# BOUNDS FOR $GL_3$ $L$-FUNCTIONS IN DEPTH ASPECT

QINGFENG SUN AND RUI ZHAO

Abstract. Let $f$ be a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and $\chi$ a primitive Dirichlet character of prime power conductor $\mathfrak{q} = p^\kappa$ with $p$ prime and $\kappa \geq 10$. We prove a subconvexity bound

$$L\left(\frac{1}{2}, \pi \otimes \chi\right) \ll_{p,\pi,\varepsilon} \mathfrak{q}^{3/4-3/40+\varepsilon}$$

for any $\varepsilon > 0$, where the dependence of the implied constant on $p$ is explicit and polynomial. We obtain this result by applying the circle method of Kloosterman's version, summation formulas of Poisson and Voronoi's type and a conductor lowering mechanism introduced by Munshi [14]. The main new technical estimates are the essentially square root bounds for some twisted multi-dimensional character sums, which are proved by an elementary method.

## Contents



## 1. Introduction

Let $L(s, f)$ be an $L$-function with the analytic conductor $\mathfrak{q}(s, f)$. By the functional equation and the Phragmen-Lindelöf convexity principle, we have the convexity bound $L(s, f) \ll \mathfrak{q}(s, f)^{1/4+\varepsilon}$. It is an fascinating problem to break the convexity barrier. In the $t$-aspect, one has the classical result for the Riemann zeta function $\zeta(1/2 + it) \ll_\varepsilon (1 + |t|)^{1/6+\varepsilon}$ due to Weyl [19]. For $L$-functions on $GL_2$, results of the same strength

$$L\left(\frac{1}{2} + it, f\right) \ll_{f,\varepsilon} (1 + |t|)^{1/3+\varepsilon} \tag{1.1}$$

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were proved by Good [4], Jutila [7] and Meurman [10], where $f$ is a fixed holomorphic cusp form or a Maass cusp form. For $GL_3$ $L$-functions, Munshi [15] proved that

$$L\left(\frac{1}{2}+it,\pi\right) \ll_{\pi,\varepsilon} (1+|t|)^{3/4-1/16+\varepsilon}, \tag{1.2}$$

where $\pi$ is a fixed $GL_3$ Hecke-Maass cusp form (this bound was first proved by Li [9] for $\pi$ self-dual). On the other hand, in the conductor aspect, we have the Burgess' bounds $L(1/2,\chi) \ll_\varepsilon q^{3/16+\varepsilon}$ and $L(1/2, f\otimes\chi) \ll_{f,\varepsilon} q^{3/8+\varepsilon}$ for a primitive character $\chi$ of conductor $q$, where $f$ is a fixed $GL_2$ cusp form. Interestingly, for $\chi$ quadratic, Conrey and Iwaniec [3] proved the exponent $1/3$, i.e., the quantitative analogue of (1.1). Recently, by developing a general result on $p$-adic analytic phase and a $p$-adic version of van der Corput's method for exponential sums, Blomer and Milićević [2] also proved the same exponent if the conductor of $\chi$ is a prime power $\mathfrak{q}=p^\kappa$

$$L(1/2+it, f\otimes\chi) \ll_{p,t,f,\varepsilon} \mathfrak{q}^{1/3+\varepsilon}, \tag{1.3}$$

where the implied constant on $p$ and $t$ is explicit and polynomial (Munshi and Singh proved the same result using the approach in [14]). Also see [11] and [8] for other interesting subconvexity results in the depth aspect.

Let $\pi$ be a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and $\chi$ a primitive Dirichlet character modulo $\mathfrak{q}$. Then the convexity bound for $L(1/2,\pi\otimes\chi)$ is $\mathfrak{q}^{3/4+\varepsilon}$. For $\mathfrak{q}$ prime, the subconvexity results for $L(1/2,\pi\otimes\chi)$ have recently been established in the work [1], [6] and [16]-[17]. Munshi [14] showed a subconvexity bound for $\mathfrak{q}$ square-free. In this paper, following Munshi [14], we want to prove a subconvexity bound for $L(1/2,\pi\otimes\chi)$ in the depth aspect. Our main result is the following.

**Theorem 1.** *Let $\pi$ be a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ and $\chi$ a primitive Dirichlet character of prime power conductor $\mathfrak{q}=p^\kappa$ with $\kappa\geq 3$. We have*

$$L\left(\frac{1}{2},\pi\otimes\chi\right) \ll_{\pi,\varepsilon} p^{3/4}\mathfrak{q}^{3/4-3/40+\varepsilon}$$

*for any $\varepsilon>0$.*

**Remark 1.** Our result in Theorem 1 can be compared with the $t$-aspect subconvexity in (1.2) as explained in [13]. It is worth noting that for $\pi$ the symmetric-square lifts of $GL_2$ cusp forms, Munshi [13] proved the better result $\mathfrak{q}^{3/4-1/12+\varepsilon}$ by the moment method.

**Remark 2.** We are not trying to get the best exponent in $p$. With the present exponent $3/4$, the bound in Theorem 1 breaks the convexity for $\kappa>10$.

**Notation.** Throughout the paper, the letters $q$, $m$ and $n$, with or without subscript, denote integers. The letter $\varepsilon$ is an arbitrarily small positive constant, not necessarily the same at different occurrences. The symbol $\ll_{a,b,c}$ denotes that the implied constant depends at most on $a$, $b$ and $c$. Finally, fractional numbers such as $\frac{ab}{cd}$ will be written as $ab/cd$ and $a/b+c$ or $c+a/b$ means $\frac{a}{b}+c$.

## 2. Sketch of the proof

By the functional equation we have $L\left(\frac{1}{2},\pi\otimes\chi\right) \ll N^{-1/2}\mathscr{S}(N)$, where

$$\mathscr{S}(N)=\sum_{n\sim N} A_\pi(1,n)\chi(n),$$

with $N \sim \mathfrak{q}^{3/2}$. Applying the conductor lowering mechanism introduced by Munshi [14], we have

$$\mathscr{S}(N) = \mathop{\sum\sum}_{\substack{n,m\sim N \\ n\equiv m(\mathrm{mod} p^\lambda)}} A_\pi(1,n)\chi(n)\delta\left(\frac{n-m}{p^\lambda}\right)$$

where $\delta : \mathbb{Z} \to \{0,1\}$ with $\delta(0) = 1$ and $\delta(n) = 0$ for $n \neq 0$, and $\lambda \geq 2$ is an integer to be chosen later. Using Kloosterman's circle method and removing the congruence $n \equiv m(\mathrm{mod} p^\lambda)$ by exponential sums we get

$$\mathscr{S}(N) \approx \frac{1}{p^\lambda}\sum_{q\sim Q}\frac{1}{q}\sum_{\substack{Q<a\leq q+Q \\ (a,q)=1}}\frac{1}{a}\sum_{b(\mathrm{mod}\, p^\lambda)}\mathop{\sum\sum}_{n,m\sim N} A_\pi(1,n)\chi(m)e\left(\frac{(\overline{a}+bq)(n-m)}{qp^\lambda}\right).$$

Trivially we have $\mathscr{S}(N) \ll N^2$.

For simplicity, we assume $(q,p) = 1$ and $(\overline{a}+bq,p) = 1$. Recall $\chi$ is of modulus $\mathfrak{q} = p^\kappa$. Then the conductor of the $m$-sum has the size $qp^\kappa$. Applying Poisson summation to the $m$-sum we get that the dual sum is of size $qp^\kappa/N$. The conductor for the $n$-sum has the size $qp^\lambda$ and the dual sum after $GL_3$ Voronoi summation formula is essentially supported on summation of size $q^3p^{3\lambda}/N$. Assuming square-root cancellation for the character sum, we find that we have saved

$$\frac{N}{(qp^\kappa)^{1/2}} \times \frac{N}{(qp^\lambda)^{3/2}} \times (qp^\lambda)^{1/2} \sim \mathfrak{q}^{11/8}p^{-\lambda/4}.$$

Now we arrive at an expression of the form

$$\sum_{1\leq n\ll Q^3p^{3\lambda}/N} A_\pi(n,1)\sum_{q\sim Q}\chi(q)\sum_{|m|\ll Qp^\kappa/N}\sum_{b(\mathrm{mod} p^\lambda)}\overline{\chi}(m-bp^{\kappa-\lambda})S(\overline{b},n;qp^\lambda).$$

Next we apply Cauchy-Schwartz inequality to get rid of the Fourier coefficients. Then we need to deal with

$$\sum_{1\leq n\ll Q^3p^{3\lambda}/N}\left|\sum_{q\sim Q}\chi(q)\sum_{|m|\ll Qp^\kappa/N}\sum_{b(\mathrm{mod} p^\lambda)}\overline{\chi}(m-bp^{\kappa-\lambda})S(\overline{b},n;qp^\lambda)\right|^2.$$

Opening the square and applying Poisson summation to the sum over $n$, we are able to save $Q^2p^\kappa/N \sim p^{\kappa-\lambda}$ from the diagonal term and

$$\frac{Q^3p^{3\lambda}/N}{\sqrt{Q^2p^\lambda}} \sim p^{3\lambda/2}$$

from the off-diagonal term. So the optimal choice for $\lambda$ is given by $\lambda = 2\kappa/5$. In total, we have saved

$$\mathfrak{q}^{11/8}p^{-\lambda/4} \times p^{3\lambda/4} \sim \mathfrak{q}^{3/2+3/40}.$$

It follows that

$$L\left(\frac{1}{2},\pi\otimes\chi\right) \ll N^{-1/2}\mathscr{S}(N) \ll N^{3/2}\mathfrak{q}^{-3/2-3/40} \sim \mathfrak{q}^{3/4-3/40}.$$

## 3. Proof of Theorem 1

By the approximate functional equation we have

$$L\left(\frac{1}{2},\pi\otimes\chi\right)\ll_{\pi,\varepsilon}\mathfrak{q}^{\varepsilon}\sup_{N\leq\mathfrak{q}^{3/2+\varepsilon}}\frac{|\mathscr{S}(N)|}{\sqrt{N}},\tag{3.1}$$

where

$$\mathscr{S}(N)=\sum_{n}A_\pi(1,n)\chi(n)V\left(\frac{n}{N}\right)$$

for some smooth function $V$ supported in $[1,2]$ and satisfying $V^{(j)}(y)\ll_j 1$. Note that by Cauchy's inequality and the Rankin-Selberg estimate (see [12])

$$\sum\sum_{n_1^2n_2\leq Y}|A_\pi(n_1,n_2)|^2\ll_{\pi,\varepsilon}Y^{1+\varepsilon},\tag{3.2}$$

we have the trivial bound $\mathscr{S}(N)\ll_{\pi,\varepsilon}N$. Thus Theorem 1 is true for $N\leq\mathfrak{q}^{27/20}$. In the following, we will estimate $\mathscr{S}(N)$ in the range

$$\mathfrak{q}^{27/20}<N\leq\mathfrak{q}^{3/2+\varepsilon}.\tag{3.3}$$

**Proposition 1.** *Assume $\lambda\leq 2\kappa/3$ and (3.3). Then we have*

$$\mathscr{S}(N)\ll N^{1/2+\varepsilon}(p^{3\kappa/8+3\lambda/4}+p^{7\kappa/8-\lambda/2+3/4}).$$

Take $\lambda=\lfloor 2\kappa/5\rfloor+1$, where $\lfloor x\rfloor$ denotes the largest integer which does not exceed $x$. By (3.3) and Proposition 1, we have

$$\mathscr{S}(N)\ll p^{3/4}N^{1/2+\varepsilon}\mathfrak{q}^{3/4-3/40}$$

Then Theorem 1 follows from above bound and (3.1). In the following we prove Proposition 1.

3.1. **The circle method.** Define $\delta:\mathbb{Z}\to\{0,1\}$ with $\delta(0)=1$ and $\delta(n)=0$ for $n\neq 0$. By Kloosterman's version of the circle method, for any $n\in\mathbb{Z}$ and $Q\in\mathbb{R}^+$, we have

$$\delta(n)=2\mathrm{Re}\int_0^1\sum_{1\leq q\leq Q}\ \sum_{\substack{Q<a\leq q+Q\\(a,q)=1}}\frac{1}{aq}e\left(\frac{n\overline{a}}{q}-\frac{n\zeta}{aq}\right)\mathrm{d}\zeta,\tag{3.4}$$

where throughout the paper $e(z)=e^{2\pi iz}$ and $\overline{a}(\mathrm{mod}\,q)$ denotes the multiplicative inverse of $a$ modulo $q$. Define $\mathbf{1}_{\mathscr{F}}=1$ if $\mathscr{F}$ is true, and is 0 otherwise. Following Munshi [14] we write $\delta(n)$ as $\delta(n/p^\lambda)\mathbf{1}_{p^\lambda|n}$ ($2\leq\lambda<\kappa,\lambda\in\mathbb{N}$ is a parameter to be determined later) to lower the conductor and obtain

$$\mathscr{S}(N)=\sum_n A_\pi(1,n)V\left(\frac{n}{N}\right)\sum_{p^\lambda|n-m}\chi(m)U\left(\frac{m}{N}\right)\delta\left(\frac{n-m}{p^\lambda}\right),$$

where $U$ is a smooth function supported in $[1/2,5/2]$, $U(y)=1$ for $y\in[1,2]$ and $U^{(j)}(y)\ll_j 1$. Applying (3.4) and choosing

$$Q=\sqrt{N/p^\lambda}$$

we get

$$\mathscr{S}(N)=\mathscr{S}^+(N)+\mathscr{S}^-(N),$$

where

$$\mathscr{S}^{\pm}(N) = \int_0^1 \sum_{1\le q\le Q} \sum_{\substack{Q<a\le q+Q\\(a,q)=1}} \frac{1}{aq} \sum_n A_\pi(1,n) V\left(\frac{n}{N}\right) \sum_{p^\lambda | n-m} \chi(m) U\left(\frac{m}{N}\right) e\left(\pm \frac{(n-m)\overline{a}}{qp^\lambda} \mp \frac{(n-m)\zeta}{aqp^\lambda}\right) \mathrm{d}\zeta.$$

We will only estimate $\mathscr{S}^+(N)$ (the same analysis holds for $\mathscr{S}^-(N)$) and write $\mathscr{S}^+(N)$ as $\mathscr{S}(N)$. Removing the condition $p^\lambda | n-m$ using exponential sums to separate the variables $m$ and $n$ we get

$$\mathscr{S}(N) = \int_0^1 \sum_{1\le q\le Q} \sum_{\substack{Q<a\le q+Q\\(a,q)=1}} \frac{1}{aqp^\lambda} \sum_{b (\mathrm{mod}\, p^\lambda)} \mathscr{A} \times \mathscr{B} \, \mathrm{d}\zeta, \tag{3.5}$$

where

$$\mathscr{A} = \sum_m \chi(m) e\left(-\frac{(\overline{a}+bq)m}{qp^\lambda}\right) U\left(\frac{m}{N}\right) e\left(\frac{m\zeta}{aqp^\lambda}\right)$$

and

$$\mathscr{B} = \sum_n A_\pi(1,n) e\left(\frac{(\overline{a}+bq)n}{qp^\lambda}\right) V\left(\frac{n}{N}\right) e\left(-\frac{n\zeta}{aqp^\lambda}\right).$$

3.2. **Summation formulas and Cauchy-Schwartz.** Next we transform $\mathscr{A}$ and $\mathscr{B}$ by Poisson summation formula and $GL_3$ Voronoi formula, respectively, and obtain the following results.

**Lemma 1.** *Let $q = p^s q'$, $(q', p) = 1$ and $s \ge 0$. Then we have*

$$\mathscr{A} = \frac{N\chi(q')\tau_\chi}{p^\kappa} \sum_{\substack{|m|\le N^\varepsilon Q p^\kappa/N\\ m\equiv \overline{a}p^{\kappa-\lambda} (\mathrm{mod}\, q)}} \overline{\chi}\left(\frac{m-(\overline{a}+bq)p^{\kappa-\lambda}}{p^s}\right) \mathfrak{I}(m,a,q,\zeta) + O(\mathfrak{q}^{-A})$$

*for any $A > 0$, where the integral $\mathfrak{I}(m,a,q,\zeta)$ is defined in (4.2).*

**Lemma 2.** *Let $a^* = (\overline{a}+bq)/(\overline{a}+bq, qp^\lambda)$ and $q^* = qp^\lambda/(\overline{a}+bq, qp^\lambda)$. Then we have*

$$\mathscr{B} = \frac{N^{1/2}}{q^{*1/2}} \sum_{\pm} \sum_{n_1|q^*} \sum\sum_{n_1^2 n_2 \le N^\varepsilon q^{*3} Q^3/q^3 N} \frac{A_\pi(n_2,n_1)}{\sqrt{n_2}} S\left(\overline{a^*}, \pm n_2; \frac{q^*}{n_1}\right) \times \mathfrak{J}^{\pm}\left(\frac{n_1^2 n_2}{q^{*3}}, a, q, \zeta\right) + O(\mathfrak{q}^{-A})$$

*for any $A > 0$, where $\mathfrak{J}^{\pm}(y,a,q,\zeta)$ is defined in (5.2) and satisfies*

$$\mathfrak{J}^{\pm}(y,a,q,\zeta) \ll N^\varepsilon \sqrt{\frac{Q}{q}}.$$

The details of the proof of Lemmas 1 and 2 are in Sections 4 and 5. Note that for $s \ge 1$, we have $(\overline{a}+bq, qp^\lambda) = 1$, $a^* = \overline{a}+bq$ and $q^* = qp^\lambda$. For $s = 0$, we have $(\overline{a}+bq, qp^\lambda) = p^r$, $0 \le r \le \lambda$, $a^* = (\overline{a}+bq)/p^r$ and $q^* = qp^{\lambda-r}$. Since $\overline{a}+bq \equiv 0 (\mathrm{mod}\ p^r)$, we have $b \equiv -\overline{aq} (\mathrm{mod}\ p^r)$. Denote

$\varpi_q^r := (1-q\overline{q})/p^r \in \mathbb{Z}$. We write $\overline{a}+bq = (\overline{a}\varpi_q^r + cq)p^r$ with $c(\bmod p^{\lambda-r})$. Plugging Lemmas 1 and 2 into (3.5) and reducing the $n_1, n_2$ sums into dyadic intervals, we have

$$\begin{aligned}\mathscr{S}(N) \ll & \sum_{\pm}\sum_{r=0}^{\lambda} \sum_{\substack{L_1 \ll N^{\varepsilon}p^{3\lambda-3r}Q^3/N \\ L_1 \text{ dyadic}}} |\mathscr{S}_1^{\pm}(N,L_1,r)| \\ & + \sum_{\pm}\sum_{s=1}^{\log Q/\log p} \sum_{\substack{L_2 \ll N^{\varepsilon}p^{3\lambda}Q^3/N \\ L_2 \text{ dyadic}}} |\mathscr{S}_2^{\pm}(N,L_2,s)| + \mathfrak{q}^{-2018}, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\mathscr{S}_1^{\pm}(N,L_1,r) = & \frac{N^{3/2}}{p^{(\kappa+3\lambda-r)/2}} \sum\sum_{L_1/2<n_1^2n_2\le L_1} \frac{A_\pi(n_2,n_1)}{\sqrt{n_2}} \sum_{\substack{1\le q\le Q,(q,p)=1 \\ n_1|qp^{\lambda-r}}} \frac{\chi(q)}{q^{3/2}} \\ & \times \sum_{\substack{1\le|m|\le N^{\varepsilon}Qp^{\kappa}/N \\ (m,q)=1}} \sum_{\substack{Q<a\le q+Q \\ a\equiv \overline{m}p^{\kappa-\lambda}(\bmod q)}} \frac{1}{a}\mathfrak{K}^{\pm}\left(m, \frac{n_1^2n_2}{q^3p^{3\lambda-3r}}, a, q\right) \\ & \times \sum_{c(\bmod p^{\lambda-r})} \overline{\chi}\left(m - cp^{\kappa-\lambda+r}\right) S\left(\overline{c}, \pm n_2; \frac{qp^{\lambda-r}}{n_1}\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\mathscr{S}_2^{\pm}(N,L_2,s) = & \frac{N^{3/2}}{p^{(\kappa+3\lambda+3s)/2}} \sum\sum_{L_2/2<n_1^2n_2\le L_2} \frac{A_\pi(n_2,n_1)}{\sqrt{n_2}} \sum_{\substack{1\le q\le Q/p^s,(q,p)=1 \\ n_1|qp^{\lambda+s}}} \frac{\chi(q)}{q^{3/2}} \\ & \sum_{\substack{1\le|m|\le N^{\varepsilon}Qp^{\kappa}/N \\ m\equiv 0(\bmod p^s)}} \sum_{\substack{Q<a\le qp^s+Q,(a,p)=1 \\ a\equiv \overline{m}p^{\kappa-\lambda}(\bmod q)}} \frac{1}{a}\mathfrak{K}^{\pm}\left(m, \frac{n_1^2n_2}{q^3p^{3\lambda+3s}}, a, qp^s\right) \\ & \times \sum_{b(\bmod p^{\lambda})} \overline{\chi}\left(\frac{m-(\overline{a}+bp^s)p^{\kappa-\lambda}}{p^s}\right) S\left(\overline{\overline{a}+bp^s}, \pm n_2; \frac{qp^{\lambda+s}}{n_1}\right)\end{aligned}$$

with

$$\mathfrak{K}^{\pm}(y_1,y_2,a,q) = \int_0^1 \mathfrak{I}(y_1,a,q,\zeta)\mathfrak{J}^{\pm}(y_2,a,q,\zeta)\,\mathrm{d}\zeta \ll N^{\varepsilon}\sqrt{\frac{q}{Q}}. \tag{3.8}$$

Here we have changed variables $\overline{a}\varpi_q^r + cq \to c$ and $bq \to b$.

**Remark 3.** If $m=0$, then the conditions $p^{\kappa-\lambda} \equiv 0(\bmod qp^s)$ and $((\overline{a}+bqp^s)p^{\kappa-\lambda-s},p)=1$ imply that $q=1$ and $s=\kappa-\lambda$. Thus we have $p^{\kappa-\lambda} \le Q = \sqrt{N/p^{\lambda}}$ which implies $N > p^{(3/2+\varepsilon)\kappa}$ which contradicts to the assumption (3.3). Therefore, we have $m\neq 0$.

Applying Cauchy-Schwartz inequality to $n_1,n_2$-sums in (3.7) and using the Rankin-Selberg bound (3.2), we get

$$\mathscr{S}_1^{\pm}(N,L_1,r) \ll \frac{N^{3/2}L_1^{1/2}}{p^{(\kappa+3\lambda-r)/2}}\mathscr{H}_1^{\pm}(N,L_1,r)^{1/2} \tag{3.9}$$

where

$$\begin{aligned}\mathscr{H}_1^{\pm}(N,L_1,r) &= \sum_{(n_1',p)=1}\sum_{n_1''|p^{\lambda-r}}\sum_{n_2}\frac{1}{n_2}W\left(\frac{n_1'^2n_1''^2n_2}{L_1}\right)\Bigg|\sum_{\substack{1\leq q\leq Q,(q,p)=1\\ n_1'|q}}\frac{\chi(q)}{q^{3/2}}\sum_{\substack{1\leq|m|\leq N^{\varepsilon}Qp^{\kappa}/N\\(m,q)=1}}\\ &\sum_{\substack{Q<a\leq q+Q\\ a\equiv\overline{m}p^{\kappa-\lambda}(\text{mod }q)}}\frac{1}{a}\mathfrak{C}_r(m,n_1',n_1'',\pm n_2,a,q)\mathfrak{K}^{\pm}\left(m,\frac{n_1'^2n_1''^2n_2}{q^3p^{3\lambda-3r}},a,q\right)\Bigg|^2 \end{aligned}\tag{3.10}$$

with $W(y)$ a smooth positive function, $W(y)=1$ if $y\in[1/2,1]$, and

$$\mathfrak{C}_r(m,n_1',n_1'',n_2,a,q)=S\left(a\overline{\varpi_q^r\widehat{p_r}},n_2\overline{\widehat{p_r}};\widehat{q}\right)\sum_{c(\text{mod }p^{\lambda-r})}\overline{\chi}\left(m-cp^{\kappa-\lambda+r}\right)S\left(\overline{c}\overline{\widehat{q}},n_2\overline{\widehat{q}};\widehat{p_r}\right).\tag{3.11}$$

Here $\widehat{q}=q/n_1'$ and $\widehat{p_r}=p^{\lambda-r}/n_1''$. Similarly,

$$\mathscr{S}_2^{\pm}(N,L_2,s)\ll\frac{N^{3/2}L_2^{1/2}}{p^{(\kappa+3\lambda+3s)/2}}\mathscr{H}_2^{\pm}(N,L_2,s)^{1/2}\tag{3.12}$$

where

$$\begin{aligned}\mathscr{H}_2^{\pm}(N,L_2,s) &= \sum_{(n_1',p)=1}\sum_{n_1''|p^{\lambda+s}}\sum_{n_2}\frac{1}{n_2}W\left(\frac{n_1'^2n_1''^2n_2}{L_2}\right)\Bigg|\sum_{\substack{1\leq q\leq Q/p^s,(q,p)=1\\ n_1'|q}}\frac{\chi(q)}{q^{3/2}}\sum_{\substack{1\leq|m|\leq N^{\varepsilon}Qp^{\kappa}/N\\ m\equiv0(\text{mod }p^s)}}\\ &\sum_{\substack{Q<a\leq qp^s+Q,(a,p)=1\\ a\equiv\overline{m}p^{\kappa-\lambda}(\text{mod }q)}}\frac{1}{a}\mathfrak{B}_s(m,n_1',n_1'',\pm n_2,a,q)\mathfrak{K}^{\pm}\left(m,\frac{n_1'^2n_1''^2n_2}{q^3p^{3\lambda+3s}},a,qp^s\right)\Bigg|^2\end{aligned}$$

with

$$\begin{aligned}\mathfrak{B}_s(m,n_1',n_1'',n_2,a,q) &= S\left(a\overline{p^{\lambda+s}/n_1''},n_2\overline{p^{\lambda+s}/n_1''};\frac{q}{n_1'}\right)\sum_{b(\text{mod }p^{\lambda})}\overline{\chi}\left(\frac{m-(\overline{a}+bp^s)p^{\kappa-\lambda}}{p^s}\right)\\ &\times S\left(\overline{\overline{a}+bp^s}\,\overline{q/n_1'},n_2\overline{q/n_1'};\frac{p^{\lambda+s}}{n_1''}\right).\end{aligned}\tag{3.13}$$

3.3. **Poisson summation.** Opening the square in (3.10) and switching the order of summations, we get

$$\begin{aligned}\mathscr{H}_1^{\pm}(N,L_1,r) &= \sum_{n_1'}\sum_{n_1''|p^{\lambda-r}}\sum_{\substack{1\leq q_1\leq Q,(q_1,p)=1\\ n_1'|q_1}}\frac{\chi(q_1)}{q_1^{3/2}}\sum_{\substack{1\leq q_2\leq Q,(q_2,p)=1\\ n_1'|q_2}}\frac{\overline{\chi(q_2)}}{q_2^{3/2}}\\ &\sum_{\substack{1\leq|m_1|\leq N^{\varepsilon}Qp^{\kappa}/N\\(m_1,q_1)=1}}\sum_{\substack{1\leq|m_2|\leq N^{\varepsilon}Qp^{\kappa}/N\\(m_2,q_2)=1}}\sum_{\substack{Q<a_1\leq q_1+Q\\ a_1\equiv\overline{m_1}p^{\kappa-\lambda}(\text{mod }q_1)}}\frac{1}{a_1}\sum_{\substack{Q<a_2\leq q_2+Q\\ a_2\equiv\overline{m_2}p^{\kappa-\lambda}(\text{mod }q_2)}}\frac{1}{a_2}\times\mathbf{T},\end{aligned}$$

where temporarily,

$$\mathbf{T} = \sum_{n_2} \frac{1}{n_2} W\left(\frac{n_1'^2 n_1''^2 n_2}{L_1}\right) \mathfrak{C}_r(m_1, n_1', n_1'', \pm n_2, a_1, q_1)\overline{\mathfrak{C}_r(m_2, n_1', n_1'', \pm n_2, a_2, q_2)}$$
$$\mathfrak{K}^{\pm}\left(m_1, \frac{n_1'^2 n_1''^2 n_2}{q_1^3 p^{3\lambda-3r}}, a_1, q_1\right) \overline{\mathfrak{K}^{\pm}\left(m_2, \frac{n_1'^2 n_1''^2 n_2}{q_2^3 p^{3\lambda-3r}}, a_2, q_2\right)}.$$

Applying Poisson summation with modulus $\widehat{q_1}\widehat{q_2}\widehat{p_r}$, we arrive at

$$\mathbf{T} = \frac{1}{\widehat{q_1}\widehat{q_2}\widehat{p_r}} \sum_{n_2 \in \mathbb{Z}} \mathfrak{C}^* \times \mathfrak{K}^*,$$

where

$$\mathfrak{C}^* = \sum_{\beta (\mathrm{mod}\ \widehat{q_1}\widehat{q_2}\widehat{p_r})} \mathfrak{C}_r\left(m_1, n_1', n_1'', \beta, a_1, q_1\right) \overline{\mathfrak{C}_r\left(m_2, n_1', n_1'', \beta, a_2, q_2\right)} e\left(\frac{\pm n_2 \beta}{\widehat{q_1}\widehat{q_2}\widehat{p_r}}\right) \tag{3.14}$$

and

$$\mathfrak{K}^* = \int_{\mathbb{R}} W(y) \mathfrak{K}^{\pm}\left(m_1, \frac{L_1 y}{q_1^3 p^{3\lambda-3r}}, a_1, q_1\right) \overline{\mathfrak{K}^{\pm}\left(m_2, \frac{L_1 y}{q_2^3 p^{3\lambda-3r}}, a_2, q_2\right)} e\left(-\frac{n_2 L_1 y}{q_1 q_2 p^{\lambda-r} n_1''}\right) \frac{\mathrm{d}y}{y}.$$

The integral $\mathfrak{K}^*$ gives arbitrary power saving in $\mathfrak{q}$ if $|n_2| \geq N^{\varepsilon} Q^2 p^{\lambda-r} n_1''/L_1$ for any $\varepsilon > 0$. For small values of $n_2$, by (3.8), we have

$$\mathfrak{K}^* \ll N^{\varepsilon} \frac{\sqrt{q_1 q_2}}{Q}.$$

Therefore, at the cost of a negligible error,

$$\mathbf{T} \ll N^{\varepsilon} \frac{\sqrt{q_1 q_2}}{Q} \frac{1}{\widehat{q_1}\widehat{q_2}\widehat{p_r}} \sum_{|n_2| \leq N^{\varepsilon} Q^2 p^{\lambda-r} n_1''/L_1} |\mathfrak{C}^*|$$

and

$$\mathscr{H}_1^{\pm}(N, L_1, r) \ll N^{\varepsilon} \frac{1}{Q^3} \sum_{n_1'} \sum_{n_1'' | p^{\lambda-r}} \sum_{\substack{1 \leq q_1 \leq Q \\ n_1' | q_1}} \frac{1}{q_1} \sum_{\substack{1 \leq q_2 \leq Q \\ n_1' | q_2}} \frac{1}{q_2}$$
$$\sum_{\substack{1 \leq |m_1| \leq N^{\varepsilon} Q p^{\kappa}/N \\ (m_1, q_1)=1}} \sum_{\substack{1 \leq |m_2| \leq N^{\varepsilon} Q p^{\kappa}/N \\ (m_2, q_2)=1}} \frac{1}{\widehat{q_1}\widehat{q_2}\widehat{p_r}} \sum_{|n_2| \leq N^{\varepsilon} Q^2 p^{\lambda-r} n_1''/L_1} |\mathfrak{C}^*|. \tag{3.15}$$

Similarly,

$$\mathscr{H}_2^{\pm}(N, L_2, s) \ll N^{\varepsilon} \frac{p^{3s}}{Q^3} \sum_{n_1'} \sum_{n_1'' | p^{\lambda+s}} \sum_{\substack{1 \leq q_1 \leq Q/p^s, (q_1, p)=1 \\ n_1' | q_1}} \frac{1}{q_1} \sum_{\substack{1 \leq q_2 \leq Q/p^s, (q_2, p)=1 \\ n_1' | q_2}} \frac{1}{q_2}$$
$$\sum_{\substack{1 \leq |m_1| \leq N^{\varepsilon} Q p^{\kappa}/N \\ m_1 \equiv 0 (\mathrm{mod}\ p^s)}} \sum_{\substack{1 \leq |m_2| \leq N^{\varepsilon} Q p^{\kappa}/N \\ m_2 \equiv 0 (\mathrm{mod}\ p^s)}} \frac{1}{\widehat{q_1}\widehat{q_2}\widehat{\rho_s}} \sum_{|n_2| \leq N^{\varepsilon} Q^2 p^{\lambda-s} n_1''/L_2} |\mathfrak{B}^*|, \tag{3.16}$$

where $\widehat{q}=q/n_1',\ \widehat{\rho_s}=p^{\lambda+s}/n_1''$ and

$$\mathfrak{B}^*=\sum_{\beta(\operatorname{mod}\ \widehat{q_1}\widehat{q_2}\widehat{\rho_s})}\mathfrak{B}_s\left(m_1,n_1',n_1'',\beta,a_1,q_1\right)\overline{\mathfrak{B}_s\left(m_2,n_1',n_1'',\beta,a_2,q_2\right)}e\left(\frac{\pm n_2\beta}{\widehat{q_1}\widehat{q_2}\widehat{\rho_s}}\right). \tag{3.17}$$

**Lemma 3.** *Assume $\lambda\le 2\kappa/3$. Let $p^k\|n_2$ with $k\ge 0$.*
*(1) We have*

$$\mathfrak{C}^*\ll\widehat{q_1}\widehat{q_2}(\widehat{q_1},\widehat{q_2},n_2)\widehat{p_r}^2p^{2(\lambda-r)}.$$

*Moreover, for $n_2=0$, the character sums vanish unless $q_1=q_2$ in which case*

$$\mathfrak{C}^*\ll\widehat{q_1}^2(\widehat{q_1},m_1-m_2)\widehat{p_r}p^{2(\lambda-r)}.$$

*(2) For $n_1''=p^{\lambda-r}$ or $n_1''=p^{\lambda-r-1}$ with $\lambda-r\ge 2$, we have*

$$\mathfrak{C}^*=0.$$

*(3) For $p^{\lambda-r}/n_1''\ge p^2$, we have $\mathfrak{C}^*$ vanishes unless $n_1''=1$. Moreover, let $\lambda-r=2\alpha+\delta$ with $\delta=0$ or $1$. For $n_2=0$, $\mathfrak{C}_2^*$ vanishes unless $m_1q_1^2\equiv m_2q_2^2(\operatorname{mod} p^\alpha)$. For $n_2\ne 0$, we have*

$$\mathfrak{C}^*\ll\widehat{q_1}\widehat{q_2}(\widehat{q_1},\widehat{q_2},n_2)p^{5(\lambda-r)/2+\min\{k,\alpha\}+3\delta/2}.$$

**Lemma 4.** *Assume $\lambda\le 2\kappa/3$. Let $p^k\|n_2$ with $k\ge 0$.*
*(1) We have $\mathfrak{B}^*$ vanishes unless $n_1''=1$ and*

$$\mathfrak{B}^*\ll\widehat{q_1}\widehat{q_2}(\widehat{q_1},\widehat{q_2},n_2)\widehat{\rho_s}^2p^{2\lambda}.$$

*Moreover, for $n_2=0$, the character sums vanish unless $q_1=q_2$ and $a_1\equiv a_2(\operatorname{mod} p^s)$ in which case*

$$\mathfrak{B}^*\ll\widehat{q_1}^2(\widehat{q_1},m_1-m_2)\widehat{\rho_s}p^{2\lambda+s}.$$

*(2) Let $\lambda=2\alpha+\delta$ with $\delta=0$ or $1$. For $n_2=0$, we have $\mathfrak{B}_2^*$ vanishes unless $q_1^2m_1/p^s\equiv q_2^2m_2/p^s(\operatorname{mod}\ p^\alpha)$. For $n_2\ne 0$, we have*

$$\mathfrak{B}^*\ll\widehat{q_1}\widehat{q_2}(\widehat{q_1},\widehat{q_2},n_2)p^{5\lambda/2+4s+\min\{k,\alpha\}+3\delta/2}.$$

For $r\ge\lambda-1$, by (3.15) and Lemma 3 (1), we have

$$\begin{aligned}\mathscr{H}_1^{\pm}(N,L_1,r) \ll\ & N^{\varepsilon}\frac{1}{Q^3}\sum_{n_1'}\sum_{n_1''|p^{\lambda-r}}\sum_{\substack{1\le q_1\le Q\\ n_1'|q_1}}\frac{1}{q_1}\sum_{\substack{1\le q_2\le Q\\ n_1'|q_2}}\frac{1}{q_2}\sum_{\substack{1\le|m_1|\le N^{\varepsilon}Qp^{\kappa}/N\\ (m_1,q_1)=1}}\\ &\sum_{\substack{1\le|m_2|\le N^{\varepsilon}Qp^{\kappa}/N\\ (m_2,q_2)=1}}\frac{1}{\widehat{q_1}\widehat{q_2}\widehat{p_r}}\sum_{1\le|n_2|\le N^{\varepsilon}Q^2p^{\lambda-r}n_1''/L_1}\widehat{q_1}\widehat{q_2}(\widehat{q_1},\widehat{q_2},n_2)\widehat{p_r}^2p^{2(\lambda-r)}\\ &+N^{\varepsilon}\frac{1}{Q^3}\sum_{n_1'}\sum_{n_1''|p^{\lambda-r}}\sum_{\substack{1\le q_1\le Q,(q_1,p)=1\\ n_1'|q_1}}\frac{1}{q_1^2}\sum_{\substack{1\le|m_1|\le N^{\varepsilon}Qp^{\kappa}/N\\ (m_1,q_1)=1}}\sum_{\substack{1\le|m_2|\le N^{\varepsilon}Qp^{\kappa}/N\\ (m_2,q_1)=1}}\\ &\frac{1}{\widehat{q_1}^2\widehat{p_r}}\widehat{q_1}^2(\widehat{q_1},m_1-m_2)\widehat{p_r}p^{2(\lambda-r)}\\ \ll\ & N^{\varepsilon}\left(\frac{Qp^{2\kappa+4\lambda-4r}}{N^2L_1}+\frac{p^{2\kappa+2\lambda-2r}}{QN^2}\right).\end{aligned} \tag{3.18}$$

For $r \leq \lambda - 2$, by Lemma 3, we have

$$\mathscr{H}_1^{\pm}(N, L_1, r) \ll \mathbf{R}_1 + \mathbf{R}_2, \tag{3.19}$$

where $\mathbf{R}_1$ is the contribution from $p^{\lambda-r}/n_1'' \geq p^2$ and $n_2 = 0$

$$\begin{aligned}
\mathbf{R}_1 &= N^{\varepsilon}\frac{1}{Q^3}\sum_{\delta=0,1}\sum_{n_1'}\sum_{\substack{1\leq q_1\leq Q\\ n_1'|q_1}}\frac{1}{q_1^2}\sum_{\substack{1\leq|m_1|\leq N^{\varepsilon}Qp^{\kappa}/N\\ (m_1,q_1)=1}}\sum_{\substack{1\leq|m_2|\leq N^{\varepsilon}Qp^{\kappa}/N\\ m_2\equiv m_1\left(\bmod p^{(\lambda-r-\delta)/2}\right)}}\\
&\quad \frac{1}{\widehat{q_1}^2 p^{\lambda-r}}\widehat{q_1}^2(\widehat{q_1}, m_1-m_2)p^{3(\lambda-r)}\\
&\ll N^{\varepsilon}\frac{p^{2(\lambda-r)}}{Q^3}\frac{Qp^{\kappa}}{N}\sum_{\delta=0,1}\left(1+\frac{Qp^{\kappa}}{N}p^{-\frac{\lambda-r-\delta}{2}}\right)\\
&\ll N^{\varepsilon}\left(\frac{p^{\kappa+2\lambda-2r}}{Q^2N}+p^{1/2}\frac{p^{2\kappa+3\lambda/2-3r/2}}{QN^2}\right)
\end{aligned}\tag{3.20}$$

and $\mathbf{R}_2$ is the remaining piece

$$\begin{aligned}
\mathbf{R}_2 &= N^{\varepsilon}\frac{1}{Q^3}\sum_{\delta=0,1}\sum_{n_1'}\sum_{\substack{1\leq q_1\leq Q\\ n_1'|q_1}}\frac{1}{q_1}\sum_{\substack{1\leq q_2\leq Q\\ n_1'|q_2}}\frac{1}{q_2}\sum_{\substack{1\leq|m_1|\leq N^{\varepsilon}Qp^{\kappa}/N\\ (m_1,q_1)=1}}\sum_{\substack{1\leq|m_2|\leq N^{\varepsilon}Qp^{\kappa}/N\\ (m_2,q_2)=1}}\\
&\quad \frac{1}{\widehat{q_1}\widehat{q_2}p^{\lambda-r}}\sum_{0\leq k\ll\log\mathfrak{q}}\sum_{\substack{1\leq|n_2|\leq N^{\varepsilon}Q^2p^{\lambda-r}/L_1\\ p^k|n_2}}\widehat{q_1}\widehat{q_2}(\widehat{q_1},\widehat{q_2},n_2)p^{5(\lambda-r)/2+\min\{k,(\lambda-r-\delta)/2\}+3\delta/2}\\
&\ll N^{\varepsilon}\frac{p^{3(\lambda-r)/2}}{Q^3}\left(\frac{Qp^{\kappa}}{N}\right)^2\sum_{\delta=0,1}p^{3\delta/2}\sum_{n_1'}\frac{1}{n_1'}\sum_{\substack{1\leq q_1\leq Q\\ n_1'|q_1}}\frac{1}{q_1}\left\{\sum_{0\leq k\leq(\lambda-r-\delta_1)/2}\frac{Q^2p^{\lambda-r}}{L_1}\right.\\
&\quad \left.+\sum_{(\lambda-r-\delta_1)/2<k\ll\log\mathfrak{q}}p^{(\lambda-r-\delta)/2}\frac{Q^2p^{\lambda-r-k}}{L_1}\right\}\\
&\ll p^{3/2}N^{\varepsilon}\frac{p^{2\kappa+2\lambda-5r/2}}{N^{3/2}L_1}.
\end{aligned}\tag{3.21}$$

Obviously, the second term in (3.20) is dominated by (3.21), since $N^\varepsilon Q^2 p^{\lambda-r}/L_1 \geq 1$. By (3.9) and (3.18-3.21), the contribution from $\mathscr{H}_1^{\pm}(N, L_1, r)$ to $\mathscr{S}(N)$ in (3.6) is at most

$$\begin{aligned}
& N^\varepsilon \sum_{r=\lambda-1}^{\lambda} \sum_{\substack{L_1 \ll N^\varepsilon p^{3\lambda-3r} Q^3/N \\ L_1 \text{ dyadic}}} \frac{N^{3/2} L_1^{1/2}}{p^{(\kappa+3\lambda-r)/2}} \left( \frac{Q^{1/2} p^{\kappa+2\lambda-2r}}{N L_1^{1/2}} + \frac{p^{\kappa+\lambda-r}}{Q^{1/2} N} \right) \\
& + N^\varepsilon \sum_{r=0}^{\lambda-2} \sum_{\substack{L_1 \ll N^\varepsilon p^{3\lambda-3r} Q^3/N \\ L_1 \text{ dyadic}}} \frac{N^{3/2} L_1^{1/2}}{p^{(\kappa+3\lambda-r)/2}} \left\{ \frac{p^{\kappa/2+\lambda-r}}{Q N^{1/2}} + p^{3/4} \frac{p^{\kappa+\lambda-5r/4}}{N^{3/4} L_1^{1/2}} \right\} \\
\ll \; & N^{1/2+\varepsilon} \sum_{r=\lambda-1}^{\lambda} \left( N^{1/4} p^{\kappa/2+\lambda/4-3r/2} + p^{\kappa/2+\lambda/2-2r} \right) \\
+ \; & N^{1/2+\varepsilon} \sum_{r=0}^{\lambda-2} \left( p^{3\lambda/4-2r} N^{1/4} + p^{3/4} N^{1/4} p^{\kappa/2-\lambda/2-3r/4} \right) \\
\ll \; & N^{1/2+\varepsilon} \left( N^{1/4} p^{3\lambda/4} + p^{3/4} N^{1/4} p^{\kappa/2-\lambda/2} \right) \qquad (3.22)
\end{aligned}$$

for $\lambda \geq 2$.

Similarly, by Lemma 4, the contribution from $n_2 = 0$ to $\mathscr{H}_2^{\pm}(N, L_2, s)$ is at most

$$\begin{aligned}
& N^\varepsilon \frac{p^{3s}}{Q^3} \sum_{\delta=0,1} \sum_{n_1'} \sum_{\substack{1 \leq q_1 \leq Q/p^s \\ n_1' | q_1}} \frac{1}{q_1^2} \sum_{\substack{1 \leq |m_1| \leq N^\varepsilon Q p^\kappa/N \\ p^s | m_1}} \sum_{\substack{1 \leq |m_2| \leq N^\varepsilon Q p^\kappa/N, p^s | m_2 \\ m_2/p^s \equiv m_1/p^s (\bmod p^{(\lambda-\delta)/2})}} \\
& \qquad \times \frac{1}{\widehat{q_1}^2 p^{\lambda+s}} \widehat{q_1}^2 \left( \widehat{q_1}, m_1 - m_2 \right) p^{3\lambda+2s} \\
= \; & N^\varepsilon \frac{p^{2\lambda+4s}}{Q^3} \sum_{\delta=0,1} \sum_{n_1'} \sum_{\substack{1 \leq q_1 \leq Q/p^s \\ (q_1, p)=1 \\ n_1' | q_1}} \frac{1}{q_1^2} \frac{Q p^{\kappa-s}}{N} \left( \widehat{q_1} + \frac{Q p^{\kappa-s-(\lambda-\delta)/2}}{N} \right) \\
= \; & N^\varepsilon \left( \frac{p^{\kappa+2\lambda+3s}}{Q^2 N} + p^{1/2} \frac{p^{2\kappa+3\lambda/2+2s}}{Q N^2} \right). \qquad (3.23)
\end{aligned}$$

The contribution from $n_2 \neq 0$ to $\mathscr{H}_2^{\pm}(N, L_2, s)$ is bounded by

$$\begin{aligned}
& N^{\varepsilon} \frac{p^{3s}}{Q^3} \sum_{\delta=0,1} \sum_{n_1'} \sum_{\substack{1 \leq q_1 \leq Q/p^s \\ n_1' | q_1}} \frac{1}{q_1} \sum_{\substack{1 \leq q_2 \leq Q/p^s \\ n_1' | q_2}} \frac{1}{q_2} \sum_{\substack{1 \leq |m_1| \leq N^{\varepsilon} Q p^{\kappa}/N \\ m_1 \equiv 0 (\mathrm{mod}\ p^s)}} \sum_{\substack{1 \leq |m_2| \leq N^{\varepsilon} Q p^{\kappa}/N \\ m_2 \equiv 0 (\mathrm{mod}\ p^s)}} \\
& \frac{1}{\widehat{q_1} \widehat{q_2} p^{\lambda+s}} \sum_{0 \leq k \ll \log q} \sum_{\substack{1 \leq |n_2| \leq N^{\varepsilon} Q^2 p^{\lambda-s}/L_2 \\ p^k | n_2}} \widehat{q_1} \widehat{q_2} \left(\widehat{q_1}, \widehat{q_1}, n_2\right) p^{5\lambda/2+4s+\min\{k,(\lambda-\delta)/2\}+3\delta/2} \\
\ll\ & N^{\varepsilon} \frac{p^{3\lambda/2+6s}}{Q^3} \sum_{\delta=0,1} p^{3\delta/2} \sum_{n_1'} \frac{1}{n_1'} \sum_{\substack{1 \leq q_1 \leq Q/p^s \\ n_1' | q_1}} \frac{1}{q_1} \left( \frac{Q p^{\kappa-s}}{N} \right)^2 \frac{Q^2 p^{\lambda-s}}{L_2} \\
\ll\ & p^{3/2} N^{\varepsilon} \frac{Q p^{2\kappa+5\lambda/2+3s}}{N^2 L_2}. \qquad (3.24)
\end{aligned}$$

Obviously, the second term in (3.23) is bounded by (3.24). Plugging these estimates into (3.16) and (3.12), we have that the contribution from $\mathscr{S}_2^{\pm}(N, L_2, s)$ to $\mathscr{S}(N)$ in (3.6) is bounded by

$$\begin{aligned}
& N^{\varepsilon} \sum_{s=1}^{\log Q/\log p} \sum_{\substack{L_2 \ll N^{\varepsilon} p^{3\lambda} Q^3/N \\ L_2 \text{ dyadic}}} \frac{N^{3/2} L_2^{1/2}}{p^{(\kappa+3\lambda+3s)/2}} \left( \frac{p^{(\kappa+2\lambda+3s)/2}}{Q N^{1/2}} + p^{3/4} \frac{Q^{1/2} p^{\kappa+5\lambda/4+3s/2}}{N L_2^{1/2}} \right) \\
\ll\ & N^{1/2+\varepsilon} (N^{1/4} p^{3\lambda/4} + p^{3/4} N^{1/4} p^{\kappa/2-\lambda/2}). \qquad (3.25)
\end{aligned}$$

3.4. **Conclusion.** By (3.22) and (3.25) we have

$$\mathscr{S}(N) \ll N^{1/2+\varepsilon} (N^{1/4} p^{3\lambda/4} + p^{3/4} N^{1/4} p^{\kappa/2-\lambda/2}).$$

Since $N \leq p^{3\kappa/2+\varepsilon}$, Proposition 1 follows.

## 4. Proof of Lemma 1

In this section we apply Poisson summation formula to prove Lemma 1. Precisely,

$$\begin{aligned}
\mathscr{A} &= \sum_{\beta (\mathrm{mod}\, q p^{\kappa})} \chi(\beta) e\left( -\frac{(\overline{a} + bq)\beta}{q p^{\lambda}} \right) \sum_{m \equiv \beta (\mathrm{mod}\, q p^{\kappa})} U\left( \frac{m}{N} \right) e\left( \frac{m\zeta}{a q p^{\lambda}} \right) \\
&:= \frac{N}{q p^{\kappa}} \sum_{m \in \mathbb{Z}} \mathfrak{A}(m, a, b, q) \mathfrak{I}(m, a, q, \zeta) \qquad (4.1)
\end{aligned}$$

where

$$\mathfrak{A}(m, a, b, q) = \sum_{\beta (\mathrm{mod}\, q p^{\kappa})} \chi(\beta) e\left( \frac{m - (\overline{a} + bq) p^{\kappa-\lambda}}{q p^{\kappa}} \beta \right)$$

and

$$\mathfrak{I}(m, a, q, \zeta) = \int_{\mathbb{R}} U(y) e\left( \frac{\zeta N y}{a q p^{\lambda}} \right) e\left( -\frac{m N y}{q p^{\kappa}} \right) \mathrm{d}y. \qquad (4.2)$$

4.1. **Computing the character sum $\mathfrak{A}(m,a,b,q)$.** Write $q=p^sq'$, $(q',p)=1$ and $s\geq 0$. Then

$$\begin{aligned}\mathfrak{A}(m,a,b,q) &= \sum_{\beta(\operatorname{mod} q'p^{s+\kappa})}\chi(\beta)e\left(\frac{m-(\overline{a}+bq'p^s)p^{\kappa-\lambda}}{q'p^{s+\kappa}}\beta\right)\\ &= \sum_{\beta_1(\operatorname{mod} q')} e\left(\frac{m-\overline{a}p^{\kappa-\lambda}}{q'}\overline{p^{s+\kappa}}\beta_1\right)\\ &\qquad \sum_{\beta_2(\operatorname{mod} p^{s+\kappa})}\chi(\beta_2)e\left(\frac{m-(\overline{a}+bq'p^s)p^{\kappa-\lambda}}{p^{s+\kappa}}\overline{q'}\beta_2\right),\end{aligned}$$

where the first sum over $\beta_1$ is $q'\mathbf{1}_{m\equiv\overline{a}p^{\kappa-\lambda}(\operatorname{mod} q')}$, and the second sum over $\beta_2$ is

$$\begin{aligned}&\chi(q')\sum_{\beta_2(\operatorname{mod} p^{s+\kappa})}\chi(\beta_2)e\left(\frac{m-(\overline{a}+bq'p^s)p^{\kappa-\lambda}}{p^{s+\kappa}}\beta_2\right)\\ =\ &\chi(q')\sum_{\alpha_1(\operatorname{mod} p^{s})} e\left(\frac{m-\overline{a}p^{\kappa-\lambda}}{p^s}\alpha_1\right)\sum_{\alpha_2(\operatorname{mod} p^{\kappa})}\chi(\alpha_2)e\left(\frac{m-(\overline{a}+bq'p^s)p^{\kappa-\lambda}}{p^{s+\kappa}}\alpha_2\right)\\ =\ &p^s\chi(q')\mathbf{1}_{m\equiv\overline{a}p^{\kappa-\lambda}(\operatorname{mod} p^s)}\overline{\chi}\left(\frac{m-(\overline{a}+bq'p^s)p^{\kappa-\lambda}}{p^s}\right)\tau_\chi.\end{aligned}$$

Here $\tau_\chi$ is the Gauss sum. Thus

$$\mathfrak{A}(m,a,b,q)=q\mathbf{1}_{m\equiv\overline{a}p^{\kappa-\lambda}(\operatorname{mod} q)}\chi(q')\overline{\chi}\left(\frac{m-(\overline{a}+bq)p^{\kappa-\lambda}}{p^s}\right)\tau_\chi. \tag{4.3}$$

4.2. **Bounding the integral $\mathfrak{I}(m,a,q,\zeta)$.** Integration by parts $j$ times, we get

$$\mathfrak{I}(m,a,q,\zeta)\ll\left(\frac{Qp^\kappa}{|m|N}\right)^j.$$

Thus the $m$-sum is essentially supported on $|m|\leq N^\varepsilon Qp^\kappa/N$. Then Lemma 1 follows from (4.1) and (4.2).

Furthermore, we also do repeated partial integration by integrating all the exponential factors and differentiating $U$ only to get

$$\mathfrak{I}(m,a,q,\zeta)\ll\left(\frac{N}{aqp^\lambda}\left|\zeta-\frac{ma}{p^{\kappa-\lambda}}\right|\right)^{-j}.$$

This restricts the $\zeta$-integral essentially over $\left|\zeta-ma/p^{\kappa-\lambda}\right|\leq N^\varepsilon q/Q$ for any $\varepsilon>0$.

## 5. Proof of Lemma 2

In this section we will apply the $GL_3$ Voronoi formula to transform $\mathscr{B}$, where

$$\mathscr{B}=\sum_n A_\pi(1,n)e\left(\frac{a^*n}{q^*}\right)\phi(n),$$

where $a^* = (\overline{a}+bq)/(\overline{a}+bq, qp^\lambda)$, $q^* = qp^\lambda/(\overline{a}+bq, qp^\lambda)$ and $\phi(y) = V(y/N)\, e\left(-\zeta y/aqp^\lambda\right)$. Applying the $GL_3$ Voronoi formula (see [5], [18]), we have

$$\mathscr{B} = q^* \sum_{\pm} \sum_{n_1 | q^*} \sum_{n_2=1}^{\infty} \frac{A_\pi(n_2,n_1)}{n_1 n_2} S\left(\overline{a^*}, \pm n_2; \frac{q^*}{n_1}\right) \Phi_\phi^{\pm}\left(\frac{n_1^2 n_2}{q^{*3}}\right), \tag{5.1}$$

where

$$\Phi_\phi^{\pm}(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_\pm(s) \widetilde{\phi}(-s) \mathrm{d}s, \qquad \sigma > \max_{1\le j\le 3}\{-1-\mathrm{Re}(\mu_j)\},$$

where $\mu_j$, $j=1,2,3$, are the Langlands parameters of $\pi$, $\widetilde{\phi}$ is the Mellin transform of $\phi$ and

$$\gamma_\pm(s) = \frac{1}{2\pi^{3(s+1/2)}} \left( \prod_{j=1}^{3} \frac{\Gamma\left((1+s+\mu_j)/2\right)}{\Gamma\left((-s-\mu_j)/2\right)} \mp i \prod_{j=1}^{3} \frac{\Gamma\left((2+s+\mu_j)/2\right)}{\Gamma\left((-s-\mu_j+1)/2\right)} \right).$$

First, we study the integral transform in (5.1). By Stirling's formula, for $\sigma \geq -1/2$,

$$\gamma_\pm(\sigma+i\tau) \ll_{\pi,\sigma} (1+|\tau|)^{3(\sigma+1/2)}.$$

Moreover, for $s=\sigma+i\tau$,

$$\widetilde{\phi}(-s) = N^{-s} \widetilde{V}\left(\frac{\zeta N}{aqp^\lambda}, -s\right) \ll N^{-\sigma} \min\left\{1, \left(\frac{Q}{q|\tau|}\right)^j\right\},$$

for any $j \geq 0$, where $\widetilde{V}(r,s) = \int_0^\infty V(y) e(-ry) y^{s-1} \mathrm{d}y$. Thus

$$\Phi_\phi^{\pm}(y) \ll \left(\frac{Q}{q}\right)^{5/2} \left(\frac{q^3 N y}{Q^3}\right)^{-\sigma}.$$

Thus $\Phi_\phi^{\pm}\left(n_1^2 n_2/q^{*3}\right)$ on the right hand side of (5.1) gives arbitrary power saving in $\mathfrak{q}$ if $n_1^2 n_2 \geq N^\varepsilon q^{*3} Q^3/q^3 N$ for any $\varepsilon > 0$. For small values of $n_1^2 n_2$, we move the integration line to $\sigma = -1/2$ to get

$$\Phi_\phi^{\pm}\left(\frac{n_1^2 n_2}{q^{*3}}\right) := \left(\frac{N n_1^2 n_2}{q^{*3}}\right)^{1/2} \mathfrak{J}^{\pm}\left(\frac{n_1^2 n_2}{q^{*3}}, a, q, \zeta\right),$$

where

$$\mathfrak{J}^{\pm}(y,a,q,\zeta) = \frac{1}{2\pi} \int_{\mathbb{R}} (Ny)^{-i\tau} \gamma_\pm\left(-\frac{1}{2}+i\tau\right) \widetilde{V}\left(\frac{\zeta N}{aqp^\lambda}, \frac{1}{2}-i\tau\right) \mathrm{d}\tau. \tag{5.2}$$

Therefore,

$$\begin{aligned} \mathscr{B} \;=\; & \frac{N^{1/2}}{q^{*1/2}} \sum_{\pm} \sum_{n_1|q^*} \sum_{n_1^2 n_2 \le N^\varepsilon q^{*3} Q^3/q^3 N} \frac{A_\pi(n_2,n_1)}{\sqrt{n_2}} S\left(\overline{a^*}, \pm n_2; \frac{q^*}{n_1}\right) \\ & \times \mathfrak{J}^{\pm}\left(\frac{n_1^2 n_2}{q^{*3}}, a, q, \zeta\right) + O(\mathfrak{q}^{-2018}). \end{aligned} \tag{5.3}$$

Furthermore, by the second derivative test for exponential integrals,

$$\widetilde{V}\left(\frac{\zeta N}{aqp^\lambda}, \frac{1}{2}-i\tau\right) \ll (1+|\tau|)^{-1/2}.$$

It follows that

$$\mathfrak{J}^{\pm}(y,a,q,\zeta) \ll N^{\varepsilon}\sqrt{\frac{Q}{q}}. \tag{5.4}$$

Lemma 2 follows from (5.3) and (5.4).

## 6. Character sums

In this section we estimate the character sums in (3.14) and (3.17)

$$\mathfrak{C}^* = \sum_{\beta (\mathrm{mod}\ \widehat{q_1}\widehat{q_2}\widehat{p_r})} \mathfrak{C}_r\left(m_1, n_1', n_1'', \beta, a_1, q_1\right) \overline{\mathfrak{C}_r\left(m_2, n_1', n_1'', \beta, a_2, q_2\right)} e\left(\frac{n_2\beta}{\widehat{q_1}\widehat{q_2}\widehat{p_r}}\right)$$

and

$$\mathfrak{B}^* = \sum_{\beta (\mathrm{mod}\ \widehat{q_1}\widehat{q_2}\widehat{\rho_s})} \mathfrak{B}_s\left(m_1, n_1', n_1'', \beta, a_1, q_1\right) \overline{\mathfrak{B}_s\left(m_2, n_1', n_1'', \beta, a_2, q_2\right)} e\left(\frac{n_2\beta}{\widehat{q_1}\widehat{q_2}\widehat{\rho_s}}\right),$$

where $\widehat{q} = q/n_1'$, $\widehat{p_r} = p^{\lambda - r}/n_1''$, $\widehat{\rho_s} = p^{\lambda+s}/n_1''$, $n_2 \in \mathbb{Z}$, $\mathfrak{C}_r(m, n_1', n_1'', \beta, a, q)$ and $\mathfrak{B}_s(m, n_1', n_1'', \beta, a, q)$ are defined in (3.11) and (3.13), respectively. Write $\beta = \widehat{q_1}\widehat{q_2}\overline{\widehat{q_1}\widehat{q_2}}b_1 + \widehat{p_r}\overline{\widehat{p_r}}b_2$, where $b_1\ (\mathrm{mod}\ \widehat{p_r})$ and $b_2\ (\mathrm{mod}\ \widehat{q_1}\widehat{q_2})$. We obtain

$$\mathfrak{C}^* = \mathfrak{C}_1^*\mathfrak{C}_2^*,$$

where

$$\mathfrak{C}_1^* = \sum_{b (\mathrm{mod}\ \widehat{q_1}\widehat{q_2})} S\left(a_1\overline{\varpi_{q_1}^r \widehat{p_r}}, b\overline{\widehat{p_r}}; \widehat{q_1}\right) S\left(a_2\overline{\varpi_{q_2}^r \widehat{p_r}}, b\overline{\widehat{p_r}}; \widehat{q_2}\right) e\left(\frac{n_2\overline{\widehat{p_r}}b}{\widehat{q_1}\widehat{q_2}}\right)$$

and

$$\begin{aligned}\mathfrak{C}_2^* \quad = \quad & \sum_{b (\mathrm{mod}\ \widehat{p_r})} \sum_{c_1 (\mathrm{mod}\ p^{\lambda - r})} \overline{\chi}\left(m_1 - c_1 p^{\kappa - \lambda + r}\right) S\left(\overline{c_1}\overline{\widehat{q_1}}, b\overline{\widehat{q_1}}; \widehat{p_r}\right) \\ & \sum_{c_2 (\mathrm{mod}\ p^{\lambda - r})} \chi\left(m_2 - c_2 p^{\kappa - \lambda + r}\right) S\left(\overline{c_2}\overline{\widehat{q_2}}, b\overline{\widehat{q_2}}; \widehat{p_r}\right) e\left(\frac{\overline{\widehat{q_1}\widehat{q_2}}bn_2}{\widehat{p_r}}\right).\end{aligned}$$

Similarly,

$$\mathfrak{B}^* = \mathfrak{B}_1^*\mathfrak{B}_2^*,$$

where

$$\mathfrak{B}_1^* = \sum_{b (\mathrm{mod}\ \widehat{q_1}\widehat{q_2})} S\left(a_1\overline{\widehat{\rho_s}}, b\overline{\widehat{\rho_s}}; \widehat{q_1}\right) S\left(a_2\overline{\widehat{\rho_s}}, b\overline{\widehat{\rho_s}}; \widehat{q_2}\right) e\left(\frac{n_2\overline{\widehat{\rho_s}}b}{\widehat{q_1}\widehat{q_2}}\right)$$

and

$$\begin{aligned}\mathfrak{B}_2^* \quad = \quad & \sum_{b (\mathrm{mod}\ \widehat{\rho_s})} \sum_{c_1 (\mathrm{mod}\ p^{\lambda})} \overline{\chi}\left(\frac{m_1 - (\overline{a_1} + c_1 p^s)p^{\kappa-\lambda}}{p^s}\right) S\left(\overline{\overline{a_1} + c_1 p^s}\overline{\widehat{q_1}}, b\overline{\widehat{q_1}}; \widehat{\rho_s}\right) \\ & \sum_{c_2 (\mathrm{mod}\ p^{\lambda})} \chi\left(\frac{m_2 - (\overline{a_2} + c_2 p^s)p^{\kappa-\lambda}}{p^s}\right) S\left(\overline{\overline{a_2} + c_2 p^s}\overline{\widehat{q_2}}, b\overline{\widehat{q_2}}; \widehat{\rho_s}\right) e\left(\frac{\overline{\widehat{q_1}\widehat{q_2}}bn_2}{\widehat{\rho_s}}\right).\end{aligned}$$

We quote the following estimates for $\mathfrak{C}_1^*$ and $\mathfrak{B}_1^*$ which were proved in [14] by induction.

**Lemma 5.** *We have*

$$\mathfrak{C}_1^*, \mathfrak{B}_1^* \ll \widehat{q_1}\widehat{q_2}(\widehat{q_1}, \widehat{q_2}, n_2).$$

*Moreover, for $n_2 = 0$, the character sums vanish unless $q_1 = q_2$ in which case*

$$\mathfrak{C}_1^*, \mathfrak{B}_1^* \ll \widehat{q_1}^2(\widehat{q_1}, m_1 - m_2).$$

For $\mathfrak{C}_2^*$ and $\mathfrak{B}_2^*$, we will prove the following results.

**Lemma 6.** *Assume $\lambda \leq 2\kappa/3$. Let $p^k \| n_2$ with $k \geq 0$.*
*(1) We have*

$$\mathfrak{C}_2^* \ll \widehat{p_r}^2 p^{2(\lambda - r)}. \tag{6.1}$$

*Moreover, for $n_2 = 0$, we have*

$$\mathfrak{C}_2^* \ll \widehat{p_r} p^{2(\lambda - r)}. \tag{6.2}$$

*(2) For $n_1'' = p^{\lambda - r}$ or $n_1'' = p^{\lambda - r - 1}$ with $\lambda - r \geq 2$, we have*

$$\mathfrak{C}_2^* = 0.$$

*(3) For $p^{\lambda - r}/n_1'' \geq p^2$, we have $\mathfrak{C}_2^*$ vanishes unless $n_1'' = 1$. Moreover, let $\lambda - r = 2\alpha + \delta$ with $\delta = 0$ or $1$. For $n_2 = 0$, $\mathfrak{C}_2^*$ vanishes unless $m_1 q_1^2 \equiv m_2 q_2^2 (mod\, p^\alpha)$. For $n_2 \neq 0$, we have*

$$\mathfrak{C}_2^* \ll p^{5(\lambda - r)/2 + \min\{k, \alpha\} + 3\delta/2}. \tag{6.3}$$

**Lemma 7.** *Assume $\lambda \leq 2\kappa/3$. Let $p^k \| n_2$ with $k \geq 0$.*
*(1) We have $\mathfrak{B}_2^*$ vanishes unless $n_1'' = 1$ and*

$$\mathfrak{B}_2^* \ll \widehat{\rho_s}^2 p^{2\lambda}.$$

*Moreover, for $n_2 = 0$, we have $a_2 \equiv \widehat{q_2}^2 \overline{\widehat{q_1}^2} a_1 (\text{mod } p^s)$ in which case*

$$\mathfrak{B}_2^* \ll \widehat{\rho_s} p^{2\lambda + s}.$$

*(2) Let $\lambda = 2\alpha + \delta$ with $\delta = 0$ or 1. For $n_2 = 0$, we have $\mathfrak{B}_2^*$ vanishes unless $q_1^2 m_1/p^s \equiv q_2^2 m_2/p^s (\text{mod } p^\alpha)$. For $n_2 \neq 0$, we have*

$$\mathfrak{B}_2^* \ll p^{5\lambda/2 + 4s + \min\{k, \alpha\} + 3\delta/2}.$$

Now Lemma 3 follows from Lemmas 5 and 6, and Lemma 4 follows from Lemmas 5 and 7. We only prove Lemma 6 for $\mathfrak{C}_2^*$ in detail, since the proof of Lemma 7 is very similar.

*Proof.* (1) Trivially, (6.1) follows from Weil's bound for Kloosterman sums. To prove (6.2), we open the Kloosterman sums and sum over $b$ to get

$$\begin{aligned} \mathfrak{C}_2^* &= \widehat{p_r} \sum_{c_1 (\text{mod } p^{\lambda - r})} \overline{\chi}\left(m_1 - c_1 p^{\kappa - \lambda + r}\right) \sum_{c_2 (\text{mod } p^{\lambda - r})} \chi\left(m_2 - c_2 p^{\kappa - \lambda + r}\right) \\ &\quad \sideset{}{^*}\sum_{d (\text{mod } \widehat{p_r})} e\left(\frac{\overline{\widehat{q_2} c_2} - \widehat{q_2}\overline{\widehat{q_1}\left(\widehat{q_1} + n_2 d\right) c_1}}{\widehat{p_r}} d\right). \end{aligned} \tag{6.4}$$

For $n_2 = 0$, we denote $m_0 = \overline{\widehat{q_2}c_2} - \widehat{q_2}\overline{\widehat{q_1}^2 c_1}$. Since the Ramanujan sum

$$S(m,0;c) = \sideset{}{^*}\sum_{\alpha (\bmod c)} \exp(2\pi i m\alpha/c) = \mu\left(c/(m,c)\right)\varphi(c)/\varphi\left(c/(m,c)\right),$$

where $\mu$ is the Möbius function and $\varphi$ is the Euler function, the last sum over $d$ for $n_2 = 0$ is

$$\mu\left(\frac{\widehat{p_r}}{(m_0,\widehat{p_r})}\right)\frac{\varphi(\widehat{p_r})}{\varphi\left(\widehat{p_r}/(m_0,\widehat{p_r})\right)} = \begin{cases} \widehat{p_r}(1-p^{-1}), & \text{if } (m_0,\widehat{p_r}) = \widehat{p_r} \\ -\widehat{p_r}/p, & \text{if } (m_0,\widehat{p_r}) = \widehat{p_r}/p \\ 0, & \text{otherwise.} \end{cases}$$

Thus (6.2) follows.

(2) Let $\lambda - r = 2\alpha + \delta$ with $\delta = 0$ or $1$, $\alpha \geq 1$ is a positive integer. Write $c_1 = b_1 p^{\alpha+\delta} + b_2$, $b_1 (\bmod p^\alpha)$, $b_2 (\bmod p^{\alpha+\delta})$ and $c_2 = h_1 p^{\alpha+\delta} + h_2$, $h_1 (\bmod p^\alpha)$, $h_2 (\bmod p^{\alpha+\delta})$. If $n_1'' = p^{\lambda-r-1}$, we have $\widehat{p_r} = p$ and

$$\begin{aligned} \mathfrak{C}_2^* \;=\; & p \sum_{b_2 (\bmod p^{\alpha+\delta})} \sum_{h_2 (\bmod p^{\alpha+\delta})} \overline{\chi}\left(m_1 - b_2 p^{\kappa-2\alpha-\delta}\right)\chi\left(m_2 - h_2 p^{\kappa-2\alpha-\delta}\right) \\ & \sideset{}{^*}\sum_{d (\bmod p)} e\left(\frac{\overline{\widehat{q_2}h_2} - \widehat{q_2}\overline{\widehat{q_1}\left(\widehat{q_1}+n_2 d\right)b_2}}{p} d\right) \sum_{b_1 (\bmod p^\alpha)} \chi\left(1 + \overline{m_1 - b_2 p^{\kappa-2\alpha-\delta}}\, p^{\kappa-\alpha} b_1\right) \\ & \sum_{h_1 (\bmod p^\alpha)} \chi\left(1 - \overline{m_2 - h_2 p^{\kappa-2\alpha}}\, p^{\kappa-\alpha} h_1\right). \end{aligned}$$

Recall $\chi$ is a primitive character of modulus $p^\kappa$ and $\kappa > \lambda \geq 2\alpha$. Thus $\chi(1 + zp^{\kappa-\alpha})$ is an additive character to modulus $p^\alpha$, so there exists an integer $\eta$ (uniquely determined modulo $p^\alpha$), $(\eta, p) = 1$, such that $\chi(1 + zp^{\kappa-\alpha}) = \exp(2\pi i \eta z/p^\alpha)$. Therefore, $\mathfrak{C}_2^* = 0$. For $n_1'' = p^{\lambda-r}$, the proof is similar and easier.

(3) Write $p^{\lambda-r} = p^{2\alpha+\delta_1}$, $\widehat{p_r} = p^{\lambda-r}/n_1'' = p^{2\beta+\delta_2}$, $\delta_1 = 0$ or $1$, $\delta_2 = 0$ or $1$, $\alpha \geq 1$ and $\beta \geq 1$. Write $c_1 = b_1 p^{\alpha+\delta_1} + b_2$, $b_1 (\bmod p^\alpha)$, $b_2 (\bmod p^{\alpha+\delta_1})$, $c_2 = h_1 p^{\alpha+\delta_1} + h_2$, $h_1 (\bmod p^\alpha)$, $h_2 (\bmod p^{\alpha+\delta_1})$, and $d = d_1 p^{\beta+\delta_2} + d_2$, $d_1 (\bmod p^\beta)$, $d_2 (\bmod p^{\beta+\delta_2})$. Then by (6.4), we have

$$\begin{aligned} \mathfrak{C}_2^* \;=\; & p^{2\beta+\delta_2} \sum_{b_2 (\bmod p^{\alpha+\delta_1})} \sum_{h_2 (\bmod p^{\alpha+\delta_1})} \sideset{}{^*}\sum_{d_2 \left(\bmod p^{\beta+\delta_2}\right)} \sum_{b_1 (\bmod p^\alpha)} \sum_{h_1 (\bmod p^\alpha)} \sum_{d_1 \left(\bmod p^\beta\right)} \\ & \overline{\chi}\left(m_1 - (b_2 + b_1 p^{\alpha+\delta_1}) p^{\kappa-2\alpha-\delta_1}\right)\chi\left(m_2 - (h_2 + h_1 p^{\alpha+\delta_1}) p^{\kappa-2\alpha-\delta_1}\right) \\ & e\left(\frac{\overline{\widehat{q_2}(h_2 + h_1 p^{\alpha+\delta_1})} - \widehat{q_2}\overline{\widehat{q_1}\left(\widehat{q_1} + n_2 d_2 + n_2 d_1 p^{\beta+\delta_2}\right)\left(b_2 + b_1 p^{\alpha+\delta_1}\right)}}{p^{2\beta+\delta_2}}(d_2 + d_1 p^{\beta+\delta_2})\right). \end{aligned}$$

Note that $\kappa > \lambda \geq 2\alpha + \delta_1 \geq 2\beta + \delta_2$ and $\overline{a + bp^\alpha} \equiv \overline{a}(1 - \overline{a}bp^\alpha) (\bmod p^{2\alpha})$. Thus

$$\mathfrak{C}_2^* = p^{2\alpha+3\beta+\delta_2} \sum_{b_2 (\bmod p^{\alpha+\delta_1})} \sum_{h_2 (\bmod p^{\alpha+\delta_1})} \sideset{}{^*}\sum_{\substack{d_2 \left(\bmod p^{\beta+\delta_2}\right) \\ \overline{(\widehat{q_1}+n_2 d_2)^2 b_2}\,\widehat{q_2}^2 n_2 d_2 - \overline{(\widehat{q_1}+n_2 d_2) b_2}\,\widehat{q_2}^2 + \widehat{q_1}\overline{h_2} \equiv 0 (\bmod p^\beta)}} f(b_2, h_2, d_2)\mathcal{C}_1\mathcal{C}_2, \tag{6.5}$$

where

$$\begin{aligned}\mathcal{C}_1 &= \frac{1}{p^\alpha}\sum_{b_1(\mathrm{mod}\ p^\alpha)}\chi\left(1+\overline{m_1-b_2p^{\kappa-2\alpha-\delta_1}}p^{\kappa-\alpha}b_1\right)e\left(\frac{\widehat{q_2}\overline{\widehat{q_1}(\widehat{q_1}+n_2d_2)b_2^2}d_2n_1''}{p^\alpha}b_1\right),\\ \mathcal{C}_2 &= \frac{1}{p^\alpha}\sum_{h_1(\mathrm{mod}\ p^\alpha)}\chi\left(1-\overline{m_2-h_2p^{\kappa-2\alpha-\delta_1}}p^{\kappa-\alpha}h_1\right)e\left(\frac{-\overline{\widehat{q_2}h_2^2}d_2n_1''}{p^\alpha}h_1\right)\end{aligned}$$

and

$$f(b_2,h_2,d_2)=\overline{\chi}\left(m_1-b_2p^{\kappa-2\alpha-\delta_1}\right)\chi\left(m_2-h_2p^{\kappa-2\alpha-\delta_1}\right)e\left(\frac{\overline{\widehat{q_2}h_2}-\widehat{q_2}\overline{\widehat{q_1}(\widehat{q_1}+n_2d_2)b_2}}{p^{2\beta+\delta_2}}d_2\right).$$

Since $\chi(1+zp^{\kappa-\alpha})=\exp(2\pi i\eta z/p^\alpha)$ with $(\eta,p)=1$, we have

$$\begin{aligned}\mathcal{C}_1 &= \frac{1}{p^\alpha}\sum_{b_1(\mathrm{mod}\ p^\alpha)}e\left(\frac{\overline{m_1-b_2p^{\kappa-2\alpha-\delta_1}}\eta}{p^\alpha}b_1\right)e\left(\frac{\widehat{q_2}\overline{\widehat{q_1}(\widehat{q_1}+n_2d_2)b_2^2}d_2n_1''}{p^\alpha}b_1\right)\\ &= \mathbf{1}_{\overline{m_1-b_2p^{\kappa-2\alpha-\delta_1}}\eta+\widehat{q_2}\overline{\widehat{q_1}(\widehat{q_1}+n_2d_2)b_2^2}d_2n_1''\equiv 0(\mathrm{mod}\ p^\alpha)}.\end{aligned}$$

Thus $\mathcal{C}_1$ vanishes unless $n_1''=1$ which in turn implies that $\alpha=\beta$ and $\delta_1=\delta_2$. Moreover, by taking $\lambda\le 2\kappa/3$, we have $\kappa\ge 3\alpha+2\delta_1$. Hence $\mathcal{C}_1$ vanishes unless $\overline{m_1}\eta+\widehat{q_2}\overline{\widehat{q_1}(\widehat{q_1}+n_2d_2)b_2^2}d_2\equiv 0(\mathrm{mod}\,p^\alpha)$. Similarly,

$$\mathcal{C}_2=\mathbf{1}_{\overline{m_2}\eta+\overline{\widehat{q_2}h_2^2}d_2\equiv 0(\mathrm{mod}\ p^\alpha)}.$$

Plugging these into (6.5) we obtain

$$\mathfrak{C}_2^* = p^{5\alpha+\delta_1}\sum_{\substack{b_2(\mathrm{mod}\ p^{\alpha+\delta_1})\\ \overline{(\widehat{q_1}+n_2d_2)^2b_2}\widehat{q_2}^2n_2d_2-\overline{(\widehat{q_1}+n_2d_2)b_2}\widehat{q_2}^2+\widehat{q_1}\overline{h_2}\equiv 0(\mathrm{mod}\ p^\alpha)\\ \overline{m_1}\eta+\widehat{q_2}\overline{\widehat{q_1}(\widehat{q_1}+n_2d_2)b_2^2}d_2\equiv 0(\mathrm{mod}\,p^\alpha)\\ \overline{m_2}\eta+\overline{\widehat{q_2}h_2^2}d_2\equiv 0(\mathrm{mod}\,p^\alpha)}}\ \sum_{h_2(\mathrm{mod}\ p^{\alpha+\delta_1})}\ \sideset{}{^*}\sum_{d_2\left(\mathrm{mod}\ p^{\alpha+\delta_1}\right)} f(b_2,h_2,d_2). \tag{6.6}$$

To count the numbers of $b_2, h_2$ and $d_2$, we solve the three congruence equations in (6.6).

(i) If $n_2=0$ or $n_2=p^kn_2'$ with $(n_2',p)=1$ and $p^k\ge p^\alpha$, we have

$$\begin{cases} h_2\equiv \overline{\widehat{q_2}}^2\widehat{q_1}^2b_2(\mathrm{mod}\ p^\alpha)\\ d_2\equiv -\overline{m_1}\eta\widehat{q_1}^2\overline{\widehat{q_2}}b_2^2(\mathrm{mod}\,p^\alpha)\\ d_2\equiv -\overline{m_2}\eta\widehat{q_2}h_2^2(\mathrm{mod}\,p^\alpha)\end{cases}$$

By the last two equations, one sees that $\mathfrak{C}_2^*$ vanishes unless $m_1\widehat{q_1}^2\equiv m_2\widehat{q_2}^2(\mathrm{mod}\,p^\alpha)$. Moreover, for fixed $b_2$, $h_2$ and $d_2$ are uniquely determined modulo $p^\alpha$. Therefore,

$$\mathfrak{C}_2^*\ll p^{6\alpha+4\delta_1}\ll p^{3(\lambda-r)+\delta_1}. \tag{6.7}$$

(ii) If $n_2\ne 0$, we let $n_2=p^kn_2'$ with $(n_2',p)=1$ and $p^k<p^\alpha$, and let $\gamma=\overline{\widehat{q_1}+n_2d_2}$. Then $d_2\equiv\overline{n_2'}(\overline{\gamma}-\widehat{q_1})/p^k\left(\mathrm{mod}\ p^{\alpha-k}\right)$ and the three equations give

$$\begin{cases} b_2\equiv\widehat{q_2}^2\gamma^2h_2(\mathrm{mod}\ p^\alpha),\\ \gamma\equiv\overline{\widehat{q_1}}\left(1+\overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}n_2b_2^2\right)(\mathrm{mod}\ p^\alpha),\\ \overline{\gamma}\equiv\widehat{q_1}\left(1-\overline{m_2}\eta\overline{\widehat{q_1}}\widehat{q_2}n_2h_2^2\right)(\mathrm{mod}\ p^\alpha).\end{cases} \tag{6.8}$$

Plugging the second equation into the first equation in (6.8) we get

$$b_2 \equiv \widehat{q_2}^2 \overline{\widehat{q_1}}^2 \left(1 + \overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}n_2 b_2^2\right)^2 h_2 (\bmod\ p^\alpha). \tag{6.9}$$

By (6.9) and the last two equations in (6.8) we get

$$\left(\overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}\right)^5 u^5 + 4\left(\overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}\right)^4 u^4 + 6\left(\overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}\right)^3 u^3 + 4\left(\overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}\right)^2 u^2$$
$$-\overline{m_1 m_2}\eta^2\widehat{q_1}^4\overline{\widehat{q_2}}^4 u^2 + \overline{m_1}\eta\widehat{q_1}\overline{\widehat{q_2}}u - \overline{m_2}\eta\widehat{q_1}^3\overline{\widehat{q_2}}^3 u \equiv 0 (\bmod\ p^\alpha),$$

where $u = n_2 b_2^2$. Thus there are at most 5 roots modulo $p^\alpha$ for $u$. Therefore, there are at most 10 roots modulo $p^{\alpha-k}$ for $b_2$. For fixed $u$, $\gamma$ is uniquely determined modulo $p^\alpha$ and for fixed $\gamma$ and $b_2$, $h_2$ is uniquely determined modulo $p^\alpha$ by the first equation in (6.8). Then by the last congruence equation in (6.6), $d_2$ is uniquely determined modulo $p^\alpha$. Therefore,

$$\mathfrak{C}_2^* \ll p^{5\alpha+k+4\delta_1} \ll p^{5(\lambda-r)/2+k+3\delta_1/2}.$$

By (6.7) and (6.10), the bound in (6.3) follows.

□

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School of Mathematics and Statistics, Shandong University, Weihai, Weihai, Shandong 264209, China

*E-mail address:* qfsun@sdu.edu.cn

*E-mail address:* zhaoruisdu@gmail.com