X. Suppose that for each i we are given a map $u_i : Y_i \to U_i$, such that

i) $X_i \times_X Y_i \widetilde{\longrightarrow} X_i$

ii) u_i is etale over a Zariski neighborhood of X_i.

iii) For $j \le i$ set $Y_{ji} = Y_j \times_X U_i$. Then the map $Y_{ji} \to U_i$ factors through a map $u_{ji} : Y_{ji} \to Y_i$. Moreover, for $k < j < i$ the map $Y_{ki} \to U_j \supset U_i$ factors through a map $u_{kji} : Y_{ki} \to Y_{ji}$.

Let $Y_d = \coprod_{i_1 < \dots < i_d} Y_{i_1 i_d}$. Then the diagram $Y_3 \rightrightarrows Y_2 \rightrightarrows Y_1 \to X$ satisfies descent for quasicoherent sheaves, i.e. $QCoh(X)$ is equivalent to the category of quasicoherent sheaves on Y_1 with isomorphisms of the two pull-backs to Y_2 whose pull-backs to Y_3 satisfy the natural compatibility.

Remark 3.7. To fix ideas let us first prove the Lemma for sheaves in the analytic topology assuming we work over the base field $\mathbb{C}$. Then it is easy to see that we can find an open subset $Y_i^o \subset Y_i$ for each i so that Y_i^o maps isomorphically to a neighborhood of X_i in X. Moreover, we can arrange it so that the images of Y_i^o and Y_j^o have a nonempty intersection only if $i \le j$ or $j \le i$. Replacing Y_i by Y_i^o does not affect the category of gluing data; however, (Y_i^o) is just an open covering of X, so the claim is clear.

3.2.2. *Proof of Lemma 3.6.* Let X_0 be a closed stratum. Running an inductive argument, we can assume the theorem is known for the stratified space $X \setminus X_0$. Then we are reduced to proving the claim in the situation when the stratification consists of two strata $X = X_0 \coprod X_1$. Replacing Y_0 by its open subset containing X_0 clearly does not affect the category of descent data, so we can assume without loss of generality that $Y_1 \to X$ is etale. By a standard argument the claim reduces to exactness of the complex of sheaves on X:

$$0 \to \mathcal{O} \to (Y_0 \to X)_*(\mathcal{O}) \oplus (Y_1 \to X)_*(\mathcal{O}) \to (Y_{01} \to X)_*(\mathcal{O}) \to 0.$$

The complex is clearly exact over X_1, so it is enough to show that local cohomology of this complex with support on X_1 vanishes. This reduces to showing that $\alpha^!(\mathcal{O}) \widetilde{\longrightarrow} \beta^!(\mathcal{O})$, where $\beta : X_0 \to X$, $\alpha_1 : X_0 \to Y_0$. This follows from conditions (i), (ii). □

3.2.3. *Proof of Proposition 3.4.* Proposition 3.3 implies that $X = \overline{\mathfrak{Z}}$ with the stratification of Proposition 3.3(a), and $Y_i = \mathfrak{Z}(L_i^+)$ satisfy the conditions of Lemma 3.6. It is easy to see from the definition of $\overline{Sm}^{qc}$ that the collection of quasi-coherent sheaves $Hom(M_P, N_P)$ provides gluing data described in Lemma 3.6. Also, for M, $N \in \overline{Sm}$ the module $Hom(M_P, N_P)$ is finitely generated, so the resulting quasicoherent sheaf is in fact coherent. □

3.3. **The spectral description of $\overline{Sm}$.** Recall that for every component $X \subset \mathfrak{Z}$ the choice of a sufficiently small nice (in the sense of [9]) open compact subgroup $K \subset G$ defines a coherent sheaf of algebras $\mathcal{A} = \mathcal{A}_X(K)$ on X with an equivalence between $\mathcal{A}_X(K)$ modules and the corresponding summand in smooth G-modules.

Let $\mathfrak{B}$ be the (reductive) Bruhat-Tits building of G. We fix a special vertex $x \in \mathfrak{B}$ and let $K_x \subset G$ denote the corresponding maximal compact subgroup.

We also fix a maximally split Cartan subgroup T, such that the corresponding apartment $A_T \subset \mathfrak{B}$ contains x. Thus A_T is an affine space with underlying vector space $V = X_*(T) \otimes \mathbb{R}$. Let $V_+ \subset A_T$ be a Weyl cone with vertex at x and B the corresponding minimal parabolic subgroup.

It follows from the Iwasawa decomposition (see e.g. [22, §3.3.2]) that we have a natural bijection

$$P'\backslash G/K_x \cong \check{\Lambda}_P, \tag{8}$$

where P' denotes the commutator group. Let $G^+(x, P)$ be the union of cosets corresponding to elements in $\check{\Lambda}_P^+ \subset \check{\Lambda}_P$.

Let $(U_P\backslash G)^+(x)$ be the image of $G^+(x, P)$ in $U_P\backslash G$ and $\mathcal{S}^+(U_P\backslash G)(x) \subset \mathcal{S}(U_P\backslash G)$ be the subspace of functions whose support is contained in $(U_P\backslash G)^+(x)$.

Proposition 3.8. a) The left action of L on $\mathcal{S}(U_P\backslash G)$ restrict to an L_P^+ action on $\mathcal{S}^+(U_P\backslash G)(x)$.

b) Let $K \subset K_x$ be an open subgroup. The $\mathcal{H}(L_P^+)$-module $\mathcal{S}^+(U_P\backslash G)(x)^K$ is finitely generated and projective.

c) There exists a unique object $\mathcal{P}(x)^K \in \overline{Sm}$ such that $\mathcal{P}(x)_P^K = \mathcal{S}^+(U_P\backslash G)(x)^K$ while the isomorphism (6) for $P = G$ comes from the natural arrow $\mathcal{S}^+(U_Q\backslash G)(x)^K \to \mathcal{S}(U_Q\backslash G)^K = J_Q^G(\mathcal{S}(G)^K)$.

d) For every open subgroup $K \subset K_x$ the object $\mathcal{P}(x)^K$ is locally projective, i.e. the functor $\underline{\mathrm{Hom}}(\mathcal{P}(x)^K, \ )$ is exact.

e) For every component X there exists an open subgroup $S \subset K_x$ such that for any open subgroup $K \subset S$ the object $\mathcal{P}_x^K$ is local generator of the corresponding summand $\overline{Sm}_X \subset \overline{Sm}$. The latter property means that $\underline{\mathrm{Hom}}(\mathcal{P}_\sigma(x)^K, \ )$ is conservative i.e. it $\underline{\mathrm{Hom}}(\mathcal{P}_\sigma(x)^K, \mathcal{M}) \neq 0$ for $0 \neq \mathcal{M} \in \overline{Sm}_X$.

Proof. a) Bijection (8) intertwines the left action of L on $P'\backslash G/K_x$ with the action of $\check{\Lambda}_P = L/L^0$ on itself by translations, this implies part a). The space $\mathcal{S}^+(U_P\backslash G)(x)^K$ splits as a direct sum indexed by $P\backslash G/K$, each summand is isomorphic to the space $\mathcal{S}(L_P^+)^{K_L}$ for some open compact subgroup $K_L \subset K$, this implies b).

Notice that given modules M_P and isomorphisms (6) for $P = G$ as in Definition 3.1, the rest of the isomorphisms (6) satisfying the requirements of the Definition are defined uniquely (if they exist) provided that each module M_P is torsion free as a module over $Z^+(L_P)$. This implies uniqueness in c). Existence follows from the fact that $G^+(x, P) \subset G^+(x, Q)$ for parabolic subgroups $P \subset Q$.

Statement d) follows from b).

Finally e) follows from the corresponding statement about $\mathcal{H}_K$ established in [9]. □

Set $\overline{\mathcal{A}}_X = \overline{\mathcal{A}}_X(x, K) := \underline{\mathrm{Hom}}(\mathcal{P}(x)^K, \mathcal{P}(x)^K)|_X$, this is a sheaf of algebras on X.

Corollary 3.9. Given a component $X \subset \overline{\mathfrak{Z}}$ for any small enough open subgroup $K \subset K_x$ we have a canonical equivalence between the category of (quasi)coherent sheaves of $\overline{\mathcal{A}}_X(x, K)$-modules and the summand in $\overline{Sm}$ (respectively, $\overline{Sm}^{qc}$) corresponding to X.

3.4. **Compactified category and filtered modules.** For a simple root α let P_α be the corresponding maximal proper parabolic and $\overline{\mathfrak{Z}}_\alpha$ be the closure of the corresponding stratum in $\overline{\mathfrak{Z}}$. It is easy to see that $\overline{\mathfrak{Z}}_\alpha$ is a divisor. For a weight λ set $D_\lambda = \sum_\alpha \langle \check{\alpha}, \lambda \rangle [\overline{\mathfrak{Z}}_\alpha]$ where the sum runs over the set of simple roots, let also $\mathcal{O}(\lambda) = \mathcal{O}_{\overline{\mathfrak{Z}}}(D_\lambda)$ and $\mathcal{F}(\lambda) = \mathcal{F} \otimes_{\mathcal{O}(\overline{\mathfrak{Z}})} \mathcal{O}(\lambda)$.

A coherent sheaf $\mathcal{F}$ on $\overline{\mathfrak{Z}}$ is determined by the graded module $\oplus_\lambda \Gamma(\mathcal{F}(\lambda))$, over the homogeneous coordinate ring $\oplus \Gamma(\mathcal{O}(\lambda))$.

If $\mathcal{F}$ is torsion free then the natural map $\mathcal{F}(\mu) \to \mathcal{F}(\lambda + \mu)$, $\lambda \in \Lambda^+$ is injective and $\bigcup_\lambda \mathcal{F}(\lambda) = j_* j^*(\mathcal{F})$ where $j : \mathfrak{Z} \to \overline{\mathfrak{Z}}$ is the embedding. Thus the category of torsion free coherent sheaves $Coh_{tf}(\overline{\mathfrak{Z}})$ admits a full embedding into the category of $\mathcal{O}(\mathfrak{Z})$-modules equipped with a filtration indexed by Λ^+ compatible with the natural filtration on the ring $\mathcal{O}(\mathfrak{Z})$: to a sheaf $\mathcal{F}$ it assigns the module $\Gamma(j^*(F))$ with the filtration by the subspaces $\Gamma(\mathcal{F}(\lambda))$.

Applying it to the sheaf of rings $\overline{\mathcal{A}}_X(x, K)$ we get a filtration on the Hecke algebra $F^{spec}_{\leq \lambda}(\mathcal{H}_K)$.

Thus we obtain the following:

Proposition 3.10. We have a full embedding from the category $\overline{Sm}_K^{tf}$ of torsion free objects in $\overline{Sm}_K$ to the category of modules over $\mathcal{H}_K$ equipped with a filtration compatible with the filtration F^{spec} on $\mathcal{H}_K$.

It is clear that the associated graded of the filtered module in the image of such an embedding is finitely generated; recall that a filtration with this property is called a *good filtration*.

We also have the left adjoint functor *Loc* from the category of $\mathcal{H}_K$ modules with a good filtration to $\overline{Sm}_K$.

3.4.1. *A geometric description of the filtration.* We now provide a more explicit description of the filtration $F^{spec}_{\leq\lambda}(\mathcal{H}_K)$.

Recall that the two sided cosets of K_x in G are indexed by X_B^+ (see e.g. [22, §3.3.3]), for $\lambda \in X_B^+$ let G_λ denote the corresponding coset. Let $F^{geom}_{\leq\lambda}(\mathcal{H}_K)$ be the space of functions whose support is contained in $\bigcup\limits_{\mu\leq\lambda} G_\mu$.

Proposition 3.11. For every open compact $K \subset K_x$ there exists $\lambda_0 \in X_B^+$ such that:

$$F^{spec}_{\leq\lambda}(\mathcal{H}_K) = \{h \in \mathcal{H}_K \mid \forall\mu \in \lambda_0 + X_B^+ : \quad h * F^{geom}_{\leq\mu}(\mathcal{H}_K) \subset F^{geom}_{\leq\mu+\lambda}(\mathcal{H}_K).\} \tag{9}$$

Proof It follows from the definition that

$$h \in F^{spec}_{\leq\lambda}(\mathcal{H}_K) \iff \forall P, \mu \ : \ \mathcal{S}((U_P\backslash G)_{\leq\mu}) * h \subset \mathcal{S}((U_P\backslash G)_{\leq\bar{\lambda}_P+\mu}), \tag{10}$$

where $\bar{\lambda}_P$ is the image of λ in $\check{\Lambda}_P$ and $(U_P\backslash G)_{\leq\mu} = \bigcup\limits_{\nu\leq\mu}(U_P\backslash G)_\nu$ for the standard partial order $\leq$ on $\check{\Lambda}_P$; here $(U_P\backslash G)_\nu$ is the image of the coset corresponding to ν under bijection (8).) Let $X_P = (G/U_P \times G/U_{P^-})/L$. We have $K_x\backslash X_P/K_x \cong \check{\Lambda}_P$, let $(X_P)_\mu$ denote the two-sided coset corresponding to $\mu \in \check{\Lambda}_P$. Then the condition in the right hand side of (10) is equivalent to:

$$\forall P, \mu \ : \ h * \mathcal{S}((X_P)_{\leq\mu}) \subset \mathcal{S}((X_P)_{\leq\lambda+\mu}), \tag{11}$$

where $(X_P)_{\leq\mu} = \bigcup\limits_{\nu\leq\mu}(X_P)_\nu$. this is clear by considering the projection $X \to G/P^-$ with fiber G/U_P. Assume now that h lies in the set in the right hand side of (9). Applying the map B_I of [8, Definition 5.3] and using [8, Lemma 5.5], we see that h also satisfies (11). This shows that the right hand side of (9) is contained in the left hand side. We proceed to check the opposite inclusion. The Rees ring $\oplus_\lambda F^{spec}_{\leq\lambda}(\mathcal{H}_K)$ is finite over its center which is a finitely generated commutative ring, hence the Rees ring is finitely generated. Thus it suffices to check that for a finite set of generators $h_i \in F^{spec}_{\leq\lambda_i}(\mathcal{H}_K)$ we have:

$$h_i * F^{geom}_{\leq\mu}(\mathcal{H}_K) \subset F^{geom}_{\leq\mu+\lambda_i}(\mathcal{H}_K) \ \ \forall\mu \in \lambda_0 + \Lambda^+$$

for some $\lambda_0 \in \Lambda^+$. Existence of such a λ_0 for a given h_i follows from [8, Lemma 5.5]. Since we consider a finite set of h_i, there exists λ_0 which satisfies the requirement for all h_i. □)

Remark 3.12. It easily follows from the construction that $gr(F^{spec})$ is a Noetherian ring.

Notice that the geometric filtration on $\mathcal{H}$ is also compatible with the algebra structure; however, its associated graded is neither Noetherian, nor finitely generated in general. This is closely related to the fact that the intertwining operator acting on the space of functions over $\underline{(G/U)}(O/\pi^n O)$ is not an isomorphism for $n > 1$ (here $\underline{(G/U)}$ is a scheme over O coming from the O-group scheme with generic fiber G corresponding to $x \in \mathfrak{B}$).

Example 3.13. In view of Proposition 3.11 the geometric filtration F^{geom} on $\mathcal{H}$ makes it into a filtered module over the filtered algebra $(\mathcal{H}, F^{spec})$. Applying the functor *Loc* to that filtered module we get an object in $\overline{Sm}$. We denote it by $\overline{\mathcal{H}}'$ and call it the *intertwining* object in $\overline{Sm}$.

Lemma 3.14. Assume that $G = SL(2)$.

The intertwining object $\overline{\mathcal{H}}'$ is equivalently described by $\overline{\mathcal{H}}'_G = \mathcal{H}$,

$$\overline{\mathcal{H}}'_B = \{f \in \mathcal{S}(G/U) \mid supp(I^{-1}(f)) \subset (G/U_-)^+\},$$

where I^{-1} denotes the inverse *intertwining operator* taking values in functions of bounded support.[3]

Proof. Without loss of generality we can assume that x is the standard vertex, so that $K_x = SL(2, O)$, we will write K_0 instead of K_x.

We need to check that for $h \in \mathcal{H}_K$ and large $\lambda \in \Lambda$ we have:

$$h \in F^{geom}_{\leq \lambda} \iff supp(I_P^{-1} A_P(h)) \subset (X_P)_{\leq \lambda},$$

where $X_{\leq \lambda}$ is the union of two-sided K_0 cosets corresponding to weights $\mu \leq \lambda$ and $A : \mathcal{H} \to \mathcal{S}(Y_P)$. By [8, Theorem 7.6] $I^{-1}A = B^\star$ (notations of *loc. cit.*). It follows from [8, Lemma 5.5] that for large λ we have

$$supp(h) \subset G_{\leq \lambda} \iff supp(B^\star(f)) \subset X_{\leq \lambda}.$$

The claim follows. □

Remark 3.15. A similar statement can also be checked for an arbitrary reductive group G based on a generalization of [8, Theorem 7.6] to an arbitrary parabolic subgroup.

Remark 3.16. We expect the local trace formula [1] to be closely related to the computation of Chern character of $\overline{\mathcal{H}}'$ taking values in the appropriate Hochschild homology group.

4. Hochschild homology and character values

Recall the notion of *Hochschild homology* of an abelian category [18], [16]. For a coherent sheaf of algebras A over a quasi-projective algebraic variety X over a field we have $HH_*(X; A) \cong HH_*(A - mod)$, where $A - mod$ is the abelian category of coherent sheaves of A-modules and $HH_*(X; A)$ is defined as the derived global section of a naturally defined object $R\underline{Hom}_{A \otimes A^{op}}(A, A)$ in the derived category of sheaves on X, here the isomorphism is shown in [17].

One can also define the *compactly supported* Hochschild homology $HH^c_*(X; A)$ as the derived global sections with compact support in the sense of [12] of $R\underline{Hom}_{A \otimes A^{op}}(A, A)$. If X is projective then we also have

$$HH^c_*(X; A) \cong HH_*(X; A) \cong HH_*(A - mod),$$

where the first isomorphism is clear since $R\Gamma = R\Gamma_c$ for a projective scheme. Also, for an open subscheme $U \subset X$ we have a natural push forward map $HH^c_*(U; A|_X) \to HH^c_*(X, A)$.

Assume that the sheaf of algebras A (locally) has finite homological dimension. Then we have the *Chern character* map $ch : K^0(A - mod) \to HH_0(A - mod)$, see e.g. [17, §4.2] for the definition (it is called the Euler class map in *loc. cit.*).

Example 4.1. For future reference we spell out this general construction in some simple special cases. We leave the proofs of these standard facts to the interested reader.

(1) Assume X is affine, so that $A - mod$ is the category of finitely generated modules over a Noetherian ring which we also denote by A. Then $HH_*(X; A) = HH_*(A)$ is computed by the bar complex of A, in particular $HH_0(X; A) = HH_0(A) = A/[A, A]$. A finitely generated projective module M is isomorphic to $A^{\oplus n}e$ for an idempotent $e \in Mat_n(A)$ for some n. Then $ch(M) = \sum(e_{ii}) \mod [A, A]$.

[3]Here by a bounded set in $G/U_- = F^2 \setminus \{0\}$ we mean a subset with compact closure in F^2.

(2) Assume that $X = U_1 \cup U_2$ for affine open subsets U_1, U_2. Let $A_i = \Gamma(U_i, A)$, $i = 1, 2$ and $A_{12} = \Gamma(U_1 \cap U_2, A)$. Then $HH_*(X; A)$ is computed by the complex

$$Cone\,(Bar(A_1) \oplus Bar(A_2) \to Bar(A_{12}))\,[-1].$$

For a locally projective module coherent sheaf M of A-modules we can find integers n, m and idempotents $e^1 \in Mat_n(A_1)$, $e^2 \in Mat_m(A_2)$ together with isomorphisms $\Gamma(U_1, M) \cong A_1^{\oplus n} e^1$, $\Gamma(U_2, M) \cong A_2^{\oplus m} e^2$ and matrices $a \in Mat_{nm}(A_{12})$, $b \in Mat_{mn}(A_{21})$ which induce the inverse isomorphisms between $A_{12}^{\oplus n} e^1$ and $A_{12}^{\oplus m} e^2$ coming from the identification of both with $\Gamma(U_1 \cap U_2, M)$. In this setting $ch(M)$ is represented by the cocycle $(\sum e^1_{ii}, \sum e^2_{jj}, \sum a_{ij} \otimes b_{ji}) \in A_1 \oplus A_2 \oplus A_{12}^{\otimes 2}$.

(3) Let $U \subset X$ be an affine open subset such that its complement Z is also affine. Let $A_1 = \Gamma(U, A)$, and let A_1, A_{12} be the algebras of sections of A on the formal neighborhood and the punctured formal neighborhood of Z respectively. Then the statements of part (b) continue to hold *mutatis mutandis*.

Definition 4.2. In the above setting, for an object $\mathcal{M} \in A - mod$ with *proper* support we have the Chern character $ch^c(\mathcal{M}) \in HH_0^c(X; A)$ as follows. Fix a proper subscheme $Z \subset X$ containing support of cohomology of $\mathcal{M}$. Then [17, §5.7] yields $ch_Z(\mathcal{M}) \in R\Gamma_Z(X, R\underline{Hom}_{A \otimes A^{op}}(A, A))$. We define $ch^c(\mathcal{M})$ to be the image of $ch_Z(\mathcal{M})$ under the canonical map $R\Gamma_Z(X, F) \to R\Gamma_c(X, F)$, $F = R\underline{Hom}_{A \otimes A^{op}}(A, A)$.

It is immediate to check that $ch^c(\mathcal{M})$ is independent of the auxiliary choice of Z.

In particular, we have Chern character map $\bar{ch} : K^0(\overline{Sm}) \to HH_0(\overline{Sm})$ and $ch^c : K^0(Adm) \to HH_0^c(Sm)$, where $Adm \subset Sm$ is the subcategory of admissible modules.

Let $\mathcal{K}^c \subset \mathcal{K}$, $\mathcal{K}^{nc} \subset \mathcal{K}$ be the subspace of measures supported on compact (respectively, noncompact) elements.

Conjecture 4.3. a) We have a canonical isomorphism:
$\mathcal{K}^c_G \cong Im(HH_0^c(Sm) \to HH_0(\overline{Sm}))$.

b) For $\rho \in Adm$ and $g \in G$ we have $WO_g(\bar{ch}(\rho)) = \chi_\rho(g)$ if g is compact regular semisimple. (Here we use that $\bar{ch}(\rho)$ is the image of $ch^c(\rho)$ thus it belongs to $Im(HH_0^c(Sm) \to HH_0(\overline{Sm}))$ which we identify with $\mathcal{K}^c_G$ by a).

The proof of Conjecture for $G = SL(2)$ is presented in the next section; we plan to present the proof in the general case in a later publication.

5. $SL(2)$ calculations

Theorem 5.1. Conjecture 4.3 holds for $G = SL(2)$.

The rest of the section is devoted to the proof of the Theorem. From now on set $G = SL(2)$.

5.1. **Explicit complexes for Hochschild homology.** Applying the general construction of section 3.3 with $\sigma = \{x\}$ where x is the vertex with stabilizer $G(O)$ we arrive at a sheaf of algebras $\mathcal{A}$ on $\overline{\mathfrak{Z}}$ such that a direct summand in $\overline{Sm}$ is canonically identified with the category of coherent sheaves of $\mathcal{A}$-modules. It is easily seen to coincide with the sheaf $\overline{\mathcal{H}}_K$ introduced in Example 3.2. We keep notations of that Example.

We also let $\hat{\mathcal{H}}^+_K$ denote its sections on the formal neighborhood $\widehat{\partial \mathfrak{Z}}$ of $\partial\mathfrak{Z} = \overline{\mathfrak{Z}} \setminus \mathfrak{Z}$ and $\hat{\mathcal{H}}_K$ the sections on the punctured formal neighborhood of $\partial\mathfrak{Z}$. We set $\hat{\mathcal{H}}^+ =$

$\bigcup_K \hat{\mathcal{H}}_K^+$ and $\hat{\mathcal{H}} = \bigcup_K \hat{\mathcal{H}}_K$. We also let $X = (G/U \times G/U^-)/T$ be the set of rank one matrices in $Mat_2(F)$ and $X_+ = X \cap Mat_2(O)$.

Then we have $\hat{\mathcal{H}}_K^+ = End_{T^+}(\mathcal{S}(G/U)_+^K) \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$, $\hat{\mathcal{H}}_K = \hat{\mathcal{H}}_+^K \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t))$. We have $\hat{\mathcal{H}}_K = End_{\hat{\mathcal{H}}(T)}(\mathcal{S}_b^K(G/U))$, where $\mathcal{S}_b(G/U)^K \cong \mathcal{S}(G/U) \otimes_{\mathbb{C}[t,t^{-1}]} \mathbb{C}((t))$ is the space of functions on G/U with bounded support.

Also notice that $\hat{\mathcal{H}}$ (though not $\hat{\mathcal{H}}_+$) carries a $G \times G$ action.

Lemma 5.2. a) $HH_*(\overline{Sm})$ is computed by the complex

$$Bar(\overline{Sm}) := cone[Bar(\mathcal{H}) \oplus Bar(\hat{\mathcal{H}}^+) \to Bar(\hat{\mathcal{H}})][-1]. \tag{12}$$

b) $HH_*^c(Sm)$ is computed by the complex

$$Bar^c(Sm) := cone[Bar(\mathcal{H}) \to Bar(\hat{\mathcal{H}})][-1]. \tag{13}$$

c) $HH_*^c(Sm)$ is canonically isomorphic to derived G coinvariants in the two-term complex

$$\mathcal{H} \to \hat{\mathcal{H}} \tag{14}$$

(placed in degrees 0,1).

Proof. For a complex $\mathcal{F}$ of coherent sheaves on $\overline{\mathfrak{Z}}$ the complexes

$$cone[\Gamma(\mathfrak{Z}, \mathcal{F}) \oplus \hat{\Gamma}^+(\mathcal{F}) \to \hat{\Gamma}(\mathcal{F})][-1],$$

$$cone[\Gamma(\mathfrak{Z}, \mathcal{F}) \to \hat{\Gamma}(\mathcal{F})][-1]$$

compute, respectively, $R\Gamma(\mathcal{F})$ and $R\Gamma_c(\mathfrak{Z}, \mathcal{F})$, where $\hat{\Gamma}^+(\mathcal{F})$ and $\hat{\Gamma}(\mathcal{F})$ denote, respectively, sections of $\mathcal{F}$ on the formal neighborhood and on the punctured formal neighborhood of $\partial\mathfrak{Z}$. Applying this to a complex representing $R\underline{Hom}_{\overline{\mathcal{H}}_K \otimes \overline{\mathcal{H}}_K^{op}}(\overline{\mathcal{H}}_K, \overline{\mathcal{H}}_K)$ and observing that we have canonical quasi-isomorphisms $\hat{\Gamma}^+(\mathcal{F}) \to Bar(\hat{\mathcal{H}}^+)$, $\hat{\Gamma}(\mathcal{F}) \to Bar(\hat{\mathcal{H}})$ we get statements (a,b).

Statement (c) follows from isomorphisms $HH_*(M) \cong H_*(G, M)$, $HH_*^{\hat{\mathcal{H}}}(N) \cong HH_*^{\mathcal{H}}(N)$ for an $\mathcal{H}$-bimodule M and an $\hat{\mathcal{H}}$-bimodule N. □

5.2. **Calculation of HH_0.** In this subsection we prove part (a) of the Theorem.

Proposition 5.3. a) We have a short exact sequence:

$$0 \to \mathcal{K}_G \to HH_0^c(Sm) \to \widehat{\mathcal{S}(T)}/\mathcal{S}(T) \to 0. \tag{15}$$

b) We have a natural isomorphism $\mathcal{K}_G^c \cong Im(HH_0^c(Sm) \to HH_0(\overline{Sm}))$.

We start the proof with the following

Lemma 5.4. G acts trivially on the cokernel of the map $H \to \mathcal{S}(Y_\Delta)$.

Proof follows from Lemma 2.25. □

Corollary 5.5. We have:
$H_1(G, CoKer(H \to \mathcal{S}(Y_\Delta))) = 0$.
$H_1(G, CoKer(H \to \widehat{\mathcal{S}(Y_\Delta)})) = \widehat{\mathcal{S}(T)}/\mathcal{S}(T)$,
where $\widehat{\mathcal{S}(T)}$ is the Tate completion of $\mathcal{S}(T)$ (functions on the torus).

Proof of Proposition 5.3. By Lemma 5.2 (c), $HH_*^c(Sm)$ is identified with the derived G-coinvariants in the complex $\mathcal{H} \to \widehat{\mathcal{S}(Y)}$ which is clearly isomorphic to (14). We have a short exact sequence

$$0 \to \mathcal{S}(Y_0) \to \mathcal{S}(Y) \to \mathcal{S}(Y_\Delta) \to 0,$$

where $Y_0 = Y \setminus Y_\Delta$ is the open set of non-colinear pairs of vectors. Thus the action of G on Y_0 is almost free, i.e. the stabilizer of every point in $\{\pm 1\}$, and the orbits of G on Y_0 are indexed by the finite set $F^\times/(F^\times)^2$. This shows that $H_1(G, \mathcal{S}(Y_0)) = 0$ and $H_0(G, \mathcal{S}(Y_0))$ is finite dimensional. Denoting as above $\widehat{\mathcal{S}(Y)} = \mathcal{S}(Y) \otimes_{\mathbb{C}[\varpi,\varpi^{-1}]} \mathbb{C}((\varpi))$ and similarly for $\widehat{\mathcal{S}(Y_0)}$ we see $H_i(G, \mathcal{S}(Y_0)) = 0$ for all i. Thus derived G coinvariants of the complex (14) are canonically isomorphic to derived G-coinvariants of the quotient complex $\mathfrak{C} = \mathcal{H} \to \widehat{\mathcal{S}(Y_\Delta)}$. Notice that $H^0(\mathfrak{C}) = \mathcal{K}$, $H^1(\mathfrak{C}) = CoKer(\mathcal{H} \to \widehat{\mathcal{S}(Y_\Delta)}))$ and $\mathfrak{C}$ is quasi-isomorphic to the complex with zero differential since the category of G-modules has homological dimension one. Thus part(a) of the Proposition follows from Corollary 5.5.

Lemma 5.2 shows that $Bar^c(Sm)$ is canonically quasiisomorphic to $\mathbb{C} \overset{\mathrm{L}}{\otimes}_G \mathfrak{C}$. One the other hand, we have a short exact sequence of complexes

$$0 \to Bar^c(Sm) \to Bar(\overline{Sm}) \to Bar(\widehat{H(T^+)}) \to 0,$$

which yields a long exact sequence on cohomology:

$$\cdots \to HH_{i+1}(\widehat{H(T^+)}) \to HH_i^c(Sm) \to HH_i(\overline{Sm}) \to \cdots$$

We are interested in $i = 0$. We have $HH_1(\widehat{H(T^+)}) \cong \widehat{H(T^+)}$. To finish the proof we need another

Lemma 5.6. a) Let us identify

$$HH_1(\widehat{H(T)}) = H_1(G, \widehat{\mathcal{S}(Y_\Delta)}) = \widehat{\mathcal{S}(T)}.$$

Then the image of the natural map $HH_1(\widehat{H(T^+)}) \to HH_1(\widehat{H(T)})$ coincides with $\varpi\widehat{\mathcal{S}(T^+)}$, i.e. with the space of functions supported on ϖT^+, where $\varpi \in F$ is a uniformizer.

b) In view of (15), part (a) yields a map $\varpi\widehat{\mathcal{S}(T^+)} \cap \mathcal{S}(T) \to \mathcal{K}_G \subset HH_0^c(Sm)$. This map induces an isomorphism onto $(\mathcal{K}_{nc})_G$ where $\mathcal{K}_{nc} \subset \mathcal{K}$ is the space of measures supported on noncompact elements.

Proof of the Lemma. By Hochschild-Kostant-Rosenberg, we can identify Hochschild homology with forms. It is easy to see that elements in $\mathcal{S}(T^0) \subset \widehat{\mathcal{S}(T)}$ (where T^0 is the maximal compact subgroup in the torus T) locally on each component of Bernstein center are proportional to $\frac{dz}{z}$ for a local coordinate coming from a global coordinate on $\mathbb{C}^\times$. On the other hand, the image of $HH_1(\widehat{H(T^+)})$ equals $\mathbb{C}[[z]]dz$, this proves (a).

To prove (b) observe that the map in question can be described as follows.

We have a short exact sequence of G-modules

$$0 \to \mathcal{K} \to \mathcal{H} \to I \to 0,$$

where $I = Im(\mathcal{H} \to \mathcal{S}(Y_\Delta))$ which yields the Bockstein homomorphism $\phi : H_1(G, I) \to H_0(G, \mathcal{K})$. By Corollary 5.5 we have $H_1(G, I) \cong H_1(G, \mathcal{S}(Y_\Delta)) = H_1(G, \mu_{\mathbb{P}^1}) \otimes \mathcal{S}(T) = \mathcal{S}(T)$. In view of (a) it suffices to check that $\phi : \mathcal{S}(\varpi T^+) \widetilde{\longrightarrow} H_0(G, \mathcal{K}^{nc})$.

We have a direct sum decomposition $\mathcal{H} = \mathcal{H}^c \oplus \mathcal{H}^{nc}$ compatible with the decomposition $\mathcal{S}(Y_\Delta) = \mathcal{S}(Y_\Delta^c) \oplus \mathcal{S}(Y_\Delta^{nc})$, where $Y_\Delta^c = \mathbb{P}^1 \times O^\times$ and $Y_\Delta^{nc} = Y_\Delta \setminus Y_\Delta^{nc}$. Let us further decompose $Y_\Delta^{nc} = Y_\Delta^+ \cup Y_\Delta^-$, where $Y_\Delta^+ = \mathbb{P}^1 \times (\varpi O \setminus \{0\})$, $Y_\Delta^- = \mathbb{P}^1 \times (F \setminus O)$.

The image of ϕ clearly coincides with the image of $H_1(G, \mathcal{S}(Y_\Delta^+))$ under the map coming from the class in $Ext^1(\mathcal{S}(Y_\Delta^+), \mathcal{K})$ induced from the above short exact sequence. It is clear that it is contained in $H_0(G, \mathcal{K}^{nc})$. It coincides with $H_0(G, \mathcal{K}^{nc})$ for the following reason. Consider a regular orbit $O = G/T$, then the image $\mathcal{K}_O$ of $\mathcal{K}$ in $\mathcal{S}(O)$ is the kernel of the pushforward map $\mathcal{S}(G/T) \to \mu_{G/B} \oplus \mu_{G/B^-}$. Then

$(\mathcal{K}_O)_G$ is one dimensional and the Bockstein map $H_1(G,\mu_{G/B}) \to H_0(G,\mathcal{K}_O)$ is nonzero. $\square$

The Lemma clearly implies statement (b) of Proposition. $\square$

5.3. **Cocycles for Chern character and Euler characteristic.** Let now M be a locally projective object of $\overline{Sm}$.

Thus there exist idempotents $e \in Mat_n(\mathcal{H})$, $e_+ \in Mat_m(\hat{\mathcal{H}}_+)$ and $a, b \in Mat_{n,m}(\hat{\mathcal{H}})$, such that $ab = e$, $ba = e_+$ with isomorphisms $M|_{\mathfrak{Z}} \cong \mathcal{H}^n e$, $M_{\widehat{\partial \mathfrak{Z}}} \cong \hat{\mathcal{H}}_+^m e_+$, so that a, b induces the two inverse isomorphisms between the restrictions of $\mathcal{H}^n e$ and $\hat{\mathcal{H}}_+ e_+$ to the punctured formal neighborhood of $\partial\mathfrak{Z}$ coming from the identification with the restriction of M.

Lemma 5.7. In the above notation, $\bar{ch}(M)$ is represented by the cocycle for $Bar(\overline{Sm})$ $c = (c_{\mathcal{H}}, c_+, \hat{c})$, where
$c = \sum\limits_i e_{ii} \in \mathcal{H}$, $c_+ = \sum_j (e_+)_{jj} \in \hat{\mathcal{H}}_+$, $\hat{c} = \sum\limits_{i,j} a_{ij} \otimes b_{ji} \in \hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$.

Proof. This is a special case of Example 4.1(c). $\square$

To state the next result we need the following notation. Fix $g \in G(O)$ normalizing K thus g acts by conjugation on the sheaf of algebras $\mathcal{A}$ and on global sections of an object in $\mathcal{A} - mod$.

We introduce a zero-cochain τ_g for the dual complex $Bar(\overline{Sm})^*$, $\tau_g = (\tau_{\mathcal{H}}(g), \tau_+(g), \hat{\tau}(g))$, where $\tau_{\mathcal{H}}(g) = WO_g \in \mathcal{H}^*$ (the weighted orbital integral), $\tau_+(g) = 0 \in \hat{\mathcal{H}}_+^*$ and $\hat{\tau}(g) : \alpha \otimes \beta \mapsto Tr([\alpha, \Pi]\beta \circ g, \widehat{\mathcal{S}(X)})$, where $\alpha, \beta \in \hat{\mathcal{H}}$ act on $\widehat{\mathcal{S}(X)}$ by left multiplication, Π is the operator of multiplication by the characteristic function of X_+ acting on $\widehat{\mathcal{S}(X)}$ and g is acting on the right.

Proposition 5.8. a) The cochain τ_g is a cocycle. It satisfies:

$$Tr(g, RHom(\overline{\mathcal{H}}', \mathcal{M})) = \langle \tau_g, \bar{ch}(M) \rangle,$$

for $M \in \overline{Sm}$, where $\overline{\mathcal{H}}'$ is the intertwining bimodule introduced at the end of section 3.

b) For $\phi \in \mathcal{K}_G^c = Im(HH_0^c(Sm) \to HH_0(\overline{Sm}))$ we have:

$$\langle \phi, \tau_g \rangle = WO_g(\phi).$$

c) The image of τ_g in $HH_0^c(Sm)$ is independent of the auxiliary choices in the definition of τ_g.

Proof. Part (c) clearly follows from (b).

We deduce (b) from Lemma 3.14. Let us present the map $\mathcal{K}_G^c \to HH_0(Bar^c(Sm))$ by an explicit cocycle. Fix $f \in \mathcal{K} \cap \mathcal{H}_K$. Consider the decomposition of $\mathcal{S}(G/U)^K$ as a direct sum of spaces of functions on preimages of K orbits in G/B. Then the action of f on this space has zero block diagonal components with respect to this decomposition. Also, the image of f in $\hat{\mathcal{H}} = \widehat{\mathcal{S}(Y)}$ is contained in $\mathcal{S}(Y_0)$, while $\widehat{\mathcal{S}(Y_0)}_G = 0$ by the proof of Proposition 5.3. Thus $f = \sum h_i -^{g_i} h_i$, $g_i \in G$, $h_i \in \widehat{\mathcal{S}(Y_0)}$. It is easy to check that the image of $\bar{f}$ in $HH_0(Bar^c(Sm))$ is represented by $(f, \sum g_i \delta_K \otimes h_i g^{-1}) \in \mathcal{H} \oplus \hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$.

It remains to check that for h_i as above we have

$$Tr([g_i, \Pi] h_i g_i^{-1} \circ g, \widehat{\mathcal{S}(X)}) = 0. \tag{16}$$

We claim that the following stronger statement holds: the operator $[g_i, \Pi] h_i g_i^{-1}$ acting on $\mathcal{S}(G/U)$ has zero diagonal blocks with respect to the above block decomposition. To see this pick an open compact subgroup K' such that $h_i \in \mathcal{H}_{g_i^{-1}K'}$ for all i. Then g_i^{-1} sends a summand of the decomposition corresponding to a

K'-orbit $U \subset G/B$ to a summand corresponding to the ${}^{g_i^{-1}}K'$ orbits $g_i^{-1}(U)$, the endomorphism h_i acts by an operator with zero diagonal blocks with respect to the decomposition corresponding to ${}^{g_i^{-1}}K'$ orbits, while $[g_i, \Pi]$ sends the summand corresponding to an orbit $g_i^{-1}(U)$ to the summand corresponding to the orbit U (this is clear since Π preserves all the direct sum decompositions above). Equality (16) and hence part (b) of the Proposition follows.

The proof of part (a) occupies the next subsection.

5.4. **Tate cocyle and weighted orbital integral.**

5.4.1. *The Tate cocycle.* A general reference for the following material is [5] or [4], especially [4], §4.2.13, p 142; §7.13.18, p 344, developing the theme started in [21].

Let V be a Tate vector space, then the Lie algebra $End(V)$ of continuous endomorphism of V has a canonical central extension, which we will denote by $\widetilde{End}(V)$.

Let $End^b(V) \subset End(V)$ denote the subspace of endomorphisms with bounded image, and $End^d(V) \subset End(V)$ be the subspace of endomorphisms having open kernel (here b stands for bounded, and d for discrete). Then $End^b(V)$, $End^d(V)$ are two-sided ideals in the associative algebra $End(V)$, and hence ideals in the Lie algebra $End(V)$. Further, the trace functional is defined on the intersection $End^{bd} = End^b(V) \cap End^d(V)$, $tr : End^{bd} \to k$; it satisfies

$$tr([E, E^{bd}]) = 0; \qquad (17)$$
$$tr([E^b, E^d]) = 0$$

for $E^b \in End^b(V)$, $E^d \in End^d(V)$, $E^{bd} \in End^{bd}(V)$, $E \in End(V)$.

The central extension $\widetilde{End}(V) \to End(V)$ is specified by the requirement that it is trivialized on the ideals $End^b(V)$, $End^d(V)$, i.e. the embeddings $End^b(V) \hookrightarrow End(V)$, $End^d(V) \hookrightarrow End(V)$ are lifted to (fixed) homomorphisms $s_b : End^b(V) \hookrightarrow \widetilde{End}(V)$, $s_d : End^d(V) \hookrightarrow \widetilde{End}(V)$, where s_b, s_d intertwine the adjoint action of $End(V)$; and for $E^{bd} \in End^{bd}(V)$ we have

$$s_b(E^{bd}) - s_d(E^{bd}) = tr(E^{bd}) \cdot c, \qquad (18)$$

where $c \in \widetilde{End}(V)$ is the generator of the kernel of the projection $\widetilde{End}(V) \to End(V)$.

Suppose now that a decomposition of V

$$V = V^+ \oplus V^- \qquad (19)$$

into a sum of a bounded open and a discrete subspaces is given, and let Π denote the projection to the bounded open V^+ along the discrete V^-. Then for any $E \in End(V)$ in the right-hand side of

$$E = E \cdot \Pi + E \cdot (Id - \Pi)$$

the first summand lies in $End^b(V)$, and the second one in $End^d(V)$. Thus we get a splitting $s = s_\Pi : End(V) \to \widetilde{End}(V)$,

$$s_\Pi(E) = s_b(E \cdot \Pi) + s_d(E \cdot (Id - \Pi)). \qquad (20)$$

It is also easy to see that for E preserving V_- (respectively, V_-) the element $s_\Pi(E)$ is independent of the choice of the complement V_- (respectively, V_+). Thus we get a canonical splitting s_{V^+} (respectively, s_{V^-}) of the central extension on the subalgebras $End_{V_+}(V)$, $End_{V^-}(V)$ consisting of endomorphisms preserving V_+ (respectively, V_-).

Denote by $C(E_1, E_2)$ the corresponding 2-cocycle of $End(V)$, i.e. a bi-linear functional, such that

$$[s(E_1), s(E_2)] = s([E_1, E_2]) + C(E_1, E_2) \cdot c.$$

Then we have

$$C(E_1, E_2) = Tr(E_1 \circ \Pi \circ E_2 \circ (Id - \Pi) - E_2 \circ \Pi \circ E_1 \circ (Id - \Pi)). \tag{21}$$

Finally, suppose that another discrete cobounded space $W \subset V$ is fixed. The splitting s_W of the central extension on $End_W(V)$ yields a linear functional $\sigma_W = s_W - s_\Pi|_{End_W(V)}$ on $End_W(V)$.

Example 5.9. We have

$$\sigma_W(Id) = \dim(V^+ \cap W) - \operatorname{co}\dim_V(V^+ + W). \tag{22}$$

Also, suppose that F is an automorphism of V such that either $F(V^+) \subset V^+$, or $V^+ \subset F(V^+)$; set $d_{V^+}(F) = -\dim(V^+/F(V^+))$ in the former and
$d_{V_+}(F) = \dim(F(V^+)/V^+)$ in the latter case. Then

$$C(F, F^{-1}) = d_{V^+}(F). \tag{23}$$

Both equalities follow directly from the definitions.

Consider now the complex

$$Cone\,(Bar(End_{V^+}(V)) \oplus Bar(End_W(V)) \to Bar(End(V)))\,[-1] \tag{24}$$

and define a zero-cochain for the dual complex $\epsilon = (0, \sigma_W, C)$.

It is easy to check that the zero-cochain ϵ is in fact a cocycle whose cohomology class does not depend on the choice of V^- for a fixed V^+.

5.4.2. *Weighted orbital integral and Tate cocycle.* We now apply this in the following example. Consider the Tate space

$$V = \hat{\mathcal{H}} \cong \widehat{\mathcal{S}(Y)} \cong \widehat{\mathcal{S}(X)} \tag{25}$$

where X was defined in the second paragraph of §5.1, the first isomorphism was discussed above and the second one is induced by the inverse intertwining operator. Let W be the image of $\mathcal{H}$ in the space (25) and $\widehat{\mathcal{S}(X)^+}$ be the space of functions supported on X_+ while $\widehat{\mathcal{S}(X)^-}$ is the space of functions supported on the complement of X_+.

The group K_0 acts on all these spaces. Fix a representation ρ of K_0 and set $V_\rho = Hom_{K_0}(\rho, V)$, $W_\rho = Hom_{K_0}(\rho, \mathcal{H})$ and $V_\rho^\pm = Hom_{K_0}(\rho, \widehat{\mathcal{S}(X)}^\pm)$.

The complex (12) maps naturally to the complex (24) constructed from $V = V_\rho$ $V^\pm = V_\rho^\pm$ and $W = W_\rho$, let a_ρ denote that map.

Lemma 5.10. We have $a_\rho^*(\epsilon) = \int_{K_0} \tau_g Tr(g, \rho) dg$.

Proof. Equality of components in $(\hat{\mathcal{H}}^{\otimes 2})^*$ follows by inspection.

It remains to check equality of components in $\mathcal{H}^*$. Recall (see e.g. [3, §I.11]) that $WO_g(f) = \phi(0)$, where ϕ is a linear function on dominant weights such that for large λ we have

$$\phi(\lambda) = Tr(f \circ \Pi_\lambda \circ g, \mathcal{H}).$$

Here f acts on $\mathcal{H}$ by convolution on the left, g acts by right translation and Π_λ is the characteristic function of $G_{\leq \lambda}$ which is the union of two sided $K_0 = G(O)$ cosets corresponding to $\mu \leq \lambda$.

It follows that for a locally constant function ψ on K_0 supported on regular semisimple elements there exists an affine linear function $\phi_\psi(\lambda)$, such that

$$\phi_\psi(\lambda) = Tr(f \circ \Pi_\lambda \circ \psi, \mathcal{H}) \qquad \text{for} \quad \lambda \gg 0,$$

$$\int_{K_0} WO_g(f)\psi(g)dg = \phi_\psi(0),$$

where ψ acts by convolution on the right.

Comparing this with the definition of Tate cocycle ϵ and of the intertwining bimodule $\overline{\mathcal{H}}'$ (see Example 3.13) and using Lemma 3.14, we see that for ψ as above:

$$\int_{K_0} WO_g(f)\psi(g)dg = \sigma_{\mathcal{H}}(f\otimes\psi),$$

where $f\otimes\psi$ acts on the Tate space (25) via its natural $\mathcal{H}$-bimodule structure. Both sides of the last equality are continuous in ψ with respect to the L^1 norm: this is clear for the right hand side and it follows from Theorem 2.14 for the left hand side. Thus validity of the equality for ψ supported on regular semisimple elements implies its validity for all ψ. □

5.4.3. *Sheaves of algebras on curves.* We now apply the above construction in the following setting. Let C be a smooth[4] complete curve with a finite collection of points $\mathbf{x}=\{x_1,\dots,x_n\}$. Let $\hat{C}_{\mathbf{x}}$, $\hat{C}^0_{\mathbf{x}}$ be the formal neighborhood and the punctured formal neighborhood of $\mathbf{x}$ respectively. Let $\mathcal{V}$ be a torsion free coherent sheaf on C, and set $V=\Gamma(\hat{C}^0_{\mathbf{x}},\mathcal{V})$, $W=\Gamma(C\setminus\mathbf{x},\mathcal{V})$, $V^+=\Gamma(\hat{C}_{\mathbf{x}},\mathcal{V})$. Fixing formal coordinates z_i at x_i and trivializations of $\mathcal{V}$ on the formal neighborhood of x_i we get an isomorphism $V\cong\mathbb{C}((z))^N$ sending V_+ to $\mathbb{C}[[z]]^N$, thus we get a splitting $V=V^+\oplus V^-$ where $V^-\cong z^{-1}\mathbb{C}[z^{-1}]^N$.

Let $\mathcal{A}$ be a torsion free coherent sheaf of algebras on C with a right action on V; we assume also that $\mathcal{A}$ has finite homological dimension (i.e. that the algebra of sections of $\mathcal{A}$ on an affine open set has this property).

Let $A=\Gamma(C\setminus\mathbf{x},\mathcal{A})$, $\hat{A}^+=\Gamma(\hat{C}_{\mathbf{x}},\mathcal{A})$ and $\hat{A}=\Gamma(\hat{C}^0_{\mathbf{x}},\mathcal{A})$. Then the complex

$$Cone\left(Bar(A)\oplus Bar(\hat{A}^+)\to Bar(\hat{A})\right)[-1] \tag{26}$$

computes Hochschild homology of the category $\mathcal{A}-mod$. On the other hand, it maps naturally to the complex (24), let α denote this map.

Lemma 5.11. Suppose that $\mathcal{V}$ is a locally projective sheaf of right $\mathcal{A}$-modules. Then for a coherent sheaf $\mathcal{M}$ of (left) $\mathcal{A}$-modules we have:

$$\langle ch(\mathcal{M}),\alpha^*(\epsilon)\rangle=\chi(\mathcal{V}\otimes_{\mathcal{A}}\mathcal{M}), \tag{27}$$

where χ denotes Euler characteristic.

Proof. Both sides of (27) do not change if we replace $\mathcal{A}$ by $\underline{End}_{\mathcal{O}_C}(\mathcal{V})^{op}$ and $\mathcal{M}$ by $\underline{End}_{\mathcal{O}_C}(\mathcal{V})^{op}\otimes_{\mathcal{A}}\mathcal{M}$. Thus we can assume without loss of generality that $\mathcal{A}=\underline{End}_{\mathcal{O}_C}(\mathcal{V})^{op}$. In that case we have a canonical equivalence of categories $Coh(C)\cong\mathcal{A}-mod$, $\mathcal{F}\mapsto\mathcal{F}\otimes_{\mathcal{O}_C}\mathcal{V}^*$.

Furthermore, the operator $[\mathcal{F}]\mapsto[\mathcal{F}\otimes\mathcal{V}]$ induces an automorphism of the rational Grothendieck group $K^0(Coh(C))\otimes\mathbb{Q}$, thus it suffices to prove (27) for $\mathcal{A}=\underline{End}_{\mathcal{O}_C}(\mathcal{V})^{op}$, $\mathcal{M}=\mathcal{F}\otimes\mathcal{V}^*$, where $\mathcal{F}=\mathcal{V}\otimes\mathcal{F}'$, thus $\mathcal{F}=\mathcal{F}'\otimes_{\mathcal{O}_C}\mathcal{A}$. Now, both sides of (27) do not change if we replace $\mathcal{A}$ by $\mathcal{O}_C$ and $\mathcal{M}$ by $\mathcal{F}'$; the locally projective module $\mathcal{V}$ is replaced by the corresponding coherent sheaf $\mathcal{V}|_{\mathcal{O}_C}$. Thus we have reduced to the case $\mathcal{A}=\mathcal{O}_C$.

It is easy to see that both sides of (27) are additive on short exact sequences as a function of both $\mathcal{V}$ and $\mathcal{F}$. A locally free sheaf on a curve admits a filtration whose associated graded is a sum of line bundles; also, since C is smooth the Grothendieck group $K^0(Coh(C))$ is generated by line bundles. Thus we can assume without loss of generality that both $\mathcal{M}$ and $\mathcal{V}$ are line bundles.

If $\mathcal{M}$ is a trivial line bundle then $ch(\mathcal{M})$ is represented by the cocycle $(1,1,1\otimes 1)$, so (27) follows from (22). Notice also that both sides of (27) for a fixed $\mathcal{V}$ depend only on the degree of the line bundle $\mathcal{M}$ (here we assume without loss of generality that the curve C is irreducible): this is clear for the right hand side, while for the

[4]This assumption is likely unnecessary but it allows one to simplify the statements and the proofs.

left hand side it follows from the fact that a regular function on an abelian variety is constant. Thus it suffices to consider the case when $\mathcal{M} = \mathcal{O}(nx)$, $x \in \mathbf{x}$. In this case $ch(\mathcal{M})$ is represented by the cocycle $(1, 1, f \otimes f^{-1})$ where f is a function on the formal punctured neighborhood of $\mathbf{x}$ having order n at x and order 0 at other points. In this case (27) follows from (22) and (23). □

Corollary 5.12. a) Suppose that $\mathcal{V} \cong \mathbf{e}\mathcal{A}$ is a direct summand in a free rank one module for an idempotent $\mathbf{e} \in \Gamma(C, \mathcal{A})$. Then for a coherent sheaf $\mathcal{M}$ of $\mathcal{A}$-modules we have:

$$\langle ch(\mathcal{M}), \alpha^*(\epsilon)\rangle = \chi(\mathbf{e}\mathcal{M}),$$

where χ denotes the Euler characteristic.

b) Suppose that $\mathcal{M}$ has finite support contained in $C \setminus \mathbf{x}$. Then the equality in (a) holds under a weaker assumption that $\mathcal{V}|_{C\setminus\mathbf{x}} \cong \mathcal{A}\mathbf{e}|_{C\setminus\mathbf{x}}$.

5.4.4. *Proof of Proposition 5.8(a).* It suffices to check that we get an equality upon averaging against a character of any representation ψ of $K_0 = G(O)$.

In view of Corollary 5.12(b) it suffices to check that the cochain $\int_{K_0} \tau_g Tr(g, \psi) dg$ is obtained as in Corollary 5.12 for an appropriate choice of trivialization for $\mathcal{V}$ on the formal neighborhood of $\mathbf{x}$, where $C \subset \overline{\mathfrak{Z}}$ is the union of one dimensional components containing representations with nonzero K-invariant vectors, $\mathbf{x} = (\overline{\mathfrak{Z}} \setminus \mathfrak{Z}) \cap \mathbf{C}$, $\mathcal{A} = \overline{\mathcal{H}}$, z_i are the natural local coordinates and $\mathcal{V} = Hom_{K_0}(\psi, \overline{\mathcal{H}}')$. Applying Lemma 5.10 we reduce to showing that $a_\psi^*(\epsilon)$ has the required form.

It follows from Lemma 3.14 that the space V_+ of sections of $\overline{\mathcal{H}}'$ on the formal neighborhood of $\overline{\mathfrak{Z}} \setminus \mathfrak{Z}$ is identified with $\widehat{\mathcal{S}(X_+)}^{K\times K}$ (notation introduced in the second paragraph of §(5.1)); moreover, we can choose a trivialization of the sheaf $\overline{\mathcal{H}}$ on the formal neighborhood of $\overline{\mathfrak{Z}} \setminus \mathfrak{Z}$ so that constant sections correspond to functions supported on the set $X(O) = X_+ \setminus (tX_+)$. Then the space V_- is identified with $\mathcal{S}(X \setminus X_+)^{K\times K}$. The desired equality now follows by inspecting the definitions. □

5.5. **Proof of part (b) of Theorem 5.1.** For $g \in K_0$ the formula follows from Proposition 5.8. The general case of a compact element follows by conjugating with an element of $GL(2, F)$. □

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