DIFFERENTIAL PROJECTIVE MODULES OVER ALGEBRAS WITH RADICAL SQUARE ZERO

DAWEI SHEN

Abstract. Let Q be a finite quiver and Λ be the radical square zero algebra of Q over a field. We give a full and dense functor from the category of reduced differential projective modules over Λ to the category of representations of the opposite of Q. If moreover Q has oriented cycles and Q is not a basic cycle, we prove that the algebra of dual numbers over Λ is not virtually Gorenstein.

1. Introduction

Given a ring Λ, recall that a *differential* Λ*-module* is a pair (M, d), where M is a Λ-module and d is an endomorphism of M such that $d^2 = 0$. If $(M, d), (N, \delta)$ are differential Λ-modules, a *differential* Λ*-module map* $f\colon (M, d) \to (N, \delta)$ is a Λ-module map $f\colon M \to N$ such that $fd = \delta f$. A Λ-module map $f\colon (M, d) \to (N, \delta)$ is *null homotopic* if there exists a Λ-module map $r\colon M \to N$ such that $f = rd + \delta r$. Note that differential modules over Λ are just modules over the ring $\Lambda[\epsilon]$ of dual numbers over Λ.

Let (M, d) be a differential Λ-module. The *shift* $\Sigma(M, d)$ of (M, d) is $(M, -d)$. Recall that (M, d) is *contractible* if the identity map of M is null homotopic, (M, d) is *reduced* if it has no nonzero contractible direct summands, and (M, d) is *exact* if $\operatorname{Ker} d = \operatorname{Im} d$. Every contractible differential Λ-module is exact.

By the term *differential projective* Λ*-modules*, we mean differential Λ-modules which are projective as Λ-modules. In [20] differential finitely generated projective Λ-module are called perfect differential Λ-modules.

The graded differential modules, namely the complexes, have been studied by many authors. However, few papers investigate the differential modules in detail. L. L. Avramov, R.-O. Buchweitz and S. Iyengar [4] study the projective class as well as free class and flat class for differential modules. C. M. Ringel and P. Zhang [20] investigate the perfect differential modules over path algebras, they prove that the homology functor gives a bijection from the reduced perfect differential modules to the finitely generated modules over path algebras. J. Wei [23] studies the Gorenstein homological theory for differential modules and extends their bijection.

The study of differential modules is related to the Gorenstein homological theory. M. Auslander and M. Bridger [1] introduce the notion of modules of G-dimension zero over two-sided Noetherian rings. This kind of modules are also called totally reflexive modules [5]. E. E. Enochs and O. M. G. Jenda [12, 13] extend their ideas and introduce the notion of Gorenstein projective modules, Gorenstein injective modules and Gorenstein flat modules for arbitrary rings. In particular, the totally reflexive modules are just the finitely generated Gorenstein projective modules for two-sided Noetherian rings.

Date: May 28, 2022.

2010 *Mathematics Subject Classification.* Primary 16G10; Secondary 16G50, 18G25.

Key words and phrases. Gorenstein projective module, representation of quivers, differential module, algebra with radical square zero, virtually Gorenstein algebra.

The Gorenstein projective modules over algebras with radical square zero have been well studied. X.-W. Chen [11] shows that a connected Artin algebra with radical square zero is either selfinjective or CM-free. Recall that an Artin algebra is called CM-free if every totally reflexive module is projective. C. M. Ringel and B.-L. Xiong [19] extend this result to arbitrary Gorenstein projective modules.

However, the Gorenstein projective modules over algebras with radical cubic zero are quite complicated. Y. Yoshino [24] studies a class of commutative local Artin algebras with radical cubic zero, over these algebras the simple module has no right approximations by the totally reflexive modules. They are not virtually Gorenstein algebras in the sense of [8].

The totally reflexive modules over $S_n = k[X, Y_1, \cdots, Y_n]/(X^2, Y_iY_j)$ are studied by D. A. Rangel Tracy [18], where k is a field, $n \geq 2$ and $1 \leq i, j \leq n$. The main result gives a bijection from the reduced totally reflexive modules over S_n to the finite-dimensional modules over the free algebra of n variables.

Inspired their works, we investigate the differential projective modules over Artin algebras with radical square zero.

Let k be a field and Q be a finite quiver. Denote by kQ the path algebra of Q. Let J be the arrow ideal of kQ, then the quotient algebra kQ/J^2 is an Artin algebra with radical square zero. Let $Q^{\rm op}$ be the opposite quiver of Q.

We construct a "Koszul dual functor" F from the category of reduced differential projective kQ/J^2-modules to the category of $kQ^{\rm op}$-modules.

Denote by $\mathrm{Diff}(kQ/J^2\text{-Proj})$ the Frobenius category of all differential projective kQ/J^2-modules [15]. The homotopy category $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$ of all differential projective kQ/J^2-modules is a triangulated category [22].

Denote by $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$ the full subcategory of $\mathrm{Diff}(kQ/J^2\text{-Proj})$ formed by reduced differential projective kQ/J^2-modules. We recall the abelian category $kQ^{\rm op}\text{-Mod}$ of all $kQ^{\rm op}$-modules.

The following is the main result of this paper; see also [18], compare [7, 20].

Theorem 1.1. *Let k be a field and Q be a finite quiver. Then taking the top makes a functor $F\colon \mathrm{Diff}_0(kQ/J^2\text{-Proj}) \to kQ^{\rm op}\text{-Mod}$ from the category of reduced differential projective kQ/J^2-modules to the category of $kQ^{\rm op}$-modules. Moreover,*

(1) *F is full, dense, and detects the isomorphisms;*
(2) *F is exact and commutes with all small coproducts;*
(3) *F vanishes on all null-homotopic maps;*
(4) *For any M, N in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$, there is an isomorphism*

$$\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(M,N) \simeq \mathrm{Hom}_{kQ^{\rm op}}(F(M),F(N)) \coprod \mathrm{Ext}^1_{kQ^{\rm op}}(F(M),F\Sigma(N)).$$

The "Koszul dual functor" F has a good restriction on some full subcategories. More precisely, we have the following.

Proposition 1.2. *Let M be a reduced differential projective kQ/J^2-module in the homotopy category $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.*

(1) *M is finite dimensional if and only if $F(M)$ is finite dimensional.*
(2) *M is compact if and only if $F(M)$ is finitely presented.*
(3) *M is exact if and only if $\mathrm{Ext}^n_{kQ^{\rm op}}(kQ_0, F(M)) = 0$ for $n = 0, 1$.*

We give a compact generator for the homotopy category $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$ as follows. Let C be the kQ/J^2-module $kQ/J^2 \otimes_{kQ_0} kQ^{\rm op}$ with a differential d given by $d(y \otimes z) = \sum_{\alpha \in Q_1} y\alpha \otimes \alpha^* z$ for $y \in kQ/J^2, z \in kQ^{\rm op}$. Here, $\alpha^* \in Q^{\rm op}$ is the reversed arrow of α.

We have the following.

Theorem 1.3. *The above C is a compact generator for $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.*

Recall that a finite connected quiver Q is a *basic cycle* if the number of vertices is equal to the number of arrows in Q and all arrows form an oriented cycle.

The following gives a class of noncommutative algebras with radical cubic zero which are not virtually Gorenstein algebras; compare [24].

Theorem 1.4. *If Q is a finite connected quiver with oriented cycles and Q is not a basic cycle. Then the algebra $kQ/J^2[\epsilon]$ of dual number over kQ/J^2 is not virtually Gorenstein.*

The present paper is organized as follows. In Section 2 and Section 3, we recall some required facts about quivers and radical square zero algebras, respectively. In Section 4, we construct the functor F and prove Theorem 1.1. In Section 5, we study the restriction of F and give the proofs of Proposition 1.2 and Theorem 1.3. In Section 6, we study virtual Gorensteinness of algebras and prove Theorem 1.4.

2. Quivers and representations

In this section, we recall some facts about quivers and their representations. We refer to [3, III.1] for more details.

A *finite quiver* Q is a quadruple $(Q_0, Q_1; s, t)$, where Q_0 is the finite set of vertices, Q_1 is the finite set of arrows, and $s, t\colon Q_1 \to Q_0$ are the source map and the target map, respectively. Denote by e_i the *trivial path* at i for $i \in Q_0$, where $s(e_i) = t(e_i) = i$. A *nontrivial path* p is a sequence $\alpha_l \cdots \alpha_2\alpha_1$ of arrows, where $l \geq 1$ and $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq l-1$. Here, $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_l)$. A nontrivial path p is called an *oriented cycle* if $s(p) = t(p)$. A finite quiver Q is said to be *acyclic* if it has no oriented cycles.

Let kQ be the path algebra of Q over a field k. Recall that kQ is a hereditary algebra and kQ is finite dimensional if and only if Q is acyclic.

A *representation* of Q is a collection $(M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$, where M_i is a k-vector space and M_α is a k-linear map from $M_{s(\alpha)}$ to $M_{t(\alpha)}$. If M, N are representations of Q, a morphism from M to N is a collection $(f_i)_{i\in Q_0}$, where f_i is a k-linear map from M_i to N_i such that $f_{t(\alpha)}M_\alpha = N_\alpha f_{s(\alpha)}$ for each $\alpha \in Q_1$.

Recall that the category of representations of Q is equivalent to the category of kQ-modules. We identify a kQ-module with the associated representation of Q.

Let J be the *arrow ideal* of kQ; it is the ideal generated by all arrows in Q. For an *admissible* ideal I of kQ satisfying $J^n \subseteq I \subseteq J^2$ for some $n \geq 2$, the quotient algebra kQ/I is finite dimensional over k. Recall that a kQ/I-module is a kQ-module M such that $wM = 0$ for every $w \in I$.

3. Projective modules for radical square zero algebras

Let k be a field and Q be a finite quiver. Recall that the unit element of the path algebra kQ is $\sum_{i\in Q_0} e_i$, where e_i the trivial path at vertex i.

Let J be arrow ideal of kQ, then kQ/J^2 is the radical square zero algebra of Q. In this section, we study some facts about the projective kQ/J^2-modules.

Lemma 3.1. *Let M be a projective kQ/J^2-module. Then we have*
(1) $\operatorname{rad}^2(M) = 0$;
(2) $(\operatorname{rad} M)_i = \coprod_{\alpha\in Q_1, t(\alpha)=i} \operatorname{Im} M_\alpha$ *for each* $i \in Q_0$;
(3) $(\operatorname{rad} M)_{s(\alpha)} = \operatorname{Ker} M_\alpha$ *for each* $\alpha \in Q_1$;
(4) $(M/\operatorname{rad} M)_{s(\alpha)} = M_{s(\alpha)}/\operatorname{Ker} M_\alpha$ *for each* $\alpha \in Q_1$.

Proof. Observe that (1)–(4) hold for the regular module over kQ/J^2. Since M is projective over kQ/J^2, it is a direct summand of direct sums of copies of kQ/J^2. Since taking the radicals, kernels and images commute with all small coproducts, we infer that (1)–(4) also hold for M. □

Given a module M, recall that the *radical* $\operatorname{rad} M$ of M is the intersection of all maximal submodules of M, and the *top* of M is the quotient module $M/\operatorname{rad} M$.

Let $f\colon M \to N$ be a kQ/J^2-module map between projective kQ/J^2-modules. Recall that f is *radical* if $\operatorname{Im} f \subseteq \operatorname{rad} N$. Denote by $F(f)\colon M/\operatorname{rad} M \to N/\operatorname{rad} N$ the induced map of f. It follows that f is radical if and only if $F(f) = 0$. Denote by $\operatorname{Rad}(M, N)$ the subspace of $\operatorname{Hom}_{kQ/J^2}(M, N)$ formed by radical maps.

Lemma 3.2. *For projective kQ/J^2-modules M, N, there is a short exact sequence*

$$0 \to \operatorname{Rad}(M, N) \xrightarrow{\subset} \operatorname{Hom}_{kQ/J^2}(M, N) \xrightarrow{F} \operatorname{Hom}_{kQ_0}(M/\operatorname{rad} M, N/\operatorname{rad} N) \to 0.$$

Proof. Let $g\colon M/\operatorname{rad} M \to N/\operatorname{rad} N$ be a kQ_0-module map. Since M is projective over kQ/J^2, we have $g = F(f)$ for some kQ/J^2-module map $f\colon M \to N$. Then the map F is surjective. Since $\operatorname{Ker} F = \operatorname{Rad}(M, N)$, this gives rise to the desired short exact sequence. □

For kQ_0-modules X and Y, denote by $\mathrm{E}(X, Y)$ the k-vector space consisting of the collections $(f_{\alpha^*})_{\alpha\in Q_1}$, where $f_{\alpha^*}\colon X_{t(\alpha)} \to Y_{s(\alpha)}$ is a k-linear map.

Lemma 3.3. *For projective kQ/J^2-modules M, N, there is an isomorphism*

$$\gamma\colon \operatorname{Rad}(M, N) \xrightarrow{F} \operatorname{Hom}_{kQ_0}(M/\operatorname{rad} M, \operatorname{rad} N) \xrightarrow{G} \mathrm{E}(M/\operatorname{rad} M, N/\operatorname{rad} N).$$

Proof. We identify $\operatorname{Rad}(M, N)$ with $\operatorname{Hom}_{kQ/J^2}(M, \operatorname{rad} N)$. Then the map F is well defined since $\operatorname{rad}^2(N)$ is zero. By Lemma 3.2 the map F is surjective and $\operatorname{Ker} F = 0$. Then F is an isomorphism.

We recall Lemma 3.1(1)–(4). Denote by $p_\alpha\colon (\operatorname{rad} N)_{t(\alpha)} \to \operatorname{Im} N_\alpha$ the natural projection and by $i_\alpha\colon \operatorname{Im} N_\alpha \to (\operatorname{rad} N)_{t(\alpha)}$ the natural inclusion. Let us denote by $\overline{N_\alpha}\colon N_{s(\alpha)}/\operatorname{Ker} N_\alpha \to \operatorname{Im} N_\alpha$ the induced isomorphism of N_α.

Let $g \in \operatorname{Hom}_{kQ_0}(M/\operatorname{rad} M, \operatorname{rad} N)$. Define $G(g)_{\alpha^*} = (\overline{N_\alpha})^{-1} p_\alpha g_{t(\alpha)}$ for $\alpha \in Q_1$. Let $h \in \mathrm{E}(M/\operatorname{rad} M, N/\operatorname{rad} N)$. Define $G^{-1}(h)_i = \sum_\alpha i_\alpha \overline{N_\alpha} h_{\alpha^*}$ for $i \in Q_0$, where α runs through all arrows terminating at i. One checks that G and G^{-1} are mutually inverse isomorphisms.

Therefore, the composite $\gamma = G \circ F$ is an isomorphism. □

Recall the *opposite* quiver Q^{op} of Q. The underlying graph of Q^{op} is the same as Q, but the orientations are all reversed. We denote by α^* the reversed arrow in Q^{op} for each arrow α in Q.

If X, Y are kQ^{op}-modules, let $\mathrm{E}_0(X, Y)$ be the subspace of $\mathrm{E}(X, Y)$ formed by $(h_{\alpha^*})_{\alpha\in Q_1}$, where $h_{\alpha^*} = \theta_{s(\alpha)} X_{\alpha^*} - Y_{\alpha^*}\theta_{t(\alpha)}$ for some kQ_0-module map $\theta\colon X \to Y$.

We need the following lemmas.

Lemma 3.4. *For kQ^{op}-modules X, Y, there is an isomorphism*

$$\operatorname{Ext}^1_{kQ^{\mathrm{op}}}(X, Y) \simeq \mathrm{E}(X, Y)/\mathrm{E}_0(X, Y).$$

Proof. This follows from [14, 7.2]. □

Lemma 3.5. *Let X be a kQ^{op}-module. Then X is finitely presented if and only if the functors $\operatorname{Hom}_{kQ^{\mathrm{op}}}(X, -)$ and $\operatorname{Ext}^1_{kQ^{\mathrm{op}}}(X, -)$ commute with all small coproducts.*

Proof. Since the path algebra kQ^{op} is hereditary, the projective dimension of X is no more than one. Then the statement follows from [21, 1.4 Corollary 2]. □

4. Construction of the "Koszul dual functor"

In this section, for a finite quiver Q we show that taking the top makes a full and dense functor from the category of reduced differential projective kQ/J^2-modules to the category of $kQ^{\rm op}$-modules. Here, $Q^{\rm op}$ is the opposite of Q.

Let M and N be differential projective kQ/J^2-modules. A kQ/J^2-module map $f\colon M \to N$ is said to be null homotopic if there is a kQ/J^2-module map $r\colon M \to N$ such that $f = rd + \delta r$. Here, d and δ are the differentials of M and N, respectively. Denote by $\mathrm{Hpt}(M, N)$ the subspace of $\mathrm{Hom}_{kQ/J^2}(M, N)$ formed by null-homotopic maps.

Let M be a differential projective kQ/J^2-module. Recall that M is said to be contractible if the identity map of M is null homotopic, and M is said to be reduced if M has no nonzero contractible direct summands.

We need the following.

Lemma 4.1. *Let M be a differential projective kQ/J^2-module.*

(1) *M is reduced if and only if the differential of M is a radical map.*
(2) *There exists a decomposition $M = M' \coprod M''$ such that M' is contractible and M'' is reduced. Moreover, this decomposition is unique up to isomorphism.*

Proof. View M as a one-periodic complex $\cdots \xrightarrow{d} M \xrightarrow{d} M \xrightarrow{d} \cdots$, where d is the differential. Then (1) and (2) follow from the dual versions of [16, Appendix B]. □

Let us recall some notations. We denote by $\mathrm{Diff}(kQ/J^2\text{-Proj})$ the category of all differential projective kQ/J^2-modules. It is a Frobenius category and it admits all small products. Denote by $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$ the full subcategory consisting of reduced differential projective kQ/J^2-modules.

Recall the homopopy category $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$ of all differential projective kQ/J^2-modules. The objects are all differential projective kQ/J^2-modules. The morphisms are obtained from differential kQ/J^2-module maps by factoring out the null homotopic maps.

Note that $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$ is not extension closed in $\mathrm{Diff}(kQ/J^2\text{-Proj})$. However, the homotopy categories of these two categories are equivalent.

Let (M, d) and (N, δ) be reduced differential projective kQ/J^2-modules. Then we have inclusions

$$\mathrm{Hpt}(M, N) \subseteq \mathrm{Rad}(M, N) \subseteq \mathrm{Diff}(kQ/J^2\text{-Proj})(M, N).$$

In fact, since M and N are reduced, by Lemma 4.1(1) d and δ are radical maps. If $f\colon M \to N$ is radical, then $fd = 0 = \delta f$ and thus f is a differential map. Then we obtain the inclusion on the right hand side. Similarly, the inclusion on the left hand side also holds.

We now prove the following key lemma. Here, we recall the maps F and γ from Lemma 3.3.

Lemma 4.2. *Let $f\colon M \to N$ be a kQ/J^2-module map between reduced differential projective kQ/J^2-modules. Then we have*

(1) *f is a differential map if and only if $F(f)\gamma(d) = \gamma(\delta)F(f)$;*
(2) *f is null homotopic if and only if $F(f) = 0$ and there exists a kQ_0-module map $\theta\colon M/\mathrm{rad}\, M \to N/\mathrm{rad}\, N$ such that $\gamma(f) = \theta\gamma(d) + \gamma(\delta)\theta$.*

Proof. (1) Since the map γ is an isomorphism by Lemma 3.3, we infer that $fd = \delta f$ if and only if $\gamma(fd) = \gamma(\delta f)$. Note that $\gamma(fd) = F(f)\gamma(d)$ and $\gamma(\delta f) = \gamma(\delta)F(f)$. Then f is a differential map if and only if $F(f)\gamma(d) = \gamma(\delta)F(f)$.

(2) "$\Longrightarrow$" Since f is null homotopic, there is a kQ/J^2-module map $r\colon M \to N$ such that $f = rd + \delta r$. Then $F(f) = 0$ and $\gamma(f) = F(r)\gamma(d) + \delta F(r)$.

" $\Longleftarrow$ " Since M is projective, there is a kQ/J^2-module map $r\colon M \to N$ such that $\theta = F(r)$. Note that f is radical and $\gamma(f) = \gamma(rd + \delta r)$. Then $f = rd + \delta r$ by Lemma 3.3. We infer that f is null homotopic. □

We now construct the "Koszul dual functor" F from the category of reduced differential projective kQ/J^2-modules to the category of kQ^{op}-modules.

For any object (M, d) in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$, set $F(M,d)_i = (M/\operatorname{rad} M)_i$ for $i \in Q_0$ and $F(M,d)_{\alpha^*} = \gamma(d)_{\alpha^*}$ for $\alpha \in Q_1$. Then $F(M,d)$ is a kQ^{op}-module.

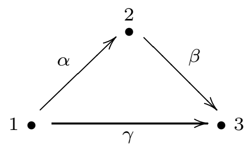
For a morphism f in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$, recall that $F(f)$ is a kQ_0-module map. It follows from Lemma 4.2(1) that $F(f)$ is a kQ^{op}-module map.

Let σ be the algebra isomorphism of kQ^{op} induced by $\sigma(q) = (-1)^l q$ for every path q in Q^{op}, where l is the length of q. Note that σ^2 is the identity map.

For a kQ^{op}-module X, let ${}^{\sigma}X$ be the *twisted module* of X. Here, ${}^{\sigma}X$ is equal to X as k-vector spaces, and the action $\circ$ is given by $w \circ x = \sigma(w)x$ for $w \in kQ^{\mathrm{op}}, x \in X$.

We see that ${}^{\sigma}({}^{\sigma}X)$ is the same as X. However, the twisted module ${}^{\sigma}X$ and the original module X need not be isomorphic. The following is an example.

Example 4.3. Let k be field and Q be the following quiver.



Let X be the kQ-module with $X_1 = X_2 = X_3 = k$, $X_\alpha = X_\beta = X_\gamma = 1_k$, where 1_k is the identity map. Then ${}^{\sigma}X_1 = {}^{\sigma}X_2 = {}^{\sigma}X_3 = k$, ${}^{\sigma}X_\alpha = {}^{\sigma}X_\beta = {}^{\sigma}X_\gamma = -1_k$.

If the characteristic of k is not equal to 2, then the two kQ-modules X and ${}^{\sigma}X$ are not isomorphic.

Let M be a reduced differential projective kQ/J^2-module. The shift $\Sigma(M)$ of M is the equal to M as kQ/J^2-modules, while the differential of $\Sigma(M)$ is the negative of the differential of M. Observe that the kQ^{op}-modules ${}^{\sigma}F(M)$ and $F\Sigma(M)$ are isomorphic.

We have the following.

Lemma 4.4. *For any M, N in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$, there is an isomorphism*

$$\underline{\mathrm{Rad}}(M,N) \simeq \mathrm{Ext}^1_{kQ^{\mathrm{op}}}(F(M), F\Sigma(N)).$$

Here, we write $\underline{\mathrm{Rad}}(M,N) = \mathrm{Rad}(M,N)/\mathrm{Htp}(M,N)$.

Proof. It follows from Lemma 3.3 that there is an isomorphism

$$\gamma\colon \mathrm{Rad}(M,N) \xrightarrow{\sim} \mathrm{E}(F(M), F(N)) = \mathrm{E}(F(M), F\Sigma(N)).$$

The image of $\mathrm{Hpt}(M,N)$ under γ is $\mathrm{E}_0(F(M), F\Sigma(N))$ by Lemma 4.2(2). We infer from Lemma 3.4 that $\underline{\mathrm{Rad}}(M,N)$ and $\mathrm{Ext}^1_{kQ^{\mathrm{op}}}(F(M), F\Sigma(N))$ are isomorphic. □

The following is the main result of this section.

Theorem 4.5. *Let k be a field and Q be a finite quiver. Then taking the top makes a functor $F\colon \mathrm{Diff}_0(kQ/J^2\text{-Proj}) \to kQ^{\mathrm{op}}\text{-Mod}$ from the category of reduced differential projective kQ/J^2-modules to the category of kQ^{op}-modules. Moreover,*

(1) *F is full, dense, and detects the isomorphisms;*
(2) *F is exact and commutes with all small coproducts;*
(3) *F vanishes on all null-homotopic maps;*
(4) *For any M, N in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$, there is an isomorphism*

$$\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(M,N) \simeq \mathrm{Hom}_{kQ^{\mathrm{op}}}(F(M), F(N)) \coprod \mathrm{Ext}^1_{kQ^{\mathrm{op}}}(F(M), F\Sigma(N)).$$

Proof. (1) Let M and N be in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$ and let $g\colon F(M)\to F(N)$ be a kQ^{op}-module map. By Lemma 3.2 there is a kQ/J^2-module map $f\colon M\to N$ such that $g=F(f)$. Since g is a kQ^{op}-module map, it follows from Lemma 4.2(1) that f is a differential map . This shows that the functor F is full.

For a kQ^{op}-module X, set $G(X)$ be the kQ/J^2-module $kQ/J^2\otimes_{kQ_0}X$ with a differential d given by

$$d(\sum_{i\in Q_0} y_i\otimes x_i)=\sum_{\alpha\in Q_1} y_{t(\alpha)}\alpha\otimes X_{\alpha^*}(x_{t(\alpha)})$$

for $y_i\in(kQ/J^2)e_i$ and $x_i\in X_i$. Here, we recall the target $t(\alpha)$ of the arrow α.

Note that $G(X)$ is a reduced differential projective kQ/J^2-module and $F(G(X))$ is isomorphic to X. It follows that the functor F is dense.

It remains to show that F detects the isomorphisms. Suppose that $f\colon M\to N$ is a morphism in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$ with $F(f)$ being an isomorphism.

Let g be the inverse of $F(f)$. Since N is projective, there exists a morphism $h\colon N\to M$ such that $F(h)=g$. Then $F(1_M)=F(hf)$, $F(1_N)=F(fh)$, where 1_M and 1_N are the identity maps. By Lemma 3.2 both 1_M-hf and 1_N-fh are radical maps. Since $\mathrm{rad}^2(M)=0$, $\mathrm{rad}^2(N)=0$, we have $(1_M-hf)^2=0$, $(1_N-fh)^2=0$. Then hf and fh are isomorphisms. It follows that f is an isomorphism.

(2) For every reduced differential projective kQ/J^2-module M, recall that $F(M)$ is isomorphic to $kQ_0\otimes_{kQ/J^2}M$ as k-vector spaces. It follows that F is exact and commutes with all small coproducts.

(3) Since every null-homotopic map is radical and F vanishes on all radical maps, it follows that F vanishes on all null-homotopic maps.

(4) Let M and N be in $\mathrm{Diff}_0(kQ/J^2\text{-Proj})$. Since the functor F is full by (1), there is a short exact sequence

$$0\to \mathrm{Rad}(M,N)\to \mathrm{Hom}_{\mathrm{Diff}(kQ/J^2\text{-Proj})}(M,N)\to \mathrm{Hom}_{kQ^{\mathrm{op}}}(F(M),F(N))\to 0.$$

Since $\mathrm{Hpt}(M,N)\subseteq \mathrm{Rad}(M,N)$, it yields a short exact sequence

$$0\to \underline{\mathrm{Rad}}(M,N)\to \mathrm{Hom}_{\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})}(M,N)\to \mathrm{Hom}_{kQ^{\mathrm{op}}}(F(M),F(N))\to 0.$$

Since k is a field, the previous exact sequence is split. Then the desired isomorphism follows from Lemma 4.4. □

By Theorem 4.5 we have the following; compare [23, Corollary 4.10].

Corollary 4.6. *The functor F gives a bijection from the isoclasses of objects in the homotopy category of differential projective kQ/J^2-modules to the isoclasses of objects in the category of kQ^{op}-modules, which carries indecomposable objects to indecomposable objects.*

5. Compact generator

Let k be a field and Q be a finite quiver. Recall the opposite quiver Q^{op} and the radical square algebra kQ/J^2 of Q.

Let C be the kQ/J^2-module $kQ/J^2\otimes_{kQ_0}kQ^{\mathrm{op}}$ with a differential d given by

$$d(y\otimes z)=\sum_{\alpha\in Q_1} y\alpha\otimes\alpha^* z$$

for $y\in kQ/J^2$ and $z\in kQ^{\mathrm{op}}$. Here, we recall that α^* is the reversed arrow of α.

Observe that C is a reduced differential projective kQ/J^2-module. Recall the functor F from the previous section. It is routine to show that $F(C)$ is isomorphic to the regular module over kQ^{op}.

Recall the homotopy category $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$ of differential projective kQ/J^2-module. We will later show C is a compact generator for this triangulated category.

Let M be a differential projective kQ/J^2-module. By Lemma 4.1(2) there exists a decomposition $M = M' \coprod M''$ such that M' is contractible and M'' is reduced. Recall that the *cohomology group* $H(M)$ of M is $\mathrm{Ker}\, d/\, \mathrm{Im}\, d$, where d is differential of M. We also recall that M is said to be exact if its cohomology group is zero. Note that $H(M') = 0$ and $H(M) = H(M'')$.

Lemma 5.1. *Let M be a differential projective kQ/J^2-module. Then we have*

(1) $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(C, M) \simeq F(M'')$*;*

(2) $H(M) \simeq \mathrm{Hom}_{kQ^{\mathrm{op}}}(kQ_0, F(M'')) \coprod \mathrm{Ext}^1_{kQ^{\mathrm{op}}}(kQ_0, F(M''))$*.*

Proof. Note that $H(M)$ is isomorphic to $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(kQ/J^2, M)$. Here we denote by kQ/J^2 the differential module kQ/J^2 with vanishing differential. Then (1) and (2) follow from Theorem 4.5(4). □

The "Koszul dual functor" F has a good restriction on some full subcategories. More precisely, we have the following result.

Proposition 5.2. *Let M be a reduced differential projective kQ/J^2-module in the stable category $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.*

(1) *M is finite dimensional if and only if $F(M)$ is finite dimensional.*

(2) *M is compact if and only if $F(M)$ is finitely presented.*

(3) *M is exact if and only if $\mathrm{Ext}^n_{kQ^{\mathrm{op}}}(kQ_0, F(M)) = 0$ for $n = 0, 1$.*

Proof. (1) Since $F(M)$ is the top of the projective module M, it follows that M is finitely generated if and only if $F(M)$ is finitely generated. Since finitely generated kQ/J^2-modules are exactly finite-dimensional kQ/J^2-modules, we infer that M is finite dimensional if and only if $F(M)$ is finite dimensional.

(2) "$\Longrightarrow$" Assume that M is compact. Let $\{Y_\lambda\}_{\lambda \in L}$ be a set of kQ^{op}-modules. Since the functor F is dense by Theorem 4.5(1), every Y_λ is isomorphic to $F(T_\lambda)$ for some reduced differential projective kQ/J^2-module T_λ.

By Theorem 4.5(2) and (4) there are isomorphisms

$$\mathrm{Hom}_{kQ^{\mathrm{op}}}(F(M), \coprod_{\lambda \in L} Y_\lambda) \simeq \coprod_{\lambda \in L} \mathrm{Hom}_{kQ^{\mathrm{op}}}(F(M), Y_\lambda),$$

and

$$\mathrm{Ext}^1_{kQ^{\mathrm{op}}}(F(M), \coprod_{\lambda \in L} {}^{\sigma}Y_\lambda) \simeq \coprod_{\lambda \in L} \mathrm{Ext}^1_{kQ^{\mathrm{op}}}(F(M), {}^{\sigma}Y_\lambda).$$

It follows from Lemma 3.5 that $F(M)$ is finitely presented.

"$\Longleftarrow$" Assume that $X = F(M)$ is a finitely presented kQ^{op}-module. Let us take a set $\{T_\lambda\}_{\lambda \in L}$ of differential projective kQ/J^2-modules. By Lemma 4.1(2) we have $T_\lambda = T'_\lambda \coprod T''_\lambda$ such that T'_λ is contractible and T''_λ is reduced.

By Lemma 3.5 and Theorem 4.5(4) we have isomorphisms

$$\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(M, \coprod_{\lambda \in L} T_\lambda) \simeq \coprod_{\lambda \in L} \underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(M, T_\lambda).$$

Then M is compact in $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.

(3) This follows directly from Theorem 4.5(4) and Lemma 5.1(2). □

We have two full subcategories of $\mathcal{T} = \underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$. Denote by $\mathcal{T}^c$ is the full subcategory formed by compact objects and by $\mathcal{T}^{fd}$ is the full subcategory formed by objects M such that its reduced part M'' is finite dimensional.

Corollary 5.3. *We have an inclusion $\mathcal{T}^{fd} \subseteq \mathcal{T}^c$. Moreover, the equality holds if and only if the quiver Q is acyclic.*

Proof. Recall that the category of all finite-dimensional $kQ^{\rm op}$-modules is contained in the category of all finitely presented $kQ^{\rm op}$-modules, these two categories coincide if and only if the quiver Q is acyclic. Then the corollary follow directly from Proposition 5.2. □

By Theorem 4.5 we have the following; compare [20, Theorem 2].

Corollary 5.4. *The bijection in Corollary 4.6 carries finite-dimensional objects to finite-dimensional objects and carries compact objects to finitely presented objects.*

In particular, if we take Q to be the n-loop quiver with $n \geq 2$, then the bijection between finite-dimensional objects has already studied in [18, Theorem 3.6].

Let $\mathcal{T}$ be a triangulated category admitting all small coproducts. An object S in $\mathcal{T}$ is said to be *compact* [17] if for any set $\{T_\lambda\}_{\lambda\in L}$ of objects in $\mathcal{T}$, the natural monomorphism

$$\coprod_{\lambda\in L} \mathrm{Hom}_{\mathcal{T}}(S, T_\lambda) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(S, \coprod_{\lambda\in L} T_\lambda)$$

is an epimorphism (and thus isomorphism).

Recall that a triangulated category $\mathcal{T}$ is said to be *compactly generated* [17] if $\mathcal{T}$ admits all small coproducts, and there exists a set $\mathcal{S}$ of objects in $\mathcal{T}$ such that

(1) Given $T \in \mathcal{T}$, if $\mathcal{T}(\Sigma^n S, T) = 0$ for every $S \in \mathcal{S}$ and $n \in \mathbb{Z}$, then $T \simeq 0$;
(2) Every object $S \in \mathcal{S}$ is compact.

Here, Σ denotes the translation functor of $\mathcal{T}$. The set $\mathcal{S}$ is called a *compact generating set* for $\mathcal{T}$. In particular, if $\mathcal{S} = \{S_0\}$ is a singleton, then S_0 is called a *compact generator* for $\mathcal{T}$.

We now prove that C is a compact generator for $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.

Theorem 5.5. *The homotopy category* $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$ *is compactly generated, where C is a compact generator for it.*

Proof. Let T be an object in $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$. We have $T = T' \coprod T''$ such that T' is contractible and T'' is reduced by Lemma 4.1(2).

Suppose $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(C, T) = 0$. Then $F(T'') = 0$ by Lemma 5.1(1). Since T'' is projective over kQ/J^2, we have $T'' = 0$ and $T = T'$ is contractible. Then $T \simeq 0$ in $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$. We infer that C is a generator for $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.

Take a set $\{T_\lambda\}_{\lambda\in L}$ of objects in $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$. Then Lemma 4.1(2) yields that $T_\lambda = T'_\lambda \coprod T''_\lambda$, where T'_λ is contractible and T''_λ is reduced.

By Lemma 5.1(1) there are isomorphisms

$$\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(C, \coprod_{\lambda\in L} T_\lambda) \simeq F(\coprod_{\lambda\in L} T''_\lambda)$$

and

$$\coprod_{\lambda\in L} \underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})(C, T_\lambda) \simeq \coprod_{\lambda\in L} F(T''_\lambda).$$

Recall from Theorem 4.5(2) that F commutes with all small coproducts. It follows that C is a compact object in $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$.

Therefore C is a compact generator for $\underline{\mathrm{Diff}}(kQ/J^2\text{-Proj})$. □

6. Virtually Gorenstein algebras

In this section, we study the virtue Gorensteiness of algebras. Here, we recall the notion of Gorenstein projective modules and Gorenstein injective modules; see [12] for more details.

Given a ring Λ, a complex $P^\bullet$ of projective Λ-modules is said to be *totally acyclic* if $P^\bullet$ is acyclic and $\mathrm{Hom}_\Lambda(P^\bullet, T)$ is acyclic for every projective Λ-module T. A Λ-module M is said to be *Gorenstein projective* if there exists a totally acyclic complex

$P^\bullet$ of projective Λ-modules such that M is isomorphic to $\mathrm{Coker}(d^{-1}\colon P^{-1} \to P^0)$. The complex $P^\bullet$ is called a *complete projective resolution* of M.

Dually, a complex $I^\bullet$ of injective Λ-modules is said to be *totally acyclic* if $I^\bullet$ is acyclic and $\mathrm{Hom}_\Lambda(T, I^\bullet)$ is acyclic for every injective Λ-module T. A Λ-module M is said to be *Gorenstein injective* if there exists a totally acyclic complex $I^\bullet$ of injective Λ-modules such that M is isomorphic to $\mathrm{Ker}(d^0\colon I^0 \to I^1)$. The complex $I^\bullet$ is called a *complete injective resolution* of M.

Denote by Λ-Mod the category of all Λ-modules. Let us denote by Λ-Proj the full subcategory of projective Λ-modules and denote by Λ-GProj the full subcategory of Gorenstein projective Λ-modules.

We need the following facts; see [10, 13] for more details.

(1) Λ-GProj is a Frobenius category with $\mathrm{proj}(\Lambda\text{-GProj}) = \Lambda\text{-Proj}$.
(2) The stable category $\Lambda\text{-}\underline{\mathrm{GProj}}$ is a triangulated category.
(3) Λ is a quasi-Frobenius ring if and only if $\Lambda\text{-GProj} = \Lambda\text{-Mod}$.
(4) If the left global dimension of Λ is finite, then $\Lambda\text{-GProj} = \Lambda\text{-Proj}$.

Let $\Lambda[\epsilon] = \Lambda[T]/\langle T^2\rangle$ be the ring of dual numbers over Λ. Note that differential modules over Λ are just modules over $\Lambda[\epsilon]$. In particular, if Λ is an algebra over a field k, then the algebra $\Lambda[\epsilon]$ is isomorphic to the tensor product $\Lambda \otimes_k k[\epsilon]$ of algebras. Here, we recall that $k[\epsilon] = k[T]/\langle T^2\rangle$ is the algebra of dual numbers over k; it is a selfinjective algebra.

By [23, Theorem 1.1] a differential Λ-module (M, d) is Gorenstein projective if and only if the underlying Λ-module M is Gorenstein projective. Then we know that every differential projective Λ-module is Gorenstein projective since every projective module is Gorenstein projective.

Let Λ be an Artin algebra. Recall that Λ is said to *Gorenstein* [2] if the injective dimension of Λ and the injective dimension of Λ^{op} are both finite. Algebras of finite global dimension and selfinjective algebras are Gorenstein algebras.

We also recall that Λ is said to be *virtually Gorenstein* [8] if for every Λ-module M, the functor $\mathrm{Ext}^1_\Lambda(-, M)$ vanishes on all Gorenstein projective Λ-modules if and only if the functor $\mathrm{Ext}^1_\Lambda(M, -)$ vanishes on all Gorenstein injective Λ-modules. Algebras of finite representation type and Gorenstein algebras are virtually Gorenstein algebras. Examples of non-virtually Gorenstein algebras can be founded in [9, 24].

We need the following.

Lemma 6.1. *Let Λ and Γ be finite-dimensional algebras over a field k.*

(1) *Λ is virtually Gorenstein if and only if every reduced compact object in the stable category $\Lambda\text{-}\underline{\mathrm{GProj}}$ is finite dimensional.*
(2) *$\Lambda \otimes_k \Gamma$ is Gorenstein if and only if Λ and Γ are Gorenstein.*
(3) *$\Lambda \otimes_k \Gamma$ is selfinjective if and only if Λ and Γ are selfinjective.*

Proof. (1) This follows [8, Theorem 8.2]; see also [9, Theorem 4].

(2) and (3) These are taken from [2, Proposition 2.2]. □

Let k be a field and Q be a finite quiver. We investigate the Gorenstein projective differential modules over the radical square zero algebra kQ/J^2 of Q. Recall that a finite connected quiver Q is a basic cycle if the number of vertices is equal to the number of arrows in Q and all arrows form an oriented cycle.

Proposition 6.2. *Let k be a field and Q be a finite connected quiver.*

(1) *If Q is not a basic cycle, then Gorenstein projective differential kQ/J^2-modules are just differential projective kQ/J^2-modules.*
(2) *Otherwise, every differential kQ/J^2-module is Gorenstein projective.*

Proof. (1) Since Q is not a basic cycle, by [19, Theorem 2] all Gorenstein projective modules over kQ/J^2 are projective. By [23, Theorem 1.1] Gorenstein projective differential kQ/J^2-modules are just differential projective kQ/J^2-modules.

(2) Since Q is a basic cycle, the algebra kQ/J^2 is selfinjective. Then $kQ/J^2[\epsilon]$ is selfinjective and thus every differential kQ/J^2-module is Gorenstein projective. □

The following theorem provides a class of noncommutative Artin algebras that are not virtually Gorenstein; compare [24, Theorem 6.1].

Theorem 6.3. *Let k be a field and Q be a finite connected quiver.*

(1) *If Q is acyclic, then the algebra $kQ/J^2[\epsilon]$ is Gorenstein.*
(2) *If Q is a basic cycle, then the algebra $kQ/J^2[\epsilon]$ is selfinjective.*
(3) *Otherwise, the algebra $kQ/J^2[\epsilon]$ is not virtually Gorenstein.*

Proof. (1) If Q is acyclic, then the algebra kQ/J^2 has finite global dimension. By Lemma 6.1(2) the algebra $kQ/J^2[\epsilon]$ is Gorenstein.

(2) If Q is a basic cycle, then the algebra kQ/J^2 is selfinjective. By Lemma 6.1(3) the algebra $kQ/J^2[\epsilon]$ is selfinjective.

(3) Since Q is not a basic cycle, by Proposition 6.2(1) the homotopy category of differential projective kQ/J^2-modules is exactly the stable category of Gorenstein projective $kQ/J^2[\epsilon]$-modules.

Since Q has oriented cycles, the compact generator in Theorem 5.5 is reduced and not finite dimensional. It follows from Lemma 6.1(1) that the algebra $kQ/J^2[\epsilon]$ is not virtually Gorenstein. □

Remark 6.4. Following [19] we know that the radical square zero algebra kQ/J^2 is virtually Gorenstein for every finite quiver Q.

Let k be a field of characteristic 2, then the algebra $kQ/J^2[\epsilon]$ is isomorphic to the group algebra $(kQ/J^2)C_2$ where C_2 is the cyclic group of order 2. Now if Q is a quiver in Theorem 6.3(3), then the group algebra $(kQ/J^2)C_2$ is not virtually Gorenstein; compare [6, Proposition 3.1].

We end this section by an example.

Example 6.5. Let k be a field and Q, Q' be the following quivers.

Let I be the ideal of kQ' generated by $\{\alpha^2, \beta\alpha, \epsilon_1^2, \epsilon_2^2, \alpha\epsilon_1 - \epsilon_1\alpha, \beta\epsilon_1 - \epsilon_2\beta\}$. Then the algebras kQ'/I and $kQ/J^2[\epsilon]$ are isomorphic.

By Theorem 6.3(2) the algebra kQ'/I is not virtually Gorenstein.

Acknowledgements. The author is very grateful to Prof. Xiao-Wu Chen and Prof. Yu Ye for numerous inspiring ideas. The author thanks Dr. Zhe Han and Dr. Bo Hou for many discussions in the seminar at Henan University. The work is supported by the Natural Science Foundation of China (No. 11571329).

References

[1] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, **94**. American Mathematical Society, Providence, R. I., 1969.

[2] M. Auslander and I. Reiten, *Cohen-Macaulay and Gorenstein Artin algebras*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 221–245, Progress in Mathematics, **95**. Birkhäuser, Basel, 1991.

[3] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**. Cambridge University Press, Cambridge, 1995.

[4] L. L. Avramov, R.-O. Buchweitz and S. Iyengar, *Class and rank of differential modules*, Invent. Math. **169** (2007), no. 1, 1–35.

[5] L. L. Avramov and A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), no. 2, 393–440.

[6] A. Bahlekeh and Sh. Salarian, *Virtual Gorensteinness over group algebras*, Kyoto J. Math. 55 (2015), no. 1, 129–141.

[7] A. A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), no. 2, 473–527.

[8] A. Beligiannis, *Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras*, J. Algebra **288** (2005), no. 1, 137–211.

[9] A. Beligiannis and H. Krause, *Thick subcategories and virtually Gorenstein algebras*, Illinois J. Math. **52** (2008), no. 2, 551–562.

[10] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*. Unpublished Manuscript, Available at: http://hdl.handle.net/1807/16682, 1987.

[11] X.-W. Chen, *Algebras with radical square zero are either self-injective or CM-free*, Proc. Amer. Math. Soc. **140** (2012), no. 1, 93–98.

[12] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633.

[13] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, De Gruyter Expositions in Mathematics, **30**. Walter de Gruyter & Co., Berlin, 2000.

[14] P. Gabriel, and A. V.Roiter, *Representations of finite-dimensional algebras*, Encyclopaedia Math. Sci., **73**, Algebra, VIII. Springer, Berlin, 1992.

[15] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, **119**. Cambridge University Press, Cambridge, 1988.

[16] H. Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162.

[17] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

[18] D. A. Rangel Tracy, *A description of totally reflexive modules for a class of non-Gorenstein rings*, arXiv preprint (2015), arXiv:1510.04922.

[19] C. M. Ringel and B.-L. Xiong, *On radical square zero rings*, Algebra Discrete Math. **14** (2012), no. 2, 297–306.

[20] C. M. Ringel and P. Zhang, *Representations of quivers over the algebra of dual numbers*, J. Algebra **475** (2017), 327–360.

[21] R. Strebel, *A homological finiteness criterion*, Math. Z. **151** (1976), no. 3, 263–275.

[22] J.-L. Verdier *Des catgories drives des catgories abliennes*, Astrisque, **239**. 1996.

[23] J. Wei, *Gorenstein homological theory for differential modules*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), 639–655.

[24] Y. Yoshino, *Modules of G-dimension zero over local rings with the cube of maximal ideal being zero*, Commutative algebra, singularities and computer algebra (Sinaia, 2002), 255–273, NATO Sci. Ser. II Math. Phys. Chem., **115**. Kluwer Acad. Publ., Dordrecht, 2003.

School of Mathematics and Statistics, Henan University, Kaifeng, Henan 475004, P. R. China

E-mail address: dwshen@vip.henu.edu.cn