## NONCOMMUTATIVE HOMOLOGICAL PROJECTIVE DUALITY

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Abstract. We generalize Kuznetsov's theory of homological projective duality to the setting of noncommutative algebraic geometry. Simultaneously, we develop the theory over general base schemes, and remove the usual smoothness, properness, and transversality hypotheses.

## **CONTENTS**



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## 1. INTRODUCTION

<span id="page-1-0"></span>The purpose of this paper is twofold. The first goal is to set up a robust framework for noncommutative algebraic geometry, using recent advances in higher category theory. The second and main goal, which builds on the first, is to generalize Kuznetsov's theory [\[9\]](#page-71-1) of homological projective duality (HPD) to this setting. These results are crucial in our work [\[15\]](#page-71-2) with Kuznetsov on categorical joins, which was one of the main motivations for this paper.

In this introduction, we focus on the HPD part of our work.

<span id="page-1-1"></span>1.1. Kuznetsov's HPD. The input for HPD is a smooth projective *Lefschetz variety*, which consists of the following data:

- $-$  A smooth projective variety X over an algebraically closed field of characteristic 0, with a morphism  $X \to \mathbf{P}(V)$  to a projective space.
- A right Lefschetz decomposition of the category  $\text{Perf}(X)$  of perfect complexes.

Let  $\mathcal{O}_X(H)$  denote the pullback of  $\mathcal{O}_{\mathbf{P}(V)}(1)$ . A right Lefschetz decomposition of Perf $(X)$  is a semiorthogonal decomposition of the form

$$
Perf(X) = \langle A_0, A_1(H), \ldots, A_{m-1}((m-1)H) \rangle,
$$

where  $A_0 \supset A_1 \supset \cdots \supset A_{m-1}$  is a descending chain of categories and  $A_i(iH)$  denotes the image of  $A_i$  under the autoequivalence of Perf(X) given by tensoring with  $O_X(iH)$ . Many varieties admit interesting decompositions of this form, see [\[13\]](#page-71-3) for a survey.

The key property of a Lefschetz decomposition — from which its name derives, by analogy with the Lefschetz hyperplane theorem — is that it induces semiorthogonal decompositions of the base changes of X along linear subspaces  $L \subset V$ . More precisely, let s denote the codimension of L in V. Then if  $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$  has the expected dimension dim(X) – s, pullback along  $X \times_{\mathbf{P}(V)} \mathbf{P}(L) \to X$  embeds the categories  $\mathcal{A}_i(iH)$  into  $\text{Perf}(X \times_{\mathbf{P}(V)} \mathbf{P}(L))$ for  $i \geq s$ , and there is a semiorthogonal decomposition

<span id="page-2-0"></span>
$$
\text{Perf}(X \times_{\mathbf{P}(V)} \mathbf{P}(L)) = \langle \mathcal{K}_L(X), \mathcal{A}_s(H), \mathcal{A}_{s+1}(2H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle. \tag{1.1}
$$

The category  $\mathcal{K}_L(X)$  should be thought of as the "interesting component" of the category of perfect complexes on  $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$  — it is what is left after removing the pieces coming from the ambient variety X.

Given the above data, Kuznetsov constructs the *HPD category*  $\text{Perf}(X)^{\sharp}$ , which is a  $\mathbf{P}(V^{\vee})$ linear category that can be thought of as a total space for the categories  $\mathcal{K}_L(X)$  as L ranges over the hyperplanes in V. The main theorem of [\[9\]](#page-71-1) shows that if  $\text{Perf}(X)^{\natural}$  is geometric, i.e. if there exists a variety Y together with a morphism  $Y \to \mathbf{P}(V^{\vee})$  and a  $\mathbf{P}(V^{\vee})$ -linear equivalence  $\text{Perf}(Y) \simeq \text{Perf}(X)^{\natural}$ , then  $\text{Perf}(Y)$  has a natural Lefschetz decomposition such that the "interesting components" of orthogonal linear sections of  $X$  and  $Y$  are equivalent. To be precise, we must also assume that the Lefschetz decomposition of X is *moderate*, in the sense that its length satisfies m satisfies  $m < dim(V)$ . This condition is quite mild and essentially always satisfied in practice, see Remark [6.12.](#page-40-1) Then if  $\mathcal{O}_Y(H')$  denotes the pullback of  $\mathcal{O}_{\mathbf{P}(V^{\vee})}(1)$ , the theorem is as follows.

<span id="page-2-2"></span>**Theorem 1.1** ([\[9\]](#page-71-1)). *The category*  $\text{Perf}(Y)$  *admits a natural left Lefschetz decomposition, i.e. a semiorthogonal decomposition*

$$
Perf(Y) = \langle \mathcal{B}_{1-n}((1-n)H'), \ldots, \mathcal{B}_{-1}(-H'), \mathcal{B}_0 \rangle
$$

*where*  $B_{1-n}$  ⊂ · · · ⊂  $B_{-1}$  ⊂  $B_0$ *, such that the following holds. Let*  $L \subset V$  *be a linear subspace,*  $let L^{\perp} = \text{ker}(V^{\vee} \to L^{\vee})$  *be its orthogonal, and let*  $r = \dim(L)$  *and*  $s = \dim(L^{\perp})$ *. Assume the fiber products*  $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$  *and*  $Y \times_{\mathbf{P}(V^{\vee})} \mathbf{P}(L^{\perp})$  *are of expected dimension. Let* 

<span id="page-2-1"></span>
$$
\operatorname{Perf}(Y \times_{\mathbf{P}(V^{\vee})} \mathbf{P}(L^{\perp})) = \langle \mathcal{B}_{1-n}((r-n)H'), \dots, \mathcal{B}_{-r-1}(-2H'), \mathcal{B}_{-r}(-H'), \mathcal{K}'_{L^{\perp}}(Y) \rangle \tag{1.2}
$$

*be the induced semiorthogonal decomposition analogous to* [\(1.1\)](#page-2-0)*. Then there is an equivalence of categories*

$$
\mathcal{K}_L(X) \simeq \mathcal{K}'_{L^{\perp}}(Y).
$$

<span id="page-2-3"></span>Remark 1.2. Kuznetsov also proves a version of the theorem for the bounded derived categories of coherent sheaves on  $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$  and  $Y \times_{\mathbf{P}(V^{\vee})} \mathbf{P}(L^{\perp})$ . Namely, these categories have semiorthogonal decompositions analogous to  $(1.1)$  and  $(1.2)$ , with equivalent "interesting components".

Theorem [1.1](#page-2-2) is the source of many semiorthogonal decompositions and derived equivalences in algebraic geometry, see [\[13,](#page-71-3) [25\]](#page-71-4) for surveys.

Finally, as the terminology suggests, HPD is closely related to classical projective duality. Namely, define the projective dual  $X^{\vee} \subset \mathbf{P}(V^{\vee})$  of  $X \to \mathbf{P}(V)$  to be the locus of hyperplanes  $H \in \mathbf{P}(V^{\vee})$  such that the base change  $X \times_{\mathbf{P}} H$  is singular; this reduces to the usual notion

when  $X \to \mathbf{P}(V^{\vee})$  is a closed immersion. Then Kuznetsov shows [\[9,](#page-71-1) Theorem 7.9] that for Y as above, the locus of critical values of the morphism  $Y \to \mathbf{P}(V^{\vee})$  coincides with  $X^{\vee}$ .

<span id="page-3-0"></span>1.2. **Noncommutative HPD.** In practice, the HPD category  $\text{Perf}(X)^{\natural}$  is often *not* geometric in the above sense. Instead, usually the best one can hope for is that  $\text{Perf}(X)^{\natural}$  is "close to geometric", e.g. equivalent to the bounded derived category of coherent R-modules  $D_{\text{coh}}^{\text{b}}(Y,\mathcal{R})$  on a variety Y equipped with a sheaf of finite (noncommutative)  $\mathcal{O}_Y$ -algebras R (see [\[13\]](#page-71-3) for examples), or to the derived category of a gauged Landau-Ginzburg model (see [\[2\]](#page-71-5) for examples).

In general, the HPD category  $\text{Perf}(X)^{\natural}$  only has the structure of a "noncommutative scheme" over  $\mathbf{P}(V^{\vee})$ , in the sense of Kontsevich. More precisely, using Lurie's foundational work [\[18\]](#page-71-6), we define a "noncommutative scheme" over a scheme  $S$  to be a small idempotentcomplete stable  $\infty$ -category C, equipped with a Perf(S)-module structure; for short, we say C is an S*-linear category*. To orient the reader, we note that the category of perfect complexes on an S-scheme X is an S-linear category, where  $\text{Perf}(S)$  acts on  $\text{Perf}(X)$  via pullback followed by tensor product.

The first goal of this paper is to develop some foundational material about linear categories. We leave the detailed discussion of our results in this direction to the main text.

The second and main goal, which builds on the first, is to generalize HPD to the case of categories linear over a fixed (quasi-compact and separated) base scheme S. Namely, for a vector bundle  $V$  on  $S$ , we develop a theory of HPD where:

(1) The input variety X itself is replaced with a  $P(V)$ -linear category.

(2) There are no geometricity assumptions on the  $\mathbf{P}(V^{\vee})$ -linear HPD category.

The existence of such a version of HPD is essential for our work [\[15\]](#page-71-2) with Kuznetsov, which was one of the main motivations for this paper.

The input for our theory is a *Lefschetz category over*  $P(V)$ , which is a  $P(V)$ -linear category A equipped with a Lefschetz structure. By definition, a Lefschetz structure consists of a pair of right and left Lefschetz decompositions

$$
\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle,
$$
  

$$
\mathcal{A} = \langle \mathcal{A}_{1-m}((1-m)H), \dots, \mathcal{A}_{-1}(-H), \mathcal{A}_0 \rangle,
$$

where

$$
0\subset \mathcal{A}_{1-m}\subset \cdots \subset \mathcal{A}_{-1}\subset \mathcal{A}_0\supset \mathcal{A}_1\supset \cdots \supset \mathcal{A}_{m-1}\supset 0.
$$

**Remark 1.3.** Technically, we also require the subcategory  $A_i \subset A$  to be right admissible for  $i \geq 0$  and left admissible for  $i \leq 0$ , see Definitions [6.1](#page-36-3) and [6.9.](#page-40-2) If  $\mathcal{A} = \text{Perf}(X)$  for a smooth projective variety X over  $P(V)$ , the above data is in fact equivalent to giving only one of the above Lefschetz decompositions, so a Lefschetz variety in the sense of §[1.1](#page-1-1) gives rise to a Lefschetz category. The two Lefschetz decompositions in the data of a Lefschetz structure give rise to "right" and "left" versions of HPD. We focus on the "right" one below.

Given any subbundle  $L \subset V$ , Lurie's work [\[20,](#page-71-7) [18\]](#page-71-6) can be used to make sense of the base change category

# $\mathcal{A} \otimes_{\text{Perf}(\mathbf{P}(V))} \text{Perf}(\mathbf{P}(L)),$

which plays the role of the linear section of A by  $P(L)$ . Indeed, the results of [\[3\]](#page-71-8) imply that if there exists a morphism of schemes  $X \to \mathbf{P}(V)$  and a  $\mathbf{P}(V)$ -linear equivalence Perf $(X) \simeq A$ ,

then the above base changed category recovers  $\text{Perf}(X \times_{\mathbf{P}(V)} \mathbf{P}(L))$ .<sup>[1](#page-4-0)</sup> Further, there is an induced semiorthogonal decomposition

<span id="page-4-1"></span>
$$
\mathcal{A} \otimes_{\text{Perf}(\mathbf{P}(V))} \text{Perf}(\mathbf{P}(L)) = \langle \mathcal{K}_L(\mathcal{A}), \mathcal{A}_s(H), \mathcal{A}_{s+1}(2H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \tag{1.3}
$$

where s denotes the corank of  $L \subset V$ .

We construct a  $\mathbf{P}(V^{\vee})$ -linear HPD category  $\mathcal{A}^{\natural}$  in the above setup, and prove the following version of Theorem [1.1.](#page-2-2) The "right strong" and "left strong" conditions appearing in the theorem are mild assumptions on the existence of certain adjoints (see Definitions [6.5](#page-39-0) and [6.9\)](#page-40-2), which are automatic if  $\mathcal{A} = \text{Perf}(X)$  for a smooth projective variety over  $\mathbf{P}(V)$ . As in §[1.1,](#page-1-1) we say  $A$  is moderate if its Lefschetz decompositions have length less than rank(V).

<span id="page-4-2"></span>**Theorem 1.4.** Assume A is right strong and moderate. Then the category  $A^{\dagger}$  admits a left *strong, moderate Lefschetz structure, such that if*

$$
\mathcal{A}^\natural=\left<\mathcal{A}^\natural_{1-n}((1-n)H'),\ldots,\mathcal{A}^\natural_{-1}(-H'),\mathcal{A}^\natural_0\right>
$$

*is its left Lefschetz decomposition, then the following holds. Let*  $L \subset V$  *be a subbundle, let*  $L^{\perp} = \text{ker}(V^{\vee} \to L^{\vee})$  *be its orthogonal, and let*  $r = \text{rank}(L)$  *and*  $s = \text{rank}(L^{\perp})$ *. Let* 

 $\mathcal{A}^\natural\otimes_{\operatorname{Perf}({\bf P}(V^\vee))}\operatorname{Perf}({\bf P}(L^\perp))=\left\langle \mathcal{A}_1^\natural\right\rangle$  $\notag_{1-n}^{\natural}((r-n)H'),\ldots,\mathcal{A}_-^{\natural}$  $_{-r-1}^{\natural}(-2H'), \mathcal{A}_{-}^{\natural}$  $\overline{\mathbb{E}}_r(-H'), \mathfrak{K}'_{L^\perp}(\mathcal{A}^\natural)\Big>$ 

*be the induced semiorthogonal decomposition analogous to* [\(1.3\)](#page-4-1)*. Then there is an equivalence of categories*

$$
\mathcal{K}_L(\mathcal{A}) \simeq \mathcal{K}'_{L^{\perp}}(\mathcal{A}^{\natural}).
$$

If the categories A and  $A^{\dagger}$  are geometric, i.e. if  $A \simeq \text{Perf}(X)$  for a  $\mathbf{P}(V)$ -scheme X and  $\mathcal{A}^{\natural} \simeq \text{Perf}(Y)$  for a  $\mathbf{P}(V^{\vee})$ -scheme Y, then Theorem [1.4](#page-4-2) recovers Theorem [1.1.](#page-2-2) In fact, even in this situation, our result is more general than Theorem [1.1](#page-2-2) in several respects. First, we work over a general base S. Second, we do not need any smoothness or projectivity hypotheses on X. Third, using [\[3\]](#page-71-8), Theorem [1.4](#page-4-2) implies Theorem [1.1](#page-2-2) holds without any expected dimension assumptions, provided the fiber products  $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$  and  $Y \times_{\mathbf{P}(V^{\vee})} \mathbf{P}(L^{\perp})$  are taken in the derived sense; this answers a question left open in [\[9\]](#page-71-1). Further, using [\[4\]](#page-71-9) we show that when the base scheme  $S$  is noetherian and defined over a field of characteristic 0, Theorem [1.4](#page-4-2) also implies a similar result for bounded derived categories of coherent sheaves, recovering the result mentioned in Remark [1.2.](#page-2-3)

As in [\[9\]](#page-71-1), we deduce Theorem [1.4](#page-4-2) from a stronger result (Theorem [8.7\)](#page-55-1), which roughly says Theorem [1.4](#page-4-2) holds in families as L varies in  $Gr(r, V)$ . As a consequence, we also prove that HPD is a duality.

Our method of proof is closely modeled on [\[9\]](#page-71-1), but there are several difficulties to overcome in our setting. First, it takes some work to even formulate the objects appearing in the proof. For instance, in the framework of [\[9\]](#page-71-1) where  $A \simeq \text{Perf}(X)$  is geometric, a key role is played by the spaces of universal linear sections of  $X \to \mathbf{P}(V)$ . Our basic observation here is that all of the necessary constructions can be made in the case where  $\mathcal{A} = \text{Perf}(\mathbf{P}(V))$ , and then transported to general A by base change. A more serious obstacle is that Kuznetsov uses Fourier–Mukai kernels in an essential way. Namely, all of the functors arising in his proof are of the form  $\Phi_{\mathcal{E}}$ :  $\text{Perf}(Z_1) \to \text{Perf}(Z_2)$ , where  $Z_1$  and  $Z_2$  are schemes over a base scheme T and  $\Phi_{\mathcal{E}}$  is the functor given by a Fourier–Mukai kernel  $\mathcal{E} \in \text{Perf}(Z_2 \times_T Z_1)$ . There

<span id="page-4-0"></span><sup>&</sup>lt;sup>1</sup>Here,  $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$  denotes the derived fiber product, which agrees with the usual scheme-theoretic fiber product if X and  $P(L)$  are Tor-independent over  $P(V)$ .

is a precise relationship between various operations on the kernel  $\mathcal{E}$  (e.g. pullback along a morphism  $f \times id$ :  $Z'_2 \times_T Z_1 \to Z_2 \times_T Z_1$  where  $f: Z'_2 \to Z_2$  is a morphism) and operations on the associated functor  $\Phi_{\mathcal{E}}$  (e.g. composition with pullback along a morphism  $f: Z_2' \to Z_2$ ). Kuznetsov uses this to deduce facts about functors  $\Phi_{\varepsilon}$  via the geometry of the kernel spaces  $Z_2 \times_T Z_1$ . To import these ideas into our setting, we develop a robust formalism of Fourier– Mukai kernels for certain categories arising from base change. Throughout, we crucially use recent advances in higher category theory and derived algebraic geometry, especially [\[18,](#page-71-6) [3\]](#page-71-8).

Finally, we show that our version of HPD is closely related to classical projective duality. Namely, for any category C linear over a base T, we introduce the notion of its *critical locus*  $Crit_T(\mathcal{C})$ , which is the locus of points in T parameterizing the singular fibers of C. For any  ${\bf P}(V)$ -linear category A which is smooth and proper over S, we use this notion to define the *classical projective dual* CPD( $A$ )  $\subset$  **P**( $V^{\vee}$ ). If A is a Lefschetz category which is smooth and proper, we prove  $CPD(\mathcal{A}) = \text{Crit}_{\mathbf{P}(V^{\vee})}(\mathcal{A}^{\natural}).$ 

<span id="page-5-0"></span>1.3. Further directions. We regard our results as part of a larger program of *homological projective geometry*, whose goal is to find categorical analogs of results from classical projective geometry, and to bring them to bear on the structure of derived categories. In this theory, Lefschetz categories over  $P(V)$  play the role of projective varieties embedded in  $P(V)$ . The prototype of homological projective geometry is HPD. In [\[15\]](#page-71-2) we show the classical notion of the join of projective varieties also fits into this framework. Namely, we introduce categorical joins, and show they are related to HPD in the same way classical joins are related to classical projective duality. An interesting feature of homological projective geometry is that all of the operations which are known so far (i.e. HPD and categorical joins) preserve smoothness of the objects involved, whereas in classical projective geometry this is far from true (i.e. projective duals and joins of smooth varieties are usually singular).

In this paper, we show that HPD can be formulated over quite general base schemes, which need not be defined over a field. Working over a field, there are a number of examples where the HPD category admits a close-to-geometric model [\[13\]](#page-71-3); we believe many of these results should extend to more general base schemes.

Finally, we note that the theory developed here may hold over even more general bases, namely over spectral schemes in the sense of [\[19\]](#page-71-10), e.g. over the sphere spectrum.

<span id="page-5-1"></span>1.4. Related work. In his unpublished habilitation thesis [\[10\]](#page-71-11), Kuznetsov developed a version of HPD which works for input categories  $C$  that are realizable (compatibly with their  ${\bf P}(V)$ -linear structure) as an admissible subcategory of the derived category of a variety. Due to my inadequate Russian I was not able to read [\[10\]](#page-71-11), but its existence served as an inspiration for this paper.

More recently, HPD has been revisited from several points of view. In [\[2\]](#page-71-5), Ballard–Deliu– Favero–Isik–Katzarkov use variation of GIT to realize in certain cases the HPD category as the derived category of a gauged Landau–Ginzburg model, and to give a new proof of Theorem [1.1](#page-2-2) in these cases.

In [\[25\]](#page-71-4), Thomas reproved Theorem [1.1](#page-2-2) using a reinterpretation of Kuznetsov's original proof. He handles the case where  $\text{Perf}(X)^{\natural}$  need not be geometric, but X is required to be a genuine variety (not "noncommutative") and he works with a special class of Lefschetz decompositions (the "rectangular" ones).

During the preparation of this paper, two other works on HPD appeared. In [\[8\]](#page-71-12), Jiang– Leung–Xie build on the argument from [\[25\]](#page-71-4) to prove a generalization of Theorem [1.1,](#page-2-2) where

the the pair  $(\mathbf{P}(L), \mathbf{P}(L^{\perp}))$  is replaced by any HPD pair. We independently discovered (a more general form of) this result, which appears as an application in [\[15\]](#page-71-2).

Finally, in [\[23\]](#page-71-13) Rennemo builds on [\[2\]](#page-71-5) to develop a version of HPD for categories C that are admissible subcategories in the derived category of a smooth quotient stack.

Our results are stronger than the above in several respects: we handle linear categories without any geometricity assumptions; we work over general base schemes and make no smoothness or properness assumptions on our categories; and we work at the enhanced level of stable ∞-categories (which allows us to give natural derived algebraic geometry interpretations in non-transverse situations).

<span id="page-6-0"></span>1.5. Organization of the paper. Part [I](#page-7-0) of this paper is dedicated to foundational material on noncommutative algebraic geometry. We begin in §[2](#page-7-1) by introducing our framework for linear categories. In §[3](#page-12-0) we discuss semiorthogonal decompositions of such categories. In §[4](#page-20-0) we define the notions of smoothness and properness of a linear category, and among other things discuss their behavior under base change and semiorthogonal decompositions. We also introduce the notion of the critical locus of a linear category. In §[5](#page-30-0) we develop our formalism of Fourier–Mukai kernels.

In Part [II](#page-36-0) of the paper, we use the material from Part [I](#page-7-0) to formulate and prove our results on HPD. First, in §[6](#page-36-1) we define Lefschetz categories and describe their behavior under passage to linear sections. In §[7](#page-44-0) we define the HPD category, show that it has a natural Lefschetz structure modulo a generation statement, and explain its relation to classical projective duality. Finally, in §[8](#page-53-0) we prove the main theorem of HPD, i.e. Theorem [1.4](#page-4-2) from above.

<span id="page-6-1"></span>1.6. **Conventions.** All schemes are assumed quasi-compact and separated, and S denotes a base scheme which is fixed throughout the paper. A vector bundle on a scheme  $X$  is a finite locally free  $\mathcal{O}_X$ -module. Given a vector bundle V on X, we denote by

$$
\operatorname{Gr}(r, V) \to X
$$

the Grassmannian parameterizing rank r *subbundles* of V, and we set  $P(V) = Gr(1, V)$ . In particular, the pushforward of  $\mathcal{O}_{\mathbf{P}(V)}(1)$  to X is  $V^{\vee}$ . The categorical conventions used in this paper are discussed in §[2.1-](#page-7-2)[2.3.](#page-9-0) In particular, we note that we always consider derived functors (pullbacks, pushforward, tensor products, etc.), but we write them with underived notation.

A remark on notation: we tend to denote general linear categories by the letters  $C$  or  $D$ , and Lefschetz categories or categories which are linear over a projective bundle by the letters A or B.

<span id="page-6-2"></span>1.7. Acknowledgements. I thank Sasha Kuznetsov for introducing me to the theory of homological projective duality, for many useful conversations surrounding this topic, and for comments on this paper. Part [II](#page-36-0) of this paper owes a great intellectual debt to his work [\[9\]](#page-71-1). I am also grateful to Daniel Halpern-Leistner for helpful discussions about ∞-categories and derived algebraic geometry.

# Part I. Noncommutative algebraic geometry

## 2. Linear categories

<span id="page-7-1"></span><span id="page-7-0"></span>In this section, we introduce the notion of an S-linear category. These categories are discussed in §[2.3,](#page-9-0) after some remarks on derived algebraic geometry in §[2.1](#page-7-2) and a review of some relevant aspects of the theory of stable  $\infty$ -categories in §[2.2.](#page-7-3) We also discuss "large" (i.e. presentable) versions of S-linear categories, which are useful for technical reasons.

<span id="page-7-2"></span>2.1. Derived algebraic geometry. We work in the setting of derived algebraic geometry, see [\[19,](#page-71-10) [7\]](#page-71-14). This means we regard schemes as objects of the  $\infty$ -category of derived schemes. In particular given morphisms of schemes  $X \to S$  and  $Y \to S$ , the symbol

$$
X\times_S Y
$$

denotes their *derived* fiber product. We note that this agrees with the usual fiber product of schemes whenever the morphisms  $X \to S$  and  $Y \to S$  are Tor-independent over S. In fact, the only time we need to leave the category of ordinary schemes is in §[5](#page-30-0) and Remark [8.8,](#page-56-0) and even there derived schemes can be avoided at the cost of requiring Tor-independence of all fiber products.

<span id="page-7-3"></span>2.2. Stable  $\infty$ -categories. We work with stable  $\infty$ -categories, as developed in [\[18\]](#page-71-6). Here we briefly review some key facts. We consider several classes of stable  $\infty$ -categories:

- The ∞-category Cat<sub>st</sub> of small idempotent-complete stable  $\infty$ -categories, with morphisms the exact functors. For  $\mathcal{C}, \mathcal{D} \in \text{Cat}_{\text{st}}$ , we denote by Fun<sup>ex</sup>( $\mathcal{C}, \mathcal{D}$ ) the  $\infty$ -category of exact functors from C to D.
- The  $\infty$ -category PrCat<sub>st</sub> of presentable stable  $\infty$ -categories, with morphisms the cocontinuous functors (i.e. the functors that preserve small colimits). For  $\mathcal{C}, \mathcal{D} \in \text{PrCat}_{st}$ , we denote by Fun<sup>L</sup>(C, D) the  $\infty$ -category of cocontinuous functors from C to D.
- The ∞-category PrCat<sup> $\omega$ </sup> of compactly generated presentable stable ∞-categories, with morphisms the cocontinuous functors that preserve compact objects. For  $\mathcal{C}, \mathcal{D} \in \text{PrCat}_{st}$ , we denote by Fun<sup>L, $\omega$ </sup>(C, D) the  $\infty$ -category of cocontinuous functors from C to D that preserve compact objects.

**Remark 2.1.** By definition,  $PrCat_{st}^{\omega}$  is a (non-full) subcategory of  $PrCat_{st}$ .

We are primarily interested in  $Cat_{st}$ , but for technical reasons it is often convenient to work with  $PrCat_{st}$ ; the category  $PrCat_{st}^{\omega}$  mediates between these two. Namely, there is an *Ind-completion* functor [\[20,](#page-71-7) Section 5.5.7]

$$
Ind\colon \mathrm{Cat}_{\mathrm{st}}\to \mathrm{PrCat}_{\mathrm{st}}
$$

which by construction factors through the inclusion  $PrCat_{st}^{\omega} \rightarrow PrCat_{st}$ , and induces an equivalence

$$
\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{st}}\to \mathrm{PrCat}_{\mathrm{st}}^\omega
$$

with inverse the functor

$$
(-)^c\colon\mathrm{PrCat}^\omega_{\mathrm{st}}\to\mathrm{Cat}_{\mathrm{st}}
$$

taking  $\mathcal{C} \in \mathrm{PrCat}_{\mathrm{st}}^{\omega}$  to its subcategory  $\mathcal{C}^c \subset \mathcal{C}$  of compact objects.

There is a natural way to form the "tensor product" of categories in  $Cat_{st}$ ,  $PrCat_{st}$ , or  $PrCat_{st}^{\omega}$  [\[18,](#page-71-6) Sections 4.8.1-4.8.2]. More precisely,  $PrCat_{st}$  is a closed symmetric monoidal  $\infty$ category, with product denoted  $\mathcal{C}\otimes\mathcal{D}$  and internal mapping objects given by Fun<sup>L</sup>( $\mathcal{C}, \mathcal{D}$ ). The tensor product  $C \otimes D$  is characterized by the universal property

$$
\text{Fun}^{\mathcal{L}}(\mathcal{C}\otimes \mathcal{D},\mathcal{E})\simeq \text{Fun}'(\mathcal{C}\times \mathcal{D},\mathcal{E}),
$$

where  $\text{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$  is the full subcategory of  $\text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$  (the category of all functors) spanned by functors which preserve small colimits separately in each variable. The category  $PrCat_{st}^{\omega}$  inherits a closed symmetric monoidal structure from  $PrCat_{st}$ , with internal mapping objects given by Fun<sup>L,ω</sup>(C, D). Via the equivalence Cat<sub>st</sub>  $\simeq$  PrCat<sub>st</sub>, we use this to equip Catst with a closed symmetric monoidal structure, with internal mapping objects given by Fun<sup>ex</sup>(C, D). Explicitly, for  $C, D \in \text{Cat}_{st}$ , the tensor product is given by the formula

 $\mathcal{C} \otimes \mathcal{D} \simeq (\text{Ind}(\mathcal{C}) \otimes \text{Ind}(\mathcal{D}))^c$ 

and is characterized by the universal property

 $\text{Fun}^{\text{ex}}(\mathcal{C}\otimes\mathcal{D},\mathcal{E})\simeq \text{Fun}'(\mathcal{C}\times\mathcal{D},\mathcal{E}),$ 

where  $\text{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$  is the full subcategory of  $\text{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$  spanned by functors that are exact separately in each variable.

2.2.1. *Derived categories of schemes.* Derived categories of derived schemes give examples of categories in  $PrCat_{st}$  and  $Cat_{st}$ . Namely, for X a derived scheme, there is a category  $D_{qc}(X) \in \text{PrCat}_{st}$  of unbounded complexes of quasi-coherent sheaves, and a full subcategory  $\text{Perf}(X) \in \text{Cat}_{\text{st}}$  of perfect complexes (see [\[3\]](#page-71-8)); if X is a classical scheme, the homotopy categories of  $D_{\text{qc}}(X)$  and  $\text{Perf}(X)$  agree with their classical triangulated versions. Further,  $D_{\text{qc}}(X)$  and  $\text{Perf}(X)$  have natural symmetric monoidal structures corresponding to tensor products of sheaves, by which  $D_{qc}(X)$  has the structure of a commutative algebra object in  $PrCat_{st}$  and  $Perf(X)$  has the structure of a commutative algebra object in Cat<sub>st</sub>. We note that if X is quasi-compact and separated, then by [\[3,](#page-71-8) Proposition 3.19] the category  $D_{\text{qc}}(X)$ is compactly generated and there are equivalences

$$
Perf(X) \simeq D_{qc}(X)^c \quad \text{and} \quad Ind(Perf(X)) \simeq D_{qc}(X),
$$

induced by the inclusion  $\text{Perf}(X) \subset D_{\text{qc}}(X)$ .

By convention, all functors between derived categories (e.g. pushforward, pullback, tensor product) are derived, but will be written with underived notation. For example, for a morphism of schemes  $f: X \to Y$  we write  $f^* \colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$  for the derived pullback functor, and similarly for the functors  $f_*$  and  $\otimes$ . We always work with functors defined between categories of perfect complexes. Note that in general,  $f_*$  may not preserve perfect complexes, but it does if  $f: X \to Y$  is a perfect (i.e. pseudo-coherent of finite Tor-dimension) proper morphism [\[16,](#page-71-15) Example 2.2(a)]. This assumption will be satisfied in all of the cases where we use  $f_*$  in Part [II](#page-36-0) of the paper.

Recall that  $f_*$  is right adjoint to  $f^*$ . Sometimes, we need other adjoint functors as well. Provided they exist, we denote by  $f'$  the right adjoint of  $f_*\colon \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$  and by  $f_!$ the left adjoint of  $f^* \colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$ , so that  $(f_1, f^*, f_*, f^!)$  is an adjoint sequence. For instance, if  $f: X \to Y$  is a perfect proper morphism and a relative dualizing complex  $\omega_f$  exists and is a perfect complex on X, then  $f^!$  and  $f_!$  exist and are given by

$$
f^{!}(-) \simeq f^{*}(-) \otimes \omega_{f} \quad \text{and} \quad f_{!}(-) \simeq f_{*}(- \otimes \omega_{f}). \tag{2.1}
$$

Indeed, the functor  $f_* \colon D_{\text{qc}}(X) \to D_{\text{qc}}(Y)$  between the unbounded derived categories of quasicoherent sheaves admits a right adjoint  $f' : D_{\text{qc}}(Y) \to D_{\text{qc}}(X)$ , the relative dualizing complex of f is by definition  $\omega_f = f^!(\mathcal{O}_Y)$ , and the above formulas for  $f^!$  and  $f_!$  hold on quasi-coherent categories (the first holds by [\[16,](#page-71-15) Proposition 2.1] and implies the second). Hence if  $f$  is a perfect proper morphism and  $\omega_f$  is perfect, it follows that all of these functors and adjunctions restrict to categories of perfect complexes. In all of the cases where we need  $f'$  and  $f_!$  in the paper, the following stronger assumptions will be satisfied.

**Remark 2.2.** Suppose  $f: X \to Y$  is a morphism between schemes which are smooth, projective, and of constant relative dimension over  $S$ . Then  $f$  is perfect, projective, and has a relative dualizing complex, which is a shift of a line bundle:

$$
\omega_f = \omega_{X/S} \otimes f^*(\omega_{Y/S})^\vee,\tag{2.2}
$$

where  $\omega_{X/S}$  and  $\omega_{Y/S}$  denote the relative dualizing *complexes*. In particular, for such an f, all of the functors  $f_!, f^*, f_*, f^!$  are defined and adjoint between categories of perfect complexes.

<span id="page-9-0"></span>2.3. Categories linear over a base. In this paper, the main objects of study are categories which are linear over a base scheme  $S$  in the following sense.

**Definition 2.3.** Let Cat<sub>S</sub> be the  $\infty$ -category of modules over the commutative algebra object  $\text{Perf}(S) \in \text{Cat}_{\text{st}}$ , i.e

$$
Cat_S = Mod_{Perf(S)}(Cat_{st})
$$

in the notation of [\[18,](#page-71-6) Section 4.5]. An object  $C \in \text{Cat}_S$  is called an *S*-linear category. A morphism  $C \to \mathcal{D}$  in Cat<sub>S</sub> is an *S*-linear functor; these morphisms form the objects of an  $\infty$ -category denoted  $\text{Fun}_{\text{Perf}(S)}(\mathcal{C}, \mathcal{D})$ .

Given an S-linear category, we use the notation

$$
\mathcal{C} \times \text{Perf}(S) \to \mathcal{C}
$$

$$
(C, F) \mapsto C \otimes F
$$

for the action functor of  $\text{Perf}(S)$  on C.

**Remark 2.4.** If  $\pi: X \to S$  is a morphism of schemes, then  $\mathcal{C} = \text{Perf}(X)$  is naturally an S-linear category, with action functor given by  $(C, F) \mapsto C \otimes \pi^*(F)$ . This is the primordial example of an S-linear category. The philosophy of noncommutative algebraic geometry is to think of an arbitrary S-linear category  $\mathfrak C$  as a "noncommutative scheme with a morphism to  $S$ " (although we will not use that terminology). Throughout the paper, we will see that a number of operations associated to  $X \to S$  have analogues for an arbitrary "noncommutative scheme" C.

As above, we can also consider presentable versions of S-linear categories.

**Definition 2.5.** Let  $PrCat_S$  be the category of modules over the commutative algebra object  $D_{\mathrm{qc}}(S) \in \mathrm{PrCat}_{\mathrm{st}}, \text{ i.e.}$ 

$$
\mathrm{PrCat}_S = \mathrm{Mod}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{PrCat}_{\mathrm{st}}).
$$

An object  $C \in \text{PrCat}_S$  is called a *presentable* S-linear category. A morphism  $C \to \mathcal{D}$  in  $\text{PrCat}_S$ is a *cocontinuous*  $S$ -linear functor; these morphisms form the objects of an  $\infty$ -category denoted  $\text{Fun}_{\text{D}_{\text{qc}}(S)}(\mathcal{C}, \mathcal{D}).$ 

Recall that  $D_{\text{qc}}(S)$  is compactly generated because S is quasi-compact and separated by our standing assumptions. Hence we can also make the following definition.

**Definition 2.6.** Let  $PrCat_S^{\omega}$  be the category of modules over the commutative algebra object  $D_{\mathrm{qc}}(S) \in \mathrm{PrCat}_{\mathrm{st}}^{\omega}$ , i.e.

$$
\mathrm{PrCat}_{S}^{\omega} = \mathrm{Mod}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{PrCat}_{\mathrm{st}}^{\omega}).
$$

An object  $C \in \text{PrCat}_{S}^{\omega}$  is called a *compactly generated presentable S-linear category*. A morphism  $C \to D$  in PrCat<sup> $\omega$ </sup> is a *cocontinuous S-linear functor preserving compact objects*; these morphisms form the objects of an  $\infty$ -category denoted  $\text{Fun}^{\omega}_{\text{D}_{\text{qc}}(S)}(\mathcal{C}, \mathcal{D})$ .

The categories  $\text{Cat}_S$ ,  $\text{PrCat}_S$ , and  $\text{PrCat}_{S}^{\omega}$  are related in the same way as the categories Cat<sub>st</sub>, PrCat<sub>st</sub>, and PrCat<sup> $\omega$ </sup><sub>st</sub>. Namely, PrCat<sup> $\omega$ </sup><sub>S</sub> is a non-full subcategory of PrCat<sub>S</sub>, and Indcompletion induces an equivalence

$$
\mathrm{Ind}\colon \mathrm{Cat}_S\rightarrow \mathrm{PrCat}_S^\omega
$$

with inverse the functor

 $(-)^c \colon \mathrm{PrCat}_S^{\omega} \to \mathrm{Cat}_S$ 

induced by taking compact objects.

Further, the categories  $\text{Cat}_S$ ,  $\text{PrCat}_S$ , and  $\text{PrCat}_S^{\omega}$  admit symmetric monoidal structures (see [\[18,](#page-71-6) Theorem 4.5.2.1]), with units respectively given by  $\text{Perf}(S)$ ,  $D_{\text{qc}}(S)$ , and  $D_{\text{qc}}(S)$ . These monoidal structures are closed with internal mapping objects given by the functor categories introduced above. For  $\mathcal{C}, \mathcal{D} \in \text{PrCat}_S$ , we denote their tensor product by

$$
\mathcal{C} \otimes_{D_{qc}(S)} \mathcal{D} \in \mathrm{PrCat}_S.
$$

For  $\mathcal{C}, \mathcal{D} \in \text{Cat}_S$ , we denote their tensor product by

 $\mathfrak{C} \otimes_{\mathrm{Perf}(S)} \mathfrak{D} \in \mathrm{Cat}_S$ 

which can be described by the formula

$$
\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \simeq (\mathrm{Ind}(\mathcal{C}) \otimes_{\mathrm{D}_{\mathrm{qc}}(S)} \mathrm{Ind}(\mathcal{D}))^c.
$$

The tensor product of categories in  $\text{Cat}_S$  or  $\text{PrCat}_S$  can be characterized by a universal property. Namely, if  $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \text{Cat}_S$ , then S-linear functors  $\mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D} \to \mathcal{E}$  classify the bilinear maps  $C \times D \to \mathcal{E}$  (as defined in [\[18,](#page-71-6) Section 4.4.1]); a similar statement holds for  $PrCat_S$ . In particular, there is a canonical functor

$$
\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}.
$$

Given objects  $C \in \mathcal{C}$  and  $D \in \mathcal{D}$ , we denote by  $C \boxtimes D \in \mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}$  their image under this functor.

The following result gives generating objects for a tensor product of S-linear categories. We will use the following terminology. A *stable subcategory* of a stable ∞-category is a full subcategory which is also stable. If C is a category in Cat<sub>st</sub> and  $\Sigma$  is a set of objects, then we say Σ *thickly generates* C if the smallest idempotent-complete stable subcategory containing  $\Sigma$  is  $\mathcal C$  itself.

<span id="page-10-0"></span>**Lemma 2.7.** Let C and D be S-linear categories. Then  $\mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}$  is thickly generated by *objects of the form*  $C \boxtimes D$  *for*  $C \in \mathcal{C}$  *and*  $D \in \mathcal{D}$ *.* 

*Proof.* Equivalently,  $\text{Ind}(\mathcal{C}) \otimes_{D_{qc}(S)} \text{Ind}(\mathcal{D})$  is compactly generated by objects of the form  $C \boxtimes D$ for  $C \in \mathcal{C}$  and  $D \in \mathcal{D}$ . This can be proved as in [\[7,](#page-71-14) Chapter I.1, Proposition 7.4.2], which treats the analogous statement for stable  $\infty$ -categories without an S-linear structure.  $\Box$ 

Finally, we observe that by using tensor products, we can make sense of base changes of linear categories. Namely, if C is an S-linear category and  $S' \to S$  is a morphism of schemes, then the *base change of*  $C$  *along*  $S' \rightarrow S$  is the  $S'$ -linear category

$$
\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S') \in \mathrm{Cat}_{S'}.
$$

2.3.1. *Mapping objects.* For objects  $C, D \in \mathcal{C}$  of an  $\infty$ -category  $\mathcal{C}$ , we denote by  $\text{Map}_{\mathcal{C}}(C, D)$ the space of maps from C to D. If C is a presentable S-linear category, then there is a mapping object

$$
\mathcal{H}\!\mathit{om}_S(C,D) \in \mathcal{D}_\mathrm{qc}(S)
$$

characterized by equivalences

$$
\mathrm{Map}_{\mathrm{D}_{\mathrm{qc}}(S)}(F, \mathcal{H}om_S(C, D)) \simeq \mathrm{Map}_{\mathcal{C}}(C \otimes F, D)
$$

for  $F \in D_{\text{qc}}(S)$ . More precisely, the functor

$$
\mathrm{Map}_{\mathcal{C}}(C\otimes (-), D) \colon \mathrm{D}_{\mathrm{qc}}(S)^{\mathrm{op}} \to \mathcal{S},
$$

where S denotes the  $\infty$ -category of spaces, is representable by [\[20,](#page-71-7) Proposition 5.5.2.2], and by definition  $\mathcal{H}om_S(C, D)$  is the representing object. If C is an S-linear category, then C is a full subcategory of the presentable S-linear category  $\text{Ind}(\mathcal{C})$ ; for objects  $C, D \in \mathcal{C}$  we denote by  $\mathfrak{Hom}_S(C, D) \in D_{\text{qc}}(S)$  the mapping object between C and D regarded as objects of Ind(C).

<span id="page-11-1"></span>**Remark 2.8.** Let X be a scheme with a morphism  $\pi: X \to S$ , so that Perf(X) is S-linear. Then for  $C, D \in \text{Perf}(X)$ , we have

$$
\mathcal{H}\!\mathit{om}_S(C,D) \simeq \pi_*\mathcal{H}\!\mathit{om}_X(C,D),
$$

where  $\mathcal{H}om_X(C, D)$  denotes the derived sheaf Hom on X.

We have the following Künneth formula for mapping objects in tensor products of categories.

<span id="page-11-0"></span>**Lemma 2.9.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be S-linear categories. If  $C_1, C_2 \in \mathcal{C}$  and  $D_1, D_2 \in \mathcal{D}$ , then the  $D_{\text{qc}}(S)$ -valued mapping object between  $C_1 \boxtimes D_1$  and  $C_2 \boxtimes D_2$  in  $\mathfrak{C} \otimes_{\text{Perf}(S)} \mathfrak{D}$  satisfies

$$
\mathfrak{Hom}_S(C_1 \boxtimes D_1, C_2 \boxtimes D_2) \simeq \mathfrak{Hom}_S(C_1, C_2) \otimes \mathfrak{Hom}_S(D_1, D_2).
$$

Given an S-linear category C, its base change along a morphism  $S' \rightarrow S$  is an S'-linear category, and hence has  $D_{qc}(S')$ -valued mapping objects. They satisfy the following Künneth formula.

<span id="page-11-2"></span>**Lemma 2.10.** Let  $C$  be an S-linear category. Let  $S' \rightarrow S$  be a morphism of schemes. If  $C_1, C_2 \in \mathcal{C}$  and  $F_1, F_2 \in \text{Perf}(S')$ , then the  $D_{qc}(S')$ -valued mapping object between  $C_1 \boxtimes F_1$  $and C_2 \boxtimes F_2$  *in*  $C ⊗_{\text{Perf}(S)}$  Perf $(S')$  *satisfies* 

 $\mathfrak{Hom}_{S'}(C_1 \boxtimes F_1, C_2 \boxtimes F_2) \simeq \mathfrak{Hom}_S(C_1, C_2) \otimes \mathfrak{Hom}_{S'}(F_1, F_2) \in D_{qc}(S'),$ 

where the product on the right is taken with respect to the  $D_{qc}(S)$ -module structure on  $D_{qc}(S')$ *induced by pullback.*

2.3.2. *Adjoints*. Given a functor of stable  $\infty$ -categories  $\phi: \mathcal{C} \to \mathcal{D}$ , we typically denote by  $\phi^*$ the left adjoint whenever it exists, and by  $\phi^!$  the right adjoint whenever it exists. Further, if C is an S-linear category and  $\phi: \mathcal{D}_1 \to \mathcal{D}_2$  is an S-linear exact functor of S-linear categories, by abuse of notation we denote also by

$$
\phi \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_1 \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_2
$$

the induced functor.

<span id="page-12-2"></span>**Lemma 2.11.** Let  $\phi: \mathcal{C} \to \mathcal{D}$  be an S-linear functor between S-linear categories. If  $\phi$  has *a left adjoint*  $\phi^* \colon \mathcal{D} \to \mathcal{C}$  *(or right adjoint*  $\phi^!$ *)* when regarded as a functor of plain stable  $\infty$ -categories, then  $\phi^*$  (or  $\phi^!$ ) is naturally an *S*-linear functor.

*Proof.* In other words, we must show that  $\phi^* : \mathcal{D} \to \mathcal{C}$  canonically commutes with the Perf(S)actions on D and C. For objects  $C \in \mathcal{C}, D \in \mathcal{D}, F \in \text{Perf}(S)$ , using adjointness and the S-linearity of  $\phi$ , we find equivalences

$$
\begin{aligned}\n\mathcal{H}om_S(\phi^*(D \otimes F), C) &\simeq \mathcal{H}om_S(D \otimes F, \phi(C)) \\
&\simeq \mathcal{H}om_S(D, \phi(C) \otimes F^\vee) \\
&\simeq \mathcal{H}om_S(D, \phi(C \otimes F^\vee)) \\
&\simeq \mathcal{H}om_S(\phi^*(D), C \otimes F^\vee) \\
&\simeq \mathcal{H}om_S(\phi^*(D) \otimes F, C).\n\end{aligned}
$$

Hence by Yoneda we obtain a canonical equivalence  $\phi^*(D \otimes F) \simeq \phi^*(D) \otimes F$ , which corresponds to an S-linear structure on  $\phi^*$ . A similar argument works for  $\phi^!$ .

The proofs of the following lemmas are formal and left to the reader.

<span id="page-12-1"></span>**Lemma 2.12.** Let  $\phi_1: \mathcal{C}_1 \to \mathcal{D}_1$  and  $\phi_2: \mathcal{C}_2 \to \mathcal{D}_2$  be *S*-linear functors.

- (1) If  $\phi_1$  and  $\phi_2$  both admit left adjoints  $\phi_1^*$  and  $\phi_2^*$  (or right adjoints  $\phi_1^!$  and  $\phi_2^!$ ), then the  $functor \phi_1 \otimes \phi_2 \colon C_1 \otimes_{\text{Perf}(S)} C_2 \to D_1 \otimes_{\text{Perf}(S)} \tilde{D_2}$  *has a left adjoint given by*  $\phi_1^* \otimes \phi_2^*$  (*or right adjoint given by*  $\phi_1^! \otimes \phi_2^!$ .
- *(2)* If  $\phi_1$  and  $\phi_2$  both admit left or right adjoints and are fully faithful, then so is  $\phi_1 \otimes \phi_2$ .

**Lemma 2.13.** Let  $\phi_i: \mathcal{C} \to \mathcal{D}$  be an S-linear functor between S-linear stable  $\infty$ -categories for  $i = 1, 2, 3$ *, and let*  $\phi_1 \rightarrow \phi_2 \rightarrow \phi_3$  *be an exact triangle in* Fun<sub>Perf(S)</sub>(C, D).

- (1) If each  $\phi_i$  admits a left adjoint  $\phi_i^*$ , there is an induced exact triangle  $\phi_3^* \to \phi_2^* \to \phi_1^*$ .
- <span id="page-12-0"></span>(2) If each  $\phi_i$  admits a right adjoint  $\phi_i^!$ , there is an induced exact triangle  $\phi_3^! \to \phi_2^! \to \phi_1^!$ .

## 3. Semiorthogonal decompositions

In this section, we consider semiorthogonal decompositions of S-linear categories. In §[3.1](#page-13-0) we give the basic definitions, and in §[3.2](#page-14-0) we discuss the notions of admissible subcategories and mutation functors; the results here are standard in the triangulated setting (see  $(5, 6)$ ), so we freely omit proofs when the usual arguments work without modification. In §[3.3](#page-16-0) we describe the relation between semiorthogonal decompositions of linear and presentable linear categories. Then we discuss induced semiorthogonal decompositions of tensor products of linear categories in §[3.4,](#page-17-0) and of functor categories between linear categories in §[3.5.](#page-18-0) Finally, in §[3.6](#page-19-0) we review the notion of a splitting functor from [\[9\]](#page-71-1).

<span id="page-13-0"></span>3.1. **Basic definitions.** Given an S-linear category C, an S-linear stable subcategory  $A \subset C$ is a stable subcategory of C which is preserved by the  $\text{Perf}(S)$ -action on C; this is equivalent to the data of an S-linear category A equipped with a fully faithful S-linear functor  $A \rightarrow \mathcal{C}$ . Similarly, given a presentable S-linear category C, a *presentable* S*-linear stable subcategory*  $A \subset \mathcal{C}$  is a stable subcategory of  $\mathcal{C}$  which is closed under colimits and preserved by the  $D_{\alpha}$  $(S)$ action on  $\mathfrak{C}$ ; this is equivalent to the data of a presentable S-linear category A equipped with a fully faithful cocontinuous S-linear functor  $A \rightarrow \mathcal{C}$ .

<span id="page-13-2"></span>Definition 3.1. Let C be an S-linear (resp. presentable S-linear) category. An S*-linear (resp. presentable* S-linear) semiorthogonal decomposition of C is a sequence  $A_1, A_2, \ldots, A_n$  of Slinear (resp. presentable S-linear) stable subcategories of C — called the *components* of the decomposition — such that:

- <span id="page-13-6"></span><span id="page-13-1"></span>(1)  $\mathcal{H}om_S(C, D) \simeq 0$  for all  $C \in \mathcal{A}_i, D \in \mathcal{A}_j$ , and  $i > j$ .
- (2) For any  $C \in \mathcal{C}$ , there exists a diagram

<span id="page-13-3"></span>

where  $A_i \in \mathcal{A}_i$  and the triangles are exact.

If only condition [\(1\)](#page-13-1) is satisfied, we say the sequence  $A_1, \ldots, A_n$  is *semiorthogonal*.

**Remark 3.2.** By S-linearity of the categories  $A_i$ , in [\(1\)](#page-13-1) it is equivalent to require that the space  $\text{Map}_{\mathcal{C}}(C, D)$  is contractible.

Given a stable  $\infty$ -category C and a collection of subcategories  $A_i \subset \mathcal{C}, i = 1, \ldots, n$ , we denote by

$$
\langle A_1,\ldots,A_n\rangle\subset\mathfrak{C}
$$

the stable subcategory of  $C$  generated by the  $A_i$ . In particular, for a semiorthogonal decomposition as in Definition [3.1,](#page-13-2) we have

$$
\mathcal{C}=\langle \mathcal{A}_1,\ldots,\mathcal{A}_n\rangle.
$$

<span id="page-13-5"></span>Lemma 3.3. *Let* C *be an* S*-linear (resp. presentable* S*-linear) category, with an* S*-linear (resp. presentable* S*-linear) semiorthogonal decomposition*

$$
\mathcal{C}=\langle \mathcal{A}_1,\ldots,\mathcal{A}_n\rangle.
$$

Let  $\alpha_i: \mathcal{A}_i \to \mathcal{C}$  denote the inclusion. Then there are S-linear (resp. cocontinuous S-linear) *functors*

$$
\begin{aligned} &\text{tr}_i\colon \mathcal{C}\to \mathcal{C}, \quad 0\leq i\leq n,\\ &\text{pr}_i\colon \mathcal{C}\to \mathcal{A}_i, \quad 1\leq i\leq n, \end{aligned}
$$

<span id="page-13-4"></span>*such that:*

(1) There is a diagram in the category of functors  $\text{Fun}_{\text{Perf}(S)}(\mathcal{C}, \mathcal{C})$  (resp.  $\text{Fun}_{D_{qc}(S)}(\mathcal{C}, \mathcal{C})$ )



*where the triangles are exact, which recovers*  $(3.1)$  *when applied to any*  $C \in \mathcal{C}$ *.* 

- (2) The functor  $pr_i: \mathcal{C} \to \mathcal{A}_i$  is a retraction, i.e.  $pr_i \circ \alpha_i \simeq id_{\mathcal{A}_i}$ .
- (3) The restriction of  $pr_i: \mathcal{C} \to \mathcal{A}_i$  to  $\langle \mathcal{A}_i, \mathcal{A}_{i+1}, \ldots, \mathcal{A}_n \rangle \subset \mathcal{C}$  is left adjoint to the inclusion  $A_i \to \langle A_i, \ldots, A_n \rangle$ . In particular,  $pr_1 = \alpha_1^*$  is left adjoint to  $\alpha_1 : A_1 \to \mathcal{C}$ .
- (4) The restriction of  $pr_i: \mathcal{C} \to \mathcal{A}_i$  to  $\langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_i \rangle \subset \mathcal{C}$  is right adjoint to the inclusion  $A_i \to \langle A_1, A_2, \ldots, A_i \rangle$ . In particular,  $\mathrm{pr}_n = \alpha_n^!$  is right adjoint to  $\alpha_n : A_n \to \mathcal{C}$ .

*Proof.* This result is well-known in the triangulated setting; the same proof works in our setup. The only point which deserves explanation is that in the presentable case, the functors  $\mathrm{tr}_i$  and  $\mathrm{pr}_i$  are indeed cocontinuous. This claim reduces to the case where the length of the semiorthogonal decomposition is  $n = 2$ . Then the diagram from [\(1\)](#page-13-4) amounts to a distinguished triangle

$$
\alpha_2 \circ pr_2 \to id_{\mathcal{C}} \to \alpha_1 \circ pr_1,
$$

where  $pr_1 = \alpha_1^*$  and  $pr_2 = \alpha_2^!$ . Being a left adjoint, the functor  $pr_1$  is cocontinuous. Hence the above triangle implies  $\alpha_2 \circ pr_2$  is cocontinuous. Since  $\alpha_2$  is fully faithful and cocontinuous, we conclude  $pr_2$  is cocontinuous.

**Definition 3.4.** Given a semiorthogonal decomposition as in Lemma [3.3,](#page-13-5) the functors  $tr_i$ and pr<sub>i</sub> are called the *truncation* and *projection* functors.

## <span id="page-14-0"></span>3.2. Admissible subcategories and mutation functors.

**Definition 3.5.** Let C be an S-linear category, and let  $A \subset C$  be an S-linear stable subcategory. Let  $\alpha: \mathcal{A} \to \mathcal{C}$  denote the inclusion. Then A is called:

- *left admissible* in  $C$  if  $\alpha$  admits a left adjoint  $\alpha^*$ :  $C \to A$ ;

 $-$  *right admissible* in  $C$  if α admits a right adjoint α<sup>1</sup>:  $C → A;$ 

– *admissible* in C if it is both left and right admissible.

Admissibility of a subcategory is related to semiorthogonal decompositions as follows. Given a subcategory A of an  $\infty$ -category C, consider the full subcategories of C defined by

$$
\mathcal{A}^{\perp} = \{ C \in \mathcal{C} \mid \text{Map}_{\mathcal{C}}(D, C) \text{ is contractible for all } D \in \mathcal{A} \},
$$
  

$$
^{\perp} \mathcal{A} = \{ C \in \mathcal{C} \mid \text{Map}_{\mathcal{C}}(C, D) \text{ is contractible for all } D \in \mathcal{A} \}.
$$

We call  $A^{\perp}$  the *right orthogonal* to  $A \subset \mathcal{C}$ , and  $\perp A$  the *left orthogonal* to  $A \subset \mathcal{C}$ . If  $\mathcal{C}$ is an S-linear category and  $A \subset \mathcal{C}$  is an S-linear stable subcategory, then the orthogonal categories  $\mathcal{A}^{\perp}$  and  $^{\perp}\mathcal{A}$  are also S-linear stable subcategories of C. Clearly,  $\mathcal{A}^{\perp}$ , A and  $\mathcal{A},^{\perp}\mathcal{A}$ are semiorthogonal pairs in C.

<span id="page-14-1"></span>Lemma 3.6. *Let* C *be an* S*-linear category, and let* A, B *be a pair of* S*-linear stable subcategories. The following are equivalent:*

*(1)*  $C = \langle A, B \rangle$  *is a semiorthogonal decomposition.* 

*(2)*  $A ⊂ C$  *is left admissible and*  $B = \perp A$ *.* 

*(3)*  $B ⊂ C$  *is right admissible and*  $A = B<sup>⊥</sup>$ *.* 

**Definition 3.7.** Let  $A ⊂ C$  be an S-linear stable subcategory of an S-linear category. If  $A \subset \mathcal{C}$  is left admissible, then by Lemma [3.6](#page-14-1) the inclusion  $i: \perp \mathcal{A} \to \mathcal{C}$  admits a right adjoint  $i^!$ ; in this case the functor

$$
R_{\mathcal{A}} = i \circ i^! \colon \mathcal{C} \to \mathcal{C}
$$

is called the *right mutation functor* through A. Similarly, if  $A \subset \mathcal{C}$  is right admissible, then the inclusion  $j: \mathcal{A}^{\perp} \to \mathcal{C}$  admits a left adjoint  $j^*$ , and the functor

$$
L_{\mathcal{A}} = j \circ j^* \colon \mathcal{C} \to \mathcal{C}
$$

is called the *left mutation functor* through A.

<span id="page-15-3"></span>**Lemma 3.8.** Let  $A \subset \mathcal{C}$  be an *S*-linear stable subcategory of an *S*-linear category.

- *(1)* If  $A ⊂ C$  *is left admissible, then the right mutation functor*  $R_A$  *kills*  $A$  *and is fully faithful on the subcategory*  $A^{\perp} \subset \mathcal{C}$ *.*
- *(2)* If  $A ⊂ C$  *is right admissible, then the left mutation functor*  $L_A$  *kills*  $A$  *and is fully faithful on the subcategory*  $^{\perp}A \subset \mathcal{C}$ *.*
- *(3)* If  $A ⊂ ℂ$  *is admissible, then*  $R_A$  *and*  $L_A$  *induce mutually inverse equivalences*

$$
R_{\mathcal{A}}|_{\mathcal{A}^{\perp}}: \mathcal{A}^{\perp} \xrightarrow{\sim} {}^{\perp} \mathcal{A} \quad and \quad L_{\mathcal{A}}|_{\perp \mathcal{A}}: \xrightarrow{\perp} \mathcal{A} \xrightarrow{\sim} \mathcal{A}^{\perp}.
$$

<span id="page-15-2"></span>**Lemma 3.9.** Let  $\mathcal{C}$  be an S-linear category, and let  $\alpha: \mathcal{A} \to \mathcal{C}$  be the inclusion of an S-linear *stable subcategory.*

<span id="page-15-0"></span>*(1)* If  $A ⊂ C$  *is left admissible and*  $\perp A ⊂ C$  *is admissible, then*  $\alpha^* : C → A$  *has a left adjoint given by*

$$
R_{\perp_{\mathcal{A}}} \circ \alpha \colon \mathcal{A} \to \mathcal{C}.
$$

<span id="page-15-1"></span>(2) If  $A ⊂ C$  *is right admissible and*  $A<sup>⊥</sup> ⊂ C$  *is admissible, then*  $\alpha$ <sup>1</sup>:  $C → A$  *has a right adjoint given by*

 $L_{\Lambda\perp}\circ\alpha\colon \mathcal{A}\to\mathcal{C}.$ 

*Proof.* We prove [\(1\)](#page-15-0); the proof of [\(2\)](#page-15-1) is similar. Let  $\mathcal{B} = {}^{\perp}A$ , so that  $\mathcal{C} = \langle A, \mathcal{B} \rangle$ , and let  $\beta: \mathcal{B} \to \mathcal{C}$  be the inclusion. Note that for  $C \in \mathcal{A}$  and  $D \in \mathcal{C}$  we have exact triangles

$$
\beta \beta^!(D) \to D \to \alpha \alpha^*(D),
$$
  
\n
$$
R_{\beta} \alpha(C) \to \alpha(C) \to \beta \beta^* \alpha(C),
$$

where  $\mathcal{H}om_S(\beta\beta^*\alpha(C), \alpha\alpha^*(D)) \simeq 0$  and  $\mathcal{H}om_S(\mathrm{R}_\mathcal{B}\alpha(C), \beta\beta^!(D)) \simeq 0$ . Using this, we obtain equivalences

$$
\begin{aligned} \mathcal{H}om_S(C, \alpha^*(D)) &\simeq \mathcal{H}om_S(\alpha(C), \alpha \alpha^*(D)) \\ &\simeq \mathcal{H}om_S(\mathrm{R}_{\perp A} \alpha(C), \alpha \alpha^*(D)) \\ &\simeq \mathcal{H}om_S(\mathrm{R}_{\perp A} \alpha(C), D), \end{aligned}
$$

which proves the adjunction.  $\Box$ 

**Lemma 3.10.** Let  $C$  *be an S-linear category, and let*  $A_1, \ldots, A_n$  *be a semiorthogonal sequence of* S*-linear stable subcategories.*

*(1) If the*  $A_i$  ⊂  $C$  *are all right admissible, then* 

$$
\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \subset \mathcal{C}
$$

*is also a right admissible* S*-linear stable subcategory, with left mutation functor given by*

$$
L_{\langle A_1,...,A_n\rangle}\simeq L_{A_1}\circ L_{A_2}\circ\cdots\circ L_{A_n}.
$$

*(2)* If the  $A_i$  ⊂  $C$  *are all left admissible, then* 

$$
\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \subset \mathcal{C}
$$

*is also a left admissible* S*-linear stable subcategory, with right mutation functor given by*

$$
R_{\langle A_1,...,A_n\rangle} \simeq R_{A_n} \circ R_{A_{n-1}} \circ \cdots \circ R_{A_1}.
$$

Lemma 3.11. *Let* C *be an* S*-linear category with an* S*-linear semiorthogonal decomposition*

$$
\mathcal{C}=\langle \mathcal{A}_1,\ldots,\mathcal{A}_n\rangle\,.
$$

- *(1)* For  $1 \le i \le n-1$ , if  $\mathcal{A}_i \subset \mathcal{C}$  *is right admissible there is a semiorthogonal decomposition*  $\mathcal{C} = \langle A_1, \ldots, A_{i-1}, \mathrm{L}_{A_i}(A_{i+1}), A_i, A_{i+2}, \ldots, A_n \rangle$ .
- *(2)* For  $2 \le i \le n$ , if  $A_i \subset \mathcal{C}$  is left admissible there is a semiorthogonal decomposition  $\langle A_n \rangle$ .

$$
\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i, R_{\mathcal{A}_i}(\mathcal{A}_{i-1}), \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle
$$

## <span id="page-16-0"></span>3.3. Linear versus presentable linear semiorthogonal decompositions.

<span id="page-16-1"></span>Lemma 3.12. *Let* C *be an* S*-linear category with an* S*-linear semiorthogonal decomposition*

$$
\mathcal{C} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \,.
$$

*Then there is an induced presentable* S*-linear semiorthogonal decomposition*

$$
Ind(\mathcal{C}) = \langle Ind(\mathcal{A}_1), \ldots, Ind(\mathcal{A}_n) \rangle ,
$$

*where the embedding functors*  $\text{Ind}(\mathcal{A}_i) \to \text{Ind}(\mathcal{C})$  *preserve compact objects.* 

*Proof.* It is easy to see that  $\text{Ind}(\mathcal{A}_1), \ldots, \text{Ind}(\mathcal{A}_n)$  are semiorthogonal in  $\text{Ind}(\mathcal{C})$ . Further, the projection functors for the given semiorthogonal decomposition of C induce projection functors for Ind(C), so that Definition [3.1](#page-13-2)[\(2\)](#page-13-6) holds. Finally, since  $\text{Ind}(\mathcal{A}_i)^c = \mathcal{A}_i$  and  $\text{Ind}(\mathcal{C})^c = \mathcal{C}$ , the embeddings  $\text{Ind}(\mathcal{A}_i) \to \text{Ind}(\mathcal{C})$  preserve compact objects.

In general, given a presentable S-linear semiorthogonal decomposition, passing to categories of compact objects does not necessarily induce an S-linear semiorthogonal decomposition. However:

<span id="page-16-2"></span>Lemma 3.13. *Let* C *be a presentable* S*-linear category with a presentable* S*-linear semiorthogonal decomposition*

$$
\mathcal{C}=\langle \mathcal{A}_1,\ldots,\mathcal{A}_n\rangle.
$$

*(1) The projection functor*  $pr_1: C \rightarrow A_1$  *preserves compact objects.* 

*(2) The embedding functor*  $\alpha_n: A_n \to \mathcal{C}$  *preserves compact objects.* 

*If for all i the embedding functors*  $\alpha_i : A_i \to \mathcal{C}$  *preserve compact objects, then furthermore:* 

*(1) For all i the projection functors*  $pr_i: \mathcal{C} \to \mathcal{A}_i$  *preserve compact objects.* 

*(2) There is an induced semiorthogonal decomposition*

$$
\mathcal{C}^c = \langle \mathcal{A}_1^c, \ldots, \mathcal{A}_n^c \rangle \, .
$$

*Proof.* By induction we reduce to the case where  $n = 2$ . The functor  $pr_1 = \alpha_1^*$  preserves compact objects because its right adjoint  $\alpha_1$  is cocontinuous. Similarly,  $\alpha_2$  preserves compact objects because its right adjoint  $pr_2 = \alpha_2^!$  is cocontinuous by Lemma [3.3.](#page-13-5)

Now assume that  $\alpha_1$  preserves compact objects. Then there is a triangle of functors

$$
\alpha_2 \circ pr_2 \to id_{\mathcal{C}} \to \alpha_1 \circ pr_1,
$$

where the second and third vertices, and hence also the first, preserve compact objects. Since  $\alpha_2$  is fully faithful and cocontinuous, it follows that  $pr_2$  preserves compact objects. Since for all i the embeddings  $\alpha_i$  and the projection functors  $pr_i$  preserve compact objects, it follows formally that there is an induced semiorthogonal decomposition  $\mathcal{C}^c = \langle \mathcal{A}_1^c, \mathcal{A}_2^c \rangle$  $\overline{\phantom{a}}$ .

Remark 3.14. Lemmas [3.12](#page-16-1) and [3.13](#page-16-2) can be summarized as follows: Under the equivalence Ind: Cat<sub>S</sub>  $\rightarrow$  PrCat<sub>S</sub><sup> $\alpha$ </sup>, S-linear semiorthogonal decompostions correspond to presentable Slinear semiorthogonal decompositions such that the embedding functors of the components preserve compact objects.

## <span id="page-17-0"></span>3.4. Semiorthogonal decompositions of tensor products.

<span id="page-17-1"></span>**Lemma 3.15.** Let  $\mathcal{C} = \langle A_1, \ldots, A_m \rangle$  and  $\mathcal{D} = \langle B_1, \ldots, B_n \rangle$  be S-linear semiorthogonal de*compositions. Then the functor*  $\mathcal{A}_i \otimes_{\text{Perf}(S)} \mathcal{B}_j \to \mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}$  *is fully faithful for all i, j. Moreover, there is an* S*-linear semiorthogonal decomposition*

$$
\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} = \langle \mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathcal{B}_j \rangle_{1 \leq i \leq m, 1 \leq j \leq n}
$$

*where the ordering on the set*  $\{(i,j) | 1 \le i \le m, 1 \le j \le n\}$  *is any one which extends the coordinate-wise partial order. The projection functor onto the*  $(i, j)$ -component of this decom*position is given by*

$$
\mathrm{pr}_{\mathcal{A}_i} \otimes \mathrm{pr}_{\mathcal{B}_j} \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathcal{B}_j,
$$

where  $pr_{A_i}: \mathcal{C} \to A_i$  and  $pr_{B_j}: \mathcal{D} \to B_j$  are the projection functors for the given decomposi*tions.*

*Proof.* The result reduces to the case where  $m = 2$  and  $n = 1$ . It follows from Lemmas [3.6](#page-14-1) and [2.12](#page-12-1) that the functor

$$
\mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}
$$

is fully faithful for  $i = 1, 2$ . It follows from Lemmas [2.7](#page-10-0) and [2.9](#page-11-0) that the categories

$$
\mathcal{A}_1\otimes_{\mathrm{Perf}(S)} \mathcal{D},\ \mathcal{A}_2\otimes_{\mathrm{Perf}(S)} \mathcal{D}
$$

are semiorthogonal in  $\mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}$ . The projection functors for the original semiorthogonal decomposition induce projection functors for  $\mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}$ , so that Definition [3.1](#page-13-2)[\(2\)](#page-13-6) holds.  $\Box$ 

**Remark 3.16.** If  $S' \to S$  is a morphism of schemes, then taking  $\mathcal{D} = \text{Perf}(S')$  in Lemma [3.15](#page-17-1) gives a base change result for semiorthogonal decompositions.

<span id="page-17-3"></span>**Lemma 3.17.** Let  $\mathcal{C}$  *be an* S-linear category, and let  $\mathcal{A} \subset \mathcal{C}$  *be an* S-linear stable subcategory. *Let* D *be another* S*-linear category. Then if* A ⊂ C *is left (or right) admissible, so is*

$$
\mathcal{A} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \subset \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}.
$$

*Proof.* Follows from Lemma [2.12.](#page-12-1) □

**Lemma 3.18.** Let  $C$  be an S-linear category, and let  $A \subset C$  be a left admissible S-linear *stable subcategory.* Let  $f: S' \to S$  *be a morphism of schemes such that*  $f_*$  *preserves perfect complexes. Let*

$$
\mathcal{A}' = \mathcal{A} \otimes_{\text{Perf}(S)} \text{Perf}(S') \subset \mathcal{C}' = \mathcal{C} \otimes_{\text{Perf}(S)} \text{Perf}(S').
$$

*Then in terms of the functor*

$$
f_*\colon \mathcal{C}'=\mathcal{C}\otimes_{\mathrm{Perf}(S)}\mathrm{Perf}(S')\to \mathcal{C}\otimes_{\mathrm{Perf}(S)}\mathrm{Perf}(S)\simeq \mathcal{C},
$$

*the subcategory*  $A' \subset \mathcal{C}'$  *is given by* 

<span id="page-17-2"></span>
$$
\mathcal{A}' = \left\{ C' \in \mathcal{C}' \mid f_*(C' \otimes F) \in \mathcal{A} \text{ for all } F \in \text{Perf}(S') \right\}.
$$
 (3.2)

Proof. Clearly A' is contained in the right side of  $(3.2)$ . For the reverse inclusion, we consider the left orthogonal  $\mathcal{B} \subset \mathcal{C}$  to A and its base change

$$
\mathcal{B}' = \mathcal{B} \otimes_{\text{Perf}(S)} \text{Perf}(S').
$$

We have  $\mathcal{C} = \langle A, \mathcal{B} \rangle$  by Lemma [3.6,](#page-14-1) so  $\mathcal{C}' = \langle A', \mathcal{B}' \rangle$  by Lemma [3.15.](#page-17-1) Hence given  $C' \in \mathcal{C}'$  such that  $f_*(C' \otimes F) \in \mathcal{A}$  for all  $F \in \text{Perf}(S')$ , we must show that  $C'$  is right orthogonal to  $\mathcal{B}'$ . By Lemma [2.7](#page-10-0) it suffices to show that if  $B \in \mathcal{B}$  and  $G \in \text{Perf}(S')$ , then

$$
\mathcal{H}\!\mathit{om}_S(B \boxtimes G, C') \simeq 0.
$$

Note that we can write  $B \boxtimes G = f^*(B) \otimes G$ , and hence

$$
\mathcal{H}\!\mathit{om}_S(B \boxtimes G, C') \simeq \mathcal{H}\!\mathit{om}_S(f^*(B), C' \otimes G^\vee) \simeq \mathcal{H}\!\mathit{om}_S(B, f_*(C' \otimes G^\vee)).
$$

This vanishes since  $f_*(C' \otimes G^{\vee}) \in \mathcal{A}$  by assumption.

<span id="page-18-0"></span>3.5. Semiorthogonal decompositions of functor categories. Recall that given S-linear (resp. presentable S-linear) categories C and D, the S-linear (resp. cocontinuous S-linear) functors form the objects of an S-linear (resp. presentable S-linear) category  $\text{Fun}_{\text{Perf}(S)}(\mathcal{C}, \mathcal{D})$ (resp. Fun<sub>Dqc</sub>(s)( $(\mathcal{C}, \mathcal{D})$ ). In the following lemma, we use the uniform notation Funs( $(\mathcal{C}, \mathcal{D})$ ) to denote  $\text{Fun}_{\text{Perf}(S)}(\mathcal{C}, \mathcal{D})$  if  $\mathcal{C}, \mathcal{D} \in \text{Cat}_S$  or  $\text{Fun}_{\text{D}_{\text{qc}}(S)}(\mathcal{C}, \mathcal{D})$  if  $\mathcal{C}, \mathcal{D} \in \text{PrCat}_S$ .

<span id="page-18-5"></span><span id="page-18-1"></span>Lemma 3.19. *Let* C *and* D *be* S*-linear (resp. presentable* S*-linear) categories.*

(1) Let  $\mathcal{C} = \langle A_1, \ldots, A_n \rangle$  be an *S*-linear (resp. presentable *S*-linear) semiorthogonal decom*position. Then for every* i *the functor*

<span id="page-18-3"></span>
$$
\text{Fun}_S(\mathcal{A}_i, \mathcal{D}) \to \text{Fun}_S(\mathcal{C}, \mathcal{D})
$$
\n(3.3)

*induced by the projection*  $pr_i: \mathcal{C} \to \mathcal{A}_i$  *is fully faithful. Further, there is an S-linear (resp. presentable* S*-linear) semiorthogonal decomposition*

$$
\mathrm{Fun}_S(\mathcal{C},\mathcal{D})=\langle \mathrm{Fun}_S(\mathcal{A}_1,\mathcal{D}),\ldots,\mathrm{Fun}_S(\mathcal{A}_n,\mathcal{D})\rangle\,,
$$

*whose projection functors*

<span id="page-18-2"></span>
$$
\text{Fun}_S(\mathcal{C}, \mathcal{D}) \to \text{Fun}_S(\mathcal{A}_i, \mathcal{D})
$$
\n(3.4)

*are induced by the embeddings*  $\alpha_i : A_i \to \mathcal{D}$ .

<span id="page-18-4"></span>(2) Let  $\mathcal{D} = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle$  be an *S*-linear (resp. presentable *S*-linear) semiorthogonal decom*position. Then for every* i *the functor*

$$
\mathrm{Fun}_S(\mathcal{C}, \mathcal{B}_i) \to \mathrm{Fun}_S(\mathcal{C}, \mathcal{D})
$$

*induced by the emedding*  $\beta_i: \mathcal{B}_i \to \mathcal{D}$  *is fully faithful. Further, there is an S-linear (resp. presentable* S*-linear) semiorthogonal decomposition*

$$
\text{Fun}_S(\mathcal{C}, \mathcal{D}) = \langle \text{Fun}_S(\mathcal{C}, \mathcal{B}_1), \ldots, \text{Fun}_S(\mathcal{C}, \mathcal{B}_n) \rangle,
$$

*whose projection functors*

$$
\mathrm{Fun}_S(\mathcal{C},\mathcal{D})\to \mathrm{Fun}_S(\mathcal{C},\mathcal{B}_i)
$$

*are induced by the projections*  $pr_i: \mathcal{D} \to \mathcal{B}_i$ *.* 

*Proof.* We prove [\(1\)](#page-18-1). First assume we are in the S-linear (not presentable) situation. The result reduces to the case  $n = 2$ . One checks that the functor [\(3.4\)](#page-18-2) is left adjoint to [\(3.3\)](#page-18-3) for  $i = 1$ , right adjoint to [\(3.3\)](#page-18-3) for  $i = 2$ , and in both cases the composition

$$
\text{Fun}_S(\mathcal{A}_i, \mathcal{D}) \to \text{Fun}_S(\mathcal{C}, \mathcal{D}) \to \text{Fun}_S(\mathcal{A}_i, \mathcal{D})
$$

is equivalent to the identity. This proves the claim that  $\text{Fun}_S(\mathcal{A}_i, \mathcal{D}) \to \text{Fun}_S(\mathcal{C}, \mathcal{D})$  is fully faithful, and the adjointness of [\(3.4\)](#page-18-2) and [\(3.3\)](#page-18-3) immediately implies that

 $Fun_S(\mathcal{A}_1, \mathcal{D}),$   $Fun_S(\mathcal{A}_2, \mathcal{D})$ 

are semiorthogonal in Fun<sub>S</sub>( $\mathcal{C}, \mathcal{D}$ ). Finally, given any  $F \in \text{Fun}_{S}(\mathcal{C}, \mathcal{D})$ , the triangle

$$
\alpha_2 \circ pr_2 \to id_{\mathcal{C}} \to \alpha_1 \circ pr_1
$$

induces a triangle

$$
F \circ \alpha_2 \circ \text{pr}_2 \to F \to F \circ \alpha_1 \circ \text{pr}_1,
$$

which shows that Definition [3.1\(](#page-13-2)[2\)](#page-13-6) holds. In the presentable S-linear case the argument above works verbatim, with the additional remark that the functors [\(3.3\)](#page-18-3) and [\(3.4\)](#page-18-2) are cocontinuous, because colimits of functors are computed objectwise (see [\[20,](#page-71-7) Section 5.1.2]).

Part  $(2)$  is proved similarly.

<span id="page-19-0"></span>3.6. Splitting functors. Here we review the notion of a splitting functor from [\[9\]](#page-71-1) in the context of S-linear categories.

<span id="page-19-1"></span>**Definition 3.20.** Let  $\phi: \mathcal{C} \to \mathcal{D}$  be an *S*-linear functor between *S*-linear categories. The *kernel* of  $\phi$  is the full subcategory ker  $\phi \subset C$  spanned by objects  $C \in C$  such that  $\phi(C) \simeq 0$ , and the *image* of  $\phi$  is the full subcategory im  $\phi \subset \mathcal{D}$  spanned by the objects  $\phi(C) \in \mathcal{D}$  for  $C \in \mathcal{C}$ .

**Remark 3.21.** The kernel of a functor  $\phi$  as in Definition [3.20](#page-19-1) is automatically an S-linear stable subcategory of C. This is true of the image of  $\phi$  if for instance  $\phi$  is fully faithful, but is not true in general.

The following result is [\[9,](#page-71-1) Theorem 3.3] in our setting, which holds by the same proof.

<span id="page-19-4"></span>**Theorem 3.22.** Let  $\phi: \mathcal{C} \to \mathcal{D}$  be an S-linear functor between S-linear categories. The fol*lowing are equivalent:*

- <span id="page-19-2"></span>*(1r)* ker  $\phi$  *is a right admissible subcategory of* C, the restriction of  $\phi$  to (ker  $\phi$ )<sup> $\perp$ </sup> *is fully faithful, and*  $\text{im } \phi$  *is a right admissible subcategory of*  $\mathcal{D}$ *.*
- (2r)  $\phi$  *admits a right adjoint*  $\phi^!$ , and the composition of the canonical morphism id  $\rightarrow \phi^! \phi$ *with*  $\phi$  *gives an equivalence*  $\phi \simeq \phi \phi^{\dagger} \phi$ *.*
- <span id="page-19-3"></span> $(3r)$   $\phi$  *admits a right adjoint*  $\phi^!$ , there are semiorthogonal decompositions

$$
C = \langle \operatorname{im} \phi^!, \operatorname{ker} \phi \rangle,
$$
  

$$
\mathcal{D} = \langle \operatorname{ker} \phi^!, \operatorname{im} \phi \rangle,
$$

*and the functors*  $\phi$  *and*  $\phi$ <sup>*!*</sup> *induce mutually inverse equivalences* im  $\phi$ <sup>*!*</sup>  $\simeq$  im  $\phi$ *.* 

<span id="page-19-5"></span>*Similarly, the following are equivalent:*

*(1l)* ker  $\phi$  *is a left admissible subcategory of* C, the restriction of  $\phi$  *to* <sup>⊥</sup>(ker  $\phi$ ) *is fully faithful, and*  $\text{im } \phi$  *is a left admissible subcategory of*  $\mathcal{D}$ *.* 

- <span id="page-20-3"></span>(2l)  $\phi$  *admits* a left adjoint  $\phi^*$ , and the composition of the canonical morphism  $\phi^* \phi \to id$  *with*  $\phi$  *gives an equivalence*  $\phi \phi^* \phi \simeq \phi$ .
- <span id="page-20-2"></span>(3l)  $\phi$  *admits a left adjoint*  $\phi^*$ , there are semiorthogonal decompositions

$$
C = \langle \ker \phi, \operatorname{im} \phi^* \rangle ,
$$
  

$$
\mathcal{D} = \langle \operatorname{im} \phi, \ker \phi^* \rangle ,
$$

*and the functors*  $\phi$  *and*  $\phi^*$  *induce mutually inverse equivalences* im  $\phi^* \simeq \text{im } \phi$ .

**Definition 3.23.** Let  $\phi: \mathcal{C} \to \mathcal{D}$  be an S-linear functor between S-linear categories. We say  $\phi$ is *right splitting* if the equivalent conditions [\(1r\)](#page-19-2)[-\(3r\)](#page-19-3) of Theorem [3.22](#page-19-4) hold, and *left splitting* if the equivalent conditions [\(1l\)](#page-19-5)[-\(3l\)](#page-20-2) hold, and *splitting* if it is right and left splitting.

## 4. Smooth and proper categories

<span id="page-20-0"></span>In this section, we study the notions of smoothness and properness of S-linear categories, which are the analogues in noncommutative algebraic geometry of the usual geometric notions of smoothness and properness of a scheme over  $S$ . In  $\S 4.1$  $\S 4.1$  we define smoothness and properness of linear categories, and discuss the closely related notion of dualizability. In §[4.2](#page-22-0) we explain how smoothness or properness of an S-scheme relates to the corresponding property of its category of perfect complexes. In §[4.3](#page-23-0) we show that smoothness and properness of linear categories behave well under base change. In §[4.4](#page-23-1) we show that a functor between linear categories admits adjoints if the source is smooth and proper and the target is proper. In §[4.5](#page-24-0) we analyze the behavior of smoothness and properness under semiorthogonal decompositions. In §[4.6](#page-26-0) we define relative Serre functors, give some of their properties, and prove they exist in the smooth and proper case. In §[4.7](#page-27-0) we introduce the notion of the critical locus of a linear category. Finally, in §[4.8](#page-28-0) we discuss the bounded coherent category of an S-linear category and its behavior under semiorthogonal decompositions; under reasonable assumptions, this construction recovers the bounded derived category of coherent sheaves on an S-scheme from its category of perfect complexes.

We note that a number of the results in this section are folklore, or appear in some form in the literature, cf. [\[1,](#page-71-18) [22,](#page-71-19) [17,](#page-71-20) [14,](#page-71-21) [26\]](#page-71-22). However, we could not find an adequate reference for the point of view taken in this work.

## <span id="page-20-1"></span>4.1. Basic definitions.

**Definition 4.1.** Let C be a symmetric monoidal  $\infty$ -category with unit 1. An object  $C \in \mathcal{C}$ is called *dualizable* if there exists an object  $D \in \mathcal{C}$  and morphisms

$$
\text{ev}: D \otimes C \to \mathbf{1},
$$
  
coev:  $\mathbf{1} \to C \otimes D$ ,

such that the compositions

$$
C \xrightarrow{\text{coev}\otimes\text{id}} C \otimes D \otimes C \xrightarrow{\text{id}\otimes\text{ev}} C,
$$
  

$$
D \xrightarrow{\text{id}\otimes\text{coev}} D \otimes C \otimes D \xrightarrow{\text{ev}\otimes\text{id}} D,
$$

are equivalent to the identity. The object D is called the *dual* of C, and ev and coev are called the *evaluation* and *coevaluation* morphisms.

<span id="page-21-4"></span>**Remark 4.2.** Equivalently, an object  $C \in \mathcal{C}$  is dualizable if it is dualizable as an object of the symmetric monoidal homotopy category Ho(C). Moreover, if  $C \in \mathcal{C}$  is dualizable, then the dual D and the evaluation and coevaluation morphisms are uniquely determined in  $Ho(\mathcal{C})$ .

In particular, given a category C which is in Cat<sub>S</sub> or  $PrCat_S$ , it makes sense to ask whether C is dualizable. If so, we denote by  $D_S(\mathcal{C})$  the dual.

<span id="page-21-0"></span>Lemma 4.3. *Let* C *be a compactly generated presentable* S*-linear category. Then* C *is dualizable as an object of*  $PrCat_S$ *, with dual given by* 

$$
D_S(\mathcal{C}) \simeq \mathrm{Ind}((\mathcal{C}^c)^{\mathrm{op}}),
$$

where  $(\mathcal{C}^c)$ <sup>op</sup> denotes the opposite category of  $\mathcal{C}^c$ .

*Proof.* By [\[7,](#page-71-14) Chapter I.1, Proposition 7.3.2], the underlying stable  $\infty$ -category of C is dualizable with dual Ind( $(\mathcal{C}^c)^{\text{op}}$ ). Since by assumption S is quasi-compact and separated, an object of  $D_{\text{qc}}(S)$  is compact if and only if it is dualizable if and only if it is a perfect complex. Hence by [\[7,](#page-71-14) Chapter I.1, Lemma 9.1.5],  $D_{qc}(S)$  is a rigid symmetric monoidal  $\infty$ -category. Now the result follows from [\[7,](#page-71-14) Chapter I.1, Proposition 9.4.4].  $\Box$ 

The following is an easy consequence of the definitions.

<span id="page-21-1"></span>**Lemma 4.4.** Let  $\mathcal{C}$  be a presentable S-linear category. If  $\mathcal{C}$  is dualizable, then for any  $\mathcal{D}_1, \mathcal{D}_2 \in$ PrCat<sub>S</sub> *there is an equivalence* 

$$
\mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathcal{D}_1 \otimes_{\mathrm{D}_{\mathrm{qc}}(S)} \mathcal{C}, \mathcal{D}_2) \simeq \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathcal{D}_1, \mathrm{D}_S(\mathcal{C}) \otimes_{\mathrm{D}_{\mathrm{qc}}(S)} \mathcal{D}_2).
$$

Now let  $C$  be an S-linear category. Then by Lemma [4.3,](#page-21-0) the presentable S-linear category Ind $(\mathcal{C})$  is dualizable with dual Ind $(\mathcal{C}^{\mathrm{op}})$ . More explicitly, the evaluation morphism

<span id="page-21-2"></span>
$$
ev: Ind(\mathcal{C}^{op}) \otimes_{D_{qc}(S)} Ind(\mathcal{C}) \to D_{qc}(S)
$$
\n(4.1)

is induced by the functor

$$
\mathcal{H}om_S(-,-): \mathcal{C}^{op} \times \mathcal{C} \to D_{qc}(S).
$$

Under the equivalence

$$
\mathrm{Ind}(\mathcal{C}) \otimes_{D_{qc}(S)} \mathrm{Ind}(\mathcal{C}^{op}) \simeq \mathrm{Fun}_{D_{qc}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))
$$

deduced from Lemma [4.4,](#page-21-1) the coevaluation morphism

<span id="page-21-3"></span>
$$
coev: D_{qc}(S) \to Ind(\mathcal{C}) \otimes_{D_{qc}(S)} Ind(\mathcal{C}^{op})
$$
\n(4.2)

is identified with the canonical functor

 $D_{\mathrm{qc}}(S) \to \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ 

which sends  $\mathcal{O}_S \in D_{\mathrm{qc}}(S)$  to id  $\in \mathrm{Fun}_{D_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ .

**Definition 4.5.** Let  $C$  be an *S*-linear category. We say:

- (1) C is *proper* if the evaluation morphism [\(4.1\)](#page-21-2) is a morphism in the category  $PrCat_{S}^{\omega}$ , i.e. if this functor preserves compact objects.
- (2) C is *smooth* if the coevaluation morphism [\(4.2\)](#page-21-3) is a morphism in the category  $PrCat_{S}^{\omega}$ , i.e. if this functor preserves compact objects.

**Remark 4.6.** The condition that C is smooth or proper depends on its S-linear structure. For emphasis, we shall sometimes say C is smooth *over* S or proper *over* S. For instance, if  $T \to S$  is a morphism of schemes and C is a T-linear category, then we say C is smooth and proper over  $S$  to mean that  $C$  is smooth and proper with its induced  $S$ -linear structure.

<span id="page-22-2"></span><span id="page-22-1"></span>Lemma 4.7. *Let* C *be an* S*-linear category.*

- *(1)* C *is proper if and only if for every*  $C, D \in \mathcal{C}$  *the mapping object*  $\mathcal{H}om_S(C, D)$  *lies in*  $\text{Perf}(S) \subset D_{\text{qc}}(S)$ .
- <span id="page-22-5"></span>(2)  $\mathcal C$  *is smooth if and only if*  $id_{Ind(\mathcal C)} \in \text{Fun}_{D_{qc}(S)}(Ind(\mathcal C), Ind(\mathcal C))$  *is a compact object.*

*Proof.* Follows from the descriptions of the functors  $(4.1)$  and  $(4.2)$  above.

<span id="page-22-12"></span>Lemma 4.8. *Let* C *be an* S*-linear category. Then* C *is smooth and proper over* S *if and only if*  $C$  *is dualizable as an object of*  $Cat_S$ *. In this case, the dual of*  $C$  *is*  $C^{op}$ *.* 

*Proof.* Since Ind induces a symmetric monoidal equivalence  $Cat_S \simeq \text{PrCat}^{\omega}_{S}$ ,  $\mathcal{C}$  is dualizable as an object of  $Cat_S$  if and only if  $Ind(\mathcal{C})$  is dualizable as an object of  $PrCat_S^{\omega}$ . By definition, if C is smooth and proper over S, then  $Ind(\mathcal{C})$  is dualizable as an object of  $PrCat_{S}^{\omega}$ . Conversely, if Ind(C) is dualizable as an object of  $PrCat_S^{\omega}$ , by Remark [4.2](#page-21-4) its duality data must be given by [\(4.1\)](#page-21-2) and [\(4.2\)](#page-21-3), which are hence morphisms in  $PrCat_{S}^{\omega}$ .

<span id="page-22-0"></span>4.2. **Relation to geometry.** If X is an S-scheme, smoothness and properness of  $\text{Perf}(X)$ are related to the corresponding properties of X as follows. Recall that a morphism  $X \to S$ is called *perfect* if it is pseudo-coherent and of finite Tor-dimension, see [\[24,](#page-71-23) [Tag 0685\]](http://stacks.math.columbia.edu/tag/0685).

<span id="page-22-11"></span><span id="page-22-3"></span>**Lemma 4.9.** *Let*  $X \rightarrow S$  *be a morphism of schemes.* 

- <span id="page-22-4"></span>*(1) If*  $X \to S$  *is a perfect proper morphism, then*  $\text{Perf}(X)$  *is proper over* S.
- *(2) If* X *and* S *are noetherian,* X → S *is separated and of finite type, and* Perf(X) *is proper over* S, then  $X \to S$  *is a perfect proper morphism.*
- <span id="page-22-6"></span>*(3)* Perf(X) *is smooth over* S *if and only if*  $\Delta_*(0_X) \in D_{qc}(X \times_S X)$  *is a perfect complex, where*  $\Delta: X \to X \times_S X$  *is the diagonal morphism.*
- <span id="page-22-8"></span><span id="page-22-7"></span>(4) If X is smooth over S, then  $\text{Perf}(X)$  is smooth over S.
- <span id="page-22-9"></span>(5) If  $X \to S$  *is flat and locally of finite presentation and*  $\text{Perf}(X)$  *is smooth over* S, then X *is smooth over* S*.*
- <span id="page-22-10"></span>*(6)* If X is smooth and proper over S, then  $\text{Perf}(X)$  is smooth and proper over S.
- *(7)* If X and S are noetherian,  $X \to S$  is a flat, separated, and of finite type, and  $\text{Perf}(X)$  is *smooth and proper over* S*, then* X *is smooth and proper over* S*.*

*Proof.* By the criterion of Lemma [4.7\(](#page-22-1)[1\)](#page-22-2) and Remark [2.8,](#page-11-1) part [\(1\)](#page-22-3) follows from the fact that pushforward along a perfect proper morphism preserves perfect complexes (see [\[16,](#page-71-15) Example  $2.2(a)$ .

Part [\(2\)](#page-22-4) holds by Remark [2.8](#page-11-1) combined with [\[21,](#page-71-24) Lemma 0.20 and Proposition 0.21].

By the criterion of Lemma [4.7](#page-22-1)[\(2\)](#page-22-5),  $\text{Perf}(X)$  is smooth over S if and only if

$$
id \in \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{D}_{\mathrm{qc}}(X), \mathrm{D}_{\mathrm{qc}}(X))
$$

is a compact object. But by  $[3,$  Theorem 1.2(2)] there is an equivalence

$$
\mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{D}_{\mathrm{qc}}(X), \mathrm{D}_{\mathrm{qc}}(X)) \simeq \mathrm{D}_{\mathrm{qc}}(X \times_S X)
$$

under which id corresponds to  $\Delta_*(\mathcal{O}_X)$ . By [\[3,](#page-71-8) Proposition 3.24] the fiber product  $X \times_S X$  is a perfect derived scheme; in particular,  $D_{\text{qc}}(X \times_S X)^c = \text{Perf}(X \times_S X)$ . Hence [\(3\)](#page-22-6) holds.

If  $X \to S$  is smooth, then the derived fiber product  $X \times_S X$  agrees with the usual fiber product of schemes, and  $\Delta$  is a section of the smooth morphism  $X \times_S X \to X$ . Hence  $\Delta$  is a regular immersion by [\[24,](#page-71-23) [Tag 067R\]](http://stacks.math.columbia.edu/tag/067R), and hence a regular closed immersion by our standing separatedness assumptions. So  $\Delta_*(\mathcal{O}_X)$  is a perfect complex, which by [\(3\)](#page-22-6) proves [\(4\)](#page-22-7).

By Lemma [4.10](#page-23-2)[\(2\)](#page-23-3) below, part [\(5\)](#page-22-8) reduces to the case where  $S = \text{Spec}(k)$  for a field k. Moreover, it follows from [\(3\)](#page-22-6) that the smoothness of  $\text{Perf}(X)$  over S is local on X, so we may assume X is affine. In this case  $(5)$  holds by [\[17,](#page-71-20) Proposition 4.17].

Finally, [\(6\)](#page-22-9) follows by combining [\(1\)](#page-22-3) and [\(4\)](#page-22-7), and [\(7\)](#page-22-10) by combining [\(2\)](#page-22-4) and [\(5\)](#page-22-8).  $\Box$ 

<span id="page-23-0"></span>4.3. Behavior under base change. Smoothness and properness of linear categories are stable under base change:

<span id="page-23-4"></span><span id="page-23-2"></span>**Lemma 4.10.** Let  $C$  be an S-linear category, and let  $S' \rightarrow S$  be a morphism of schemes.

<span id="page-23-3"></span>*(1)* If  $\mathfrak C$  *is proper over S, then*  $\mathfrak C \otimes_{\operatorname{Perf}(S)} \operatorname{Perf}(S')$  *is proper over S'*.

(2) If  $C$  *is smooth over* S, then  $C \otimes_{\text{Perf}(S)} \text{Perf}(S')$  *is smooth over* S'.

*Proof.* Part  $(1)$  follows from Lemmas [4.7](#page-22-1) $(1)$ , [2.7,](#page-10-0) and [2.10.](#page-11-2) For part  $(2)$ , we note that if  $\mathcal{C}' = \mathcal{C} \otimes_{\text{Perf}(S)} \text{Perf}(S')$ , then there is an equivalence

 $\text{Fun}_{\text{D}_{\text{qc}}(S')}(\text{Ind}(\mathcal{C}'),\text{Ind}(\mathcal{C}')) \simeq \text{Fun}_{\text{D}_{\text{qc}}(S)}(\text{Ind}(\mathcal{C}),\text{Ind}(\mathcal{C})) \otimes_{\text{D}_{\text{qc}}(S)} \text{D}_{\text{qc}}(S'),$ 

under which  $id_{Ind(\mathcal{C})}$  corresponds to  $id_{Ind(\mathcal{C})} \boxtimes \mathcal{O}_{S'}$ . Now the result follows from the criterion of Lemma [4.7\(](#page-22-1)[2\)](#page-22-5).

<span id="page-23-5"></span>**Lemma 4.11.** Let  $T \rightarrow S$  be a morphism of schemes. Let  $C$  be a  $T$ -linear category which is smooth and proper over S. Let  $T' \to T$  be a smooth and proper morphism. Then the base *change*

$$
\mathfrak{C}\otimes_{\mathrm{Perf}(T)}\mathrm{Perf}(T')
$$

*is smooth and proper over* S*.*

*Proof.* By Lemma [4.9](#page-22-11) the category  $\text{Perf}(T')$  is smooth and proper over T. Now the result follows from Lemma [4.8](#page-22-12) combined with [\[7,](#page-71-14) Chapter I.1, Corollary 9.5.4].

**Remark 4.12.** Lemma [4.11](#page-23-5) is the analog of the following simple geometric fact: if  $T$  is an S-scheme, X is a T-scheme which is smooth and proper over S, and  $T' \to T$  is a smooth and proper morphism, then the base change  $X \times_T T'$  is smooth and proper over S. Indeed,  $X \times_T T' \to S$  is the composition of the smooth and proper morphisms  $X \times_T T' \to X$  and  $X \to S$ .

## <span id="page-23-1"></span>4.4. Existence of adjoints.

<span id="page-23-6"></span>**Lemma 4.13.** Let  $\phi: \mathcal{C} \to \mathcal{D}$  be an S-linear functor between S-linear categories, where  $\mathcal{C}$  is *smooth and proper over* S and D *is proper over* S. Then  $\phi$  *admits left and right adjoints.* 

*Proof.* Since  $\mathcal{D}$  is proper, we have a Yoneda functor

<span id="page-23-7"></span>
$$
\mathcal{H}om_S(\phi(-), -): \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{P}erf(S). \tag{4.3}
$$

This induces an S-linear functor

 $\mathcal{C}^{op} \otimes_{\text{Perf}(S)} \mathcal{D} \to \text{Perf}(S).$ 

Since  $C$  is dualizable, there is an equivalence

 $\text{Fun}_{\text{Perf}(S)}(\mathcal{C}^{\text{op}} \otimes_{\text{Perf}(S)} \mathcal{D}, \text{Perf}(S)) \simeq \text{Fun}_{\text{Perf}(S)}(\mathcal{D}, \mathcal{C}).$ 

Under this equivalence, the above functor corresponds to a functor  $\phi' : \mathcal{D} \to \mathcal{C}$ , which is the right adjoint to  $\phi$ ; namely, by construction we have an equivalence of functors

$$
\mathcal{H}om_S(\phi(-), -) \simeq \mathcal{H}om_S(-, \phi^!(-)) : \mathcal{C}^{op} \times \mathcal{D} \to \text{Perf}(S).
$$

A similar argument shows the existence of a left adjoint to  $\phi$ .

<span id="page-24-0"></span>4.5. Behavior under semiorthogonal decompositions. We will need the following technical result below.

<span id="page-24-4"></span><span id="page-24-1"></span>Lemma 4.14. *Let* C *and* D *be* S*-linear categories.*

- (1) Let  $\mathcal{C} = \langle A_1, \ldots, A_n \rangle$  *be an S-linear semiorthogonal decomposition whose components are admissible. Then the embedding and projection functors for the components of the semiorthogonal decomposition*
- $\text{Fun}_{\text{D}_{\text{qc}}(S)}(\text{Ind}(\mathcal{C}),\text{Ind}(\mathcal{D})) = \langle \text{Fun}_{\text{D}_{\text{qc}}(S)}(\text{Ind}(\mathcal{A}_1),\text{Ind}(\mathcal{D})),\ldots,\text{Fun}_{\text{D}_{\text{qc}}(S)}(\text{Ind}(\mathcal{A}_n),\text{Ind}(\mathcal{D})) \rangle$ *induced by Lemmas [3.12](#page-16-1) and [3.19](#page-18-5) all preserve compact objects.*
- <span id="page-24-3"></span>(2) Let  $\mathcal{D} = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle$  *be an S-linear semiorthogonal decomposition. Then the embedding and projection functors for the components of the semiorthogonal decomposition*

$$
\text{Fun}_{\text{D}_{qc}(S)}(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{D})) = \langle \text{Fun}_{\text{D}_{qc}(S)}(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{B}_1)), \dots, \text{Fun}_{\text{D}_{qc}(S)}(\text{Ind}(\mathcal{C}), \text{Ind}(\mathcal{B}_n)) \rangle
$$
  
induced by Lemmas 3.12 and 3.19 all preserve compact objects.

*Proof.* We prove [\(1\)](#page-24-1). We may assume  $n = 2$ . Let  $\widehat{pr}_i$ :  $\text{Ind}(\mathcal{C}) \to \text{Ind}(\mathcal{A}_i)$  be the functor induced by  $\text{pr}_i: \mathcal{C} \to \mathcal{A}_i$ . Then by Lemma [3.13,](#page-16-2) all we need to show is that the embedding functor

<span id="page-24-2"></span>
$$
\operatorname{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{D})) \to \operatorname{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))
$$
\n
$$
F \mapsto F \circ \widehat{\mathrm{pr}}_1 \tag{4.4}
$$

preserves compact objects. Note that we have an equivalence

$$
\operatorname{Ind}(\mathcal{A}_1^{\text{op}})\otimes_{\mathcal{D}_{\mathrm{qc}}(S)}\operatorname{Ind}(\mathcal{D})\simeq \operatorname{Fun}_{\mathcal{D}_{\mathrm{qc}}(S)}(\operatorname{Ind}(\mathcal{A}_1),\operatorname{Ind}(\mathcal{D}))
$$

induced by the functor

$$
\mathcal{A}_1^{\text{op}} \times \mathcal{D} \to \text{Fun}_{\text{D}_{\text{qc}}(S)}(\text{Ind}(\mathcal{A}_1), \text{Ind}(\mathcal{D}))
$$
  

$$
(C, D) \mapsto \text{Hom}_S(C, -) \otimes D,
$$

and similarly an equivalence

$$
\operatorname{Ind}(\mathcal{C}^\mathrm{op}) \otimes_{\mathbf{D}_{\mathrm{qc}}(S)} \operatorname{Ind}(\mathcal{D}) \simeq \operatorname{Fun}_{\mathbf{D}_{\mathrm{qc}}(S)}(\operatorname{Ind}(\mathcal{C}), \operatorname{Ind}(\mathcal{D}))
$$

induced by the functor

$$
\mathcal{C}^\mathrm{op} \times \mathcal{D} \to \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))
$$

$$
(C, D) \mapsto \mathcal{H}om_S(C, -) \otimes D.
$$

By Lemma [3.9](#page-15-2) the functor  $pr_1 = \alpha_1^* \colon \mathcal{C} \to \mathcal{A}_1$  admits a left adjoint, namely  $R_{\mathcal{A}_2} \circ \alpha_1 \colon \mathcal{A}_1 \to \mathcal{C}$ . It follows that under the above equivalences, the functor  $(4.4)$  is identified with the functor

$$
\operatorname{Ind}(\mathcal{A}^{op}_1)\otimes_{D_{qc}(S)}\operatorname{Ind}(\mathcal{D})\rightarrow \operatorname{Ind}(\mathcal{C}^{op})\otimes_{D_{qc}(S)}\operatorname{Ind}(\mathcal{D})
$$

given by Ind of the functor

$$
\mathcal{A}_1^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D}
$$

induced by

$$
(R_{\mathcal{A}_2} \circ \alpha_1) \times id_{\mathcal{D}} \colon \mathcal{A}_1^{op} \times \mathcal{D} \to \mathcal{C}^{op} \times \mathcal{D}.
$$

Hence the functor [\(4.4\)](#page-24-2) preserves compact objects.

The proof of [\(2\)](#page-24-3) is similar. Let  $\widehat{\beta}_i$ : Ind( $\mathcal{B}_i$ )  $\to$  Ind( $\mathcal{D}$ ) be the functor induced by  $\beta_i: \mathcal{B}_i \to \mathcal{D}$ . Then the functor

$$
\operatorname{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{B}_1)) \to \operatorname{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))
$$
\n
$$
F \mapsto \widehat{\beta}_1 \circ F \tag{4.5}
$$

is identified with Ind of the functor

$$
\mathcal{C}^{op} \otimes_{\mathrm{Perf}(S)} \mathcal{B}_1 \to \mathcal{C}^{op} \otimes_{\mathrm{Perf}(S)} \mathcal{D}
$$

induced by

$$
id_{\mathcal{C}^{op}} \times \beta_1 \colon \mathcal{C}^{op} \times \mathcal{B}_1 \to \mathcal{C}^{op} \times \mathcal{D}.
$$

Now we conclude by the same argument as above. Note that unlike [\(1\)](#page-24-1), we did not need the components  $\mathcal{B}_i \subset \mathcal{D}$  to be admissible.

<span id="page-25-4"></span>Lemma 4.15. *Let* C *be an* S*-linear category with an* S*-linear semiorthogonal decomposition*

$$
\mathcal{C}=\langle \mathcal{A}_1,\ldots,\mathcal{A}_n\rangle\,.
$$

- <span id="page-25-1"></span><span id="page-25-0"></span>(1) If  $C$  *is proper over*  $S$ *, then*  $A_i$  *is proper over*  $S$  *for all i*.
- <span id="page-25-3"></span>*(2)* If  $\mathbb C$  *is smooth over*  $S$ *, then*  $\mathcal A_1$  *is smooth over*  $S$ *.*
- (3) If all the components  $A_i \subset C$  are admissible, then C is smooth over S if and only if  $A_i$  is *smooth over* S *for all* i*.*
- <span id="page-25-2"></span>(4) If  $\mathcal{C}$  *is smooth and proper over* S, then  $\mathcal{A}_i \subset \mathcal{C}$  *is admissible and*  $\mathcal{A}_i$  *is smooth and proper over* S *for all* i*.*

*Proof.* Part [\(1\)](#page-25-0) is immediate from the definitions. Parts [\(2\)](#page-25-1)-[\(4\)](#page-25-2) reduce to the case  $n = 2$ , so we assume this for the rest of the proof.

Let  $\hat{\alpha}_i: \text{Ind}(\mathcal{A}_i) \to \text{Ind}(\mathcal{C})$  be the functor induced by the embedding  $\alpha_i: \mathcal{A}_i \to \mathcal{C}$ , and let  $\hat{pr}_i: \text{Ind}(\mathcal{C}) \to \text{Ind}(\mathcal{A}_i)$  be the functor induced by the projection  $pr_i: \mathcal{C} \to \mathcal{A}_i$ . Then by Lemmas [3.12](#page-16-1) and [3.19](#page-18-5) there is a semiorthogonal decomposition

$$
\mathrm{Fun}_{\mathrm{D}_{qc}(S)}(\mathrm{Ind}(\mathcal{C}),\mathrm{Ind}(\mathcal{C}))=\langle \mathrm{Fun}_{\mathrm{D}_{qc}(S)}(\mathrm{Ind}(\mathcal{A}_1),\mathrm{Ind}(\mathcal{A}_1)), \mathrm{Fun}_{\mathrm{D}_{qc}(S)}(\mathrm{Ind}(\mathcal{A}_1),\mathrm{Ind}(\mathcal{A}_2)), \\ \mathrm{Fun}_{\mathrm{D}_{qc}(S)}(\mathrm{Ind}(\mathcal{A}_2),\mathrm{Ind}(\mathcal{A}_1)), \mathrm{Fun}_{\mathrm{D}_{qc}(S)}(\mathrm{Ind}(\mathcal{A}_2),\mathrm{Ind}(\mathcal{A}_2)) \rangle
$$

with embedding functors

$$
f_{ij} \colon \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{A}_i), \mathrm{Ind}(\mathcal{A}_j)) \to \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))
$$

$$
F \mapsto \widehat{\alpha}_j \circ F \circ \widehat{\mathrm{pr}}_i
$$

and projection functors

$$
p_{ij} \colon \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})) \to \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{A}_i), \mathrm{Ind}(\mathcal{A}_j))
$$

$$
G \mapsto \widehat{\mathrm{pr}}_j \circ G \circ \widehat{\alpha}_i.
$$

Observe that  $p_{11}(\text{id}_{\text{Ind}(\mathcal{C})}) \simeq \text{id}_{\text{Ind}(\mathcal{A}_1)}$ ,  $p_{22}(\text{id}_{\text{Ind}(\mathcal{C})}) \simeq \text{id}_{\text{Ind}(\mathcal{A}_2)}$ , and there is an exact triangle

$$
f_{22}(\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_2)}) \to \mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \to f_{11}(\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_1)}).
$$

By Lemma [3.13](#page-16-2) the projection  $p_{11}$  preserves compact objects. Hence if

 $\mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \in \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ 

is compact then so is

 $\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_1)} \in \mathrm{Fun}_{\mathrm{D}_{\mathrm{qc}}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{A}_1)).$ 

By Lemma  $4.7(2)$  $4.7(2)$  this proves  $(2)$ .

If all  $A_i \subset \mathcal{C}$  are admissible, then by Lemma [4.14](#page-24-4) all of the embedding and projection functors  $f_{ij}, p_{ij}$  preserve compact objects. Hence by the observation above,  $\mathrm{id}_{\mathrm{Ind}(\mathcal{C})}$  is a compact object if and only if  $id_{Ind(A_1)}$  and  $id_{Ind(A_2)}$  are compact objects. By Lemma [4.7](#page-22-1)[\(2\)](#page-22-5) this proves [\(3\)](#page-25-3).

Finally, assume C is smooth and proper over S. Then by parts [\(1\)](#page-25-0) and [\(2\)](#page-25-1) the category  $\mathcal{A}_1$ is smooth and proper over S. By Lemma [4.13](#page-23-6) it follows that  $A_1 \subset \mathcal{C}$  is admissible. Hence by Lemma [3.6](#page-14-1) there is a decomposition  $C = \langle A_1^{\perp}, A_1 \rangle$ , and by Lemma [3.8](#page-15-3) there is an equivalence  $A_2 \simeq A_1^{\perp}$ . So we can apply the above argument to conclude  $A_2$  is also smooth and proper over S. This proves [\(4\)](#page-25-2).

<span id="page-26-0"></span>4.6. Serre functors. If  $C$  is a proper S-linear category, then by Lemma [4.7](#page-22-1) the mapping object  $\mathcal{H}om_S(C, D)$  lies in Perf(S) for every  $C, D \in \mathcal{C}$ . In this situation, we have functors

$$
\mathcal{H}om_S: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Perf}(S) \qquad \mathcal{H}om_S^{\text{op}}: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Perf}(S)^{\text{op}} (X, Y) \mapsto \mathcal{H}om_S(X, Y) \qquad (X, Y) \mapsto \mathcal{H}om_S(Y, X),
$$

where  $(-)$ <sup>op</sup> denotes the opposite category. Note that dualization gives an equivalence

$$
(-)^{\vee} \colon \operatorname{Perf}(S)^{\text{op}} \to \operatorname{Perf}(S).
$$

<span id="page-26-1"></span>Definition 4.16. Let C be a proper S-linear category. A *relative Serre functor* for C over S is an S-linear autoequivalence  $S_{\mathcal{C}/S}$ :  $\mathcal{C} \xrightarrow{\sim} \mathcal{C}$  such that there is a commutative diagram

$$
\begin{aligned}\n\mathcal{C}^{op} \times \mathcal{C} &\xrightarrow{\mathrm{id} \times \mathrm{S}_{\mathcal{C}/S}} \mathcal{C}^{op} \times \mathcal{C} \\
\downarrow^{\mathrm{Hom}^{\mathrm{op}}_S} &\downarrow^{\mathrm{Hom}_S} \\
\mathrm{Perf}(S)^{\mathrm{op}} &\xrightarrow{(-)^\vee} \mathrm{Perf}(S)\n\end{aligned}
$$

In other words, a relative Serre functor for C is an S-linear autoequivalence characterized by the existence of natural equivalences

$$
\mathcal{H}\!\mathit{om}_S(C, \mathsf{S}_{\mathcal{C}/S}(D)) \simeq \mathcal{H}\!\mathit{om}_S(D, C)^\vee
$$

for all objects  $C, D \in \mathcal{C}$ .

**Remark 4.17.** Let  $\pi: X \to S$  be a proper morphism of schemes such that  $\pi_*$  preserves perfect complexes, and such that a relative dualizing complex  $\omega_{\pi}$  exists and is a shift of a line bundle. Then by Grothendieck duality  $S_{X/S} = (-\otimes \omega_{\pi})$  is a relative Serre functor for Perf(X) over S.

The following useful result follows easily from the definitions, see [\[6,](#page-71-17) Proposition 3.6].

**Lemma 4.18.** Let C be a proper S-linear category that admits a relative Serre functor  $S_{\varrho/S}$ .

 $(1)$  If  $\mathcal{A} \subset \mathcal{C}$  is a right admissible subcategory, then  $\mathrm{S}_{\mathcal{C}/S}(\mathcal{A}) = \mathcal{A}^{\perp \perp}$ (2) If  $A \subset \mathcal{C}$  *is a left admissible subcategory, then*  $S_{\mathcal{C}/S}^{-1}(A) = {}^{\perp\perp}A$ *.* 

A relative Serre functor exists for any smooth and proper category:

Lemma 4.19. *Let* C *be a smooth and proper* S*-linear category. Then there exists a relative Serre functor*  $S_{\mathcal{C}/S}$  *of*  $\mathcal{C}$  *over S.* 

*Proof.* Let  $\tau(-,-)$ :  $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$  be the transposition functor. Then the argument of Lemma [4.13](#page-23-6) applied to

$$
\mathcal{H}\!{\it om}_S(\tau(-,-))^{\vee}\colon \mathcal{C}^{\mathrm{op}}\times \mathcal{C}\to \mathrm{Perf}(S)
$$

in place of [\(4.3\)](#page-23-7) shows that there exists a functor  $S_{\mathcal{C}/S}$ :  $\mathcal{C} \to \mathcal{C}$  such that there is a commutative diagram as in Definition [4.16.](#page-26-1) It remains to show that  $S_{\mathcal{C}/S}$  is an equivalence. The following computation for  $C, D \in \mathcal{C}$  shows that  $S_{\mathcal{C}/S}$  is fully faithful:

$$
\mathcal{H}om_S(S_{\mathcal{C}/S}(C), S_{\mathcal{C}/S}(D)) \simeq \mathcal{H}om_S(D, S_{\mathcal{C}/S}(C))^{\vee}
$$
  
\n
$$
\simeq \mathcal{H}om_S(C, D)^{\vee \vee}
$$
  
\n
$$
\simeq \mathcal{H}om_S(C, D).
$$

Next note that by Lemma [4.13](#page-23-6) there exists a left adjoint  $S_{\mathcal{C}/S}^* : \mathcal{C} \to \mathcal{C}$ . For  $C, D \in \mathcal{C}$ , we compute:

$$
\mathcal{H}om_S(C, \mathcal{S}_{\mathcal{C}/S} \mathcal{S}_{\mathcal{C}/S}^*(D)) \simeq \mathcal{H}om_S(\mathcal{S}_{\mathcal{C}/S}^*(D), C)^\vee
$$
  
\n
$$
\simeq \mathcal{H}om_S(D, \mathcal{S}_{\mathcal{C}/S}(C))^\vee
$$
  
\n
$$
\simeq \mathcal{H}om_S(C, D)^\vee
$$
  
\n
$$
\simeq \mathcal{H}om_S(C, D).
$$

It follows that  $S_{\mathcal{C}/S} S_{\mathcal{C}/S}^*(D) \simeq D$ , and hence  $S_{\mathcal{C}/S}$  is essentially surjective.

Finally, we note that if a relative Serre functor exists, it is automatically linear over any scheme over which  $\mathfrak C$  is defined.

**Lemma 4.20.** Let  $C$  be a proper S-linear category with a relative Serre functor  $S_{C/S}$ . Assume *the* S*-linear structure of* C *is induced by a* T*-linear structure, where* T *is an* S*-scheme. Then*  $S_{C/S}$  *has the structure of a T-linear functor.* 

*Proof.* Similar to the proof of Lemma [2.11.](#page-12-2) □

## <span id="page-27-0"></span>4.7. Critical loci.

**Definition 4.21.** Let C be an S-linear category. Let  $s \in S$  be a point and let  $Spec(\kappa(s)) \to S$ be the corresponding morphism from the spectrum of the residue field at  $s \in S$ . We say s is a *critical point* for C if the base change

 $C \otimes_{\text{Perf}(S)} \text{Perf}(\text{Spec}(\kappa(s)))$ 

fails to be smooth over  $Spec(\kappa(s))$ . The *critical locus* of C is the set

 $Crit_S(\mathcal{C}) = \{ s \in S \mid s \text{ is a critical point for } \mathcal{C} \} \subset S.$ 

**Remark 4.22.** Let  $X \rightarrow S$  be a morphism of schemes which is flat and locally of finite presentation. Then by Lemma [4.9](#page-22-11)[\(5\)](#page-22-8), we have  $s \in \text{Crit}_S(\text{Perf}(X))$  if and only if the fiber  $X_s \to \text{Spec}(\kappa(s))$  fails to be smooth (the flatness of  $X \to S$  guarantees  $X_s \to \text{Spec}(\kappa(s))$  is an ordinary, i.e. not derived, scheme).

**Remark 4.23.** If  $X \to S$  is a morphism of schemes which is flat, proper, and of finite presentation, then  $\mathrm{Crit}_S(\mathrm{Perf}(X))$  is a closed subset of S. We expect that, under suitable assumptions, this holds for general S-linear categories C, but we have not checked this.

**Lemma 4.24.** *Let*  $C$  *be a smooth S*-linear category. Then  $\operatorname{Crit}_S(C) = \emptyset$ .

*Proof.* Follows from Lemma [4.10.](#page-23-2) □

Lemma 4.25. *Let* C *be an* S*-linear category with an* S*-linear semiorthogonal decomposition*

$$
\mathcal{C} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle
$$

*whose components are admissible. Then we have*

$$
\mathrm{Crit}_S(\mathcal{C}) = \bigcup_{i=1}^n \mathrm{Crit}_S(\mathcal{A}_i) \subset S.
$$

<span id="page-28-0"></span>*Proof.* Follows from Lemmas [4.15\(](#page-25-4)[3\)](#page-25-3) and [3.17.](#page-17-3)

4.8. Bounded coherent categories. Given an  $S$ -scheme  $X$ , in addition to the  $S$ -linear category  $\text{Perf}(X)$ , there is another naturally associated S-linear category: the full subcategory  $D_{\text{coh}}^{b}(X) \subset D_{\text{qc}}(S)$  spanned by complexes with bounded coherent cohomology. By the results of [\[4,](#page-71-9) Theorem 1.1.3], under suitable hypotheses  $\text{Perf}(X)$  in fact determines  $D^b_{coh}(X)$ .

**Theorem 4.26.** Let  $\pi: X \to S$  be a proper morphism of finite presentation between derived *schemes over a field of characteristic* 0*, with* S *locally noetherian. Then there is an equivalence*

$$
D_{\mathrm{coh}}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Perf}(S)}\Big(\mathrm{Perf}(X)^{\mathrm{op}}, D_{\mathrm{coh}}^{\mathrm{b}}(S)\Big)
$$

$$
\mathcal{E} \mapsto \mathcal{H}om_S(-, \mathcal{E}),
$$

where  $\text{Perf}(X)$ <sup>op</sup> denotes the opposite category.

*Proof.* By [\[4,](#page-71-9) Theorem 1.1.3] there is an equivalence

$$
D_{\mathrm{coh}}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Perf}(S)}\Big(\mathrm{Perf}(X), D_{\mathrm{coh}}^{\mathrm{b}}(S)\Big)
$$

$$
\mathcal{E} \mapsto \pi_* \circ (-\otimes \mathcal{E}).
$$

Now the result follows by composing with the equivalence  $\text{Perf}(X) \simeq \text{Perf}(X)^{\text{op}}$  given by dualization.

This motivates the following definition.

**Definition 4.27.** Let  $C$  be a proper S-linear category, where S is locally noetherian over a field of characteristic 0. The *bounded coherent category* of C is

$$
\mathcal{C}^{\text{coh}} = \text{Fun}_{\text{Perf}(S)}\left(\mathcal{C}^{\text{op}}, \mathcal{D}_{\text{coh}}^{\text{b}}(S)\right).
$$

Note that in the above situation, there is a canonical functor

$$
C \to C^{coh}, \quad C \mapsto \mathcal{H}om_S(-,C),
$$

which in the geometric case  $\mathcal{C} = \text{Perf}(X)$  corresponds to the embedding  $\text{Perf}(X) \subset D^b_{\text{coh}}(X)$ . Next we show that under suitable hypotheses, a semiorthogonal decomposition of  $\mathfrak C$  induces a semiorthogonal decomposition on bounded coherent categories, compatible with the above functor. This can be used to recover the results of [\[12\]](#page-71-25) and remove the geometricity assumptions there, see Remark [4.29](#page-29-0) below.

<span id="page-28-1"></span>Proposition 4.28. *Let* C *be a proper* S*-linear category, where* S *is locally noetherian over a field of characteristic* 0*. Let*  $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$  *be an S-linear semiorthogonal decomposition with* B ⊂ C *admissible. Then there is an* S*-linear semiorthogonal decomposition*

$$
\mathcal{C}^{\mathrm{coh}}=\left\langle \mathcal{A}^{\mathrm{coh}},\mathcal{B}^{\mathrm{coh}}\right\rangle ,
$$

with the property that the canonical functor  $C \rightarrow C^{coh}$  restricts to the canonical functors  $A \rightarrow A^{\text{coh}}$  and  $B \rightarrow B^{\text{coh}}$  on each component.

*Proof.* Let  $\alpha: \mathcal{A} \to \mathcal{C}$  and  $\beta: \mathcal{B} \to \mathcal{C}$  be the inclusions. Let  $\alpha': \mathcal{A}' = {}^{\perp} \mathcal{B} \to \mathcal{C}$  be the inclusion of the orthogonal, so that by Lemma [3.6](#page-14-1) there is a semiorthogonal decomposition  $C = \langle \mathcal{B}, \mathcal{A}' \rangle$ . Passing to opposite categories, we obtain a semiorthogonal decomposition  $\mathcal{C}^{op} = \langle (\mathcal{A}')^{op}, \mathcal{B}^{op} \rangle$ , with embedding functors induced by  $\alpha'$  and  $\beta$ . Note that the functor  $\mathcal{C}^{op} \to \mathcal{B}^{op}$  induced by the left adjoint  $\beta^* \colon \mathcal{C} \to \mathcal{B}$  is the *right* adjoint to the inclusion  $\mathcal{B}^{op} \to \mathcal{C}^{op}$ ; similarly, the functor  $\mathcal{C}^{\text{op}} \to (\mathcal{A}')^{\text{op}}$  induced by  $(\alpha')^! : \mathcal{C} \to \mathcal{A}'$  is *left* adjoint to  $(\mathcal{A}')^{\text{op}} \to \mathcal{C}^{\text{op}}$ . By Lemma [3.19,](#page-18-5) we conclude that there is a semiorthogonal decomposition

$$
\mathcal{C}^{\mathrm{coh}}=\left\langle (\mathcal{A}')^{\mathrm{coh}}, \mathcal{B}^{\mathrm{coh}}\right\rangle.
$$

Note that by Lemma [3.8](#page-15-3) the functor  $R_{\mathcal{B}} \circ \alpha$  induces an equivalence  $\mathcal{A} \simeq \mathcal{A}'$ , and hence an equivalence  $\mathcal{A}^{\text{coh}} \simeq (\mathcal{A}')^{\text{coh}}$ . Hence it remains to show that the above decomposition is compatible with the canonical functors to the bounded coherent categories. By the description of the embedding functors in Lemma [3.8,](#page-15-3) we see that the composition  $\mathcal{A}' \to (\mathcal{A}')^{\text{coh}} \to \mathcal{C}^{\text{coh}}$ is given by

$$
A' \mapsto \mathcal{H}om_S(-, A') \mapsto \mathcal{H}om_S((\alpha')^!(-), A').
$$

By Lemma [3.9](#page-15-2) we have

$$
\mathcal{H}om_S((\alpha')^!(-),A') \simeq \mathcal{H}om_S(-,L_{\mathcal{B}} \circ \alpha'(A')).
$$

Hence by Lemma [3.8,](#page-15-3) it follows that under the identifications  $A \simeq A'$  and  $A^{coh} \simeq (A')^{coh}$ given by  $R_B \circ \alpha$ , the canonical functor  $C \to C^{coh}$  restricts to the canonical functor  $A \to \mathcal{A}^{coh}$ . A similar (even easier) argument shows the compatibility of the canonical functors to the bounded coherent categories for the second component  $\mathcal{B}^{\text{coh}} \subset \mathcal{C}^{\text{coh}}$ . В последните последните и производите в село <br>В село в сел

<span id="page-29-0"></span>Remark 4.29. In [\[12,](#page-71-25) Theorem 5.6], the problem of inducing a semiorthogonal decomposition of  $D^b_{coh}(X)$  from one of  $\mathrm{Perf}(X)$  is addressed under certain assumptions. Let  $\mathcal{A} \subset \mathrm{Perf}(X)$  be a component of a semiorthogonal decomposition with admissible components. Then the induced subcategory  $\hat{\mathcal{A}} \subset D^b_{coh}(X)$  defined in [\[12\]](#page-71-25) can be described as follows. By taking Ind-categories, we obtain an admissible subcategory  $\text{Ind}(\mathcal{A}) \subset D_{\text{qc}}(X)$ . Note that also  $D_{\text{coh}}^{\text{b}}(X) \subset D_{\text{qc}}(X)$ . Then  $\hat{\mathcal{A}} \subset D^b_{coh}(X)$  is the full subcategory with objects given by the intersection

$$
\hat{\mathcal{A}} = \mathrm{Ind}(\mathcal{A}) \cap D^{\mathrm{b}}_{\mathrm{coh}}(X).
$$

On the other hand, in Proposition [4.28](#page-28-1) we have defined a subcategory  $\mathcal{A}^{coh} \subset D^b_{coh}(X)$ . The categories  $\hat{\mathcal{A}}$  and  $\mathcal{A}^{\text{coh}}$  agree when the assumptions of [\[12,](#page-71-25) Theorem 5.6] and Proposition [4.28](#page-28-1) both hold. Indeed, we can describe  $\mathcal{A}^{\text{coh}} \subset D^{\text{b}}_{\text{coh}}(X)$  by the formula

$$
\mathcal{A}^{\mathrm{coh}} = \mathrm{Ind}(\mathcal{A}^{\mathrm{coh}}) \cap D_{\mathrm{coh}}^{\mathrm{b}}(X),
$$

where the intersection is taken in  $\text{Ind}(\text{D}_{\text{coh}}^{\text{b}}(X))$ . Note that the inclusions  $A \subset \mathcal{A}^{\text{coh}}$  and  $\text{Perf}(X) \subset D^{\text{b}}_{\text{coh}}(X)$  induce inclusions  $\text{Ind}(\mathcal{A}) \subset \text{Ind}(\mathcal{A}^{\text{coh}})$  and  $D_{\text{qc}}(X) \subset \text{Ind}(D^{\text{b}}_{\text{coh}}(X)).$ Hence the above formulas imply  $\hat{\mathcal{A}} \subset \mathcal{A}^{\text{coh}}$ . Writing  $\text{Perf}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ , we similarly have  $\hat{\mathcal{B}} \subset \mathcal{B}^{\text{coh}}.$  Since  $D_{\text{coh}}^{\text{b}}(X) = \langle \hat{\mathcal{A}}, \hat{\mathcal{B}} \rangle = \langle \mathcal{A}_{\cdot}^{\text{coh}}, \mathcal{B}^{\text{coh}} \rangle$ , it follows that  $\hat{\mathcal{A}} = \mathcal{A}^{\text{coh}}$  and  $\hat{\mathcal{B}} = \mathcal{B}^{\text{coh}}.$ 

An advantage of our definition of  $A^{coh}$  is that it is defined canonically in terms of A, and does not require an embedding into the derived category of a variety or the use of t-structures as in [\[12\]](#page-71-25).

## 5. A formalism of Fourier–Mukai kernels

<span id="page-30-0"></span>In this section, we introduce a formalism for describing certain functors between linear categories in terms of "Fourier–Mukai kernels". After reviewing the classical situation in §[5.1,](#page-30-1) we define our notion of Fourier–Mukai kernels in the noncommutative setting in §[5.2.](#page-30-2) In §[5.3](#page-32-0) we develop some basic properties of these functors.

<span id="page-30-1"></span>5.1. The geometric setting. Let  $X_1 \to T$  and  $X_2 \to T$  be morphisms of schemes, such that pushforward along the projection  $pr_2: X_2 \times_T X_1 \to X_2$  preserves perfect complexes.

**Remark 5.1.** By [\[16,](#page-71-15) Example 2.2(a)], the assumption on  $pr_{2*}$  is satisfied if  $pr_2$  is a perfect proper morphism.

In the above situation, we call an object

$$
\mathcal{E} \in \mathrm{Perf}(X_2 \times_T X_1)
$$

a Fourier–Mukai kernel. To any such  $\mathcal{E}$ , there is an associated T-linear functor

$$
\Phi_{\mathcal{E}}\colon \mathrm{Perf}(X_1) \to \mathrm{Perf}(X_2), \quad F \mapsto \mathrm{pr}_{2*}(\mathcal{E} \otimes \mathrm{pr}_1^* F).
$$

**Remark 5.2.** Our convention that a kernel  $\mathcal{E} \in \text{Perf}(X_2 \times_T X_1)$  corresponds to a functor  $\text{Perf}(X_2) \to \text{Perf}(X_1)$ , instead of  $\text{Perf}(X_2) \to \text{Perf}(X_1)$ , is slightly unconventional. But it will be convenient later.

This construction  $\mathcal{E} \mapsto \Phi_{\mathcal{E}}$  gives rise to a functor

<span id="page-30-3"></span>
$$
Perf(X_2 \times_T X_1) \to \text{Fun}_{\text{Perf}(T)}(\text{Perf}(X_1), \text{Perf}(X_2))
$$
\n
$$
(5.1)
$$

from the category of Fourier–Mukai kernels to the category of T-linear functors. We have the following fundamental result; the second part gives a criterion for the above functor to be an equivalence.

<span id="page-30-6"></span>**Theorem 5.3** ([\[3,](#page-71-8) Theorem 1.2]). *(1) For any*  $X_1 \to T$  *and*  $X_2 \to T$ *, there is an equivalence*  $\text{Perf}(X_1 \times_T X_2) \simeq \text{Perf}(X_1) \otimes_{\text{Perf}(T)} \text{Perf}(X_2).$ 

*(2)* If  $X_1 \rightarrow T$  *is smooth and proper,* [\(5.1\)](#page-30-3) *is an equivalence.* 

**Remark 5.4.** In [\[3\]](#page-71-8), the theorem is formulated for  $X_1, X_2$ , and T being so-called perfect stacks. Any quasi-compact and separated scheme is a perfect stack [\[3,](#page-71-8) Proposition 3.19], so with our conventions from  $\S1.6$  $\S1.6$  any scheme is perfect.

<span id="page-30-2"></span>5.2. The noncommutative setting. We consider the following situation.

<span id="page-30-4"></span>**Setup 5.5.** For  $i = 1, 2$ , assume given the following data:

– A diagram of schemes

<span id="page-30-5"></span> $Z_i$ ~~  $\diagup$  $\mathbf{r}$ ❄ ❄ ❄ ❄ ❄ ❄ ❄ ❄  $S_i$  T (5.2)

such that the pushforward along the projection  $pr_2: Z_2 \times_T Z_1 \to Z_2$  preserves perfect complexes.

- An  $S_i$ -linear category  $\mathcal{C}_i$ .

In the above setup, the base change

$$
\mathfrak C_i\otimes_{{\rm Perf}(S_i)}{\rm Perf}(Z_i)
$$

has the structure of T-linear category. In what follows, this category plays the role of the scheme  $X_i$  from our discussion in §[5.1](#page-30-1) of Fourier–Mukai functors in the geometric setting.

Definition 5.6. In Setup [5.5,](#page-30-4) we define the category of *Fourier–Mukai kernels* as

 $FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) = \text{Fun}_{\text{Perf}(S_1)}(\mathcal{C}_1, \mathcal{C}_2 \otimes_{\text{Perf}(S_2)} \text{Perf}(Z_2 \times_T Z_1)).$ 

When  $\mathcal{C}_i$  and  $S_i$  are clear from the context, we sometimes refer to an object of the above category as a  $Z_2 \times_T Z_1$ -kernel.

This definition is motivated by the following result.

<span id="page-31-0"></span>Proposition 5.7. *In Setup [5.5,](#page-30-4) there is a natural functor*

$$
\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) \to \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)),
$$

*which is an equivalence if*  $Z_1 \rightarrow T$  *is smooth and proper.* 

*Proof.* The classical functor

$$
Perf(Z_2 \times_T Z_1) \to Fun_{Perf(T)}(Perf(Z_1), Perf(Z_2))
$$

induces a functor

$$
\mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2 \times_T Z_1) \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathrm{Perf}(Z_1),\mathrm{Perf}(Z_2)).
$$

Composing with the canonical functor

$$
\text{C}_2 \otimes_{\text{Perf}(S_2)} \text{Fun}_{\text{Perf}(T)}(\text{Perf}(Z_1),\text{Perf}(Z_2)) \rightarrow \text{Fun}_{\text{Perf}(T)}\big(\text{Perf}(Z_1),\text{C}_2 \otimes_{\text{Perf}(S_2)} \text{Perf}(Z_2)\big),
$$

this induces a functor

$$
\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) \to \mathrm{Fun}_{\mathrm{Perf}(S_1)}\big(\mathcal{C}_1, \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big)\big).
$$

Composing with the canonical equivalence

$$
\text{Fun}_{\text{Perf}(S_1)}(\mathcal{C}_1, \text{Fun}_{\text{Perf}(T)}(\text{Perf}(Z_1), \mathcal{C}_2 \otimes_{\text{Perf}(S_2)} \text{Perf}(Z_2))) \\ \xrightarrow{\sim} \text{Fun}_{\text{Perf}(T)}(\mathcal{C}_1 \otimes_{\text{Perf}(S_1)} \text{Perf}(Z_1), \mathcal{C}_2 \otimes_{\text{Perf}(S_2)} \text{Perf}(Z_2)),
$$

gives the sought-after functor. It follows from Theorem 5.3 and the construction that this functor is an equivalence if  $Z_1 \to T$  is smooth and proper.

In Setup [5.5,](#page-30-4) given a kernel

$$
\mathcal{E} \in \text{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1),
$$

we denote by

$$
\Phi_{\mathcal{E}} \in \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2))
$$

the corresponding functor given by Proposition [5.7.](#page-31-0)

<span id="page-32-0"></span>5.3. Properties of kernel functors. For the rest of this section, we describe the the effect of certain operations on  $\mathcal E$  in terms of the associated functor  $\Phi_{\mathcal E}$ .

<span id="page-32-1"></span>**Setup 5.8.** In addition to the data of Setup [5.5,](#page-30-4) assume given for  $i = 1, 2$  another diagram of schemes

<span id="page-32-2"></span> $Z_i'$ ′  $\overline{ }$  $\bigg/$ ſ ❄ ❄ ❄ ❄ ❄ ❄ ❄  $S_i$   $T'$ (5.3)

such that the pushforward along the projection  $pr_2: Z'_2 \times_{T'} Z'_1 \to Z'_2$  preserves perfect complexes.

In the above setup, if we are given a functor

$$
F: \mathrm{Perf}(Z_2 \times_T Z_1) \to \mathrm{Perf}(Z_2' \times_{T'} Z_1')
$$

which is  $S_2 \times_S S_1$ -linear, we denote by the same symbol the induced functor

 $F: FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) \to FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z'_2 \times_{T'} Z'_1)$ 

on kernel categories. In particular, this construction allows us to pullback and pushforward kernels along morphisms  $Z'_2 \times_{T'} Z'_1 \to Z_2 \times_T Z_1$ , and to tensor kernels with objects of Perf( $Z_2 \times_T Z_1$ ). As we discuss below, these operations formally behave the same as in the geometric case.

The next two results follow by unwinding the definitions.

Lemma 5.9. *Assume we are in Setup [5.8.](#page-32-1) Let*

 $F_1 \rightarrow F_2 \rightarrow F_3$ 

*be an exact triangle of*  $S_2 \times_S S_1$ -linear functors  $\text{Perf}(Z_2 \times_T Z_1) \to \text{Perf}(Z_2' \times_{T'} Z_1')$ . Let

$$
\mathcal{E} \in FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)
$$

*be a kernel. Then there is an induced exact triangle of*  $Z'_2 \times_{T'} Z'_1$ -kernels

$$
F_1(\mathcal{E}) \to F_2(\mathcal{E}) \to F_3(\mathcal{E}).
$$

Lemma 5.10. *Assume we are in Setup [5.8.](#page-32-1) Let*

$$
\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3
$$

*be an exact triangle of kernels in*  $FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$ *. Let* 

$$
F\colon \mathrm{Perf}(Z_2\times_T Z_1)\to \mathrm{Perf}(Z_2'\times_{T'} Z_1')
$$

*be an*  $S_2 \times_S S_1$ -linear functor. Then there is an induced exact triangle of kernels

 $F(\mathcal{E}_1) \to F(\mathcal{E}_2) \to F(\mathcal{E}_3).$ 

*in*  $FM(C_1/S_1, C_2/S_2, Z'_2 \times_{T'} Z'_1)$ *.* 

We leave it to the reader to check that the next two results reduce to the geometric case, where they are well-known.

**Lemma 5.11.** *Assume we are in Setup [5.8.](#page-32-1)* Let  $f: Z'_2 \times_{T'} Z'_1 \to Z_2 \times_T Z_1$  *be a morphism such that* f<sup>∗</sup> *preserves perfect complexes.*

*(1) Let*

$$
\mathcal{E} \in \text{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)
$$
  
Let  $\mathcal{F}' \in \text{Perf}(Z'_2 \times_{T'} Z'_1)$ . Then there is an equivalence of kernels

 $f_*(f^*(\mathcal{E}) \otimes \mathcal{F}') \simeq \mathcal{E} \otimes f_*(\mathcal{F}').$ 

*(2) Let*

$$
\mathcal{E}' \in FM\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z'_2\times_{T'}Z'_1\big)
$$

*be a kernel. Let*  $\mathcal{F} \in \text{Perf}(Z_2 \times_T Z_1)$ *. Then there is an equivalence of kernels* 

 $f_*(\mathcal{E}' \otimes f^*(\mathcal{F})) \simeq f_*(\mathcal{E}') \otimes \mathcal{F}.$ 

Lemma 5.12. *Let*

*be a kernel.* 

$$
\mathcal{E} \in FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)
$$

*be a kernel. Let*  $\mathfrak{F}_2 \in \text{Perf}(Z_2), \mathfrak{F}_1 \in \text{Perf}(Z_1)$ *, and let*  $\mathfrak{F}_2 \boxtimes \mathfrak{F}_1 \in \text{Perf}(Z_2 \times_T Z_1)$  *be their exterior product. Then there is an equivalence*

$$
\Phi_{\mathcal{E}\otimes(\mathcal{F}_2\boxtimes\mathcal{F}_1)}\simeq(-\otimes\mathcal{F}_2)\circ\Phi_{\mathcal{E}}\circ(-\otimes\mathcal{F}_1).
$$

<span id="page-33-0"></span>**Setup 5.13.** Assume that in Setup [5.8,](#page-32-1) for  $i = 1, 2$  the diagrams [\(5.2\)](#page-30-5) and [\(5.3\)](#page-32-2) fit into a commutative diagram

<span id="page-33-2"></span>
$$
Z'_{i} \longrightarrow T'
$$
\n
$$
S_{i} \longleftarrow Z'_{i} \longrightarrow T
$$
\n
$$
S_{i} \longleftarrow Z_{i} \longrightarrow T
$$
\n
$$
(5.4)
$$

In the above setup, we have a morphism

 $f_2 \times f_1$ :  $Z'_2 \times_{T'} Z'_1 \to Z_2 \times_T Z_1$ ,

which by pushforward (when it preserves perfect complexes) and pullback induces functors between kernel categories as above. Further, the morphism  $f_i$  induces by pushforward (when it preserves perfect complexes) and pullback functors

$$
f_{i*}: \mathcal{C}_i \otimes_{\text{Perf}(S_i)} \text{Perf}(Z_i') \to \mathcal{C}_i \otimes_{\text{Perf}(S_i)} \text{Perf}(Z_i),
$$
  

$$
f_i^*: \mathcal{C}_i \otimes_{\text{Perf}(S_i)} \text{Perf}(Z_i) \to \mathcal{C}_i \otimes_{\text{Perf}(S_i)} \text{Perf}(Z_i').
$$

The next two results easily reduce to the geometric case, where they are straightforward.

**Lemma 5.14.** *In Setup* [5.13,](#page-33-0) assume further that  $(f_2 \times f_1)_*$  preserves perfect complexes. Let

$$
\mathcal{E}' \in \mathrm{FM}\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z'_2 \times_{T'} Z'_1\big)
$$

*be a kernel, and*

$$
\mathcal{E} = (f_2 \times f_1)_* \mathcal{E}' \in \text{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1).
$$

*Then there is an equivalence*

$$
\Phi_{\mathcal{E}} \simeq f_{2*} \circ \Phi_{\mathcal{E}'} \circ f_1^*.
$$

<span id="page-33-1"></span>Lemma 5.15. *Assume we are in Setup [5.13.](#page-33-0) Let*

 $\mathcal{E} \in \text{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$ 

*be a kernel, and*

$$
\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z'_2 \times_{T'} Z'_1).
$$

<span id="page-34-0"></span>*(1) If the square*



*is cartesian, then there is an equivalence*

<span id="page-34-3"></span>
$$
\Phi_{\mathcal{E}'} \circ f_1^* \simeq f_2^* \circ \Phi_{\mathcal{E}}.\tag{5.5}
$$

<span id="page-34-1"></span>*(2) If the square*

$$
Z_2' \times_{T'} Z_1' \longrightarrow Z_1'
$$
  

$$
f_2 \times f_1 \downarrow \qquad f_1
$$
  

$$
Z_2 \times_T Z_1 \longrightarrow Z_1
$$

*is cartesian, then there is an equivalence*

<span id="page-34-4"></span>
$$
f_{2*} \circ \Phi_{\mathcal{E}'} \simeq \Phi_{\mathcal{E}} \circ f_{1*}.\tag{5.6}
$$

**Remark 5.16.** For  $i = 1, 2$ , the assumption of Lemma [5.15\(](#page-33-1)i) automatically holds if the square in diagram [\(5.4\)](#page-33-2) is cartesian.

**Lemma 5.17.** In Setup [5.13,](#page-33-0) assume further that that  $T' = T$ . Let

 $\mathcal{E} \in \text{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$ 

*be a kernel, and*

$$
\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E} \in \text{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z'_2 \times_T Z'_1).
$$

*Then there is an equivalence*

$$
\Phi_{\mathcal{E}'} \simeq f_2^* \circ \Phi_{\mathcal{E}} \circ f_{1*}.\tag{5.7}
$$

*Proof.* Factor  $f_2 \times f_1$  as the composition

$$
Z'_2 \times_T Z'_1 \xrightarrow{\operatorname{id} \times f_1} Z'_2 \times_T Z_1 \xrightarrow{f_2 \times \operatorname{id}} Z_2 \times_T Z_1
$$

and apply successively Lemma [5.15](#page-33-1) parts [\(1\)](#page-34-0) and [\(2\)](#page-34-1).  $\Box$ 

<span id="page-34-6"></span>**Proposition 5.18.** In Setup [5.13,](#page-33-0) assume further that for  $i = 1, 2$  the square in diagram [\(5.4\)](#page-33-2) *is cartesian. Let*

$$
\mathcal{E} \in FM(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)
$$

*be a kernel, and let*  $\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E}$  *be its base change. Assume the associated functors* 

$$
\Phi_{\mathcal{E}} \colon \mathcal{C}_1 \otimes_{\text{Perf}(S_1)} \text{Perf}(Z_1) \to \mathcal{C}_2 \otimes_{\text{Perf}(S_2)} \text{Perf}(Z_2),
$$
  

$$
\Phi_{\mathcal{E}'} \colon \mathcal{C}_1 \otimes_{\text{Perf}(S_1)} \text{Perf}(Z'_1) \to \mathcal{C}_2 \otimes_{\text{Perf}(S_2)} \text{Perf}(Z'_2),
$$

<span id="page-34-2"></span>*admit left adjoints*  $\Phi_{\varepsilon}^*$  *and*  $\Phi_{\varepsilon}^*$ .

<span id="page-34-5"></span>*(1)* If  $\Phi_{\varepsilon}$  *is fully faithful, so is*  $\Phi_{\varepsilon'}$ *.* 

(2) If  $\Phi_{\varepsilon}^*$  is fully faithful, so is  $\Phi_{\varepsilon}^*$ .

*Proof.* For  $(1)$ , we must check that the canonical morphism

$$
\Phi_{\mathcal{E}'}^* \Phi_{\mathcal{E}'}(C) \to C
$$

is an equivalence for any  $C \in \mathcal{C}_1 \otimes_{\text{Perf}(S_1)} \text{Perf}(Z_1')$ . Note that by Theorem 5.3[\(1\)](#page-30-6), there is an equivalence

$$
Perf(Z'_1) \simeq Perf(Z_1) \otimes_{Perf(T)} Perf(T').
$$

Hence Lemma [2.7](#page-10-0) implies it is enough to check the morphism above is an equivalence for

$$
C = C_1 \boxtimes F_1 \boxtimes G,
$$

where  $C_1 \in \mathcal{C}_1, F_1 \in \text{Perf}(Z_1), G \in \text{Perf}(T')$ . This follows from T'-linearity of  $\Phi_{\mathcal{E}'}$  and  $\Phi_{\mathcal{E}'}^*$ , the equivalence [\(5.5\)](#page-34-3), and the equivalence  $\Phi_{\xi'}^* \circ f_2^* \simeq f_1^* \circ \Phi_{\xi}^*$  obtained from [\(5.6\)](#page-34-4) by taking left adjoints.

Part  $(2)$  is proved similarly.

**Proposition 5.19.** In Setup [5.13,](#page-33-0) assume further that for  $i = 1, 2$  the diagram [\(5.4\)](#page-33-2) is *cartesian. Let*

 $\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$ 

*be a kernel such that the associated functor*

$$
\Phi_{\mathcal{E}} \colon \mathfrak{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1) \to \mathfrak{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)
$$

is left splitting. Let  $\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E}$  be the base changed kernel, and assume the associated *functor*

$$
\Phi_{\mathcal{E}'}\colon \mathcal{C}_1\otimes_{\mathrm{Perf}(S_1)}\mathrm{Perf}(Z'_1)\to \mathcal{C}_2\otimes_{\mathrm{Perf}(S_2)}\mathrm{Perf}(Z'_2)
$$

*admits a left adjoint. Then*  $\Phi_{\xi'}$  *is left splitting.* 

*Proof.* By the same argument used to prove Proposition [5.18,](#page-34-6) it follows that  $\Phi_{\mathcal{E}'}$  satisfies the condition of Theorem [3.22](#page-19-4)[\(2l\).](#page-20-3)
# Part II. Homological projective duality

In this part of the paper, we develop a theory of HPD relative to a fixed base scheme S. All schemes will be defined over  $S$ , and all categories and functors will be  $S$ -linear. To simplify notation, given schemes  $X$  and  $Y$  over  $S$  we write

$$
X \times Y = X \times_S Y.
$$

Similarly, given S-linear categories  $\mathcal C$  and  $\mathcal D$  we write

$$
\mathcal{C} \otimes \mathcal{D} = \mathcal{C} \otimes_{\text{Perf}(S)} \mathcal{D}.
$$

We denote by  $V$  a fixed vector bundle on our base scheme  $S$ . We write  $N$  for the rank of V and H for the relative hyperplane class on  $\mathbf{P}(V)$  such that  $\mathcal{O}(H) = \mathcal{O}_{\mathbf{P}(V)}(1)$ . Similarly, we write H' for the relative hyperplane class on  $\mathbf{P}(V^{\vee})$ . We often use the notation  $\mathbf{P} = \mathbf{P}(V)$ and  $\mathbf{P}^{\vee} = \mathbf{P}(V^{\vee})$ . Further, given a  $\mathbf{P}(V)$ -linear category C and an object  $C \in \mathcal{C}$ , we write  $C(H) = C \otimes \mathcal{O}_{\mathbf{P}(V)}(H).$ 

## 6. Lefschetz categories

<span id="page-36-6"></span>In this section, we introduce the notion of a Lefschetz category over a projective space, which plays the role of an embedded projective variety in homological projective geometry. We start by studying in §[6.1-](#page-36-0)[6.2](#page-38-0) the notion of a Lefschetz center of a linear category equipped with an autoequivalence. A Lefschetz category over  $P(V)$  is then defined as a  $P(V)$ -linear category equipped with a Lefschetz center with respect to the autoequivalence given by tensoring with  $\mathcal{O}_{\mathbf{P}(V)}(1)$ . In §[6.3](#page-40-0) we show that there are induced semiorthogonal decompositions of the "linear sections" of a Lefschetz category over  $P(V)$ . Our definitions are modeled on those of [\[9\]](#page-71-0), which treats the geometric case.

## <span id="page-36-0"></span>6.1. Lefschetz centers.

<span id="page-36-5"></span>**Definition 6.1.** Let A be an S-linear category, and let  $T: A \rightarrow A$  be an S-linear autoequivalence. An admissible S-linear subcategory  $A_0 \subset A$  is called a *Lefschetz center* of A with respect to T if the subcategories  $A_i \subset A$ ,  $i \in \mathbb{Z}$ , determined by

<span id="page-36-3"></span>
$$
\mathcal{A}_i = \mathcal{A}_{i-1} \cap \perp^{\perp}(\mathcal{T}^{-i}(\mathcal{A}_0)), \quad i \ge 1,
$$
\n(6.1)

<span id="page-36-4"></span><span id="page-36-2"></span><span id="page-36-1"></span>
$$
\mathcal{A}_i = \mathcal{A}_{i+1} \cap (\mathbf{T}^{-i}(\mathcal{A}_0))^{\perp}, \quad i \le -1,
$$
\n(6.2)

are right admissible in A for  $i \geq 1$ , left admissible in A for  $i \leq -1$ , vanish for all i of sufficiently large absolute value, say for  $|i| \geq m$ , and provide S-linear semiorthogonal decompositions

$$
\mathcal{A} = \langle \mathcal{A}_0, \mathcal{T}(\mathcal{A}_1), \dots, \mathcal{T}^{m-1}(\mathcal{A}_{m-1}) \rangle, \tag{6.3}
$$

$$
\mathcal{A} = \langle T^{1-m}(\mathcal{A}_{1-m}), \dots, T^{-1}(\mathcal{A}_{-1}), \mathcal{A}_0 \rangle.
$$
 (6.4)

Here, T<sup>*i*</sup> denotes the *i*-fold composition of T with itself if  $i \geq 1$ , and  $T^i = (T^{-1})^{-i}$  if  $i \leq -1$ .

The categories  $A_i$ ,  $i \in \mathbb{Z}$ , are called the *Lefschetz components* of the Lefschetz center  $A_0 \subset A$ . The semiorthogonal decompositions [\(6.3\)](#page-36-1) and [\(6.4\)](#page-36-2) are called *the right Lefschetz decomposition* and *the left Lefschetz decomposition* of A. The minimal m above is called the *length* of the Lefschetz decompositions.

Note that the subcategories  $A_i \subset A_0$ ,  $i \in \mathbb{Z}$ , associated to a Lefschetz center form two (different in general) chains of subcategories

 $0 \subset A_{1-m} \subset \cdots \subset A_{-1} \subset A_0 \supset A_1 \supset \cdots \supset A_{m-1} \supset 0.$  (6.5)

Sometimes it is convenient to consider the following weakening of the notion of a Lefschetz center.

**Definition 6.2.** Let A be an S-linear category with an S-linear autoequivalence T:  $A \rightarrow A$ .

– A *right Lefschetz chain* in A with respect to T is a chain  $A_0$  ⊃  $A_1$  ⊃  $\cdots$  ⊃  $A_{m-1}$  of S-linear subcategories of A, such that the sequence

$$
\mathcal{A}_0, T(\mathcal{A}_1), \ldots, T^{m-1}(\mathcal{A}_{m-1}),
$$

is semiorthogonal in A.

 $- A$  *left Lefschetz chain* in A with respect to T is a chain  $A_{1-m} \subset A_{-1} \subset \cdots \subset A_0$  of S-linear subcategories of  $A$ , such that the sequence

$$
T^{1-m}(\mathcal{A}_{1-m}), \ldots, T^{-1}(\mathcal{A}_{-1}), \mathcal{A}_0,
$$

is semiorthogonal in A.

We say that a (right or left) Lefschetz chain is *full* if the above corresponding semiorthogonal sequence gives a semiorthogonal decomposition of A.

Note that the right and left Lefschetz components of a Lefschetz center form full right and left Lefschetz chains. Conversely, the following shows that if a full Lefschetz chain has admissible components, then its biggest component is a Lefschetz center. This is useful in practice for constructing Lefschetz centers.

<span id="page-37-2"></span><span id="page-37-0"></span>**Lemma 6.3.** Let A be an S-linear category with an S-linear autoequivalence  $T: A \rightarrow A$ .

- *(1)* Let  $A_0$  ⊃  $A_1$  ⊃  $\cdots$  ⊃  $A_{m-1}$  *be a full right Lefschetz chain in* A *with respect to* T. Then *the*  $A_i$ ,  $0 \le i \le m-1$ *, satisfy* [\(6.1\)](#page-36-3)*, and the categories defined by* (6.1*) for*  $i \ge m$  *vanish.*
- <span id="page-37-1"></span>*(2) Let* A1−<sup>m</sup> ⊂ · · · ⊂ A−<sup>1</sup> ⊂ A<sup>0</sup> *be a full left Lefschetz chain in* A *with respect to* T*. Then the*  $A_i$ ,  $1 - m \leq i \leq 0$ , satisfy [\(6.2\)](#page-36-4), and the categories defined by (6.2) for  $i \leq -m$  vanish.

*Moreover, in the situation of* [\(1\)](#page-37-0) *or* [\(2\)](#page-37-1)*, if the components of the Lefschetz chain are admissible in* A, then  $A_0 \subset A$  *is a Lefschetz center. Further, in this case the length of the Lefschetz decompositions of* A *is given by*

$$
\min\{ i \ge 0 \mid \mathcal{A}_i = 0 \} = \min\{ i \ge 0 \mid \mathcal{A}_{-i} = 0 \}.
$$

*Proof.* We prove the results in the case of a right Lefschetz chain; the arguments for left Lefschetz chains are similar. So let  $A_0 \supset \cdots \supset A_{m-1}$  be a full right Lefschetz chain in A. Then the argument of [\[11,](#page-71-1) Lemmas 2.18] shows that the  $A_i$ ,  $0 \le i \le m-1$ , satisfy [\(6.1\)](#page-36-3). To show the categories  $A_i, i \geq m$ , defined by [\(6.1\)](#page-36-3) vanish, it suffices to show

$$
\mathcal{A}_m = \mathcal{A}_{m-1} \cap {}^{\perp}(T^{-m}(\mathcal{A}_0))
$$

vanishes. Equivalently, we must show

$$
\mathrm{T}^m(\mathcal{A}_m) = \mathrm{T}^m(\mathcal{A}_{m-1}) \cap {}^{\perp}\mathcal{A}_0
$$

vanishes. First note that by applying T to the semiorthogonal decomposition given by the Lefschetz chain, we obtain a semiorthogonal decomposition

$$
\mathcal{A} = \langle T(\mathcal{A}_0), T^2(\mathcal{A}_1), \dots, T^{m-1}(\mathcal{A}_{m-2}), T^m(\mathcal{A}_{m-1}) \rangle.
$$

Since  $A_i \subset A_{i-1}$  for  $1 \leq i \leq m-1$ , this shows that

$$
T^m(\mathcal{A}_{m-1}) \subset \mathcal{L}\left(T(\mathcal{A}_1), T^2(\mathcal{A}_2), \ldots, T^{m-1}(\mathcal{A}_{m-1})\right).
$$

Hence

$$
T^m(\mathcal{A}_{m-1}) \cap \perp \mathcal{A}_0 \subset \perp \langle \mathcal{A}_0, T(\mathcal{A}_1), T^2(\mathcal{A}_2), \ldots, T^{m-1}(\mathcal{A}_{m-1}) \rangle = 0.
$$

Assume now that that  $A_i \subset A$ ,  $0 \leq i \leq m-1$ , are admissible. Then the argument of [\[11,](#page-71-1) Lemma 2.19] shows that the chain of categories  $A_{1-m} \subset \cdots \subset A_{-1} \subset A_0$  defined recursively by the formula [\(6.2\)](#page-36-4) is a full left Lefschetz chain, and that furthermore for any  $0 \le i \le m-1$ we have

<span id="page-38-1"></span>
$$
\langle \mathcal{A}_{-i}, \dots, T^{i-1}(\mathcal{A}_{-1}), T^i(\mathcal{A}_0) \rangle = \langle \mathcal{A}_0, T(\mathcal{A}_1), \dots, T^i(\mathcal{A}_i) \rangle.
$$
 (6.6)

Hence by part [\(2\)](#page-37-1), to prove  $A_0 \subset A$  is a Lefschetz center it remains to show the categories  $A_{-i}, 0 \le i \le m-1$ , are left admissible. By Lemma [3.10](#page-15-0) the subcategory [\(6.6\)](#page-38-1) of A is admissible, and by Lemma [3.6](#page-14-0) the category  $A_{-i}$  is left admissible in [\(6.6\)](#page-38-1). Hence  $A_{-i} \subset A$  is left admissible. Finally, the formula for the length of the Lefschetz decompositions of A also follows directly from  $(6.6)$ .

<span id="page-38-0"></span>6.2. Primitive Lefschetz components. Let A be an S-linear category, and  $A_0 \subset A$  a Lefschetz center with respect to an S-linear autoequivalence  $T: \mathcal{A} \to \mathcal{A}$ . The Lefschetz components  $A_i$  associated to  $A_0 \subset A$  can be broken into smaller building blocks, as follows.

For  $i \geq 1$  the *i*-th *right primitive component*  $a_i$  of a Lefschetz center is defined as the right orthogonal to  $A_{i+1}$  in  $A_i$ , i.e.

$$
\mathfrak{a}_i = \mathcal{A}_{i+1}^{\perp} \cap \mathcal{A}_i.
$$

Note that since  $A_{i+1}$  is right admissible in A (and hence also in  $A_i$ ), by Lemma [3.6](#page-14-0) we have a semiorthogonal decomposition

$$
A_i = \langle \mathfrak{a}_i, A_{i+1} \rangle = \langle \mathfrak{a}_i, \mathfrak{a}_{i+1}, \ldots, \mathfrak{a}_{m-1} \rangle.
$$

Similarly, for  $i \leq -1$  the *i*-th *left primitive component*  $a_i$  of a Lefschetz center is the left orthogonal to  $A_{i-1}$  in  $A_i$ , i.e.

$$
\mathfrak{a}_i = {}^{\perp} \mathcal{A}_{i-1} \cap \mathcal{A}_i,
$$

so that

$$
\mathcal{A}_i = \langle \mathcal{A}_{i-1}, \mathfrak{a}_i \rangle = \langle \mathfrak{a}_{1-m}, \ldots, \mathfrak{a}_{i-1}, \mathfrak{a}_i \rangle \,.
$$

For  $i = 0$ , we have both right and left primitive components, defined by

$$
\mathfrak{a}_{+0} = \mathcal{A}_1^{\perp} \cap \mathcal{A}_0 \quad \text{and} \quad \mathfrak{a}_{-0} = {}^{\perp} \mathcal{A}_{-1} \cap \mathcal{A}_0.
$$

Remark 6.4. We have

$$
\mathfrak{a}_{-0}=T(\mathfrak{a}_{+0}).
$$

Indeed, in this case the equality [\(6.6\)](#page-38-1) for  $i = 1$  gives

$$
\langle \mathcal{A}_{-1}, T(\mathcal{A}_0) \rangle = \langle \mathcal{A}_0, T(\mathcal{A}_1) \rangle \,.
$$

(Note that in the argument of [\[11,](#page-71-1) Lemma 2.19] proving [\(6.6\)](#page-38-1) for  $i = 1$  only uses the admissibility of  $A_0 \subset A$ , which holds by the definition of a Lefschetz category.) Substituting  $A_0 = \langle \mathfrak{a}_{+0}, A_1 \rangle$  on the left and  $A_0 = \langle A_{-1}, \mathfrak{a}_{-0} \rangle$  on the right, we obtain

$$
\left\langle \mathcal{A}_{-1}, \mathrm{T}(\mathfrak{a}_{+0}), \mathrm{T}(\mathcal{A}_1) \right\rangle=\left\langle \mathcal{A}_{-1}, \mathfrak{a}_{-0}, \mathrm{T}(\mathcal{A}_1) \right\rangle,
$$

so the claim follows.

If instead of a Lefschetz center  $A_0 \subset A$  we are given a (right or left) Lefschetz chain, then we define its (right or left) primitive components by the same formulas as above.

Later we will need to consider Lefschetz centers (and chains) that satisfy the following finiteness assumption.

**Definition 6.5.** Let  $A_0 \subset A$  be a Lefschetz center of an S-linear category with respect to an autoequivalence  $T: A \rightarrow A$ . We say that the center is *right strong* if all of the right primitive components  $a_{+0}$ ,  $a_i$ ,  $i \geq 1$ , are admissible in A, *left strong* if all of the left primitive components  $\mathfrak{a}_{-0}$ ,  $\mathfrak{a}_i$ ,  $i \leq -1$ , are admissible in A, and *strong* if all of the primitive components are admissible.

Similarly, a (right or left) Lefschetz chain is called *strong* if its (right or left) primitive components are admissible.

<span id="page-39-2"></span>**Remark 6.6.** By Lemma [3.10,](#page-15-0) if a Lefschetz center  $A_0 \subset A$  is (right or left) strong then all of its (right or left) Lefschetz components are automatically admissible in A. The analogous statement holds for Lefschetz chains.

<span id="page-39-3"></span>**Remark 6.7.** If A is smooth and proper over S, then by Lemma [4.15](#page-25-0) any Lefschetz center  $A_0 \subset A$  is strong.

Next we show that a Lefschetz center  $A_0 \subset A$  also admits a decomposition into certain "twisted" primitive components. This will be an essential ingredient in §[7.2](#page-45-0) when we build a Lefschetz structure on the homological projective dual category. Let

$$
\alpha_0\colon \mathcal{A}_0\to \mathcal{A}
$$

denote the inclusion functor. Since  $A_0 \subset A$  is admissible,  $\alpha_0$  admits a left adjoint  $\alpha_0^*$  and right adjoint  $\alpha_0^!$ . We define the *twisted right primitive components* by

$$
\mathfrak{a}'_{+0} = \alpha_0^*(\mathcal{T}(\mathfrak{a}_{+0}))
$$
 and  $\mathfrak{a}'_i = \alpha_0^*(\mathcal{T}^{i+1}(\mathfrak{a}_i))$  for  $i \ge 1$ . (6.7)

and the *twisted left primitive components* by

$$
\mathfrak{a}'_{-0} = \alpha_0^!(T^{-1}(\mathfrak{a}_{-0})) \quad \text{and} \quad \mathfrak{a}'_i = \alpha_0^!(T^{i-1}(\mathfrak{a}_i)) \text{ for } i \le -1.
$$
 (6.8)

<span id="page-39-4"></span>**Lemma 6.8.** Let A be an S-linear category, and  $A_0 \subset A$  a Lefschetz center with respect to *an S*-linear autoequivalence  $T: A \rightarrow A$ *.* 

<span id="page-39-0"></span>(1) The functor  $\alpha_0^* \colon \mathcal{A} \to \mathcal{A}_0$  *is fully faithful on*  $T(\mathfrak{a}_{+0})$  *and on*  $T^{i+1}(\mathfrak{a}_i)$  *for*  $i \geq 1$ *, and hence the right twisted primitive components are stable subcategories of*  $A_0$ *. Moreover, there is an* S*-linear semiorthogonal decomposition*

$$
\mathcal{A}_0 = \left\langle \mathfrak{a}'_{+0}, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_{m-1} \right\rangle.
$$

If further the Lefschetz center  $A_0$  is right strong, then the right twisted primitive compo*nents are admissible subcategories of* A*.*

<span id="page-39-1"></span>(2) The functor  $\alpha_0^!$ :  $A \to A_0$  is fully faithful on  $T(\mathfrak{a}_{-0})$  and on  $T^{i-1}(\mathfrak{a}_i)$  for  $i \leq -1$ , and hence *the left twisted primitive components are stable subcategories of* A0*. Moreover, there is an* S*-linear semiorthogonal decomposition*

$$
\mathcal{A}_0=\left\langle \mathfrak{a}'_{1-m},\ldots,\mathfrak{a}'_{-1},\mathfrak{a}'_{-0} \right\rangle.
$$

*If further the Lefschetz center*  $A_0$  *is left strong, then the left twisted primitive components are admissible subcategories of* A*.*

*Proof.* We prove [\(1\)](#page-39-0); the proof of [\(2\)](#page-39-1) is similar. Applying the autoequivalence T to the right Lefschetz decomposition [\(6.3\)](#page-36-1) and using  $A_i = \langle \mathfrak{a}_i, A_{i+1} \rangle$ , we obtain a decomposition

<span id="page-40-1"></span>
$$
\mathcal{A} = \langle T(\mathfrak{a}_{+0}), T(\mathcal{A}_1), T^2(\mathfrak{a}_1), T^2(\mathcal{A}_2), \dots, T^{m-1}(\mathfrak{a}_{m-2}), T^{m-1}(\mathcal{A}_{m-1}), T^m(\mathfrak{a}_{m-1}) \rangle. \tag{6.9}
$$

Hence using Lemma [3.11,](#page-16-0) we find a decomposition

$$
A = \langle T(\mathfrak{a}_{+0}), L_{T(\mathcal{A}_1)}(T^2(\mathfrak{a}_1)), \dots, L_{\langle T(\mathcal{A}_1), \dots, T^{m-1}(\mathcal{A}_{m-1}) \rangle}(T^m(\mathfrak{a}_{m-1})),
$$
  

$$
T(\mathcal{A}_1), \dots, T^{m-1}(\mathcal{A}_{m-1}) \rangle.
$$

Comparing with [\(6.3\)](#page-36-1), this implies there is a decomposition

 $\mathcal{A}_0 = \left\langle \mathrm{T}(\mathfrak{a}_{+0}), \mathrm{L}_{\mathrm{T}(\mathcal{A}_1)}(\mathrm{T}^2(\mathfrak{a}_1)), \ldots, \mathrm{L}_{\left\langle \mathrm{T}(\mathcal{A}_1), \ldots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \right\rangle}(\mathrm{T}^m(\mathfrak{a}_{m-1})) \right\rangle.$ 

Since by [\(6.9\)](#page-40-1) we have

$$
\mathrm{T}^{i+1}(\mathfrak{a}_i) \subset \left\langle \mathrm{T}^{i+1}(\mathcal{A}_i), \mathrm{T}^{i+2}(\mathcal{A}_{i+2}), \ldots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \right\rangle^{\perp},
$$

it follows that  $\alpha_0^*$  is given on  $T^{i+1}(\mathfrak{a}_i)$  by the mutation functor  $L_{\langle T(\mathcal{A}_1),...,T^i(\mathcal{A}_i)\rangle}$ . Hence the fully faithful claim follows from Lemma [3.8,](#page-15-1) and the claimed semiorthogonal decomposition of  $A_0$  holds.

Finally, assume the center  $A_0 \subset A$  is right strong. Then by Remark [6.6](#page-39-2) the right Lefschetz components  $A_i$ ,  $i \geq 0$ , are admissible, and so by Lemma [3.10](#page-15-0) the subcategory

$$
^{\perp}\mathcal{A}_0=\left\langle \mathrm{T}(\mathcal{A}_1),\ldots,\mathrm{T}^{m-1}(\mathcal{A}_{m-1})\right\rangle
$$

is admissible in A. Hence by Lemma [3.9](#page-15-2) the functor  $\alpha_0^*$ :  $A \to A_0$  admits a left adjoint, and it tautologically admits a right adjoint. From this, it follows that the right twisted primitive components are admissible subcategories of  $A_0$ , and hence also of A.

<span id="page-40-0"></span>6.3. Lefschetz categories. Let V be a vector bundle on  $S$ , and let H denote the relative hyperplane class on the projective bundle  $\mathbf{P}(V)$  such that  $\mathcal{O}(H) = \mathcal{O}_{\mathbf{P}(V)}(1)$ .

**Definition 6.9.** A *Lefschetz category* over  $P(V)$  is a  $P(V)$ -linear category A equipped with a Lefschetz center  $A_0 \subset A$  with respect to the autoequivalence  $-\otimes \mathcal{O}_{\mathbf{P}(V)}(H)$ :  $\overline{A} \to \overline{A}$ . We say A is (*right* or *left*) *strong* if its center  $A_0 \subset A$  is (*right* or *left*) *strong*. The *length* of A is the length of its Lefschetz decompositions, and is denoted by length(A). We say A is *moderate* if  $\text{length}(\mathcal{A}) < \text{rank}(V)$ .

If A and B are Lefschetz categories over  $P(V)$ , an *equivalence of Lefschetz categories* or a *Lefschetz equivalence* is a  $P(V)$ -linear equivalence  $A \simeq B$  which induces an S-linear equivalence  $A_0 \simeq B_0$  of centers.

**Remark 6.10.** Given a Lefschetz category A, the symbol  $A_i$  for  $i \in \mathbb{Z}$  will be used to denote the Lefschetz component associated to the center  $A_0 \subset A$ , as in Definition [6.1.](#page-36-5)

**Remark 6.11.** Analogous to Remark [6.7,](#page-39-3) if A is smooth and proper over S, then any Lefschetz structure on A is automatically strong.

<span id="page-40-2"></span>**Remark 6.12.** Moderateness of a Lefschetz category A over  $P(V)$  is a very mild condition. First of all, we can always embed V into a larger rank vector bundle  $V'$  so that A is a moderate Lefschetz category over  $P(V')$ . Second, the Lefschetz categories that arise in practice are essentially always moderate. In fact, Corollary [6.19](#page-43-0) below shows that  $\text{length}(\mathcal{A}) \leq \text{rank}(V)$ , so the only way for moderateness to fail is if equality holds. Moreover, if equality holds we show A is closely related to  $\text{Perf}(\mathbf{P}(V))$ .

Here we recall a basic example of a Lefschetz category; see [\[13\]](#page-71-2) for many more examples.

**Example 6.13.** Let  $W \subset V$  be a subbundle of rank m. The morphism  $P(W) \to P(V)$ induces a  $P(V)$ -linear structure on  $\text{Perf}(P(W))$ . Note that pullback along the projection  ${\bf P}(W) \to S$  gives an embedding  $\text{Perf}(S) \subset \text{Perf}({\bf P}(W))$ . The category  $\text{Perf}({\bf P}(W))$  is a strong Lefschetz category over  $P(V)$  with center  $\text{Perf}(S)$ ; the corresponding right and left Lefschetz decompositions are given by the standard semiorthogonal decompositions

$$
Perf(\mathbf{P}(W)) = \langle Perf(S), Perf(S)(H), \dots, Perf(S)((m-1)H) \rangle,
$$
  

$$
Perf(\mathbf{P}(W)) = \langle Perf(S)((1-m)H), \dots, Perf(S)(-H), Perf(S) \rangle.
$$

Note that  $\text{Perf}(\mathbf{P}(W))$  is a moderate Lefschetz category over  $\mathbf{P}(V)$  as long as  $W \neq V$ .

Next we describe the behavior of Lefschetz categories under passage to linear sections. From now on, we simplify our notation by writing  $\mathbf{P} = \mathbf{P}(V)$ . Further, for  $0 \le r \le N$ , let  $\mathbf{G}_r = \text{Gr}(r, V)$  be the Grassmannian of rank r subbundles of V, and let  $\mathcal{U}_r$  be the universal rank r bundle on  $G_r$ . Let  $L_r = P_{G_r}(\mathcal{U}_r)$  be the corresponding universal family of (projective) linear subspaces. These spaces fit into a commutative diagram

<span id="page-41-1"></span>

where  $\iota_r$  is a closed embedding.

Definition 6.14. Let A be a P-linear category. The *universal family of* r*-dimensional linear sections* of A is defined by

$$
\mathbf{L}_r(\mathcal{A}) = \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{L}_r).
$$

**Remark 6.15.** If there exists a morphism of schemes  $X \to \mathbf{P}$  and a **P**-linear equivalence  $A \simeq \text{Perf}(X)$ , then by Theorem 5.3[\(1\)](#page-30-0)

$$
\mathbf{L}_r(\mathcal{A}) = \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{L}_r) \simeq \mathrm{Perf}(X \times_{\mathbf{P}} \mathbf{L}_r).
$$

Hence, at the level of perfect complexes, the category  $\mathbf{L}_r(\mathcal{A})$  recovers the universal family of linear sections  $X \times_{\mathbf{P}} \mathbf{L}_r$  of X.

<span id="page-41-0"></span>**Remark 6.16.** Let  $L \subset V$  be a fixed rank r subbundle of V. Let  $x_L : S \to \mathbf{G}_r$  be the corresponding morphism, so that the pullback of  $\mathbf{L}_r \to \mathbf{G}_r$  along  $x_L$  is  $\mathbf{P}(L) \to S$ . Then by Theorem 5.3[\(1\)](#page-30-0), base change along  $x_L^* \colon \mathrm{Perf}(\mathbf{G}_r) \to \mathrm{Perf}(S)$  gives

$$
\mathbf{L}_r(\mathcal{A})\otimes_{\mathrm{Perf}(\mathbf{G}_r)}\mathrm{Perf}(S)\simeq \mathcal{A}\otimes_{\mathrm{Perf}(\mathbf{P})}\mathrm{Perf}(\mathbf{P}(L)).
$$

The category on the right should be thought of as the "linear section" of A by  $P(L)$ . Indeed, if there exists a morphism of schemes  $X \to \mathbf{P}$  and a **P**-linear equivalence  $A \simeq \text{Perf}(X)$ , then by Theorem  $5.3(1)$  $5.3(1)$ 

$$
\mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P}(L)) \simeq \mathrm{Perf}(X \times_{\mathbf{P}} \mathbf{P}(L)).
$$

Hence in general the "fibers" of the natural  $\mathbf{G}_r$ -linear structure on  $\mathbf{L}_r(\mathcal{A})$  are the "linear sections" of A.

<span id="page-41-2"></span>**Remark 6.17.** The category  $\mathbf{L}_r(\mathcal{A})$  is easy to describe for some extreme values of r:  $-\mathbf{G}_0 = S$  and  $\mathbf{L}_0 = \varnothing$ , so  $\mathbf{L}_0(\mathcal{A}) = 0$ .

 $\mathbf{G}_1 = \mathbf{L}_1 = \mathbf{P}$ , so  $\mathbf{L}_1(\mathcal{A}) = \mathcal{A}$ .

$$
- \mathbf{G}_N = S \text{ and } \mathbf{L}_N = \mathbf{P}, \text{ so } \mathbf{L}_N(\mathcal{A}) = \mathcal{A}.
$$

We can also form the product of the **P**-linear category A with  $\mathbf{G}_r$ , i.e.  $\mathcal{A} \otimes \text{Perf}(\mathbf{G}_r)$ . If A is a Lefschetz category over **P**, then by Lemma [3.15](#page-17-0) there is a  $\mathbf{G}_r$ -linear semiorthogonal decomposition

<span id="page-42-2"></span>
$$
\mathcal{A} \otimes \operatorname{Perf}(\mathbf{G}_r) = \langle \mathcal{A}_0 \otimes \operatorname{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-1)H) \otimes \operatorname{Perf}(\mathbf{G}_r) \rangle. \tag{6.11}
$$

We are going to show that part of this semiorthogonal decomposition embeds into  $\mathbf{L}_r(\mathcal{A})$ . The argument works more generally for a category equipped with a Lefschetz chain. We begin by discussing the "right" case.

Note that there is a canonical equivalence

$$
\mathcal{A} \otimes \mathrm{Perf}(\mathbf{G}_r) \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P} \times \mathbf{G}_r).
$$

induced by pullback along the projection  $P \times G_r \to G_r$ . Via this identification, we regard the functor induced by pullback along  $\iota_r$  as a functor

<span id="page-42-0"></span>
$$
\iota_r^* \colon \mathcal{A} \otimes \operatorname{Perf}(\mathbf{G}_r) \to \mathbf{L}_r(\mathcal{A}). \tag{6.12}
$$

<span id="page-42-3"></span>**Lemma 6.18.** Let A be P-linear category with a right Lefschetz chain  $A_0 \supset \cdots \supset A_{m-1}$ *with respect to*  $-\otimes \mathcal{O}_P(H)$ *. Fix nonnegative integers* r *and* s *such that*  $r + s = N$ *. Then the functor* [\(6.12\)](#page-42-0) *is fully faithful on the subcategory*

$$
\mathcal{A}_i\otimes\operatorname{Perf}(\mathbf{G}_r)\subset\mathcal{A}\otimes\operatorname{Perf}(\mathbf{G}_r)
$$

*for*  $s \leq i \leq m-1$ *, and the images under* [\(6.12\)](#page-42-0) *of the categories* 

<span id="page-42-1"></span>
$$
\mathcal{A}_s(sH) \otimes \mathrm{Perf}(\mathbf{G}_r), \mathcal{A}_{s+1}((s+1)H) \otimes \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-1)H) \otimes \mathrm{Perf}(\mathbf{G}_r) \tag{6.13}
$$

*form a semiorthogonal sequence in*  $\mathbf{L}_r(\mathcal{A})$ *. Moreover, if the subcategories*  $\mathcal{A}_i \subset \mathcal{A}$  *are (right or left)* admissible, so are the images of  $(6.13)$  in  $\mathbf{L}_r(\mathcal{A})$ .

*Proof.* By the semiorthogonal decomposition  $(6.11)$ , to prove the result it suffices to show that if  $C \in \mathcal{A}_i(iH)$ ,  $D \in \mathcal{A}_i(jH)$ ,  $s \leq i \leq j \leq m-1$ , and  $F, G$  are in the image of the pullback functor  $\text{Perf}(\mathbf{G}_r) \to \text{Perf}(\mathbf{P} \times \mathbf{G}_r)$ , then

$$
\mathcal{H}om_{\mathbf{G}_r}(\iota_r^*(D \boxtimes G), \iota_r^*(C \boxtimes F)) \simeq \mathcal{H}om_{\mathbf{G}_r}(D \boxtimes G, C \boxtimes F).
$$

For this, first observe that by adjunction

$$
\mathcal{H}om_{\mathbf{G}_r}(\iota_r^*(D \boxtimes G), \iota_r^*(C \boxtimes F)) \simeq \mathcal{H}om_{\mathbf{G}_r}(D \boxtimes G, C \boxtimes \iota_{r*}\iota_r^*F).
$$

Next note that  $\iota_r : \mathbf{L}_r \hookrightarrow \mathbf{P} \times \mathbf{G}_r$  is a codimension s subscheme, cut out by the tautological section of the vector bundle  $\mathcal{Q}_r(H)$ , where  $\mathcal{Q}_r = V/\mathcal{U}_r$  is the universal quotient bundle. Hence there is a Koszul resolution

$$
0 \to (\wedge^s \mathcal{Q}_r^{\vee})(-sH) \to \cdots \to \mathcal{Q}_r^{\vee}(-H) \to \mathcal{O}_{\mathbf{P} \times \mathbf{G}_r} \to \mathcal{O}_{\mathbf{L}_r} \to 0. \tag{6.14}
$$

In view of the equivalence  $\iota_{r\ast}\iota_r^* F \simeq F \otimes \mathcal{O}_{\mathbf{L}_r}$ , it thus suffices to show that

$$
\mathcal{H}\!\mathit{om}_{\mathbf{G}_r}(D \boxtimes G, C(-tH) \boxtimes (F \otimes \wedge^t \mathfrak{Q}_r^{\vee}))
$$

vanishes for  $1 \le t \le s$ . But  $C(-tH) \in \mathcal{A}_i((i-t)H) \subset \mathcal{A}_{i-t}((i-t)H)$ , so the vanishing holds by [\(6.11\)](#page-42-2). This proves the fully faithful and semiorthogonal statements of the lemma. The images of the subcategories [\(6.13\)](#page-42-1) are (right or left) admissible if  $A_i \subset A$  are, because then each  $A_i \otimes \text{Perf}(\mathbf{G}_r) \subset A \otimes \text{Perf}(\mathbf{G}_r)$  is (by Lemma [3.17\)](#page-17-1) and the functor [\(6.12\)](#page-42-0) has left and right adjoints (by Remark [2.2](#page-9-0) and Lemma [2.12\)](#page-12-0).  $\Box$ 

The following result was anticipated in Remark [6.12.](#page-40-2)

<span id="page-43-0"></span>**Corollary 6.19.** Let A be **P**-linear category with a right Lefschetz chain  $A_0 \supset \cdots \supset A_{m-1}$ *with respect to*  $-\otimes \mathbb{O}_P(H)$ *. Let*  $N = \text{rank}(V)$ *.* 

<span id="page-43-2"></span><span id="page-43-1"></span>*(1)*  $A_i = 0$  *for*  $i \geq N$ *.* 

*(2) The action functor*  $A \otimes \text{Perf}(\mathbf{P}) \rightarrow A$  *is fully faithful on the subcategory* 

$$
\mathcal{A}_{N-1}\otimes \operatorname{Perf}(\mathbf{P})\subset \mathcal{A}\otimes \operatorname{Perf}(\mathbf{P}).
$$

<span id="page-43-3"></span>*(3)* If  $A_i = A_0$  for  $0 \le i \le N - 1$ , then the action functor  $A \otimes \text{Perf}(\mathbf{P}) \rightarrow A$  *induces an equivalence of Lefschetz categories*

$$
\mathcal{A}_0\otimes\operatorname{Perf}(\mathbf{P})\simeq\mathcal{A},
$$

*where the Lefschetz decomposition on the left side is the one induced by the standard semiorthogonal decomposition of* Perf(P)*, namely*

$$
\mathcal{A}_0 \otimes \mathrm{Perf}(\mathbf{P}) = \langle \mathcal{A}_0, \mathcal{A}_0(H), \ldots, \mathcal{A}_0((N-1)H) \rangle.
$$

*Proof.* Taking  $r = 0$  in Lemma [6.18](#page-42-3) gives [\(1\)](#page-43-1), taking  $r = 1$  gives [\(2\)](#page-43-2), and then [\(3\)](#page-43-3) follows from the definitions.

If A is a P-linear category with a right Lefschetz chain  $A_0 \supset \cdots \supset A_{m-1}$  with respect to  $-\otimes \mathbb{O}_{\mathbf{P}}(H)$ , we denote by  $\mathcal{K}_r(\mathcal{A})$  the right orthogonal in  $\mathbf{L}_r(\mathcal{A})$  to the twist by  $\mathcal{O}(-(s-1)H)$ of the image of the sequence of categories [\(6.13\)](#page-42-1). The twist by  $\mathcal{O}(-(s-1)H)$  in our definition does not affect  $\mathcal{K}_r(\mathcal{A})$  up to equivalence, but this normalization will be convenient later. Since  $\iota_r^*$  is an equivalence when restricted to any of the categories in [\(6.13\)](#page-42-1), we will typically omit it in our notation for their images. Thus if the  $A_i \subset A$  are right admissible, we have a  $\mathbf{G}_r$ -linear semiorthogonal decomposition

<span id="page-43-4"></span>
$$
\mathbf{L}_r(\mathcal{A}) = \langle \mathcal{K}_r(\mathcal{A}), \mathcal{A}_s(H) \otimes \text{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes \text{Perf}(\mathbf{G}_r) \rangle. \tag{6.15}
$$

For  $s \geq m$  the sequence [\(6.13\)](#page-42-1) is empty, so by definition  $\mathbf{L}_r(\mathcal{A}) = \mathcal{K}_r(\mathcal{A})$  in this case. Further, if  $m < N$  then  $\mathcal{K}_1(\mathcal{A}) = L_1(\mathcal{A}) = \mathcal{A}$ .

The  $\mathbf{G}_r$ -linear category  $\mathcal{K}_r(\mathcal{A})$  should be thought of as the "interesting" component of the family of linear sections  $\mathbf{L}_r(\mathcal{A})$ , since the remaining components in [\(6.15\)](#page-43-4) come from the ambient Lefschetz chain. Our ultimate goal is to show that, when  $A$  is a right strong, moderate Lefschetz category, the category  $\mathcal{K}_{N-1}(\mathcal{A})$  controls all of the other  $\mathcal{K}_r(\mathcal{A})$ . Given a rank r subbundle  $L \subset V$ , we define

$$
\mathcal{K}_L(\mathcal{A}) = \mathcal{K}_r(\mathcal{A}) \otimes_{\text{Perf}(\mathbf{G}_r)} \text{Perf}(S) \tag{6.16}
$$

as the base change along the morphism  $S \to \mathbf{G}_r$  classifying  $L \subset V$ .

<span id="page-43-5"></span>**Lemma 6.20.** Let A be P-linear category with a right Lefschetz chain  $A_0 \supset \cdots \supset A_{m-1}$ *with respect to*  $-\otimes \mathbb{O}_P(H)$ *, with each*  $\mathcal{A}_i \subset \mathcal{A}$  *right admissible. Let*  $L \subset V$  *be a subbundle of corank* s*. Then there is a semiorthogonal decomposition*

$$
\mathcal{A} \otimes_{\text{Perf}(\mathbf{P})} \text{Perf}(\mathbf{P}(L)) = \langle \mathcal{K}_L(\mathcal{A}), \mathcal{A}_s(H), \ldots, \mathcal{A}_{m-1}((m-s)H) \rangle.
$$

*Proof.* Follows from Remark [6.16,](#page-41-0) Lemma [3.15,](#page-17-0) and the decomposition  $(6.15)$ .

Remark 6.21. Lemma [6.20](#page-43-5) says that for a range of degrees, the components of a Lefschetz sequence in A embed into any linear section of  $A$ . This is analogous to the behavior of the cohomology of a projective variety, as governed by the Lefschetz hyperplane theorem. This analogy is the source of our terminology, which goes back to [\[9\]](#page-71-0).

The following result is a combination of Lemmas [6.18](#page-42-3) and [6.20](#page-43-5) in the "left" case, and holds by the same proof.

<span id="page-44-1"></span>**Lemma 6.22.** Let A be P-linear category with a left Lefschetz chain  $A_{1-m}$  ⊂ · · · ⊂  $A_0$  with *respect to*  $-\otimes \mathbb{O}_P(H)$ *. Fix nonnegative integers* r and s such that  $r + s = N$ *.* 

*(1) The functor*  $\iota_r^* : \mathcal{A} \otimes \text{Perf}(\mathbf{G}_r) \to \mathbf{L}_r(\mathcal{A})$  *is fully faithful on the subcategory* 

 $\mathcal{A}_i \otimes \mathrm{Perf}(\mathbf{G}_r) \subset \mathcal{A} \otimes \mathrm{Perf}(\mathbf{G}_r)$ 

*for*  $1 - m \le i \le -s$ , and the image is (right or left) admissible if  $A_i \subset A$  is.

*(2) If the subcategories*  $A_i$  ⊂ A *are left admissible, then there is a*  $G_r$ -linear semiorthogonal *decomposition*

$$
\mathbf{L}_r(\mathcal{A}) = \left\langle \mathcal{A}_{1-m}((s-m)H) \otimes \mathrm{Perf}(\mathbf{G}_r), \ldots, \mathcal{A}_{-s}(-H) \otimes \mathrm{Perf}(\mathbf{G}_r), \mathcal{K}'_r(\mathcal{A}) \right\rangle.
$$

<span id="page-44-4"></span>*(3) If the subcategories* A<sup>i</sup> ⊂ A *are left admissible and* L ⊂ V *is a subbundle of corank* s*, then there is a semiorthogonal decomposition*

 $\mathcal{A} \otimes_{\text{Perf}(\mathbf{P})} \text{Perf}(\mathbf{P}(L)) = \langle \mathcal{A}_{1-m}((s-m)H), \ldots, \mathcal{A}_{-s}(-H), \mathcal{K}'_L(\mathcal{A}) \rangle$ .

# 7. The homological projective dual of a Lefschetz category

<span id="page-44-3"></span>In this section we introduce the main object of homological projective duality, namely the HPD category of a Lefschetz category  $A$  over **P**. In fact, in §[7.1](#page-44-0) we define both "right" and "left" HPD categories  $\mathcal{A}^{\natural}$  and  $^{\natural}\mathcal{A}$ , which are linear over  $\mathbf{P}^{\vee} = \mathbf{P}(V^{\vee})$ . The categories  $\mathcal{A}^{\natural}$  and ♮A are actually equivalent as P<sup>∨</sup> -linear categories (Lemma [7.2\)](#page-45-1), but it is useful to distinguish them notationally, for the following reason. Assuming  $A$  is moderate, in §[7.2](#page-45-0) we construct a left Lefschetz chain in  $\mathcal{A}^{\natural}$  and a right Lefschetz chain in  $^{\natural}\mathcal{A}$ . Under suitable hypotheses, we will later show in §[8](#page-53-0) that these Lefschetz chains give  $A^{\dagger}$  and  $^{\dagger}A$  the structure of Lefschetz categories over P<sup>∨</sup>. In general, there may not be a Lefschetz equivalence  $\mathcal{A}^{\natural} \simeq {}^{\natural} \mathcal{A}$ , but in §[7.4](#page-50-0) we show that there is under suitable finiteness conditions. Finally, in §[7.5](#page-52-0) we explain how the HPD category can be thought of as a categorification of classical projective duality.

<span id="page-44-0"></span>7.1. The HPD category. Note that there is an isomorphism  $\mathbf{G}_{N-1} \cong \mathbf{P}(V^{\vee})$ . Parallel to our notation in §[6,](#page-36-6) we write  $\mathbf{P}^{\vee} = \mathbf{P}(V^{\vee})$ , and denote by H' the relative hyperplane class on  $\mathbf{P}^{\vee}$ . Moreover, we use the notation  $\mathbf{H} = \mathbf{L}_{N-1}$  for the universal hyperplane in P. We also drop the subscripts in our notation for the maps  $p_{N-1}$ ,  $f_{N-1}$ ,  $i_{N-1}$ , so that diagram [\(6.10\)](#page-41-1) takes the form

<span id="page-44-5"></span><span id="page-44-2"></span>

Finally, given a P-linear category A, we write  $H(\mathcal{A}) = L_{N-1}(\mathcal{A})$  and call this category the *universal hyperplane section* of A.

Definition 7.1. Let A be a Lefschetz category over P. The *right* and *left HPD categories*  $\mathcal{A}^{\natural}$  and  $^{\natural}\mathcal{A}$  of  $\mathcal{A}$  are the P<sup> $\vee$ </sup>-linear categories defined by  $\mathcal{A}^{\natural} = \mathcal{K}_{N-1}(\mathcal{A})$  and  $^{\natural}\mathcal{A} = \mathcal{K}'_{N-1}(\mathcal{A})$ . Explicitly, these categories are defined by P<sup>∨</sup> -linear semiorthogonal decompositions

$$
\mathbf{H}(\mathcal{A}) = \left\langle \mathcal{A}^{\natural}, \iota^*(\mathcal{A}_1(H) \otimes \operatorname{Perf}(\mathbf{P}^{\vee})), \dots, \iota^*(\mathcal{A}_{m-1}((m-1)H) \otimes \operatorname{Perf}(\mathbf{P}^{\vee})) \right\rangle, \tag{7.1}
$$

$$
\mathbf{H}(\mathcal{A}) = \left\langle \iota^*(\mathcal{A}_{1-m}((1-m)H) \otimes \mathrm{Perf}(\mathbf{P}^{\vee})), \dots, \iota^*(\mathcal{A}_{-1}(-H) \otimes \mathrm{Perf}(\mathbf{P}^{\vee})), ^{\sharp} \mathcal{A} \right\rangle, \tag{7.2}
$$

where  $m = \text{length}(\mathcal{A})$ .

These categories can be described more symmetrically as follows.

<span id="page-45-1"></span>Lemma 7.2. *Let* A *be a Lefschetz category over* P*. Then*

<span id="page-45-2"></span>
$$
\mathcal{A}^{\natural} = \{ C \in \mathbf{H}(\mathcal{A}) \mid \iota_*(C) \in \mathcal{A}_0 \otimes \text{Perf}(\mathbf{P}^{\vee}) \},\tag{7.3}
$$

$$
\nexists \mathcal{A} = \{ C \in \mathbf{H}(\mathcal{A}) \mid \iota_!(C) \in \mathcal{A}_0 \otimes \text{Perf}(\mathbf{P}^\vee) \},
$$
\n(7.4)

*where ι*<sub>1</sub> denotes the left adjoint to *ι*<sup>\*</sup>. Further, there are **P**<sup>ν</sup>-linear semiorthogonal decompo*sitions*

$$
\mathbf{H}(\mathcal{A}) = \left\langle \iota^{!}(\mathcal{A}_{1-m}((1-m)H) \otimes \operatorname{Perf}(\mathbf{P}^{\vee})), \dots, \iota^{!}(\mathcal{A}_{-1}(-H) \otimes \operatorname{Perf}(\mathbf{P}^{\vee})), \mathcal{A}^{\natural} \right\rangle, \tag{7.5}
$$

$$
\mathbf{H}(\mathcal{A}) = \left\langle {}^{\natural}\mathcal{A}, \iota^{!}(\mathcal{A}_1(H) \otimes \operatorname{Perf}(\mathbf{P}^{\vee})), \dots, \iota^{!}(\mathcal{A}_{m-1}((m-1)H) \otimes \operatorname{Perf}(\mathbf{P}^{\vee})) \right\rangle, \tag{7.6}
$$

*where*  $m = \text{length}(A)$  *and*  $\iota^!$  *denotes the right adjoint to*  $\iota_*$ *. In particular,*  $\sharp A$  *and*  $A^{\sharp}$  *are admissible subcategories of* H(A)*. Finally, the functor*

$$
- \otimes \mathfrak{O}(H) \colon \mathbf{H}(\mathcal{A}) \to \mathbf{H}(\mathcal{A})
$$

*induces a*  $\mathbf{P}^{\vee}$ *-linear equivalence*  $^{\natural}A \simeq \mathcal{A}^{\natural}$ *.* 

*Proof.* Using the right Lefschetz decomposition of A, the first claim follows easily by adjunction. For the second claim, note that  $\iota: \mathbf{H} \to \mathbf{P} \times \mathbf{P}^{\vee}$  is the embedding of the divisor cut out by the canonical section of  $O(H + H')$  on  $\mathbf{P} \times \mathbf{P}^{\vee}$ , and thus  $\iota^{!}(C) = \iota^{*}(C) \otimes O(H + H')[-1]$ . Using this, it follows from Lemma [6.22](#page-44-1) that we have a semiorthogonal decomposition

$$
\mathbf{H}(\mathcal{A})=\left\langle \iota^!(\mathcal{A}_{1-m}((1-m)H)\otimes\mathrm{Perf}(\mathbf{P}^{\vee})),\ldots,\iota^!(\mathcal{A}_{-1}(-H)\otimes\mathrm{Perf}(\mathbf{P}^{\vee})),\mathfrak{K}_{N-1}'(\mathcal{A})(H)\right\rangle.
$$

By adjunction and the left Lefschetz decomposition of A, we find that  $\mathcal{K}'_{N-1}(\mathcal{A})(H)$  is also given by [\(7.3\)](#page-45-2). This also implies the final claim since  $^{\sharp}A = \mathcal{K}'_{N-1}(A)$  by definition.

<span id="page-45-0"></span>7.2. Construction of the Lefschetz sequence. Let  $A$  be a moderate Lefschetz category over **P** of length m. Let  $\gamma: A^{\natural} \to \mathbf{H}(\mathcal{A})$  denote the inclusion functor. Let  $\gamma^*: \mathbf{H}(\mathcal{A}) \to A^{\natural}$ denote its left adjoint, which exists by Lemma [3.6.](#page-14-0) The functor  $p^* \text{ : } \text{Perf}(\mathbf{P}) \rightarrow \text{Perf}(\mathbf{H})$ induces a functor

$$
p^*\colon \mathcal{A}\simeq \mathcal{A}\otimes_{\mathrm{Perf}(\mathbf{P})}\mathrm{Perf}(\mathbf{P})\to \mathbf{H}(\mathcal{A})
$$

which is abusively denoted by the same symbol. Further, recall that by Lemma [6.8](#page-39-4) there is a semiorthogonal decomposition

<span id="page-45-5"></span>
$$
\mathcal{A}_0 = \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_{m-1} \rangle \tag{7.7}
$$

into twisted primitive components, where for simplicity we write  $\mathfrak{a}'_0$  for  $\mathfrak{a}'_{+0}$ . The following is the key ingredient in constructing the desired Lefschetz sequence in  $\mathcal{A}^{\natural}$ . We postpone its proof until §[7.3](#page-47-0) below.

<span id="page-45-4"></span><span id="page-45-3"></span>Lemma 7.3. *Let* A *be a moderate Lefschetz category over* P*. (1) The functor*

$$
\gamma^*\circ p^*\colon \mathcal{A}\to \mathcal{A}^{\natural}
$$

*is fully faithful on the subcategory*  $A_0 \subset A$ .

<span id="page-46-0"></span>(2) Let  $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \rangle \subset \mathcal{A}_0$  for some  $0 \leq i \leq m-1$ . Then for any  $D \in \mathcal{A}$  and  $1 \leq t \leq N-2-i$ , we have

$$
\mathcal{H}\!\mathit{om}_{\mathbf{P}^\vee}(\gamma^*p^*(D), \gamma^*p^*(C)(-tH')) \simeq 0.
$$

For  $A$  a moderate Lefschetz category over  $P$ , we define

$$
\mathcal{A}_0^{\natural} = \gamma^* p^* (\mathcal{A}_0),
$$

which by virtue of Lemma [7.3](#page-45-3)[\(1\)](#page-45-4) is a stable subcategory of  $A^{\dagger}$  equivalent to  $A_0$ . This will be the biggest component in the promised left Lefschetz sequence in  $\mathcal{A}^{\natural}$ .

By moderateness of A, the number m of components in  $(7.7)$  satisfies  $m < N$ . Hence, since  $\mathfrak{a}'_i = \mathfrak{a}_i = 0$  for  $i \geq m$ , we can rewrite [\(7.7\)](#page-45-5) as

$$
\mathcal{A}_0 = \left\langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_{N-2} \right\rangle.
$$

For  $2 - N \leq j \leq 0$ , we define

$$
\mathfrak{a}_j^{\natural} = \gamma^* p^* (\mathfrak{a}_{N-2+j}').
$$

Applying  $\gamma^* p^*$  to the above semiorthogonal decomposition then gives

$$
\mathcal{A}_0^{\natural}=\left\langle \mathfrak{a}_{2-N}^{\natural},\ldots,\mathfrak{a}_{-1}^{\natural},\mathfrak{a}_0^{\natural}\right\rangle.
$$

Further, we define

$$
n = N - #\{ i \ge 0 \mid \mathcal{A}_i = \mathcal{A}_0 \}.
$$

Then it follows from the definitions that  $\mathfrak{a}_j^{\natural} = 0$  for  $j \leq -n$  and  $\mathfrak{a}_1^{\natural}$  $a_{1-n}^{\mathfrak{q}}\neq 0$ . Hence we have

$$
\mathcal{A}_0^\natural=\left\langle \mathfrak{a}_{1-n}^\natural,\ldots,\mathfrak{a}_{-1}^\natural,\mathfrak{a}_0^\natural\right\rangle.
$$

The  $\mathfrak{a}_i^{\natural}$ <sup> $\natural$ </sup> determine a sequence of *S*-linear categories  $\mathcal{A}_{1-n}^{\natural} \subset \cdots \subset \mathcal{A}_{-1}^{\natural} \subset \mathcal{A}_{0}^{\natural}$  $\frac{q}{0}$  by the formula

<span id="page-46-2"></span>
$$
\mathcal{A}_{j}^{\natural} = \left\langle \mathfrak{a}_{1-n}^{\natural}, \ldots, \mathfrak{a}_{j-1}^{\natural}, \mathfrak{a}_{j}^{\natural} \right\rangle.
$$
 (7.8)

<span id="page-46-1"></span>Proposition 7.4. *Let* A *be a moderate Lefschetz category over* P*. Then*

$$
\mathcal{A}_{1-n}^{\natural}\subset\cdots\subset\mathcal{A}_{-1}^{\natural}\subset\mathcal{A}_0^{\natural}
$$

*is a left Lefschetz chain in*  $\mathcal{A}^{\natural}$  *with respect to the autoequivalence*  $-\otimes \mathbb{O}_{\mathbf{P}^{\vee}}(H')$ *, which is strong if* A *is right strong.*

*Proof.* The fact that the  $\mathcal{A}_i^{\natural}$  $j^{\mu}_{j}$  form a left Lefschetz chain is a direct consequence of Lemma [7.3](#page-45-3)[\(2\)](#page-46-0) and the definitions. Now suppose that  $A$  is right strong. Then the categories to the right of  $A^{\dagger}$  in the defining decomposition [\(7.1\)](#page-44-2) are admissible by a combination of Lemma [6.18](#page-42-3) and Remark [6.6,](#page-39-2) and so  $^{\perp}\mathcal{A}^{\sharp}$  is admissible in  $\mathbf{H}(\mathcal{A})$  by Lemma [3.10.](#page-15-0) Hence by Lemma [3.9](#page-15-2) the functor  $\gamma^*: \mathbf{H}(\mathcal{A}) \to \mathcal{A}^{\dagger}$  admits a left adjoint, and it tautologically admits a right adjoint. Further,  $p^* \colon \text{Perf}(\mathbf{P}) \to \text{Perf}(\mathbf{H})$  admits left and right adjoints. Since  $\mathfrak{a}'_i$ ,  $i \geq 0$ , is admissible in A by Lemma [6.8,](#page-39-4) it follows that  $\mathfrak{a}_{i}^{\natural}$  $j, j \leq 0$ , is admissible in  $\mathcal{A}^{\natural}$ . This proves that the left Lefschetz chain formed by the  $\mathcal{A}_i^{\natural}$  $\frac{a}{j}$  is strong.

Let us state the analogous results for the left HPD category. We omit the proofs as they are parallel to the right case. As above  $A$  denotes a moderate Lefschetz category over  $P$ . Let

 $\lambda: {}^{\natural}A \to \mathbf{H}(\mathcal{A})$  denote the inclusion functor, and let  $\lambda^! \colon \mathbf{H}(\mathcal{A}) \to {}^{\natural}A$  denote its right adjoint. Then, analogous to Lemma [7.3](#page-45-3)[\(1\)](#page-45-4), the functor

$$
\lambda^! \circ p^* \colon \mathcal{A} \to {}^{\natural}\mathcal{A}
$$

is fully faithful on the subcategory  $A_0 \subset A$ . We set

$$
{}^{\natural}\mathcal{A}_0=\lambda^!p^*(\mathcal{A}_0).
$$

Further, we can write the decomposition of  $A_0$  into left primitive components as

$$
\mathcal{A}_0=\left\langle \mathfrak{a}'_{2-N},\ldots,\mathfrak{a}'_{-1},\mathfrak{a}'_0\right\rangle,
$$

where for simplicity we write  $\mathfrak{a}'_0$  for  $\mathfrak{a}'_{-0}$ . For  $0 \leq j \leq N-2$ , we define

$$
{}^{\natural}\mathfrak{a}_j=\lambda^!p^*(\mathfrak{a}_{2-N+j}')
$$

Then  $\mathfrak{a}_j = 0$  for  $j \geq n$  where  $n = N - \#\{i \leq 0 \mid \mathcal{A}_i = \mathcal{A}_0\}$ , and we have

$$
{}^{\natural}\mathcal{A}_0=\left\langle {}^{\natural}\mathfrak{a}_0, {}^{\natural}\mathfrak{a}_1, \ldots, {}^{\natural}\mathfrak{a}_{n-1}\right\rangle
$$

We define a sequence of categories  $\mathfrak{A}_0 \supset \cdots \supset \mathfrak{A}_{n-1}$  by the formula

$$
{}^{\natural}\mathcal{A}_j = \left\langle {}^{\natural}\mathfrak{a}_j, {}^{\natural}\mathfrak{a}_{j+1}, \dots, {}^{\natural}\mathfrak{a}_{n-1} \right\rangle.
$$
 (7.9)

.

Finally, here is the left analogue of Proposition [7.4.](#page-46-1)

<span id="page-47-6"></span>Proposition 7.5. *Let* A *be a moderate Lefschetz category over* P*. Then*

$$
{}^{\natural}\mathcal{A}_0\supset {}^{\natural}\mathcal{A}_1\supset\cdots\supset {}^{\natural}\mathcal{A}_{n-1}
$$

*is a right Lefschetz chain in*  $^{\natural}$ A *with respect to the autoequivalence*  $-\otimes \mathbb{O}_{\mathbf{P}^{\vee}}(H')$ *, which is strong if* A *is left strong.*

<span id="page-47-0"></span>7.3. Proof of the key lemma. In this subsection, we prove Lemma [7.3.](#page-45-3) We will need some auxiliary results. Let  $p_* \colon \mathbf{H}(\mathcal{A}) \to \mathcal{A}$  denote the functor induced by  $p_* \colon \mathrm{Perf}(\mathbf{H}) \to \mathrm{Perf}(\mathbf{P})$ .

<span id="page-47-5"></span><span id="page-47-1"></span>Lemma 7.6. *Let* A *be a* P*-linear category. Then:*

<span id="page-47-4"></span>*(1)* The functor  $p^*$ :  $A \to \mathbf{H}(A)$  *is fully faithful.* 

(2) The functor  $p_* \colon \mathbf{H}(\mathcal{A}) \to \mathcal{A}$  *kills the subcategory*  $p^*(\mathcal{A})(-tH')$  for  $1 \le t \le N-2$ .

*Proof.* The morphism  $p: H \to P$  is the projectivization of a rank  $N-1$  vector bundle on P. Namely,  $\mathbf{H} = \mathbf{P}(\mathcal{K})$  where  $\mathcal{K}$  is the kernel of the canonical surjection  $V^{\vee} \otimes \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(H)$ . It follows that  $p^* \colon \mathrm{Perf}(\mathbf{P}) \to \mathrm{Perf}(\mathbf{H})$  is fully faithful, and  $p_* \colon \mathrm{Perf}(\mathbf{H}) \to \mathrm{Perf}(\mathbf{P})$  kills the subcategory

$$
p^*(\mathrm{Perf}(\mathbf{P}))(-tH') \subset \mathrm{Perf}(\mathbf{H})
$$

for  $1 \le t \le N-2$ . From this, the lemma follows formally.

Lemma [7.6](#page-47-1) shows in particular that  $p^* \colon \mathcal{A} \to \mathbf{H}(\mathcal{A})$  embeds  $\mathcal{A}_k$  into  $\mathbf{H}(\mathcal{A})$  for all k; below we abusively denote the image also by  $\mathcal{A}_k \subset \mathbf{H}(\mathcal{A})$ . The following result controls the morphisms between objects in various twists of these categories.

<span id="page-47-3"></span><span id="page-47-2"></span>Lemma 7.7. *Let* A *be a Lefschetz category over* P*.*

*(1) The pair of categories*

$$
\mathcal{A}_k, \, \mathcal{A}_\ell(aH + bH')
$$

*is semiorthogonal in* H(A) *provided one of the following conditions hold:*

 $-1 \leq a \leq \ell - 1$ . *–* 1 ≤ b ≤ N − 2*.*  $- a = 0$  *and*  $b = N - 1$ .  $-a = \ell$  *and*  $b = 0$ *.* 

<span id="page-48-3"></span>*(2) The pair of categories*

$$
p^*(\left\langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \right\rangle), \, \mathcal{A}_{\ell}(\ell H+bH')
$$

*is semiorthogonal in*  $H(A)$  *provided*  $i < l$ *.* 

*Proof.* Let  $C, D \in \mathcal{A}$ . Note that we have equivalences

$$
p^*(D)(aH + bH') \simeq \iota^*(D(aH) \boxtimes \mathcal{O}(bH')),
$$
  

$$
p^*(C) \simeq \iota^*(C \boxtimes \mathcal{O}),
$$

where  $\iota: \mathbf{H} \to \mathbf{P} \times_S \mathbf{P}^{\vee}$  denotes the embedding and  $D(aH) \boxtimes \mathcal{O}(bH')$  and  $C \boxtimes \mathcal{O}$  are regarded as objects of

$$
\mathcal{A} \otimes \mathrm{Perf}(\mathbf{P}^{\vee}) \simeq \mathcal{A} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P} \times \mathbf{P}^{\vee}).
$$

Specializing the argument in Lemma [6.18](#page-42-3) to the case  $r = N - 1$ , we thus obtain an exact triangle

<span id="page-48-0"></span>
$$
\mathcal{H}om_{\mathbf{P}^{\vee}}(D((a+1)H) \boxtimes \mathcal{O}((b+1)H'), C \boxtimes \mathcal{O}) \to \mathcal{H}om_{\mathbf{P}^{\vee}}(D(aH) \boxtimes \mathcal{O}(bH'), C \boxtimes \mathcal{O})
$$
  

$$
\to \mathcal{H}om_{\mathbf{P}^{\vee}}(p^*(D)(aH + bH'), p^*(C)).
$$
 (7.10)

By Lemma [2.10](#page-11-0) the first two terms of this triangle can be written as

 $\mathcal{H}om_S(D((a+1)H), C) \otimes \mathcal{H}om_{\mathbf{P}^\vee}(\mathcal{O}((b+1)H')$  $(7.11)$ 

$$
\mathcal{H}om_S(D(aH), C) \otimes \mathcal{H}om_{\mathbf{P}^{\vee}}(\mathcal{O}(bH'), \mathcal{O}).\tag{7.12}
$$

To prove [\(1\)](#page-47-2), assume  $C \in \mathcal{A}_k$  and  $D \in \mathcal{A}_{\ell}$ . We must show that

<span id="page-48-2"></span><span id="page-48-1"></span>
$$
\mathcal{H}\!\mathit{om}_{\mathbf{P}^\vee}(p^*(D)(aH+bH'),p^*(C))
$$

vanishes for  $a, b$  satisfying any of the stated conditions. By the exact triangle  $(7.10)$ , it suffices to show that the terms [\(7.11\)](#page-48-1) and [\(7.12\)](#page-48-2) vanish. But (7.11) vanishes if either  $0 \le a \le \ell - 1$ or  $0 \le b \le N-2$ , and [\(7.12\)](#page-48-2) vanishes if either  $1 \le a \le \ell$  or  $1 \le b \le N-1$ .

To prove [\(2\)](#page-48-3), assume  $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \rangle$ ,  $D \in \mathcal{A}_{\ell}, a = \ell$ , and  $i < \ell$ . As above, it suffices to show that [\(7.11\)](#page-48-1) and [\(7.12\)](#page-48-2) vanish. This is clear for (7.12). Note that since  $C \in \mathcal{A}_k \subset \mathcal{A}_0$ , we have

<span id="page-48-4"></span>
$$
\mathcal{H}om_S(D((\ell+1)H), C) \simeq \mathcal{H}om_S(\alpha_0^*(D((\ell+1)H)), C). \tag{7.13}
$$

It suffices to show that this morphism space vanishes to show that [\(7.11\)](#page-48-1) does. Observe that  $\alpha_0^*$  kills the second term in the semiorthogonal decomposition

$$
\mathcal{A}_{\ell}((\ell+1)H) = \langle \mathfrak{a}_{\ell}((\ell+1)H), \mathcal{A}_{\ell+1}((\ell+1)H) \rangle,
$$

hence  $\alpha_0^*(D((\ell+1)H)) \in \mathfrak{a}'_{\ell}$ . Therefore [\(7.13\)](#page-48-4) vanishes by the semiorthogonal decomposition  $(7.7)$ .

For any object  $X \in H(\mathcal{A})$ , there is an exact triangle

<span id="page-48-5"></span>
$$
R_{\mathcal{A}^{\natural}}(X) \to X \to \gamma \gamma^*(X). \tag{7.14}
$$

A priori  $R_{\mathcal{A}^{\natural}}(X)$  can be any object in the subcategory generated by the categories to the right of  $\mathcal{A}^{\dagger}$  in [\(7.1\)](#page-44-2). The following lemma shows that for X pulled back from the subcategory

generated by a subset of the twisted primitive components of  $\mathcal{A}_0$ , the object  $R_{\mathcal{A}^\natural}(X)$  lies in a restricted subcategory.

<span id="page-49-2"></span>**Lemma 7.8.** Let A be a Lefschetz category over **P**. Let  $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \rangle \subset A_0$  for some  $0 \leq i \leq m-1$ . Then  $\mathrm{R}_{\mathcal{A}^{\natural}} p^*(C)$  lies in the subcategory of  $\mathbf{H}(\mathcal{A})$  generated by the categories

 $p^*(\langle A_1(H), A_2(2H), \ldots, A_{i-t+1}((i-t+1)H) \rangle) \otimes \mathcal{O}(-tH'), \quad 1 \leq t \leq i.$ 

*Proof.* Let  $X = p^*(C)$ . By definition, the functor  $\gamma \gamma^*$  coincides with the left mutation functor through the subcategory

$$
^{\perp}\mathcal{A}^{\natural}=\langle \mathcal{A}_1(H)\otimes \operatorname{Perf}(\mathbf{P}^{\vee}),\ldots,\mathcal{A}_{m-1}((m-1)H)\otimes \operatorname{Perf}(\mathbf{P}^{\vee})\rangle\subset \mathbf{H}(\mathcal{A}),
$$

so that  $R_{\mathcal{A}^{\natural}} p^*(C)$  is determined by an exact triangle

$$
\mathrm{R}_{\mathcal{A}^{\natural}}p^*(C) \to X \to \mathrm{L}_{\perp \mathcal{A}^{\natural}}(X).
$$

To prove the lemma, we factor  $L_{\perp A_{\parallel}}$  according to the above semiorthogonal decomposition of  $^{\perp}\mathcal{A}^{\natural}$ , and inductively control the corresponding exact triangle at each step.

Namely, for  $1 \leq \ell \leq m-1$ , we define

$$
\mathcal{D}_{\ell} = \langle \mathcal{A}_{\ell}(H) \otimes \mathrm{Perf}(\mathbf{P}^{\vee}), \ldots, \mathcal{A}_{m-1}((m-1)H) \otimes \mathrm{Perf}(\mathbf{P}^{\vee}) \rangle \subset \mathbf{H}(\mathcal{A}),
$$

and define  $X_{\ell}$  by the exact triangle

$$
X_{\ell} \to X \to \mathcal{L}_{\mathcal{D}_{\ell}}(X).
$$

Then we claim that  $X_{\ell}$  is contained in the subcategory of  $\mathbf{H}(\mathcal{A})$  generated by

<span id="page-49-1"></span> $A_k(kH - tH'), \ell \leq k \leq i, 1 \leq t \leq i - k + 1,$  (7.15)

where for  $\ell > i$  this subcategory is by definition 0. The case  $\ell = 1$  gives the statement of the lemma.

Note that for any  $\ell$ , there is a semiorthogonal decomposition

$$
\mathcal{A}_{\ell}(\ell H)\otimes \operatorname{Perf}(\mathbf{P}^{\vee})=\big\langle \mathcal{A}_{\ell}(\ell H),\mathcal{A}_{\ell}(\ell H+H'),\ldots,\mathcal{A}_{\ell}(\ell H+(N-1)H')\big\rangle
$$

induced via base change by the standard decomposition of  $\text{Perf}(\mathbf{P}^{\vee})$ . By Lemma [7.7](#page-47-3)[\(2\)](#page-48-3) it thus follows that for  $\ell > i$  the category  $\mathcal{D}_{\ell}$  is left orthogonal to X, hence  $\mathcal{L}_{\mathcal{D}_{\ell}}(X) = X$  and  $X_{\ell} = 0$ . This proves the claim for  $\ell > i$ .

For  $1 \leq \ell \leq i+1$ , we argue by descending induction on  $\ell$ . The base case  $\ell = i+1$  was handled above. Assume the claim holds for  $\ell$ . By Lemma [3.10,](#page-15-0) we have

<span id="page-49-0"></span>
$$
L_{\mathcal{D}_{\ell-1}}(X) \simeq L_{\mathcal{A}_{\ell-1}((\ell-1)H)\otimes \mathrm{Perf}(\mathbf{P}^{\vee})} L_{\mathcal{D}_{\ell}}(X). \tag{7.16}
$$

It follows from Lemma [7.7\(](#page-47-3)[1\)](#page-47-2) that  $\mathcal{A}_{\ell-1}((\ell-1)H+bH')$  is left orthogonal to X for  $0 \le b \le N-2$ and to  $X_{\ell}$  for  $0 \leq b \leq N-i+\ell-3$  (here we used the induction assumption). Thus, by the exact triangle defining  $X_{\ell}$ , we find that  $\mathcal{A}_{\ell-1}((\ell-1)H + bH')$  is left orthogonal to  $\mathrm{L}_{\mathcal{D}_{\ell}}(X)$  for  $0 \le b \le N - i + \ell - 3$ . That is, in the decomposition of  $\mathcal{A}_{\ell-1}((\ell-1)H) \otimes \text{Perf}(\mathbf{P}^{\vee})$  given by

$$
\langle \mathcal{A}_{\ell-1}((\ell-1)H-(i-\ell+2)H'),\ldots,\mathcal{A}_{\ell-1}((\ell-1)H-H'),\mathcal{A}_{\ell-1}((\ell-1)H),\ldots,\mathcal{A}_{\ell-1}((\ell-1)H+(N-i+\ell-3)H')\rangle,
$$

the second row is left orthogonal to  $L_{\mathcal{D}_{\ell}}(X)$ . It follows that the right side of [\(7.16\)](#page-49-0) can be rewritten as

$$
L_{\langle \mathcal{A}_{\ell-1}((\ell-1)H-(i-\ell+2)H'),..., \mathcal{A}_{\ell-1}((\ell-1)H-H') \rangle} L_{\mathcal{D}_{\ell}}(X),
$$

and therefore the cone of the canonical morphism  $L_{\mathcal{D}_{\ell}}(X) \to L_{\mathcal{D}_{\ell-1}}(X)$  is contained in the subcategory of  $H(A)$  generated by

$$
\mathcal{A}_{\ell-1}((\ell-1)H - tH'), \ 1 \le t \le i - \ell + 2.
$$

By the induction assumption, the cone of  $X \to L_{\mathcal{D}_{\ell}}(X)$  is contained in the subcategory generated by the categories [\(7.15\)](#page-49-1). We conclude that the cone of the composite morphism  $X \to L_{\mathcal{D}_{\ell-1}}(X)$  — which agrees with  $X_{\ell-1}$  up to a shift — is contained in the claimed subcategory of  $\mathbf{H}(\mathcal{A})$ .

*Proof of Lemma [7.3.](#page-45-3)* Suppose A is a moderate Lefschetz category over P. Then by adjunction, Lemma [7.3](#page-45-3) can be rephrased as saying that

$$
p_*(\gamma\gamma^*p^*(C)(-tH'))
$$

<span id="page-50-1"></span>is canonically equivalent to:

<span id="page-50-2"></span>(1) C for  $C \in \mathcal{A}_0$  and  $t = 0$ ,

(2) 0 for  $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_i \rangle, 0 \le i \le m-1$ , and  $1 \le t \le N-2-i$ .

Taking  $X = p^*(C)$  in [\(7.14\)](#page-48-5) and tensoring by  $\mathcal{O}(-tH')$ , we get an exact triangle

$$
R_{\mathcal{A}^{\natural}}(p^*(C))(-tH') \to p^*(C)(-tH') \to (\gamma\gamma^*p^*(C))(-tH').
$$

Assume  $C \in \mathcal{A}_0$  and  $t = 0$  as in [\(1\)](#page-50-1). Then by Lemma [7.8](#page-49-2) the first term of this triangle lies in the subcategory of  $\mathbf{H}(\mathcal{A})$  generated by  $p^*(\mathcal{A})(-uH')$  for  $1 \le u \le m-1$ . By moderateness of A we have  $m-1 \leq N-2$ , so by Lemma [7.6](#page-47-1)[\(2\)](#page-47-4) the functor  $p_* : H(\mathcal{A}) \to \mathcal{A}$  kills all of these categories, and thus also the first term of the above triangle. Further, by Lemma [7.6](#page-47-1)[\(1\)](#page-47-5) the functor  $p_*$  applied to the second term  $p^*(C)$  is canonically equivalent to C. Hence applying  $p_*$  to the above triangle proves [\(1\)](#page-50-1).

Assume  $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \rangle$  and  $1 \leq t \leq N-2-i$  as in [\(2\)](#page-50-2). Then arguing as above we find that  $p_*$  kills the first two terms of the above triangle, and hence also the last, proving [\(2\)](#page-50-2).  $\Box$ 

<span id="page-50-0"></span>7.4. Right versus left HPD. In §[8](#page-53-0) we will prove that the Lefschetz chains of Proposi-tions [7.4](#page-46-1) and [7.5](#page-47-6) are full if  $A_0 \subset A$  is right or left strong, respectively. Hence by Lemma [6.3,](#page-37-2) in these cases  $\mathcal{A}_0^{\natural} \subset \mathcal{A}^{\natural}$  and  $^{\natural}\mathcal{A}_0 \subset {}^{\natural}\mathcal{A}$  define Lefschetz structures. Recall that by Lemma [7.2](#page-45-1) there is a P<sup> $\vee$ </sup>-linear equivalence  $\sharp A \simeq A^{\sharp}$ . In this subsection, we consider the question of whether there exists an equivalence of Lefschetz categories.

Our first goal is to identify the image of the Lefschetz center  $^{\sharp}A_0$  under the equivalence <sup>‡</sup> $\mathcal{A} \simeq \mathcal{A}^{\sharp}$  of Lemma [7.2.](#page-45-1) This is Corollary [7.10](#page-51-0) below. The key observation is the following.

<span id="page-50-3"></span>**Lemma 7.9.** Let A be a Lefschetz category over P. Let  $\gamma: A^{\natural} \to \mathbf{H}(\mathcal{A})$  and  $\lambda: {}^{\natural}\mathcal{A} \to \mathbf{H}(\mathcal{A})$ *be the inclusion functors. Then there is an equivalence of functors*

$$
\gamma^! \circ p^! \simeq (- \otimes \omega_{\mathbf{H}/\mathbf{P}}) \circ \lambda^! \circ p^*.
$$

*Proof.* We need an explicit formula for  $p^!$ . Set

$$
\mathcal{K} = \ker(V^{\vee} \otimes_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(H)).
$$

Then it is easy to see there is an isomorphism  $\mathbf{H} \cong \mathbf{P}_{\mathbf{P}}(\mathcal{K})$ , under which H' corresponds to the tautological  $\mathcal{O}(1)$  line bundle. From this, a computation shows

<span id="page-50-4"></span>
$$
\omega_{\mathbf{H}/\mathbf{P}} = \mathcal{O}(H - (N - 1)H')[N - 2],\tag{7.17}
$$

and hence

$$
p' \simeq (-\otimes \mathcal{O}(H-(N-1)H')[N-2]) \circ p^*.
$$

Further, since by Lemma [7.2](#page-45-1) the functor  $(- \otimes \mathcal{O}(H))$  induces an equivalence  $^{\natural} \mathcal{A} \simeq \mathcal{A}^{\natural}$ , it follows that

$$
\gamma' \simeq (-\otimes \mathcal{O}(H)) \circ \lambda^! \circ (-\otimes \mathcal{O}(-H)).
$$

Combining the above and using the  $\mathbf{P}^{\vee}$ -linearity of  $\lambda^!$  proves the result.

<span id="page-51-0"></span>Corollary 7.10. *Let* A *be a moderate Lefschetz category over* P*. Then the functor*

$$
(- \otimes \omega_{\mathbf{H}/\mathbf{P}}) \colon \mathbf{H}(\mathcal{A}) \to \mathbf{H}(\mathcal{A})
$$

*induces a*  $\mathbf{P}^{\vee}$ *-linear equivalence*  $^{\natural}A \simeq A^{\natural}$ *, which takes*  $^{\natural}A_0 \subset {}^{\natural}A$  *to*  $\gamma^!p^!(A_0) \subset A^{\natural}$ *.* 

*Proof.* Combine Lemma [7.9,](#page-50-3) the formula [\(7.17\)](#page-50-4), and the last statement of Lemma [7.2.](#page-45-1)  $\square$ 

The category  $\gamma^! p^! (\mathcal{A}_0)$  does not coincide with  $\mathcal{A}_0^{\natural}$  $_{0}^{\mu}$ . However, as the following lemma shows, it is closely related. Recall from §[3.6](#page-19-0) the notion of a splitting functor.

<span id="page-51-4"></span>**Lemma 7.11.** Let A be a moderate Lefschetz category over **P**. Then the functor  $p_* \circ \gamma : A^{\dagger} \to A$ *is splitting. Moreover, the images and kernels of this functor and its left and right adjoints are given by:*

$$
\text{im}(p_* \circ \gamma) = \mathcal{A}_0, \qquad \ker(p_* \circ \gamma) = \mathcal{A}_0^{\natural \perp}, \tag{7.18}
$$

$$
\operatorname{im}(\gamma^* \circ p^*) = \mathcal{A}_0^{\natural}, \qquad \operatorname{ker}(\gamma^* \circ p^*) = {}^{\perp}\mathcal{A}_0,\tag{7.19}
$$

$$
\operatorname{im}(\gamma^! \circ p^!) = (\mathcal{A}_0^{\natural})^{\perp \perp}, \quad \operatorname{ker}(\gamma^! \circ p^!) = \mathcal{A}_0^{\perp}.
$$
 (7.20)

*Proof.* We start by proving [\(7.18\)](#page-51-1) and [\(7.19\)](#page-51-2). First we prove  $\text{im}(p_* \circ \gamma) \subset A_0$ . For  $C \in \mathcal{A}$  and  $D \in \mathcal{A}^{\natural}$ , we have

$$
\mathcal{H}om_S(C, p_*\gamma(D)) \simeq \mathcal{H}om_S(p^*(C), \gamma(D)),
$$

which vanishes for  $C \in \mathcal{A}_i(iH)$ ,  $1 \leq i \leq m-1$ , by the defining semiorthogonal decomposi-tion [\(7.1\)](#page-44-2) of  $A^{\natural}$ . Hence

$$
p_*\gamma(D) \in \langle A_1(H), \ldots, A_{m-1}((m-1)H) \rangle^{\perp} = A_0,
$$

as desired. Since  $\gamma^* \circ p^*$  is fully faithful on  $\mathcal{A}_0$  by Lemma [7.3](#page-45-3)[\(1\)](#page-45-4), it follows that in fact  $\text{im}(p_* \circ \gamma) = A_0$ . From this,  $\text{ker}(\gamma^* \circ p^*) = {}^{\perp}A_0$  follows by adjunction. Hence  $\text{im}(\gamma^* \circ p^*)$ coincides with  $\mathcal{A}_0^{\natural} = (\gamma^* \circ p^*)(\mathcal{A}_0)$ . Again by adjunction,  $\ker(p_* \circ \gamma) = \mathcal{A}_0^{\natural}$  $_{0}^{\natural}$  follows formally from this.

Next we aim to show  $p_* \circ \gamma$  is a splitting functor. By Theorem [3.22,](#page-19-1) it suffices to show its left adjoint  $\gamma^* \circ p^*$  is right splitting and its right adjoint  $\gamma^! \circ p^!$  is left splitting. Note that ker( $\gamma^* \circ p^*$ ) =  $^{\perp}A_0$  is right admissible in  $A, \gamma^* \circ p^*$  is fully faithful on the subcategory  $\ker(\gamma^* \circ p^*)^{\perp} = \mathcal{A}_0$  by Lemma [7.3,](#page-45-3) and  $\operatorname{im}(\gamma^* \circ p^*) = \mathcal{A}_0^{\natural}$ <sup> $\natural$ </sup> is right admissible since  $\gamma^*$  and  $p^*$ admit right adjoints. Therefore  $\gamma^* \circ p^*$  verifies condition [\(1r\)](#page-19-2) of Theorem [3.22,](#page-19-1) and so is right splitting.

By Lemma [7.9](#page-50-3) the functor  $\gamma' \circ p'$  differs from  $\lambda' \circ p^*$  by the autoequivalence  $-\otimes \omega_{\mathbf{H}/\mathbf{P}}$ , and hence is left splitting if and only if  $\lambda^! \circ p^*$  is. By an argument analogous to the one above for  $\gamma^* \circ p^*$ , we find that  $\ker(\lambda^! \circ p^*) = A_0^{\perp}$  and  $\lambda^! \circ p^*$  is left splitting. This completes the proof that  $p_* \circ \gamma$  is a splitting functor. All that remains is to check that  $\text{im}(\gamma^! \circ p^!) = (\mathcal{A}_0^{\natural}$  $_{0}^{\natural})^{\perp\perp}$ . This follows from condition [\(3r\)](#page-19-3) of Theorem [3.22](#page-19-1) and the equality  $\ker(p_* \circ \gamma) = \mathcal{A}_0^{\natural}$  $\begin{array}{ccc} \n\frac{1}{2} \downarrow & \qquad & \square \ \n0 & \cdots & \end{array}$ 

Now we can give a criterion under which the right and left HPD categories are identified as Lefschetz categories.

<span id="page-51-3"></span><span id="page-51-2"></span><span id="page-51-1"></span>

<span id="page-52-1"></span>Proposition 7.12. *Let* A *be a strong, moderate Lefschetz category over* P*, which is proper over* S. Assume  $A^{\dagger}$  *admits a relative Serre functor*  $S_{A^{\dagger}/S}$  *over* S. Then there is a  $\mathbf{P}^{\vee}$ -linear *equivalence*  $^{\natural}A \simeq A^{\natural}$  *which induces an equivalence*  $^{\natural}A_0 \simeq A_0^{\natural}$ 0 *.*

*Proof.* By Corollary [7.10](#page-51-0) there is a  $\mathbf{P}^{\vee}$ -linear equivalence  $^{\sharp}A \simeq A^{\sharp}$  taking  $^{\sharp}A_0$  to  $\gamma^!p^!(A_0)$ . By [\(7.20\)](#page-51-3) the latter coincides with  $(A_0^{\dagger})$  $\downarrow{4}$   $\downarrow$  ⇒  $\perp$ . So it suffices to show that  $A^{\natural}$  admits a  $\mathbf{P}^{\vee}$ -linear autoequivalence that takes  $\mathcal{A}_0^{\natural}$  $_0^\natural$  to  $({\cal A}_0^\natural$  $\frac{1}{0}$ <sup> $\uparrow$ </sup> ⊥⊥. By Lemmas [4.18](#page-26-0) and [4.20,](#page-27-0) the Serre functor S<sub>A</sub><sup> $\uparrow$ </sup>/*S* is precisely such an autoequivalence.

In Theorem [8.7](#page-55-0) we show that  ${}^{\natural}A_0 \subset {}^{\natural}A$  and  $A_0^{\natural} \subset A^{\natural}$  are Lefschetz centers as soon as A is strong and moderate. Thus the conclusion of Proposition [7.12](#page-52-1) can be rephrased as saying there is an equivalence of Lefschetz categories  $^{\natural}A \simeq \mathcal{A}^{\natural}$ .

**Remark 7.13.** The assumption that  $A^{\dagger}$  admits a relative Serre functor  $S_{A^{\dagger}/S}$  over S holds for instance if  $A^{\dagger}$  is smooth and proper over S, by Lemma [4.19.](#page-26-1) This in turn holds if A is smooth and proper over S, by Lemma [7.18](#page-53-1) below.

<span id="page-52-0"></span>7.5. Relation to classical projective duality. Recall that in Definition [4.21](#page-27-1) we introduced the notion of the critical locus of a linear category.

<span id="page-52-2"></span>**Definition 7.14.** Let  $A$  be a **P**-linear category which is smooth and proper over  $S$ . The *classical projective dual*  $CPD(\mathcal{A}) \subset \mathbf{P}^{\vee}$  is the set

$$
CPD(\mathcal{A}) = Crit_{\mathbf{P}^{\vee}}(\mathbf{H}(\mathcal{A})) \subset \mathbf{P}^{\vee}.
$$

**Remark 7.15.** Definition [7.14](#page-52-2) can also be made without the assumption that  $A$  is smooth and proper over S. However, these hypotheses guarantee that our definition recovers the classical notion in the geometric case (Corollary [7.17](#page-52-3) below).

<span id="page-52-4"></span>**Lemma 7.16.** Assume  $S = \text{Spec}(k)$  where k is an algebraically closed field. Let X be an *integral scheme of finite type over* k *equipped with a closed immersion*  $X \to \mathbf{P}$ *, such that* X *is not contained in any hyperplane in* P*. Then*

$$
Crit_{\mathbf{P}^{\vee}}(\mathbf{H}(\text{Perf}(X))) = \{ H \in \mathbf{P}^{\vee} \mid X \times_{\mathbf{P}} H \text{ is singular} \}.
$$

*Proof.* Since X is not contained in any hyperplane in  $P$ , the universal hyperplane section  $X \times_{\mathbf{P}} \mathbf{H}$  of X is flat over  $\mathbf{P}^{\vee}$ . So since  $\mathbf{H}(\text{Perf}(X)) \simeq \text{Perf}(X \times_{\mathbf{P}} \mathbf{H})$ , the result follows from Remark [4.22.](#page-27-2)

Recall that if X as in Lemma [7.16](#page-52-4) is *in addition smooth*, then its *classical projective dual* is given by

$$
X^{\vee} = \{ H \in \mathbf{P}^{\vee} \mid X \times_{\mathbf{P}} H \text{ is singular } \}.
$$

Hence we have:

<span id="page-52-3"></span>**Corollary 7.17.** Assume  $S = \text{Spec}(k)$  where k is an algebraically closed field. Let X be a *smooth integral scheme of finite type over* k *equipped with a closed immersion*  $X \to \mathbf{P}$ , such *that* X *is not contained in any hyperplane in* P*. Then*

$$
\text{CPD}(\text{Perf}(X)) = X^{\vee} \subset \mathbf{P}^{\vee}.
$$

Classical projective typically does not preserve smoothness of a variety. An interesting feature of HPD is that it does:

<span id="page-53-1"></span>Lemma 7.18. *Let* A *be a Lefschetz category over* P*, which is smooth and proper over* S*. Then the universal hyperplane section category*  $H(A)$  *and the HPD categories*  $\mathcal{A}$  *A and*  $A^{\dagger}$  *are all smooth and proper over* S*.*

*Proof.* The claim for  $H(\mathcal{A})$  follows from Lemma [4.11,](#page-23-0) since the morphism  $H \to P$  is smooth and proper. Then the claim for the HPD categories follows from Lemma [4.15](#page-25-0)[\(4\)](#page-25-1).  $\Box$ 

Proposition 7.19. *Let* A *be a Lefschetz category over* P*, which is smooth and proper over* S*. Then*

$$
\mathrm{CPD}(\mathcal{A}) = \mathrm{Crit}_{\mathbf{P}^\vee}(\mathcal{A}^\natural) = \mathrm{Crit}_{\mathbf{P}^\vee}(\mathsf{A}^\natural \mathcal{A}) \subset \mathbf{P}^\vee.
$$

*Proof.* The equality Crit<sub>P</sub>∨ ( $\mathcal{A}^{\dagger}$ ) = Crit<sub>P</sub>∨ ( $^{\dagger}$  $\mathcal{A}$ ) holds because by Lemma [7.2](#page-45-1) there is a P<sup>∨</sup>-linear equivalence  $A^{\natural} \simeq {}^{\natural}A$ . By Lemmas [4.15\(](#page-25-0)[4\)](#page-25-1) and [7.18](#page-53-1) the components of the semiorthogonal decomposition [\(7.1\)](#page-44-2) are admissible. Hence by Lemma [4.25](#page-28-0) we have

$$
\mathrm{CPD}(\mathcal{A}) = \mathrm{Crit}_{\mathbf{P}^\vee}(\mathcal{A}^\natural) \cup \bigcup_{i=1}^m \mathrm{Crit}_{\mathbf{P}^\vee}(\mathcal{A}_i(iH) \otimes \mathrm{Perf}(\mathbf{P}^\vee)).
$$

But Lemma [4.15](#page-25-0) also implies that the components  $A_i(iH)$  of the Lefschetz decomposition of A are smooth and proper over S. Hence by Lemma [4.10](#page-23-1) their base changes

$$
\mathcal{A}_i(iH)\otimes\operatorname{Perf}(\mathbf{P}^{\vee})
$$

are smooth and proper over P<sup>∨</sup> . Hence

$$
\mathrm{Crit}_{\mathbf{P}^\vee}(\mathcal{A}_i(iH) \otimes \mathrm{Perf}(\mathbf{P}^\vee)) = \varnothing
$$

<span id="page-53-0"></span>by Lemma [4.24.](#page-27-3)  $\Box$ 

### 8. The main theorem of HPD

Let A be a moderate Lefschetz category over **P**. In  $\S7$  $\S7$  we constructed **P**<sup> $\vee$ </sup>-linear right and left HPD categories  $A^{\natural}$  and  $^{\natural}A$ , which are equipped with natural Lefschetz chains. In this section, we show that if A is right strong then this gives  $A^{\dagger}$  the structure of a left strong Lefschetz category over  $\mathbf{P}^{\vee}$ , and if A is left strong this gives <sup> $\sharp$ </sup>A the structure of a right strong Lefschetz category over  $\mathbf{P}^{\vee}$ . Further, we prove that taking the HPD once more recovers A, i.e. we prove there are Lefschetz equivalences  $^{\natural}(\mathcal{A}^{\natural}) \simeq \mathcal{A}$  and  $(^{\natural}\mathcal{A})^{\natural} \simeq \mathcal{A}$ .

In fact, we will deduce these statements from a significantly stronger result (Theorem [8.4\)](#page-55-1), which describes the series of categories  $\mathcal{K}_r(\mathcal{A})$  defined by [\(6.15\)](#page-43-4) in terms of an analogous series of categories associated to  $\mathcal{A}^{\natural}$ .

Our arguments generalize those of [\[9,](#page-71-0) §6], by placing them in the framework set up in the previous sections.

8.1. The main theorem and its corollaries. Throughout this section, we work in the setup where A is assumed right strong, and then we work with the right HPD category  $\mathcal{A}^{\sharp}$ . The reader may check that all of the analogous results hold with "right" replaced by "left".

# <span id="page-53-4"></span><span id="page-53-3"></span>Setup 8.1.

- (1) A is a right strong, moderate Lefschetz category over **P** of length m.
- <span id="page-53-2"></span>(2) B is a P<sup> $\vee$ </sup>-linear category and  $\phi: \mathcal{B} \to \mathcal{A}^{\dagger}$  is a P<sup> $\vee$ </sup>-linear functor such that: (a)  $\phi$  is fully faithful and admits left and right adjoints  $\phi^*$  and  $\phi^!$ .

<span id="page-54-0"></span>(b) the categories  $\mathcal{B}_j = \phi^* \mathcal{A}_j^{\natural}$  $j^{\natural}, j \leq 0$ , where  $\mathcal{A}_j^{\natural}$  $j$  is as in [\(7.8\)](#page-46-2), form a strong left Lefschetz chain

$$
\mathcal{B}_{1-n} \subset \cdots \subset \mathcal{B}_{-1} \subset \mathcal{B}_0
$$

- in B with respect to  $-\otimes \mathcal{O}_{\mathbf{P}^\vee}(H'),$
- <span id="page-54-1"></span>(c) the image of the composition  $p_* \circ \gamma \circ \phi : \mathcal{B} \to \mathcal{A}$  is  $\mathcal{A}_0$ , where  $\gamma : \mathcal{A}^{\dagger} \to \mathbf{H}(\mathcal{A})$  is the inclusion functor.

**Remark 8.2.** The category  $\mathcal{B} = \mathcal{A}^{\dagger}$  satisfies the assumptions with  $\phi = id_{\mathcal{A}^{\dagger}}$ . Indeed, [\(2a\)](#page-53-2) is automatic, [\(2b\)](#page-54-0) holds by Proposition [7.4,](#page-46-1) and [\(2c\)](#page-54-1) holds by Proposition [7.11.](#page-51-4) In fact, we will see below that in Setup [8.1,](#page-53-3) the functor  $\phi$  is automatically an equivalence. This gives useful criteria for checking that a category  $\mathcal B$  is equivalent to the HPD category  $\mathcal A^{\natural}$ .

<span id="page-54-2"></span>Remark 8.3. Consider the following criterion:

(2c') the functor  $\phi^* \circ \gamma^* \circ p^* \colon \mathcal{A} \to \mathcal{B}$  is fully faithful on the right twisted components  $\mathfrak{a}'_i$ ,  $i \geq 0$ , of A, and their images give a semiorthogonal decomposition

$$
\mathcal{B}_j = \left\langle \phi^* \gamma^* p^* (\mathfrak{a}'_0), \phi^* \gamma^* p^* (\mathfrak{a}'_1), \dots, \phi^* \gamma^* p^* (\mathfrak{a}'_{N-2+j}) \right\rangle
$$

for all  $j \leq 0$ .

In Lemma [8.29,](#page-69-0) we will see that conditions  $(1)$ ,  $(2a)$ ,  $(2b)$ , and  $(2c')$  together imply  $(2c)$ . This is useful because the criterion  $(2c')$  is sometimes more convenient to check than  $(2c)$ , see for instance the proof of [\[15,](#page-71-3) Theorem 9.5].

Let  $\mathbf{G}_s^{\vee} = \text{Gr}(s, V^{\vee})$  be the Grassmannian of rank s subbundles of  $V^{\vee}$ , and let  $\mathbf{L}_s^{\vee}$  be the corresponding projectivized universal family. Then Lemma [6.22](#page-44-1) (with P replaced with the dual P<sup> $\vee$ </sup>) applies to B, and we get  $\mathbf{G}_s^{\vee}$ -linear categories  $\mathcal{K}_s(\mathcal{B})$  for  $0 \leq s \leq N$ , characterized by semiorthogonal decompositions

<span id="page-54-3"></span>
$$
\mathbf{L}_s^{\vee}(\mathcal{B}) = \left\langle \mathcal{B}_{1-n}((r-n)H') \otimes \mathrm{Perf}(\mathbf{G}_s^{\vee}), \dots, \mathcal{B}_{-r}(-H') \otimes \mathrm{Perf}(\mathbf{G}_s^{\vee}), \mathcal{K}_s'(\mathcal{B}) \right\rangle
$$
(8.1)

where  $r = N - s$ . There is an identification  $G_r \cong G_s^{\vee}$ , by which we regard  $\mathcal{K}_s(\mathcal{B})$  as a  $G_r$ -linear category. We have canonical morphisms



We aim to prove that there is a  $G_r$ -linear equivalence  $\mathcal{K}'_s(\mathcal{B}) \simeq \mathcal{K}_r(\mathcal{A})$ . By definition we have  $\mathcal{K}_r(\mathcal{A}) \subset \mathbf{L}_r(\mathcal{A})$  and  $\mathcal{K}'_s(\mathcal{B}) \subset \mathbf{L}_s^{\vee}(\mathcal{B})$ . The desired equivalence will be induced by a functor  $\mathbf{L}_{s}^{\vee}(\mathcal{B}) \to \mathbf{L}_{r}(\mathcal{A})$ , which can be described using the kernel formalism of §[5](#page-30-1) as follows. Consider the P<sup> $\vee$ </sup>-linear composition  $\gamma \circ \phi : \mathcal{B} \to \mathbf{H}(\mathcal{A})$ , where recall  $\mathbf{H} = \mathbf{L}_{N-1}$  and  $\gamma : \mathcal{A}^{\dagger} \to \mathbf{H}(\mathcal{A})$  is the inclusion. By Proposition [5.7,](#page-31-0) there is a kernel

$$
\mathcal{E} \in \mathrm{FM}(\mathcal{B}/\mathbf{P}^{\vee}, \mathcal{A}/\mathbf{P}, \mathbf{H} \times_{\mathbf{P}^{\vee}} \mathbf{P}^{\vee})
$$

such that

<span id="page-54-4"></span>
$$
\Phi_{\mathcal{E}} = \gamma \circ \phi. \tag{8.2}
$$

Let

 $\zeta_r\colon\mathbf{L}_r\times_{\mathbf{G}_r}\mathbf{L}^\vee_s\to\mathbf{H}$ 

be the natural morphism. Set

$$
\mathcal{E}_r = \zeta_r^* \mathcal{E} \in \mathrm{FM}(\mathcal{B}/\mathbf{P}^{\vee}, \mathcal{A}/\mathbf{P}, \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee})
$$

and define

$$
\Phi_r=\Phi_{\mathcal{E}_r}\colon \mathbf{L}^\vee_s(\mathfrak{B})\to \mathbf{L}_r(\mathcal{A})
$$

to be the associated  $\mathbf{G}_r$ -linear functor. This is the functor we are after. Note that  $\Phi_{\mathcal{E}} = \Phi_{N-1}$ under the identifications  $\mathbf{P}^{\vee} = \mathbf{L}_1^{\vee}$ ,  $\mathbf{H} = \mathbf{L}_{N-1}$ .

The key result of this section is the following.

<span id="page-55-1"></span>**Theorem 8.4.** In Setup [8.1,](#page-53-3) for all nonnegative integers r and s such that  $r + s = N$ , there *is a* Gr*-linear equivalence*

$$
\phi_r \colon \mathcal{K}'_s(\mathcal{B}) \xrightarrow{\sim} \mathcal{K}_r(\mathcal{A})
$$

*induced by the restriction of*  $\Phi_r$  *to*  $\mathcal{K}'_s(\mathcal{B})$ *.* 

The proof of Theorem [8.4](#page-55-1) will be given below. Here we derive some consequences.

<span id="page-55-5"></span>**Corollary 8.5.** In Setup [8.1,](#page-53-3) the functor  $\phi: \mathcal{B} \to \mathcal{A}^{\dagger}$  is an equivalence.

*Proof.* By construction, the functor  $\phi : \mathcal{B} \to \mathcal{A}^{\dagger}$  coincides with the equivalence  $\phi_{N-1}$  of The-orem [8.4](#page-55-1) (note that  $\mathcal{K}'_1(\mathcal{B}) = \mathbf{L}'_1(\mathcal{B}) = \mathcal{B}$ ).

<span id="page-55-2"></span>Corollary 8.6. *In Setup [8.1,](#page-53-3) the Lefschetz chain*  $B_{1-n}$  ⊂ · · · ⊂  $B_{-1}$  ⊂  $B_0$  *in*  $B$  *is full.* 

*Proof.* The defining semiorthogonal decomposition [\(8.1\)](#page-54-3) of  $\mathcal{K}'_s(\mathcal{B})$  for  $s = N$  can be written as

$$
\mathcal{B} = \left\langle \mathcal{B}_{1-n}(-nH'), \ldots, \mathcal{B}_{-1}(-2H'), \mathcal{B}_{0}(-H'), \mathcal{K}'_{N}(\mathcal{B}) \right\rangle.
$$

Twisting by  $O(H')$ , we get

$$
\mathcal{B} = \left\langle \mathcal{B}_{1-n}((1-n)H'), \ldots, \mathcal{B}_{-1}(-H'), \mathcal{B}_0, \mathcal{K}'_N(\mathcal{B})(H') \right\rangle.
$$

Hence the Lefschetz chain  $\mathcal{B}_{1-n} \subset \cdots \subset \mathcal{B}_{-1} \subset \mathcal{B}_0$  in  $\mathcal{B}$  is full if and only if  $\mathcal{K}'_N(\mathcal{B}) \simeq 0$ . But by Theorem [8.4](#page-55-1) we have  $\mathcal{K}'_N(\mathcal{B}) \simeq \mathcal{K}_0(\mathcal{A})$ , and  $\mathcal{K}_0(\mathcal{A}) \simeq 0$  by Remark [6.17.](#page-41-2)

By combining the above results in the case  $\mathcal{B} = \mathcal{A}^{\dagger}$ , we obtain the main theorem of HPD.

<span id="page-55-3"></span><span id="page-55-0"></span>Theorem 8.7. *Let* A *be a right strong, moderate Lefschetz category over* P*. Then:*

*(1)*  $A^{\natural}$  *is a left strong, moderate Lefschetz category over*  $\mathbf{P}^{\vee}$ *, with components*  $A_j^{\natural} \subset A^{\natural}$  *given by* [\(7.8\)](#page-46-2) *and length given by*

$$
length(\mathcal{A}^{\natural}) = N - \#\{ i \ge 0 \mid \mathcal{A}_i = \mathcal{A}_0 \}.
$$

<span id="page-55-4"></span>(2) Let  $L \subset V$  be a subbundle and let  $L^{\perp} = \text{ker}(V^{\vee} \to L^{\vee})$  be its orthogonal. Set

 $r = \text{rank}(L)$ ,  $s = \text{rank}(L^{\perp})$ ,  $m = \text{length}(\mathcal{A})$ ,  $n = \text{length}(\mathcal{A}^{\natural})$ .

*Then there are semiorthogonal decompositions*

$$
\mathcal{A} \otimes_{\text{Perf}(\mathbf{P})} \text{Perf}(\mathbf{P}(L)) = \langle \mathcal{K}_L(\mathcal{A}), \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle, \mathcal{A}^{\natural} \otimes_{\text{Perf}(\mathbf{P}^{\vee})} \text{Perf}(\mathbf{P}(L^{\perp})) = \left\langle \mathcal{A}_{1-n}^{\natural}((r-n)H'), \dots, \mathcal{A}_{-r}^{\natural}(-H'), \mathcal{K}'_{L^{\perp}}(\mathcal{A}^{\natural}) \right\rangle,
$$

*and an S*-linear equivalence  $\mathcal{K}_L(\mathcal{A}) \simeq \mathcal{K}'_{L^{\perp}}(\mathcal{A}^{\dagger}).$ 

*Proof.* The categories  $\mathcal{A}_i^{\natural}$ <sup> $\natural$ </sup> form a strong left Lefschetz chain in  $\mathcal{A}^{\natural}$  by Proposition [7.4.](#page-46-1) Hence they are the components of a left strong Lefschetz structure over P<sup>∨</sup> by Corollary [8.6](#page-55-2) combined with Lemma [6.3.](#page-37-2) By construction, the length of  $\mathcal{A}^{\sharp}$  with this Lefschetz structure is as stated. This proves [\(1\)](#page-55-3). For part [\(2\)](#page-55-4), note that there are semiorthogonal decompositions of the claimed form by Lemma [6.20](#page-43-5) and Lemma [6.22\(](#page-44-1)[3\)](#page-44-4). The equivalence  $\mathcal{K}_L(\mathcal{A}) \simeq \mathcal{K}'_{L^{\perp}}(\mathcal{A}^{\dagger})$  is the base change of the equivalence  $\mathcal{K}_r(\mathcal{A}) \simeq \mathcal{K}'_s(\mathcal{A}^{\natural})$  given by Theorem [8.4](#page-55-1) along the morphism  $S \to \mathbf{G}_r$ classifying  $L \subset V$ .

**Remark 8.8.** Specializing Theorem [8.7](#page-55-0) to the case where S is the spectrum of a field and A and  $A^{\natural}$  are geometric, i.e. of the form  $A = \text{Perf}(X)$  and  $A^{\natural} = \text{Perf}(Y)$  for a **P**-scheme X and a P∨-scheme Y, we recover Theorem [1.1](#page-2-0) from §[1.](#page-1-0) In fact, even in this case our result is more general than Theorem [1.1](#page-2-0) in several respects. First, we do not impose any smooth and properness assumptions. Second, we do not need any transversality assumption about the fiber products  $X \times_{\mathbf{P}} \mathbf{P}(L)$  and  $Y \times_{\mathbf{P}'} \mathbf{P}(L^{\perp})$ , which are taken in the derived sense, according to our conventions from §[2.1.](#page-7-0) This addresses the question of [\[9,](#page-71-0) Remark 6.26], i.e. removes the "admissibility" assumption on  $L \subset V$  from [\[9,](#page-71-0) Theorem 6.3]. Finally, we note that by applying Theorem [4.26](#page-28-1) and Proposition [4.28](#page-28-2) we obtain a version of Theorem [8.7](#page-55-0) for bounded derived categories of coherent sheaves, recovering the result of Remark [1.2.](#page-2-1)

Using the above, we can prove that HPD is indeed a duality.

Theorem 8.9. *Let* A *be a right strong, moderate Lefschetz category over* P*. Then the functor*  $\phi_1$ :  $\sharp$ ( $A^{\sharp}$ )  $\xrightarrow{\sim}$  A *is an equivalence of Lefschetz categories over* **P**.

*Proof.* First note that  $\mathcal{K}'_{N-1}(\mathcal{A}^{\natural}) = {}^{\natural}(\mathcal{A}^{\natural})$  by definition and  $\mathcal{K}_1(\mathcal{A}) = \mathcal{A}$ , so that  $\phi_1$  is indeed a P-linear equivalence between the stated categories. It remains to check that this equivalence takes  $^{\natural}(\mathcal{A}^{\natural})_0$  to  $\mathcal{A}_0$ . For this, let us describe  $^{\natural}(\mathcal{A}^{\natural})_0 \subset {}^{\natural}(\mathcal{A}^{\natural})$  more explicitly. The universal hyperplane **H** in **P** is simultaneously the universal hyperplane in  $\mathbf{P}^{\vee}$ , via the projection  $f: \mathbf{H} \to \mathbf{P}^{\vee}$ . The category  $^{\natural}(\mathcal{A}^{\natural})$  is defined by the **P**-linear semiorthogonal decomposition

<span id="page-56-0"></span>
$$
\mathbf{H}(\mathcal{A}^{\natural}) = \left\langle \iota^*(\mathcal{A}_{1-n}^{\natural}((1-n)H') \otimes \mathrm{Perf}(\mathbf{P}))), \dots, \iota^*(\mathcal{A}_{-1}^{\natural}(-H') \otimes \mathrm{Perf}(\mathbf{P})),^{\natural}(\mathcal{A}^{\natural}) \right\rangle, \tag{8.3}
$$

where  $\mathbf{H}(\mathcal{A}^{\natural}) = \mathcal{A}^{\natural} \otimes_{\text{Perf}(\mathbf{P}^{\vee})} \text{Perf}(\mathbf{H}),$  see [\(7.2\)](#page-44-5). Further, by definition

$$
{}^{\natural}(\mathcal{A}^{\natural})_0=\lambda^!f^*(\mathcal{A}_0^{\natural}),
$$

where  $\lambda: {}^{\natural}(\mathcal{A}^{\natural}) \to \mathbf{H}(\mathcal{A}^{\natural})$  is the inclusion functor. By Propositions [8.28](#page-68-0) and [8.27](#page-67-0) (with  $r=1$ and  $s = N - 1$ ) proved below, we see that the functor  $\Phi_1: H(\mathcal{A}^{\dagger}) \to \mathcal{A}$  kills the categories to the left of  $\sharp(\mathcal{A}^{\natural})$  in [\(8.3\)](#page-56-0). It follows that

<span id="page-56-1"></span>
$$
\phi_1(\hbar^{\natural}(\mathcal{A}^{\natural})_0) = \Phi_1(\lambda \lambda^! f^*(\mathcal{A}_0^{\natural})) = \Phi_1(f^*(\mathcal{A}_0^{\natural})). \tag{8.4}
$$

We claim there is an equivalence of functors

$$
\Phi_1 \circ f^* \simeq p_* \circ \gamma \colon \mathcal{A}^{\natural} \to \mathcal{A}.
$$

Indeed, there is a tautological commutative diagram

 $\mathbf{P}\times_{\mathbf{P}}\mathbf{H} \mathbf{=}\frac{\zeta_1}{\sim}$  $\frac{1}{\sim}$  → id×f  $\vdash$  $\mathbf{H} \times_{\mathbf{P}^\vee} \mathbf{P}^\vee$  $\bigvee$  p×id  $\mathbf{P}\times\mathbf{P}^{\vee}$ 

Hence there is an equivalence of kernels  $(\mathrm{id} \times f)_*(\mathcal{E}_1) \simeq (p \times \mathrm{id})_*(\mathcal{E})$ , where recall  $\mathcal{E}_1 = \zeta_1^*(\mathcal{E})$ is the kernel for the functor  $\Phi_1$ . So the above claim follows from Lemma [5.14,](#page-33-0) since  $\mathcal E$  is the kernel for the functor  $\gamma$ . Combining the claim with [\(8.4\)](#page-56-1) and [\(7.18\)](#page-51-1), we conclude  $\phi_1$  takes ♮ (A♮ )<sub>0</sub> to  $A_0$ .

8.2. Notation. Throughout, r and s will denote nonnegative integers such that  $r + s = N$ . We will prove Theorem [8.4](#page-55-1) by an induction argument. For this, we need to relate the functors

$$
\Phi_r: \mathbf{L}_s^{\vee}(\mathcal{B}) \to \mathbf{L}_r(\mathcal{A}),
$$
  

$$
\Phi_{r-1}: \mathbf{L}_{s+1}^{\vee}(\mathcal{B}) \to \mathbf{L}_{r-1}(\mathcal{A}).
$$

Here and below, when we consider the functor  $\Phi_{r-1}$  or the categories  $\mathbf{L}_{s+1}^{\vee}(\mathcal{B})$  and  $\mathbf{L}_{r-1}(\mathcal{A}),$ we implicitly assume  $r \geq 1$ .

To this end, we introduce some auxiliary spaces. Let  $\mathbf{Fl}_{r-1,r}$  denote the variety of flags of subspaces  $L_{r-1} \subset L_r \subset V$  where  $\dim L_i = i$ . This variety comes with forgetful maps

<span id="page-57-1"></span>

Further, define

$$
\mathbf{L}_{r-1}^+ = \mathbf{L}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{Fl}_{r-1,r},
$$
  

$$
\mathbf{L}_r^- = \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{Fl}_{r-1,r}.
$$

These spaces fit into a commutative diagram

$$
\mathbf{Fl}_{r-1,r} \overset{f_{r-1}^+}{\longleftarrow} \mathbf{L}_{r-1}^+ \overset{a_r}{\longrightarrow} \mathbf{L}_r^- \overset{f_r^-}{\longrightarrow} \mathbf{Fl}_{r-1,r}
$$
\n
$$
\pi_{r-1} \overset{\pi_{r-1}}{\longleftarrow} \begin{bmatrix} \pi_{r-1}^+ & \pi_r^- \\ \pi_{r-1}^+ & \pi_r^- \end{bmatrix} \begin{bmatrix} \pi_r \\ \pi_r \end{bmatrix}
$$
\n
$$
\mathbf{G}_{r-1} \overset{f_{r-1}}{\longleftarrow} \mathbf{L}_{r-1} \qquad \mathbf{L}_r \overset{f_r}{\longrightarrow} \mathbf{G}_r \qquad (8.6)
$$

where the squares are cartesian and  $a_r$  is a closed embedding. We also introduce the notation

$$
\alpha_r = \pi_r^- \circ a_r \colon \mathbf{L}_{r-1}^+ \to \mathbf{L}_r,
$$
  

$$
p_{r-1}^+ = p_{r-1} \circ \pi_{r-1}^+ \colon \mathbf{L}_{r-1}^+ \to \mathbf{P},
$$
  

$$
p_r^- = p_r \circ \pi_r^- \colon \mathbf{L}_r^- \to \mathbf{P},
$$

where recall  $p_r: \mathbf{L}_r \to \mathbf{P}$  denotes the projection.

Dually, let  $\mathbf{Fl}^{\vee}_{s,s+1}$  denote the the variety of flags of subspaces  $M_s \subset M_{s+1} \subset V^{\vee}$ , where  $\dim M_i = i$ . This variety comes with forgetful maps

<span id="page-57-0"></span>

As above, we define

$$
\begin{aligned} \mathbf{L}_{s+1}^{\vee,-} &= \mathbf{L}_{s+1}^{\vee} \times_{\mathbf{G}_{s+1}^{\vee}} \mathbf{F} \mathbf{l}_{s,s+1}^{\vee}, \\ \mathbf{L}_{s}^{\vee,+} &= \mathbf{L}_{s}^{\vee} \times_{\mathbf{G}_{s}^{\vee}} \mathbf{F} \mathbf{l}_{s,s+1}^{\vee}. \end{aligned}
$$

Under the canonical isomorphisms  $\mathbf{Fl}^{\vee}_{s,s+1} \cong \mathbf{Fl}_{r-1,r}$ ,  $\mathbf{G}^{\vee}_{s+1} \cong \mathbf{G}_{r-1}$ , and  $\mathbf{G}^{\vee}_{s} \cong \mathbf{G}_{r}$ , the diagram [\(8.7\)](#page-57-0) is identified with [\(8.5\)](#page-57-1). Hence the above spaces fit into a commutative diagram

$$
\mathbf{Fl}_{r-1,r} \overset{g_{s+1}^{-}}{\leftarrow} \mathbf{L}_{s+1}^{\vee,-} \overset{b_{s+1}}{\leftarrow} \mathbf{L}_{s}^{\vee,+} \overset{g_{s}^{+}}{\leftarrow} \mathbf{Fl}_{r-1,r} \overset{g_{r-1}^{+}}{\leftarrow} \mathbf{Fl}_{r-1,r} \overset{g_{r-1}^{-}}{\leftarrow} \overset{g_{s+1}^{-}}{\leftarrow} \overset{g_{s+1}^{-
$$

As above, we also set

$$
\begin{aligned} \beta_{s+1} &= \pi_{s+1}^{\vee,-} \circ b_{s+1} \colon \mathbf{L}_s^{\vee,+} \to \mathbf{L}_{s+1}^{\vee}, \\ q_{s+1}^- &= q_{s+1} \circ \pi_{s+1}^{\vee,-} \colon \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{P}^{\vee}, \\ q_s^+ &= q_s \circ \pi_s^{\vee,+} \colon \mathbf{L}_s^{\vee,+} \to \mathbf{P}^{\vee}, \end{aligned}
$$

where recall  $q_s \colon L_s^{\vee} \to \mathbf{P}^{\vee}$  denotes the projection.

We have morphisms

$$
\begin{aligned} \pi_r^- \times \pi_s^{\vee, +} \colon \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_s^{\vee, +} &\to \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee}, \\ \pi_{r-1}^+ \times \pi_{s+1}^{\vee, -} \colon \mathbf{L}_{r-1}^+ \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee, -} &\to \mathbf{L}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{L}_{s+1}^{\vee}.\end{aligned}
$$

Pulling back the kernel  $\mathcal{E}_r$  for  $\Phi_r$  along these morphisms, we obtain kernels

$$
\mathcal{E}_r^- = (\pi_r^- \times \pi_s^{\vee,+})^* \mathcal{E}_r \in \text{FM}(\mathcal{B}/\mathbf{P}^{\vee}, \mathcal{A}/\mathbf{P}, \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_s^{\vee,+}),
$$
  

$$
\mathcal{E}_{r-1}^+ = (\pi_{r-1}^+ \times \pi_{s+1}^{\vee,-})^* \mathcal{E}_{r-1} \in \text{FM}(\mathcal{B}/\mathbf{P}^{\vee}, \mathcal{A}/\mathbf{P}, \mathbf{L}_{r-1}^+ \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}).
$$

We denote by

$$
\begin{aligned} \Phi_r^-&\colon \mathbf L_s^{\vee,+}(\mathcal B) \to \mathbf L_r^-(\mathcal A),\\ \Phi_{r-1}^+&\colon \mathbf L_{s+1}^{\vee,-}(\mathcal B) \to \mathbf L_{r-1}^+(\mathcal A), \end{aligned}
$$

the associated  $\mathbf{Fl}_{r-1,r}$ -linear functors.

Finally, we denote by  $\mathcal{U}_r$  the rank r tautological subbundle of  $V \otimes \mathcal{O}$  on  $\mathbf{G}_r$ , and by  $\mathcal{W}_s$ the rank s tautological subbundle of  $V^{\vee} \otimes \mathcal{O}$  on  $\mathbf{G}_{s}^{\vee}$ . By abuse of notation, we use the same symbol to denote the pullback of  $\mathcal{U}_r$  or  $\mathcal{W}_s$  to any space mapping to  $\mathbf{G}_r$  or  $\mathbf{G}_s^{\vee}$ . Note that under the isomorphism  $G_r \cong G_s^{\vee}$ , the bundle  $\mathcal{W}_s$  corresponds to the orthogonal bundle  $\mathcal{U}_r^{\perp}$ .

8.3. Geometric lemmas. Here we gather some results describing the geometry of the spaces introduced above. The proofs are left to the reader.

**Lemma 8.10.** (1) The morphism  $a_r: \mathbf{L}_{r-1}^+ \to \mathbf{L}_r^-$  embeds  $\mathbf{L}_{r-1}^+$  as a divisor, cut out by a *section of the line bundle*  $(\mathcal{U}_r/\mathcal{U}_{r-1})(H)$ *.* 

- <span id="page-58-0"></span>(2) The morphism  $b_{s+1}: L_s^{\vee,+} \to L_{s+1}^{\vee,-}$  embeds  $L_s^{\vee,+}$  as a divisor, cut out by a section of the *line bundle*  $(\mathcal{U}_r/\mathcal{U}_{r-1})^{\vee}(H')$ *.*
- <span id="page-58-1"></span>**Lemma 8.11.** (1) The morphism  $(\zeta_r, pr_2)$ :  $\mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee} \to \mathbf{H} \times_{\mathbf{P}^{\vee}} \mathbf{L}_s^{\vee}$  is a closed immersion, with image cut out by a regular section of the vector bundle  $(\mathcal{W}_s / \mathcal{O}(-H'))^{\vee}(H)$ .

(2) The morphism  $(pr_1, \zeta_r): L_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee} \to L_r \times_{\mathbf{P}} \mathbf{H}$  *is a closed immersion, with image cut out by a regular section of the vector bundle*  $(\mathcal{U}_r/\mathcal{O}(-H))^{\vee}(H')$ *.* 

<span id="page-59-1"></span>**Lemma 8.12.** The morphism  $(p_r^-, f_r^-): \mathbf{L}_r^- \to \mathbf{P} \times \mathbf{Fl}_{r-1,r}$  is a closed immersion, with image *cut out by a regular section of the vector bundle*  $((V \otimes 0)/\mathfrak{U}_r)(H)$ *.* 

<span id="page-59-2"></span>**Lemma 8.13.** The morphism  $\beta_{s+1} = \pi_{s+1}^{\vee,-} \circ b_{s+1} : L_s^{\vee,+} \to L_{s+1}^{\vee}$  is the projectivization of the *vector bundle*  $(\mathcal{W}_{s+1}/\mathcal{O}(-H'))^{\vee}$ *.* 

8.4. Relations between the functors. We aim here to relate the various kernel functors introduced above. The main statement we are after is Proposition 8.18. We start with some preparations.

<span id="page-59-0"></span>Lemma 8.14. *The functors*

$$
\Phi_r: \mathbf{L}_s^{\vee}(\mathcal{B}) \to \mathbf{L}_r(\mathcal{A}), \qquad \Phi_{r-1}: \mathbf{L}_{s+1}^{\vee}(\mathcal{B}) \to \mathbf{L}_{r-1}(\mathcal{A}),
$$
  

$$
\Phi_r^-: \mathbf{L}_s^{\vee,+}(\mathcal{B}) \to \mathbf{L}_r^-(\mathcal{A}), \qquad \Phi_{r-1}^+: \mathbf{L}_{s+1}^{\vee,-}(\mathcal{B}) \to \mathbf{L}_{r-1}^+(\mathcal{A}),
$$

 $admit$  left adjoints  $\Phi_r^*, (\Phi_r^-)^*, \Phi_{r-1}^*, (\Phi_{r-1}^+)^*,$  and right adjoints  $\Phi_r^!, (\Phi_r^-)^!, \Phi_{r-1}^!, (\Phi_{r-1}^+)^!$ .

*Proof.* We show the claim for  $\Phi_r$ . The arguments for the other functors are similar and left to the reader. The proof consists of two steps: first we write down a functor which is easily seen to admit adjoints, and then we prove this functor is equivalent to  $\Phi_r$ .

Consider the universal space of linear sections  ${\bf L}_s^{\vee}({\bf H}) = {\bf H} \times_{\bf P} {\bf v} {\bf L}_s^{\vee}$  of  ${\bf H}$  with respect to the morphism  $\mathbf{H} \to \mathbf{P}^{\vee}$ . We define a space  $\mathbf{LL}_r(\mathbf{H})$  by the fiber product diagram



By base changing the  $P$ -linear category  $A$  along the top of this diagram, we obtain a diagram of  $\mathbf{G}_r$ -linear functors



We denote by

 $\mathbf{L}^\vee_s(\Phi_\mathcal{E})\colon \mathbf{L}^\vee_s(\mathcal{B})\to \mathbf{L}^\vee_s(\mathbf{H}(\mathcal{A}))$ 

the  $\mathbf{L}_s^{\vee}$ -linear functor induced by  $\Phi_{\mathcal{E}}: \mathcal{B} \to \mathbf{H}(\mathcal{A})$  on the linear section categories. Set

$$
\Psi_r = a_* \circ b^* \circ \mathbf{L}_s^{\vee}(\Phi_{\mathcal{E}}) \colon \mathbf{L}_s^{\vee}(\mathcal{B}) \to \mathbf{L}_r(\mathcal{A}).
$$

The functor  $\Psi_r$  is  $\mathbf{G}_r$ -linear, since it is a composition of such functors. The functors  $a_*$  and  $b^*$  in this composition admit left and right adjoints by Remark [2.2.](#page-9-0) Further, recall  $(8.2)$  that  $\Phi_{\mathcal{E}} = \gamma \circ \phi$  by definition. The functor  $\phi$  admits adjoints by Setup [8.1\(](#page-53-3)[2a\)](#page-53-2), and so does  $\gamma$ by Lemma [7.2.](#page-45-1) It follows that  $\Phi_{\mathcal{E}}$  and hence also  $\mathbf{L}_s^{\vee}(\Phi_{\mathcal{E}})$  admits adjoints. In conclusion,  $\Psi_r$ admits left and right adjoints, being a composition of functors with this property.

Now we prove that  $\Psi_r$  is equivalent to  $\Phi_r$ . Consider the commutative diagram



where  $\psi$  is the evident morphism. We have written the spaces in the above diagram with redundant fiber products because we want to think of them as morphisms between kernel spaces. Recall that by definition  $\mathcal{E}_r = \zeta_r^* \mathcal{E}$  is the kernel for the functor  $\Phi_r$ . It is easy to see that  $\psi^*\mathcal{E}$  is a kernel for the functor  $\mathbf{L}_s^{\vee}(\Phi_{\mathcal{E}})$ . Then by Lemma [5.17](#page-34-0) we find  $(b \times id)^*\psi^*\mathcal{E}$  is a kernel for the functor  $b^* \circ L_s^{\vee}(\Phi_{\varepsilon}) \colon L_s^{\vee}(\mathcal{B}) \to LL_r(\mathbf{H}(\mathcal{A}))$ , and then further by Lemma [5.14](#page-33-0) we find that  $(a \times id)_*(b \times id)^*\psi^*\mathcal{E}$  is a kernel for  $\Psi_r$ . Hence to prove  $\Psi_r \simeq \Phi_r$ , it suffices to show

$$
(a \times id)_* \circ (b \times id)^* \circ \psi^* \simeq \zeta_r^* \colon \mathrm{Perf}(\mathbf{H} \times_{\mathbf{P}^\vee} \mathbf{P}^\vee) \to \mathrm{Perf}(\mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee).
$$

By the above diagram, the left side can be rewritten as

 $(a \times id)_* \circ (b \times id)^* \circ \psi^* \simeq (a \times id)_* \circ (a \times id)^* \circ \zeta_r^*$ .

But  $a \times id$ :  $\mathbf{LL}_r(\mathbf{H}) \times_{\mathbf{L}_s^{\vee}} \mathbf{L}_s^{\vee} \to \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee}$  is a projective bundle, being a base change of the projective bundle  $\mathbf{H} \to \mathbf{P}^{\vee}$ , and hence  $(a \times id)_* \circ (a \times id)^* \simeq id$ . This finishes the proof that  $\Psi_r \simeq \Phi_r$ . Since we already showed that  $\Psi_r$  admits left and right adjoints, we are done.  $\square$ 

<span id="page-60-0"></span>Lemma 8.15. *There are equivalences of functors*

$$
\begin{aligned}\n\Phi_r^- \circ (\pi_s^{\vee,+})^* &\simeq (\pi_r^-)^* \circ \Phi_r, &\Phi_{r-1}^+ \circ (\pi_{s+1}^{\vee,-})^* &\simeq (\pi_{r-1}^+)^* \circ \Phi_{r-1}, \\
(\pi_r^-)_* \circ \Phi_r^- &\simeq \Phi_r \circ (\pi_s^{\vee,+})_*, &(\pi_{r-1}^+)_* \circ \Phi_{r-1}^+ &\simeq \Phi_{r-1} \circ (\pi_{s+1}^{\vee,-})_*, \\
(\pi_s^{\vee,+})_* \circ (\Phi_r^-)^* &\simeq (\Phi_r)^* \circ (\pi_r^-)_*, &(\pi_{s+1}^{\vee,-})_* \circ (\Phi_{r-1}^+)^* &\simeq \Phi_{r-1}^* \circ (\pi_{r-1}^+)_*, \\
(\pi_s^{\vee,+})_* \circ (\Phi_r^-)^* &\simeq (\Phi_r)^* \circ (\pi_r^-)_!, &(\pi_{s+1}^{\vee,-}) \circ (\Phi_{r-1}^+)^* &\simeq \Phi_{r-1}^* \circ (\pi_{r-1}^+)_!,\n\end{aligned}
$$

*where*  $(\pi_s^{\vee,+})$  *denotes the left adjoint of*  $(\pi_s^{\vee,+})^*$ *, and similarly for*  $(\pi_r^-)$ *,*  $(\pi_{s+1}^{\vee,-})$ *,*  $(\pi_{r-1}^+)$ *.* 

*Proof.* The first two rows follow from Lemma [5.15](#page-33-1) and Remark [5.16.](#page-34-1) By adjunction the third row is equivalent to the assertion

$$
\Phi_r^- \circ (\pi_s^{\vee,+})^! \simeq (\pi_r^-)^! \circ \Phi_r, \qquad \Phi_{r-1}^+ \circ (\pi_{s+1}^{\vee,-})^! \simeq (\pi_{r-1}^+)^! \circ \Phi_{r-1}.
$$

We have

$$
(\pi_r^{-})^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}) \circ (\pi_r^{-})^*,
$$
  
\n
$$
(\pi_s^{\vee,+})^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}) \circ (\pi_s^{\vee,+})^*,
$$
  
\n
$$
(\pi_{r-1}^+)^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}) \circ (\pi_{r-1}^+)^*,
$$
  
\n
$$
(\pi_{s+1}^{\vee,-})^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}) \circ (\pi_{s+1}^{\vee,-})^*.
$$

Hence by  $\mathbf{Fl}_{r-1,r}$ -linearity the assertion reduces to the first row. Similarly, we have

$$
(\pi_r^-)_{!} \simeq (\pi_r^-)_{*} \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}),
$$
  
\n
$$
(\pi_s^{\vee,+})_{!} \simeq (\pi_s^{\vee,+})_{*} \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}),
$$
  
\n
$$
(\pi_{r-1}^+)_{!} \simeq (\pi_{r-1}^+)_{*} \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}),
$$
  
\n
$$
(\pi_{s+1}^{\vee,-})_{!} \simeq (\pi_{s+1}^{\vee,-})_{*} \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}).
$$

Hence by  $\mathbf{Fl}_{r-1,r}$ -linearity the fourth row follows from the third.  $\Box$ 

We define

$$
\mathbf{M}_{r}^{-} = \mathbf{L}_{r}^{-} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s}^{\vee,+},
$$
  

$$
\mathbf{M}_{r-1}^{+} = \mathbf{L}_{r-1}^{+} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}.
$$

Then by definition  $\Phi_r^-$  is defined by an  $M_r^-$ -kernel and  $\Phi_{r-1}^+$  by an  $M_{r-1}^+$ -kernel. These spaces fit into a cartesian commutative diagram

$$
\mathbf{L}_{r-1}^{+} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s}^{\vee,+} \xrightarrow{\mathrm{id} \times b_{s+1}} \mathbf{M}_{r-1}^{+}
$$
\n
$$
a_{r} \times \mathrm{id} \downarrow \qquad a_{r} \times \mathrm{id} \downarrow \qquad a_{r} \times \mathrm{id} \qquad (8.9)
$$
\n
$$
\mathbf{M}_{r}^{-} \xrightarrow{\mathrm{id} \times b_{s+1}} \mathbf{L}_{r}^{-} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}
$$

By Lemma 8.10 all of the morphisms in this diagram are divisorial embeddings.

The projections  $p_r^-: \mathbf{L}_r^- \to \mathbf{P}^{\vee}$  and  $q_{s+1}^-: \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{P}^{\vee}$  induce a morphism

<span id="page-61-4"></span> $\nu_r = p_r^- \times q_{s+1}^- \colon \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{P} \times \mathbf{P}^{\vee}$  $(8.10)$ 

Let  $\mathbf{M}_r$  be defined by the fiber product diagram

<span id="page-61-3"></span>
$$
\widetilde{M}_r \xrightarrow{\iota_r} L_r^- \times_{\mathbf{Fl}_{r-1,r}} L_{s+1}^{\vee,-}
$$
\n
$$
\widetilde{\zeta}_r \downarrow \qquad \qquad \downarrow \qquad \downarrow
$$
\n
$$
\mathbf{H} \xrightarrow{\iota} \mathbf{P} \times \mathbf{P}^{\vee}
$$
\n(8.11)

The compositions

$$
\begin{aligned}\n\zeta_r^-: \mathbf{M}_r^- & \xrightarrow{\pi_r^- \times \pi_s^{\vee,+}} \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee} \xrightarrow{\zeta_r} \mathbf{H}, \\
\zeta_{r-1}^+: \mathbf{M}_{r-1}^+ & \xrightarrow{\pi_{r-1}^+ \times \pi_{s+1}^{\vee,-}} \mathbf{L}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{L}_{s+1}^{\vee} \xrightarrow{\zeta_{r-1}} \mathbf{H},\n\end{aligned}
$$

factor through closed embeddings

<span id="page-61-2"></span><span id="page-61-1"></span>
$$
\delta_r^-: \mathbf{M}_r^- \hookrightarrow \widetilde{\mathbf{M}}_r \tag{8.12}
$$

$$
\delta_{r-1}^+ \colon \mathbf{M}_{r-1}^+ \hookrightarrow \widetilde{\mathbf{M}}_r. \tag{8.13}
$$

<span id="page-61-0"></span>Lemma 8.16. *We have*

$$
\widetilde{\mathbf{M}}_r = \mathbf{M}_r^- \cup \mathbf{M}_{r-1}^+,
$$

*where the right side is the scheme-theoretic union inside*  $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$ .

*Proof.* Note that by definition  $\widetilde{\mathbf{M}}_r$  is cut out in  $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$  by a section of the line bundle  $O(H + H')$ . On the other hand, by Lemma 8.10 the union  $M_r^- \cup M_{r-1}^+$  is cut out in  $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$  by a section of the line bundle  $(\mathcal{U}_r/\mathcal{U}_{r-1})(H) \otimes (\mathcal{U}_r/\mathcal{U}_{r-1})^{\vee}(H') \cong \mathcal{O}(H+H').$ These two sections of  $O(H + H')$  coincide, cf. [\[9,](#page-71-0) Lemma 6.13].

Let  $\widetilde{\mathcal{E}}_r = \widetilde{\zeta}_r^* \mathcal{E}$  be the  $\widetilde{\mathbf{M}}_r$ -kernel obtained by pulling back the **H**-kernel  $\mathcal{E}$ . We denote by  $D_r$ the Cartier divisor on  $\mathbf{L}_{s+1}^{\vee,-}$  corresponding to the line bundle  $(\mathcal{U}_r/\mathcal{U}_{r-1})^\vee(H')$ . By Lemma 8.10 the scheme  $\mathbf{M}_r^-$  is cut out in  $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$  by a section of  $\mathcal{O}(D_r)$ .

<span id="page-62-0"></span>Lemma 8.17. *(1) There is an exact triangle*

$$
(a_r \times id)_* \mathcal{E}_{r-1}^+ \otimes \mathcal{O}(-D_r) \to \iota_{r*} \widetilde{\mathcal{E}}_r \to (\mathrm{id} \times b_{s+1})_* \mathcal{E}_r^-
$$

 $of \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \textit{kernels}.$ 

<span id="page-62-1"></span>*(2) There is an equivalence*

$$
(a_r \times id)^* \mathcal{E}_r^- \simeq (id \times b_{s+1})^* \mathcal{E}_{r-1}^+
$$

- $of \mathbf{L}_{r-1}^+ \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_s^{\vee,+} \text{-}kernels.$
- <span id="page-62-2"></span>*(3) There is an equivalence*

$$
\iota_{r*}\widetilde{\mathcal{E}}_r \simeq \nu_r^* \iota_* \mathcal{E}
$$

 $of \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \textit{-kernels}.$ 

*Proof.* It follows from Lemma [8.16](#page-61-0) that there is an exact sequence

$$
0\rightarrow (\delta_{r-1}^+)_*\mathfrak{O}_{\mathbf{M}_{r-1}^+}(-D_r)\rightarrow \mathfrak{O}_{\widetilde{\mathbf{M}}_r}\rightarrow (\delta_r^-)_*\mathfrak{O}_{\mathbf{M}_r^-}\rightarrow 0
$$

of sheaves on  $\widetilde{M}_r$ , where  $\delta_{r-1}^+$  and  $\delta_r^-$  are the embeddings [\(8.13\)](#page-61-1) and [\(8.12\)](#page-61-2). Now [\(1\)](#page-62-0) follows by kernel formalism. More precisely, tensoring the above exact sequence by  $\tilde{\mathcal{E}}_r$  gives an exact triangle of kernels by Lemma [5.9.](#page-32-0) But by definition  $(\delta^+_{r-1})^*(\tilde{\mathcal{E}}_r) \simeq \mathcal{E}^+_{r-1}$  and  $(\delta^-_r)^*(\tilde{\mathcal{E}}_r) \simeq \mathcal{E}^-_r$ , so by the projection formula for kernels (Lemma [5.11\)](#page-32-1), the resulting exact triangle can be written

$$
(\delta_{r-1}^+)_{*}\mathcal{E}_{r-1}^+\otimes\mathcal{O}(-D_r)\to\widetilde{\mathcal{E}}_r\to(\delta_r^-)_{*}\mathcal{E}_r^-
$$

Now the result follows by pushing forward via  $\iota_r \colon \widetilde{\mathbf{M}}_r \to \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$  and using the projection formula once more.

It follows from the definitions that both kernels appearing in [\(2\)](#page-62-1) are equivalent to the pullback of  $\widetilde{\mathcal{E}}_r$  along the natural map  $\mathbf{L}_r^+ \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,+} \to \widetilde{\mathbf{M}}_r$ , hence [\(2\)](#page-62-1) holds.

Finally, [\(3\)](#page-62-2) follows from the cartesian square  $(8.11)$  and base change.

<span id="page-62-3"></span>Proposition 8.18. *(1) There is an exact triangle*

$$
\Phi_r^-\circ b_{s+1}^! \to (a_r)_*\circ \Phi_{r-1}^+ \to \Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r\otimes \mathcal{O}(D_r)}
$$

 $\mathbf{L}_{s+1}^{\vee,-}(\mathcal{B}) \to \mathbf{L}_{r}^{-}(\mathcal{A}),$  where  $b_{s+1}^{!} : \mathbf{L}_{s+1}^{\vee,-}(\mathcal{B}) \to \mathbf{L}_{s}^{\vee,+}(\mathcal{B})$  is the functor induced by *the right adjoint to*  $(b_{s+1})_*$ : Perf $(\mathbf{L}_s^{\vee,+}) \to \mathrm{Perf}(\mathbf{L}_{s+1}^{\vee,-})$ 

<span id="page-62-4"></span>*(2) There is an exact triangle*

$$
(\Phi_{\iota_{r*}\widetilde{\mathcal E}_r\otimes\mathcal O(D_r)})^*\to (\Phi_{r-1}^+)^*\circ a_r^*\to (b_{s+1})_*\circ (\Phi_r^-)^*
$$

*of functors*  $\mathbf{L}_r^-(\mathcal{A}) \to \mathbf{L}_{s+1}^{\vee,-}(\mathcal{B})$ *.* 

<span id="page-62-5"></span>*(3) There is an equivalence*

 $a_r^* \circ \Phi_r^- \simeq \Phi_{r-1}^+ \circ (b_{s+1})_*$ 

*of functors*  $\mathbf{L}_{s}^{\vee,+}(\mathfrak{B}) \to \mathbf{L}_{r-1}^{+}(\mathfrak{A})$ .

*Proof.* It follows from Lemma 8.10[\(2\)](#page-58-0) that  $b_{s+1}^! = b_{s+1}^* \circ (- \otimes \mathcal{O}(D_r))[-1]$ . Now [\(1\)](#page-62-3) follows by rotating the triangle of kernels from Lemma 8.17[\(1\)](#page-62-0), twisting by  $\mathcal{O}(D_r)$ , passing to the associated kernel functors, and using Lemmas [5.12](#page-33-2) and [5.14.](#page-33-0) By Lemmas [2.13](#page-12-1) and [8.14,](#page-59-0) [\(2\)](#page-62-4) follows from [\(1\)](#page-62-3) by passing to left adjoints. Finally, [\(3\)](#page-62-5) follows from Lemma 8.17[\(2\)](#page-62-1) by passing to the associated kernel functors and using Lemma [5.15.](#page-33-1)

8.5. Semiorthogonal sequences in ker  $\Phi_r^*$  and ker  $\Phi_r$ . Our next goal is to show that the semiorthogonal sequence to the right of  $\mathcal{K}_r(\mathcal{A})$  in the decomposition [\(6.15\)](#page-43-4) is contained in ker  $\Phi_r^*$ , and the semiorthogonal sequence to the left of  $\mathcal{K}'_s(\mathcal{B})$  in  $(8.1)$  is contained ker  $\Phi_r$ . Later we will combine this result with the fact that the  $\Phi_r$  are left splitting (Proposition [8.27\)](#page-67-0) to reduce Theorem [8.4](#page-55-1) to a statement about the generation of  $\mathbf{L}_r(\mathcal{A})$  and  $\mathbf{L}_s^{\vee}(\mathcal{B})$  by certain semiorthogonal sequences.

<span id="page-63-3"></span>**Lemma 8.19.** *For*  $0 \le i \le s - 1$  *there are kernels* 

$$
\mathcal{F}_{r,i},\,\mathcal{K}_{r,i}\in \mathrm{FM}(\mathcal{B}/\mathbf{P}^{\vee},\mathcal{A}/\mathbf{P},\mathbf{P}\times_{S}\mathbf{L}^{\vee}_{s})
$$

*such that:*

 $(1) \Phi_{\mathcal{F}_{r,0}} \simeq p_{r*} \circ \Phi_r.$ 

*(2) For all* i *there is an exact triangle*

$$
\mathcal{F}_{r,i+1} \to \mathcal{K}_{r,i} \to \mathcal{F}_{r,i}
$$

*where for*  $i = s - 1$  *we set*  $\mathcal{F}_{r,s} = 0$ *.* 

<span id="page-63-1"></span>*(3) There is an equivalence*

$$
\Phi_{\mathcal{K}_{r,i}} \simeq (-\otimes 0(-iH)) \circ p_* \circ \Phi_{\mathcal{E}} \circ q_{s*} \circ (-\otimes \wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))).
$$

*Proof.* By definition the functor  $\Phi_r$  is given by the kernel

$$
\mathcal{E}_r = \zeta_r^* \mathcal{E} \in \mathrm{FM}(\mathcal{B}/\mathbf{P}^{\vee}, \mathcal{A}/\mathbf{P}, \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee}).
$$

Hence by Lemma [5.14](#page-33-0) the functor  $p_{r*} \circ \Phi_r$  is given by the pushforward of this kernel along

$$
p_r \times \mathrm{id} \colon \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee \to \mathbf{P} \times \mathbf{L}_s^\vee,
$$

i.e. by the kernel  $(p_r \times id)_* \mathcal{E}_r$ . To understand this kernel, we consider the commutative diagram

<span id="page-63-0"></span>
$$
\mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^{\vee}
$$
\n
$$
\langle \zeta_r, \text{pr}_2 \rangle \downarrow \qquad p_r \times \text{id}
$$
\n
$$
\mathbf{H} \times_{\mathbf{P}^{\vee}} \mathbf{L}_s^{\vee} \xrightarrow{p \times \text{id}} \mathbf{P} \times \mathbf{L}_s^{\vee}
$$
\n
$$
\text{pr}_1 \downarrow \qquad \qquad \downarrow \text{id} \times q_s
$$
\n
$$
\mathbf{H} \xrightarrow{\iota} \mathbf{P} \times \mathbf{P}^{\vee}
$$
\n
$$
(8.14)
$$

where the square is cartesian. We have

$$
(p_r \times id)_* \mathcal{E}_r \simeq (p \times id)_*(\zeta_r, \mathrm{pr}_2)_*(\zeta_r, \mathrm{pr}_2)^* \mathrm{pr}_1^* \mathcal{E}
$$
  

$$
\simeq (p \times id)_*(\mathrm{pr}_1^* \mathcal{E} \otimes (\zeta_r, \mathrm{pr}_2)_* 0).
$$

It follows from Lemma 8.11[\(1\)](#page-58-1) that there is a resolution of  $(\zeta_r, pr_2)_*$  of the form

<span id="page-63-2"></span>
$$
0 \to \wedge^{s-1}(\mathcal{W}_s/\mathcal{O}(-H'))(-(s-1)H) \to \cdots \to (\mathcal{W}_s/\mathcal{O}(-H'))(-H) \to \mathcal{O} \to (\zeta_r, \text{pr}_2)_*\mathcal{O} \to 0. \tag{8.15}
$$

Let

$$
\mathcal{K}_{r,i} = (p \times id)_*(\text{pr}_1^* \mathcal{E} \otimes \wedge^i (\mathcal{W}_s/\mathcal{O}(-H'))(-iH))
$$

denote the kernel corresponding to the  $i$ -th term in this resolution. Then

$$
\mathcal{K}_{r,i} \simeq ((p \times id)_* \text{pr}_1^*\mathcal{E}) \otimes \wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))(-iH)
$$
  
\simeq ((id \times q\_s)^\* \iota\_\*\mathcal{E}) \otimes \wedge^i(\mathcal{W}\_s/\mathcal{O}(-H'))(-iH)

where the first line holds since  $\wedge^{i}(\mathcal{W}_s/\mathcal{O}(-H'))$  is pulled back from  $\mathbf{L}_s^{\vee}$  and  $\mathcal{O}(-iH)$  from  $\mathbf{P},$ and the second since the square in [\(8.14\)](#page-63-0) is cartesian. The kernel formalism then shows that  $\mathcal{K}_i$  satisfies part [\(3\)](#page-63-1) of the lemma, and the rest follows by splitting the resolution [\(8.15\)](#page-63-2) into short exact sequences.

<span id="page-64-1"></span>Proposition 8.20. *There is an inclusion*

$$
\langle A_s(H) \otimes \mathrm{Perf}(\mathbf{G}_r), \ldots, A_{m-1}((m-s)H) \otimes \mathrm{Perf}(\mathbf{G}_r) \rangle \subset \ker \Phi_r^*.
$$

*Proof.* By adjunction, the assertion is equivalent to

$$
\operatorname{im} \Phi_r \subset \langle A_s(H) \otimes \operatorname{Perf}(\mathbf{G}_r), \ldots, A_{m-1}((m-s)H) \otimes \operatorname{Perf}(\mathbf{G}_r) \rangle^{\perp}
$$

where the orthogonal is taken inside  $\mathbf{L}_r(\mathcal{A})$ . By  $\mathbf{G}_r$ -linearity of  $\Phi_r$ , this is equivalent to

$$
\operatorname{im}(p_{r*} \circ \Phi_r) \subset \langle A_s(H), \ldots, A_{m-1}((m-s)H) \rangle^{\perp}
$$

where the orthogonal is taken inside A. The functor  $\Phi_{\mathcal{K}_r,i}$  from Lemma [8.19](#page-63-3) satisfies

$$
\operatorname{im} \Phi_{\mathcal{K}_{r,i}} \subset \mathcal{A}_0(-iH). \tag{8.16}
$$

,

Indeed, using the description of this functor from Lemma [8.19](#page-63-3)[\(3\)](#page-63-1), the claim follows from Lemma [7.11.](#page-51-4) Hence we find

$$
\mathrm{im}(p_{r*} \circ \Phi_r) \subset \langle A_0(-(s-1)H), A_0(-(s-2)) \dots, A_0 \rangle,
$$

so we are done by Lemma [8.21](#page-64-0) below.

<span id="page-64-0"></span>**Lemma 8.21.** For any  $i \geq 0$  the following subcategories of A coincide:

$$
\langle A_0, A_0(H), \dots, A_0((i-1)H) \rangle ,
$$
  

$$
\langle A_0, A_1(H), \dots, A_{i-1}((i-1)H) \rangle ,
$$
  

$$
\langle A_i(iH), A_{i+1}((i+1)H), \dots, A_{m-1}((m-1)H) \rangle^{\perp} .
$$

*Proof.* The semiorthogonal decomposition

$$
\mathcal{A} = \langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle
$$

implies the last two categories coincide, and the first category contains the second and is contained in the third.

<span id="page-64-3"></span>Proposition 8.22. *There is an inclusion*

$$
\big\langle \mathfrak{B}_{1-n}((r-n)H')\otimes \operatorname{Perf}(\mathbf{G}_s^{\vee}),\ldots,\mathfrak{B}_{-r}(-H')\otimes \operatorname{Perf}(\mathbf{G}_s^{\vee})\big\rangle\subset \ker \Phi_r.
$$

*Proof.* Analogous to Proposition [8.20](#page-64-1) and left to the reader.

8.6. Technical results about the functor  $\Phi_{\iota_{r*}\widetilde{\mathcal E}_r\otimes\mathcal O(D_r)}.$  Here we prove two related results about the (left adjoint of the) functor  $\Phi_{\iota_{r*}\tilde{\mathcal{E}}_r\otimes\mathcal{O}(D_r)}$  appearing in Proposition 8.18, which will be needed later in our induction arguments. Namely, we describe the image of a certain subcategory under  $(\Phi_{\iota_{r*}\tilde{\mathcal{E}}_r\otimes\mathcal{O}(D_r)})^*$  (Proposition [8.23\)](#page-64-2) and show that certain composite functors involving  $(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_{r}\otimes\mathcal{O}(D_r)})^*$  vanish (Proposition [8.26\)](#page-66-0).

<span id="page-64-2"></span>Proposition 8.23. *The image*

$$
(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r\otimes\mathcal{O}(D_r)})^*\left( (\langle \mathcal{A}_s(H),\ldots,\mathcal{A}_{m-1}((m-s)H)\rangle\otimes\mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^\perp\right)
$$

*lies inside the subcategory*

$$
\mathcal{B}_{1-r}(-H') \otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}) \subset \mathbf{L}_{s+1}^{\vee,-}(\mathcal{B}).
$$

*Proof.* By Lemma [5.12](#page-33-2) we have

$$
(\Phi_{\iota_{r*}\widetilde{\mathcal E}_r\otimes\mathcal O(D_r)})^*\simeq (-\otimes \mathcal O(-D_r))\circ (\Phi_{\iota_{r*}\widetilde{\mathcal E}_r})^*.
$$

Since by definition  $\mathcal{O}(D_r) = (\mathcal{U}_r / \mathcal{U}_{r-1})^{\vee}(H')$ , it therefore suffices to show

$$
(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_{r}})^*\left((\langle \mathcal{A}_s(H),\ldots,\mathcal{A}_{m-1}((m-s)H)\rangle\otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^{\perp}\right) \subset \mathcal{B}_{1-r}\otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}).
$$

For this, we consider the diagram

$$
\begin{array}{ccc}\n&\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} &\longrightarrow \mathbf{L}_{s+1}^{\vee,-} \\
&\mathrm{id} \times j &j &j \\
&\mathbf{L}_r^- \longleftarrow \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} (\mathbf{Fl}_{r-1,r} \times \mathbf{P}^{\vee}) \longrightarrow \mathbf{Fl}_{r-1,r} \times \mathbf{P}^{\vee} \\
&p_r^- &p_r^- \times \mathrm{pr}_2 \\
&\mathbf{P} \longleftarrow &\mathbf{P} \times \mathbf{P}^{\vee}\n\end{array}
$$

where  $j: \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{Fl}_{r-1,r} \times \mathbf{P}^{\vee}$  is the embedding and the squares are cartesian. By definition the morphism  $\nu_r : \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{P} \times \mathbf{P}^{\vee}$  defined by [\(8.10\)](#page-61-4) satisfies  $\nu_r = (p_r^- \times pr_2) \circ (\mathrm{id} \times j)$ . Hence by Lemma 8.17[\(3\)](#page-62-2) there is an equivalence

$$
\iota_{r*}\widetilde{\mathcal{E}}_r \simeq \nu_r^* \iota_* \mathcal{E} \simeq (\mathrm{id} \times j)^* (p_r^- \times \mathrm{pr}_2)^* \iota_* \mathcal{E}.
$$

It follows from Lemma [5.15](#page-33-1) that

$$
\Phi_{\iota_r\ast\widetilde{\mathcal E}_r}\simeq\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal E}\circ j_*
$$

and hence

$$
(\Phi_{\iota_{r*}\widetilde{\mathcal E}_{r}})^*\simeq j^*\circ(\Phi_{(p^-_r\times\mathrm{pr}_2)^*\iota_*\mathcal E})^*
$$

To prove the lemma, it thus suffices to show the image

$$
(\Phi_{(p_r^-\times \text{pr}_2)^*\iota_*\mathcal{E}})^*\left((\langle \mathcal{A}_s(H),\ldots,\mathcal{A}_{m-1}((m-s)H)\rangle \otimes \text{Perf}(\mathbf{Fl}_{r-1,r}))^\perp\right)
$$

is contained in

$$
\mathcal{B}_{1-r} \otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}) \subset \mathcal{B} \otimes_{\mathrm{Perf}(\mathbf{P}^{\vee})} \mathrm{Perf}(\mathbf{Fl}_{r-1,r} \times \mathbf{P}^{\vee}).
$$

Since the image in question is  $\mathbf{Fl}_{r-1,r}$ -linear, it suffices to show

<span id="page-65-3"></span>
$$
\mathrm{pr}_{2*}(\Phi_{(p_r^{-} \times \mathrm{pr}_2)^* \iota_* \mathcal{E}})^* \left( (\langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^{\perp} \right) \subset \mathcal{B}_{1-r} \quad (8.17)
$$

as subcategories of B. By the above diagram and Lemma [5.15,](#page-33-1) there is an equivalence

<span id="page-65-2"></span><span id="page-65-1"></span>
$$
\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}}\circ\mathrm{pr}_2^*\simeq (p_r^-)^*\circ\Phi_{\iota_*\mathcal{E}}.
$$

Hence, if  $(pr_2)$  and  $(p_r^-)$  denote the left adjoints of the pullback functors of the morphisms  $\text{pr}_2\colon \mathbf{Fl}_{r-1,r}\times \mathbf{P}^{\vee}\to \mathbf{P}^{\vee}$  and  $p_r^-:\mathbf{L}_r^-\to \mathbf{P}$ , then we have

<span id="page-65-0"></span>
$$
(\text{pr}_2)_{!} \circ (\Phi_{(p_r^- \times \text{pr}_2)^* \iota_* \mathcal{E}})^* \simeq (\Phi_{\iota_* \mathcal{E}})^* \circ (p_r^-)_{!}.\tag{8.18}
$$

It is easy to see

$$
(\text{pr}_2)_{!} \simeq \text{pr}_{2*} \circ (-\otimes \omega_{\mathbf{Fl}_{r-1,r}/S}) \tag{8.19}
$$

$$
(p_r^-)_{!} \simeq (p_r^-)_{*} \circ (-\otimes \mathcal{O}(sH) \otimes \det(\mathcal{U}_r)^{-1} \otimes \omega_{\mathbf{Fl}_{r-1,r}/S}[-s])
$$
(8.20)

where we have used Lemma [8.12](#page-59-1) in deriving the second equivalence. Combining [\(8.18\)](#page-65-0), [\(8.19\)](#page-65-1), [\(8.20\)](#page-65-2), and using that  $\omega_{\mathbf{Fl}_{r-1,r}/S}$  and  $\det(\mathfrak{U}_r)^{-1}$  are pulled back from  $\mathbf{Fl}_{r-1,r}$ , the claim [\(8.17\)](#page-65-3) reduces to showing

$$
(\Phi_{\iota_*\mathcal{E}})^* \left( \langle A_s((s+1)H),\ldots,A_{m-1}(mH) \rangle^{\perp} \right) \subset \mathcal{B}_{1-r}.
$$

But  $(\Phi_{\iota_*\mathcal{E}})^* \simeq \phi^* \circ \gamma^* \circ p^*$  and

$$
\langle A_s((s+1)H),\ldots,A_{m-1}(mH)\rangle^{\perp}=\langle A_0(H),\ldots,A_{s-1}(sH)\rangle,
$$

so the desired inclusion follows from Lemma [8.24](#page-66-1) below and the definition of  $\mathcal{B}_{1-r}$ .  $\Box$ 

<span id="page-66-1"></span>**Lemma 8.24.** For any  $i \geq 0$  the functor  $\gamma^* \circ p^* : A \to A^{\dagger}$  satisfies

$$
\gamma^* p^* \left( \langle A_0(H), \ldots, A_{i-1}(i) \rangle \right) \subset A_{i+1-N}^{\natural}.
$$

*Proof.* By Lemma [7.11](#page-51-4) the functor  $\gamma^* \circ p^*$  kills  $^{\perp}A_0$ . Hence  $\gamma^* p^* (C) \simeq \gamma^* p^* \alpha_0^* (C)$  for any  $C \in \mathcal{A}$ , where recall  $\alpha_0: \mathcal{A}_0 \to \mathcal{A}$  is the inclusion. But it follows immediately from the definitions that

$$
\alpha_0^* \left( \langle A_0(H), \ldots, A_{i-1}(iH) \rangle \right) = \left\langle \mathfrak{a}'_0, \ldots, \mathfrak{a}'_{i-1} \right\rangle,
$$

and then the result follows from the definition of  $\mathcal{A}_i^{\natural}$  $i+1-N$ 

We will need the following observation.

<span id="page-66-2"></span>Lemma 8.25. *There are inclusions*

$$
\operatorname{im} \Phi_r^- \subset (\langle A_s(H), \dots, A_{m-1}((m-s)H) \rangle \otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}))^\perp
$$
  

$$
\langle \mathfrak{B}_{1-n}((r-n-1)H'), \dots, \mathfrak{B}_{1-r}(-H') \rangle \otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}) \subset \ker \Phi_{r-1}^+.
$$

*Proof.* By Lemma [8.15](#page-60-0) we have

$$
(\pi_r^-)_* \circ \Phi_r^- \simeq \Phi_r \circ (\pi_s^{\vee,+})_*,
$$
  

$$
\Phi_{r-1}^+ \circ (\pi_{s+1}^{\vee,-})^* \simeq (\pi_{r-1}^+)^* \circ \Phi_{r-1},
$$

from which the result follows using  $\mathbf{Fl}_{r-1,r}$ -linearity and Propositions [8.20](#page-64-1) and [8.22.](#page-64-3) □

<span id="page-66-0"></span>Proposition 8.26. *We have*

<span id="page-66-3"></span>
$$
\Phi_{r-1}^+ \circ (\Phi_{\iota_{r*}\tilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \circ \Phi_r^- \simeq 0. \tag{8.21}
$$

*Moreover, we have the stronger vanishings*

$$
(\Phi_{\iota_{r*}\tilde{\mathcal{E}}_{r}\otimes\mathcal{O}(D_r)})^* \circ \Phi_r^- \simeq 0 \quad \text{if } r-1 \ge n,
$$
\n
$$
(8.22)
$$

$$
\Phi_{r-1}^+ \circ (\Phi_{\iota_{r*}\tilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \simeq 0 \quad \text{if } r \le N-m. \tag{8.23}
$$

*Proof.* Follows from Lemmas [8.25](#page-66-2) and [8.23](#page-64-2) and the observations that  $B_{1-r} = 0$  if  $r - 1 \geq n$ and

 $(\langle \mathcal{A}_s(H), \ldots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^{\perp} = \mathbf{L}_r^{-}(\mathcal{A})$ 

if  $s \geq m$  or equivalently  $r \leq N - m$ . □

.

<span id="page-66-4"></span>

# 8.7. The functors  $\Phi_r$  are left splitting.

<span id="page-67-0"></span>**Proposition 8.27.** The functor  $\Phi_r : \mathbf{L}_s^{\vee}(\mathcal{B}) \to \mathbf{L}_r(\mathcal{A})$  is left splitting for any r, and hence

$$
\begin{aligned} \mathbf{L}_r(\mathcal{A})&=\left\langle\operatorname{im}\Phi_r,\operatorname{ker}\Phi_r^*\right\rangle,\\ \mathbf{L}_s^\vee(\mathcal{B})&=\left\langle\operatorname{ker}\Phi_r,\operatorname{im}\Phi_r^*\right\rangle. \end{aligned}
$$

*Proof.* We argue by descending induction on r. For  $r = N$  the result is trivial as  $\mathbf{L}_0(\mathcal{B}) = 0$ , and for  $r = N - 1$  the result holds since by assumption the functor  $\Phi_{N-1}$  is fully faithful with admissible image. Now assume the result holds for  $\Phi_r$ . We will prove that  $\Phi_{r-1}$  satisfies criterion [\(2l\)](#page-20-0) of Theorem [3.22.](#page-19-1)

Note that it follows from Lemma [8.13](#page-59-2) that the functor

<span id="page-67-1"></span>
$$
\beta_{s+1}^* \colon \mathbf{L}_{s+1}^{\vee}(\mathfrak{B}) \to \mathbf{L}_{s}^{\vee,+}(\mathfrak{B})
$$

is fully faithful, and hence

$$
(\pi_{s+1}^{\vee,-})_*(b_{s+1})_*\beta_{s+1}^*\simeq(\beta_{s+1})_*\beta_{s+1}^*\simeq\mathrm{id}.
$$

Thus applying Lemma [8.15](#page-60-0) we find

$$
\Phi_{r-1}\Phi_{r-1}^* \Phi_{r-1} \simeq \Phi_{r-1}\Phi_{r-1}^* \Phi_{r-1}(\pi_{s+1}^{\vee,-})_*(b_{s+1})_* \beta_{s+1}^*
$$
\n
$$
\simeq (\pi_{r-1}^+)_* \Phi_{r-1}^+(\Phi_{r-1}^+)^* \Phi_{r-1}^+(b_{s+1})_* \beta_{s+1}^*.
$$
\n(8.24)

Now we examine the inner term  $\Phi^+_{r-1}(\Phi^+_{r-1})^* \Phi^+_{r-1}(b_{s+1})^*$ . First from Proposition 8.18[\(3\)](#page-62-5) we deduce

$$
\Phi^+_{r-1}(\Phi^+_{r-1})^* \Phi^+_{r-1}(b_{s+1})_* \simeq \Phi^+_{r-1}(\Phi^+_{r-1})^* a_r^* \Phi^-_r.
$$

By [\(8.21\)](#page-66-3) if we compose the the triangle of Proposition 8.18[\(2\)](#page-62-4) on the left with  $\Phi_{r-1}^+$  and on the right with  $\Phi_r^-$ , then the first term vanishes and we obtain an equivalence

$$
\Phi^+_{r-1}(\Phi^+_{r-1})^*a_r^*\Phi^-_r \simeq \Phi^+_{r-1}(b_{s+1})_*(\Phi^-_r)^*\Phi^-_r.
$$

Applying Proposition 8.18[\(3\)](#page-62-5) again, we find

$$
\Phi_{r-1}^+(b_{s+1})_*(\Phi_r^-)^*\Phi_r^- \simeq a_r^*\Phi_r^-(\Phi_r^-)^*\Phi_r^-
$$

Since by assumption  $\Phi_r$  is left splitting, by Proposition [5.19](#page-35-0) so is  $\Phi_r^-$ , and hence

$$
a_r^*\Phi_r^-(\Phi_r^-)^*\Phi_r^-\simeq a_r^*\Phi_r^-.
$$

Combining the above and applying Proposition 8.18[\(3\)](#page-62-5) one more time, all together we have shown

$$
\Phi^+_{r-1}(\Phi^+_{r-1})^* \Phi^+_{r-1}(b_{s+1})_* \simeq \Phi^+_{r-1}(b_{s+1})_*.
$$

Returning to [\(8.24\)](#page-67-1), we have thus shown

$$
\Phi_{r-1}\Phi_{r-1}^*\Phi_{r-1} \simeq (\pi_{r-1}^+)_{*}\Phi_{r-1}^+(b_{s+1})_{*}\beta_{s+1}^*.
$$

So using Lemma [8.15](#page-60-0) we conclude

$$
\Phi_{r-1}\Phi_{r-1}^*\Phi_{r-1} \simeq \Phi_{r-1}(\pi_{s+1}^{\vee,-})_*(b_{s+1})_*\beta_{s+1}^* \simeq \Phi_{r-1}.
$$

8.8. Proof of Theorem [8.4.](#page-55-1) By combining Proposition [8.27,](#page-67-0) Theorem [3.22,](#page-19-1) and Propositions [8.20](#page-64-1) and [8.22,](#page-64-3) we obtain semiorthogonal sequences

$$
\mathbf{L}_r(\mathcal{A}) \supset \langle \text{im } \Phi_r, \mathcal{A}_s(H) \otimes \text{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes \text{Perf}(\mathbf{G}_r) \rangle ,
$$
  

$$
\mathbf{L}_s^{\vee}(\mathcal{B}) \supset \langle \mathcal{B}_{1-n}((r-n)H') \otimes \text{Perf}(\mathbf{G}_s^{\vee}), \dots, \mathcal{B}_{-r}(-H') \otimes \text{Perf}(\mathbf{G}_s^{\vee}), \text{im } \Phi_r^{\ast} \rangle ,
$$

such that the functors  $\Phi_r$  and  $\Phi_r^*$  induce mutually inverse equivalences im  $\Phi_r^* \simeq \text{im } \Phi_r$ . Hence to prove Theorem [8.4,](#page-55-1) it remains to prove that the above semiorthogonal sequences generate  $\mathbf{L}_r(\mathcal{A})$  and  $\mathbf{L}_s^{\vee}(\mathcal{B})$ . We do this in Propositions [8.28](#page-68-0) and [8.31](#page-70-0) via an inductive argument. We start with  $\mathbf{L}_{s}^{\vee}(\mathcal{B})$ , where the base case of the induction holds by our assumptions on  $\mathcal{B}$ .

<span id="page-68-0"></span>Proposition 8.28. *We have*

$$
\mathbf{L}_s^{\vee}(\mathcal{B}) = \left\langle \mathcal{B}_{1-n}((r-n)H') \otimes \mathrm{Perf}(\mathbf{G}_s^{\vee}), \ldots, \mathcal{B}_{-r}(-H') \otimes \mathrm{Perf}(\mathbf{G}_s^{\vee}), \mathrm{im} \, \Phi_r^{\ast} \right\rangle.
$$

*Proof.* If  $r \geq n$  the right side is simply im  $\Phi_r^*$ . Since  $\Phi_r$  is left splitting by Proposition [8.27,](#page-67-0) the result amounts to the claim that ker  $\Phi_r = 0$ , i.e. that  $\Phi_r$  is fully faithful. This can be proved by descending induction in r. The base case  $r = N - 1$  holds by Setup [8.1](#page-53-3)[\(2a\)](#page-53-2). We leave it to the reader to check that the same induction argument from Proposition [8.27,](#page-67-0) but using [\(8.22\)](#page-66-4) in place of [\(8.21\)](#page-66-3) and Proposition [5.18](#page-34-2) in place of Proposition [5.19,](#page-35-0) shows that  $\Phi_r^* \Phi_r \simeq$  id, i.e.  $\Phi_r$  is fully faithful, for all  $r \geq n$ .

For the case  $r \leq n$ , we again use descending induction. Assume the result holds for some  $r \leq n$ . To prove the result for  $r - 1$ , suppose  $D \in L_{s+1}^{\vee}(\mathcal{B})$  lies in

<span id="page-68-2"></span>
$$
\left\langle \mathcal{B}_{1-n}((r-n-1)H')\otimes \mathrm{Perf}(\mathbf{G}_s^{\vee}),\ldots,\mathcal{B}_{1-r}(-H')\otimes \mathrm{Perf}(\mathbf{G}_s^{\vee}),\mathrm{im}\,\Phi_{r-1}^*\right\rangle^{\perp}.
$$
 (8.25)

We must show  $D \simeq 0$ . For this, it is enough to show  $b_{s+1}^{!}(\pi_{s+1}^{\vee,-})^*D \simeq 0$ . Indeed, the functor  $b_{s+1}^{\dagger} \circ (\pi_{s+1}^{\vee,-})^*$  differs from  $\beta_{s+1}^* = b_{s+1}^* \circ (\pi_{s+1}^{\vee,-})^*$  by the twist by a line bundle by Lemma 8.10[\(2\)](#page-58-0), and the functor  $\beta_{s+1}^*$  is fully faithful by Lemma [8.13.](#page-59-2) By base change the induction hypothesis implies

$$
\mathbf{L}_{s}^{\vee,-}(\mathcal{B}) = \langle \mathcal{B}_{1-n}((r-n)H') \otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}), \dots
$$

$$
\dots, \mathcal{B}_{-r}(-H') \otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}), \operatorname{im} (\Phi_{r}^{-})^{\ast} \rangle. \tag{8.26}
$$

Hence it suffices to prove  $b_{s+1}^{\dagger}(\pi_{s+1}^{\vee,-})^*D$  lies in the right orthogonal to the right side.

First we claim

$$
(\pi_{s+1}^{\vee,-})^* D \in \langle \mathcal{B}_{1-n}((r-n-1)H') \otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}), \dots
$$
  
...,  $\mathcal{B}_{1-r}(-H') \otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}), \operatorname{im} (\Phi_{r-1}^+)^* \rangle^{\perp}.$  (8.27)

If  $(\pi_{s+1}^{\vee,-})$  denotes the left adjoint of  $(\pi_{s+1}^{\vee,-})^*$ , then by Lemma [8.15](#page-60-0) we have

<span id="page-68-3"></span><span id="page-68-1"></span>
$$
(\pi_{s+1}^{\vee,-})_! \circ (\Phi_{r-1}^+)^\ast \simeq \Phi_{r-1}^\ast \circ (\pi_{r-1}^+)_!.
$$

Now the claim  $(8.27)$  follows by adjunction and our assumption that D lies in  $(8.25)$ .

It follows from [\(8.27\)](#page-68-1) and Proposition [8.23](#page-64-2) that  $(\pi_{s+1}^{\vee,-})^*D$  lies in the right orthogonal to the images of the subcategory

$$
(\langle \mathcal{A}_s(H),\ldots,\mathcal{A}_{m-1}((m-s)H)\rangle \otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^{\perp}
$$

under the first two terms of the triangle from Proposition 8.18[\(2\)](#page-62-4), and hence  $(\pi_{s+1}^{\vee,-})^*D$  lies in the right orthogonal to

$$
(b_{s+1})_*(\Phi_r^-)^* \left( (\langle \mathcal{A}_s(H), \ldots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^{\perp} \right).
$$

It then follows from Lemma [8.25](#page-66-2) that in fact  $(\pi_{s+1}^{\vee,-})^*D$  lies in the right orthogonal to the entire image of the functor  $(b_{s+1})_*(\Phi^-_r)^*$ . Hence

<span id="page-69-1"></span>
$$
b_{s+1}^!(\pi_{s+1}^{\vee,-})^*D \in (\text{im } (\Phi_r^-)^*)^{\perp}.
$$
\n(8.28)

Next note that by Lemma 8.10[\(2\)](#page-58-0) there is an exact sequence of sheaves on  $\mathbf{L}_{s+1}^{\vee,-}$ 

$$
0 \to (\mathfrak{U}_r/\mathfrak{U}_{r-1})(-H') \to \mathfrak{O}_{\mathbf{L}_{s+1}^{\vee,-}} \to (b_{s+1})_*\mathfrak{O}_{\mathbf{L}_s^{\vee,+}} \to 0.
$$

This implies that the image of the functor  $(b_{s+1})_* \colon L_s^{\vee,+}(\mathcal{B}) \to L_{s+1}^{\vee,-}(\mathcal{B})$  applied to

$$
\langle \mathcal{B}_{1-n}((r-n)H') \otimes \text{Perf}(\mathbf{Fl}_{r-1,r}), \ldots, \mathcal{B}_{-r}(-H') \otimes \text{Perf}(\mathbf{Fl}_{r-1,r}) \rangle
$$

is contained in

$$
\left\langle \mathcal{B}_{1-n}((r-n-1)H')\otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r}),\ldots,\mathcal{B}_{1-r}(-H')\otimes \mathrm{Perf}(\mathbf{Fl}_{r-1,r})\right\rangle.
$$

By [\(8.27\)](#page-68-1) it thus follows that  $b_{s+1}^{\dagger}(\pi_{s+1}^{\vee,-})^*D$  lies in

$$
\left\langle \mathcal{B}_{1-n}((r-n)H')\otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r}),\ldots \ldots \mathcal{B}_{-r}(-H')\otimes \operatorname{Perf}(\mathbf{Fl}_{r-1,r})\right\rangle^{\perp}.
$$

Together with [\(8.28\)](#page-69-1), this shows  $b_{s+1}^{\dagger}(\pi_{s+1}^{\vee,-})^*D$  lies in the right orthogonal to [\(8.26\)](#page-68-3), as required.  $\Box$ 

Up to now, we have not used the condition [\(2c\)](#page-54-1) of Setup [8.1.](#page-53-3) We pause here to prove the claim of Remark [8.3](#page-54-2) that this condition can be replaced with [\(2c](#page-54-2)′ ).

<span id="page-69-0"></span>**Lemma 8.29.** *The conditions* [\(1\)](#page-53-4), [\(2a\)](#page-53-2), [\(2b\)](#page-54-0) of Setup [8.1](#page-53-3) together with the condition  $(2c')$ *of Remark [8.3](#page-54-2) imply* [\(2c\)](#page-54-1)*.*

Proof. Assume [\(1\)](#page-53-4), [\(2a\)](#page-53-2), [\(2b\)](#page-54-0), and [\(2c](#page-54-2)') hold. We show the claim of Corollary [8.5](#page-55-5) holds. Then by Lemma [7.11,](#page-51-4) the condition [\(2c\)](#page-54-1) follows.

We may freely use everything proved so far, since we only invoked [\(1\)](#page-53-4), [\(2a\)](#page-53-2), and [\(2b\)](#page-54-0) up to now. First note that  $\mathbf{G}_N^{\vee} = S$ ,  $\mathbf{L}_N^{\vee}(\mathcal{B}) = \mathcal{B}$ , and  $\mathbf{L}_0(\mathcal{A}) = 0$ . Hence the claim of Corollary [8.6](#page-55-2) holds by Proposition [8.28.](#page-68-0)

To prove the claim of Corollary [8.5,](#page-55-5) note that by [\(2a\)](#page-53-2) the functor  $\phi: \mathcal{B} \to \mathcal{A}^{\dagger}$  is fully faithful and admits a left adjoint  $\phi^*$ . Thus there is a semiorthogonal decomposition  $\mathcal{A}^\sharp = \langle \text{im } \phi, \ker \phi^* \rangle$ , so it suffices to show ker  $\phi^* = 0$ . By the above, we may apply Corollary [8.6](#page-55-2) in the case  $\mathcal{B} = \mathcal{A}^{\dagger}$  to conclude that the categories  $\mathcal{A}_{j}^{\dagger} \subset \mathcal{A}^{\dagger}$  defined by [\(7.8\)](#page-46-2) form a full Lefschetz chain. By definition, the primitive components of this Lefschetz chain are given by the categories  $\mathfrak{a}_j^{\natural} = \gamma^* p^* (\mathfrak{a}_{N-2+j}'), 2-N \leq j \leq 0.$  On the other hand, by [\(2b\)](#page-54-0), [\(2c](#page-54-2)'), and the claim of Corollary [8.6,](#page-55-2) the category  $\mathcal B$  has a full Lefschetz chain with primitive components given by the images of the  $\mathfrak{a}_i^{\natural}$ <sup> $\frac{a}{j}$ </sup> under the functor  $\phi^*$ . It follows that the functor  $\phi^*$  has trivial kernel if and only if its restriction to  $\mathfrak{a}_{i}^{\natural}$ <sup>4</sup>/<sub>j</sub> has trivial kernel for all  $2 - N \leq j \leq 0$ . But by Lemma [7.3](#page-45-3)[\(1\)](#page-45-4) combined with  $(2c')$ , the restriction of  $\phi^*$  to  $\mathfrak{a}_i^{\natural}$  $\frac{a}{j}$  is fully faithful.

The following gives the base case for the induction in the proof of generation of  $\mathbf{L}_r(\mathcal{A})$ .

**Lemma 8.30.** We have  $\text{im } \Phi_1 = \mathbf{L}_1(\mathcal{A})$ .

*Proof.* By definition  $\mathbf{L}_1 = \mathbf{P}$  and  $p_1: \mathbf{L}_1 \to \mathbf{P}$  is the identity morphism. In particular,  $\Phi_1$ coincides with the functor  $p_{1*} \circ \Phi_1$ , which we will analyze using Lemma [8.19.](#page-63-3) Note that  $q_{N-1} : L_{N-1}^{\vee} \to \mathbf{P}^{\vee}$  is isomorphic to the projective bundle  $\mathbf{P}(\mathcal{S}) \to \mathbf{P}^{\vee}$ , where S is the vector bundle on  $\mathbf{P}^{\vee}$  defined by the short exact sequence

$$
0 \to S \to V \otimes \mathcal{O} \to \mathcal{O}(H') \to 0.
$$

Under this identification, the bundle  $W_{N-1}/\mathcal{O}(-H')$  on  ${\bf L}_{N-1}^{\vee}$  corresponds to  $\Omega_{{\bf P}(\mathcal{S})/{\bf P}^{\vee}}(H)$ . By a computation on the projective bundle  $\mathbf{P}(\mathcal{S}) \to \mathbf{P}^{\vee}$ , we thus obtain for  $0 \leq i, i \leq N-2$ an isomorphism

$$
(q_{N-1})_*(\wedge^i(\mathcal{W}_{N-1}/\mathcal{O}(-H'))(-tH))\simeq\begin{cases} \mathcal{O}_{\mathbf{P}^\vee}[-i] & \text{ if $i=t$,} \\ 0 & \text{ else.} \end{cases}
$$

It follows that for  $0 \le i \le N-2$  the functor  $\Phi_{\mathcal{K}_{1,i}}$  from Lemma [8.19](#page-63-3) satisfies

$$
\Phi_{\mathcal{K}_{1,i}} \circ (-\otimes \mathcal{O}(-tH)) \circ q_{N-1}^* \simeq \begin{cases} [-i] \circ (-\otimes \mathcal{O}(-iH)) \circ p_* \circ \Phi_{\mathcal{E}} & \text{if } i = t, \\ 0 & \text{else,} \end{cases}
$$

and hence

$$
\Phi_1 \circ (-\otimes \mathcal{O}(-iH)) \circ q_{N-1}^* \simeq (-\otimes \mathcal{O}(-iH)) \circ p_* \circ \Phi_{\mathcal{E}}.
$$

Since  $\Phi_{\mathcal{E}} = \gamma \circ \phi$  it follows from Setup [8.1](#page-53-3)[\(2c\)](#page-54-1) that im  $\Phi_1$  contains the categories  $\mathcal{A}_0(-iH)$  for  $0 \leq i \leq N-2$ . The stable subcategory of A generated by these categories is all of A by the assumption that A is a moderate Lefschetz category. But im  $\Phi_1 \subset A$  is a stable subcategory since  $\Phi_1$  is left splitting, so we conclude im  $\Phi_1 = A$ .

# <span id="page-70-0"></span>Proposition 8.31. *We have*

$$
\mathbf{L}_r(\mathcal{A}) = \langle \text{im } \Phi_r, \mathcal{A}_s(H) \otimes \text{Perf}(\mathbf{G}_r), \ldots, \mathcal{A}_{m-1}((m-s)H) \otimes \text{Perf}(\mathbf{G}_r) \rangle.
$$

*Proof.* Analogous to Proposition [8.28](#page-68-0) and left to the reader. □

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