Rotation number values at a discontinuity

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Abstract

In the space of orientation-preserving circle maps that are not necessarily surjective nor injective, the rotation number does not vary continuously. Each map where one of these discontinuities occurs is itself discontinuous and we can consider the possible values of the rotation number when we modify this map only at its discontinuities. These values are always rational numbers that necessarily obey a certain arithmetic relation. In this paper we show that in several examples this relation totally characterizes the possible values of the rotation number on its discontinuities, but we also prove that in certain circumstances this relation is not sufficient for this characterization.

1 Introduction, notations, examples and result

We shall consider the space $\mathcal{M}$ of lifts of orientation-preserving circle maps, that is, the set of functions $f : \mathbb{R} \longmapsto \mathbb{R}$ which satisfy the conditions

$$y > 0 \Rightarrow f(x+y) \geqslant f(x) \quad \text{and} \quad f(x+1) = f(x) + 1.$$

It should be noted that $\mathcal{M}$ contains functions which are not continuous or strictly increasing (not surjective nor injective). For each $f \in \mathcal{M}$ the limit (rotation number of f)

$$\nu(f) = \lim_{n\to\infty} \frac{f^n(x)}{n}$$

exists and is independent of $x \in \mathbb{R}$ [2].

In $\mathcal{M}$ the rotation number is an increasing functional, $f \leqslant g \;\Rightarrow\; \nu(f) \leqslant \nu(g)$, and we may have $\nu(f^-) < \nu(f^+)$, where

$$f^-(x) = \lim_{\substack{\delta\to 0\\ \delta>0}} f(x-\delta) \quad \text{and} \quad f^+(x) = \lim_{\substack{\delta\to 0\\ \delta>0}} f(x+\delta).$$

On this space $\mathcal{M}$ we shall consider the Lévy distance:

$$d_H(f,g) = \inf\{\varepsilon > 0 \;:\; f(x-\varepsilon) - \varepsilon \leqslant g(x) \leqslant f(x+\varepsilon) + \varepsilon, \quad \forall x \in \mathbb{R}\ \}.$$

Observe that we may have $d_H(f,g) = 0$ with $f \neq g$. In fact it is easy to verify that

$$d_H(f,g) = 0 \quad \Leftrightarrow \quad f^+ = g^+ \quad \Leftrightarrow \quad f^- = g^-.$$

Endowed with this distance $\mathcal{M}$ is a pseudometric space. The following theorem is known.

Theorem 1.1 ([3, 1]) *Let $f_0 \in \mathcal{M}$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $f \in \mathcal{M}$ satisfying $d_H(f, f_0) < \delta$ we have*

$$\nu\left(f_0^-\right) - \varepsilon \leqslant \nu\left(f\right) \leqslant \nu\left(f_0^+\right) + \varepsilon.$$

Therefore, the set of discontinuities of the rotation number is

$$\mathcal{D} \equiv \left\{ f \in \mathcal{M} : \; \nu\left(f^-\right) < \nu\left(f^+\right) \right\}.$$

In [1] it is shown that if $f \in \mathcal{D}$, then there exists $m \in \mathbb{Z}^+$ such that f^m is a step function, that is to say, the image $f^m([0,1])$ is a finite set. As a direct consequence of this fact we have that if $f \in \mathcal{M}$ is continuous or strictly increasing, then $f \notin \mathcal{D}$. Observe, however, that $f \in \mathcal{D}$ itself does not have to be a step function, see Examples 1.6 and 1.8.

On the other hand Theorem 1.1 cannot be improved as the following proposition shows.

Proposition 1.2 *Given $f \in \mathcal{D}$, $\delta > 0$ and $\nu \in \mathbb{R}$ satisfying $\nu\left(f^-\right) < \nu < \nu\left(f^+\right)$, there is a homeomorphism $g \in \mathcal{M}$ such that*

$$d_H(g, f) < \delta \quad \textit{and} \quad \nu\left(g\right) = \nu.$$

Proof. For each $\delta > 0$ there exist homeomorphisms $h_{-\delta}$ and $h_\delta \in \mathcal{M}$ such that

$$h_{-\delta} \leqslant f \leqslant h_\delta \quad \text{and} \quad d_H(h_{\pm\delta}, f) < \delta.$$

We can then construct a family of homeomorphisms $g_\lambda \in \mathcal{M}$, with $\lambda \in [0,1]$, by the formula

$$g_\lambda = (1-\lambda)\, h_{-\delta} + \lambda h_\delta.$$

Therefore $d_H(g_\lambda, f) < \delta$, for all $\lambda \in [0,1]$, and $\nu\left(g_0\right) = \nu\left(h_{-\delta}\right) \leqslant \nu\left(f^-\right) < \nu\left(f^+\right) \leqslant \nu\left(h_\delta\right) = \nu\left(g_1\right)$. Since the rotation number is continuous in the subspace of the homeomorphisms of $\mathcal{M}$, we have that, if $\nu\left(f^-\right) < \nu < \nu\left(f^+\right)$, then there exists $\lambda_0 \in [0,1]$ such that $\nu\left(g_{\lambda_0}\right) = \nu$. ■

Although in an arbitrary neighborhood of $f \in \mathcal{D}$ we have an interval of possible values of the rotation number, the same is not true if we consider only the functions that are at a null distance from f. In [1] is given a characterization of the possible rotation numbers for functions in these circumstances:

Theorem 1.3 ([1]) *Let $f_0, f_1 \in \mathcal{D}$ be such that $d_H(f_0, f_1) = 0$ and $\nu\left(f_0\right) < \nu\left(f_1\right)$, then $\nu\left(f_0\right) = \frac{p_0}{q_0}$ and $\nu\left(f_1\right) = \frac{p_1}{q_1}$ are rationals that, when represented as irreducible fractions, satisfy the condition*

$$\frac{p_1 - 1}{q_1} \leqslant \frac{p_0}{q_0} < \frac{p_1}{q_1} \leqslant \frac{p_0 + 1}{q_0}.$$

In particular, if we know the values of $\nu\left(f^-\right)$ and $\nu\left(f^+\right)$ we have only a finite set of possible values for $\nu\left(f\right)$. In the next proposition we give a (non-injective) parameterization of this set where we use the floor and the ceiling integer functions respectively defined by

$$\lfloor x \rfloor = \max\left\{n \in \mathbb{Z} : \quad n \leqslant x\right\} \quad \text{and} \quad \lceil x \rceil = \min\left\{n \in \mathbb{Z} : \quad x \leqslant n\right\}.$$

Proposition 1.4 *If $f \in \mathcal{D}$ is such that $\nu^- \equiv \nu\left(f^-\right) < \nu\left(f\right) < \nu\left(f^+\right) \equiv \nu^+$, then $\nu\left(f\right)$ belongs to the following finite set*

$$\nu\left(f\right) \in S_{\nu^-,\nu^+} \equiv \left\{ \frac{\lfloor q\,\nu^- \rfloor + 1}{q} : \quad \lceil q\,\nu^+ \rceil = \lfloor q\,\nu^- \rfloor + 2, \quad q \in \mathbb{Z}^+ \right\}.$$

Proof. If $\nu(f) = \frac{p}{q}$ (irreducible fraction), we know from Theorem 1.3 (applied to pairs (f^-, f) and (f, f^+)) that we have, with $\frac{p_-}{q_-} = \nu^-$ and $\frac{p_+}{q_+} = \nu^+$,

$$p - 1 \leqslant q\frac{p_-}{q_-} < p < q\frac{p_+}{q_+} \leqslant p + 1.$$

Equivalently $p = \lfloor q\,\nu^- \rfloor + 1 = \lceil q\,\nu^+ \rceil - 1$ and the condition $\lceil q\,\nu^+ \rceil = \lfloor q\,\nu^- \rfloor + 2$ can only be true for a finite number of values of $q \in \mathbb{Z}^+$ since $\nu^- < \nu^+$ (in fact we can even show that $(q_- + q_+)/\Delta \leqslant q \leqslant 2q_-q_+/\Delta$, where $\Delta = p_+q_- - p_-q_+$). ■

The purpose of this article is to evaluate the extent to which Theorem 1.3 is insightful in describing the set of rotation number values at a discontinuity $f \in \mathcal{D}$ that we can define symbolically by

$$V(f) \equiv \{\nu(g) : \ g \in \mathcal{M} \text{ and } d_H(f,g) = 0\}.$$

With this notation Theorem 1.3 states

$$V(f) \subset \left\{\nu^-, \nu^+\right\} \cup S_{\nu^-,\nu^+}, \tag{1}$$

with $\nu^- = \nu(f^-)$, $\nu^+ = \nu(f^+)$ and S_{ν^-,ν^+} defined in Proposition 1.4. The question that arises is whether we can replace the inclusion by an equality. We will see in the following examples that the answer may be affirmative, but it may be negative as well. Let us start by looking at an example where $V(f) = \{\nu^-, \nu^+\} \cup S_{\nu^-,\nu^+}$.

Example 1.5 *Let* $f = \frac{\lceil 2x \rceil}{2}$, $g_1 = \frac{1+\lceil 2x \rceil + \lfloor 2x \rfloor}{4}$ *and* $g_2 = \frac{1+\lceil x \rceil + \lfloor 2x \rfloor + \lfloor x+\frac{1}{2} \rfloor}{4}$ *(see Figure 1). We have* $f^+ = \frac{1+\lfloor 2x \rfloor}{2}$, $f^- = g_1^- = g_2^- = f$ *and* $g_1^+ = g_2^+ = f^+$*. Since* $f(0) = 0$, $g_1^4(0) = 1$,

Figure 1: Graph of f, g_1, g_2 and f^+ from Example 1.5. Their rotation numbers are 0, 1/4, 1/3 and 1/2, respectively.

$g_2^3(0) = 1$, ${f^+}^2(0) = 1$, *we obtain* $\nu(f) = 0$, $\nu(g_1) = \frac{1}{4}$, $\nu(g_2) = \frac{1}{3}$ *and* $\nu(f^+) = \frac{1}{2}$*. On the other hand* $S_{0,\frac{1}{2}} = \left\{\frac{1}{4}, \frac{1}{3}\right\}$, *therefore* $V(f) = \left\{0, \frac{1}{2}\right\} \cup S_{0,\frac{1}{2}}$.

The next example shows that we can also have $V(f) \neq \{\nu^-, \nu^+\} \cup S_{\nu^-,\nu^+}$.

Example 1.6 *Let* $f(x) = \min\left(x + \frac{1}{2}, \lceil x \rceil\right)$ *(see Figure 2). We have* $f^- = f$, $f(0) = 0$, ${f^+}^2(0) = f^+\left(\frac{1}{2}\right) = 1$, *therefore* $\nu(f^-) = 0$ *and* $\nu(f^+) = \frac{1}{2}$*. But if* $g \in \mathcal{M}$ *is such that* $d_H(f,g) = 0$ *and* $f^- \neq g \neq f^+$, *then* $0 < g(0) < \frac{1}{2}$ *and* $g^3(0) = g\left(g(0) + \frac{1}{2}\right) = 1$*; so that* $\nu(g) = \frac{1}{3}$*. Hence* $V(f) = \left\{0, \frac{1}{3}, \frac{1}{2}\right\} \neq \left\{0, \frac{1}{2}\right\} \cup S_{0,\frac{1}{2}} = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$.

The preceding examples are very particular (for being simple) and suggest several conjectures that are not true; so it is convenient to give two less trivial examples.

Figure 2: Graph of functions $f(x)$, $g = \frac{1}{2}(f + f^+)$ and f^+ from Example 1.6. Their rotation numbers are $0, 1/3$ and $1/2$, respectively.

Example 1.7 *Let$f(x)$ be defined by the expression*

$$\frac{1}{10}\left(4 + 2\lceil x\rceil + \lceil x - \tfrac{1}{10}\rceil + \lceil x - \tfrac{1}{5}\rceil + 2\lceil x - \tfrac{2}{5}\rceil + \lceil x - \tfrac{1}{2}\rceil + \lceil x - \tfrac{3}{5}\rceil + 2\lceil x - \tfrac{4}{5}\rceil\right)$$

and f_1, f_2, f_3 and f_4 according to Figure 3. Then ($s = 1, 2, 3, 4$)

Figure 3: Graph of the functions f_1, f_2, f_3 and f_4 from Example 1.7. Their rotation numbers are 2/7, 3/10, 1/3 and 3/8, respectively.

$$f^- = f_s^- = f \quad \text{and} \quad f_s^+ = f^+.$$

By calculating the successive iterates of the point 0 by each of these functions we obtain

$$\nu(f) = \tfrac{1}{4}\ , \quad \nu(f_1) = \tfrac{2}{7}\ , \quad \nu(f_2) = \tfrac{3}{10}\ , \quad \nu(f_3) = \tfrac{1}{3}\ , \quad \nu(f_4) = \tfrac{3}{8}\ , \quad \nu\left(f^+\right) = \tfrac{2}{5}.$$

On the other hand $S_{\frac{1}{4},\frac{2}{5}} = \left\{\frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}\right\}$, so in this case $V(f) = \left\{\frac{1}{4}, \frac{2}{5}\right\} \cup S_{\frac{1}{4},\frac{2}{5}}$.

The next example also shows that, in general,

$$V(f) \neq \left\{\nu\left((1-\lambda) f^- + \lambda f^+\right) : \lambda \in [0,1]\right\}.$$

Example 1.8 *Given $(\alpha, \beta) \in [0,1]^2$, let $f_{\alpha,\beta} \in \mathcal{M}$ be defined on $[0,1)$ by*

$$f_{\alpha,\beta}(x) = \begin{cases} (1+2\alpha)/6 & \text{if} \quad x = 0 \\ 1/2 & \text{if} \quad 0 < x < 1/3 \\ (1+\beta)/2 & \text{if} \quad x = 1/3 \\ 1 & \text{if} \quad 1/3 < x \leqslant 5/6 \\ x + 1/6 & \text{if} \quad 5/6 \leqslant x < 1 \end{cases}$$

and by $f_{\alpha,\beta}(x) = f_{\alpha,\beta}(x - \lfloor x \rfloor) + \lfloor x \rfloor$ *on the remaining points (see Figure 4). By calculating*

Figure 4: Graph of the function $f_{\alpha,\beta}(x)$ with $\alpha \in \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$ and $\beta = \frac{5}{6}$. Their rotation numbers are 1/3, 2/5 and 1/2, respectively.

the successive iterates of the point 0 *we obtain*

$$\nu(f_{\alpha,\beta}) = \begin{cases} 1/3 & \text{if} \quad \alpha < 1/2 \text{ or } (\alpha = 1/2 \text{ and } \beta \leqslant 2/3) \\ 2/5 & \text{if} \quad \alpha = 1/2 \text{ and } 2/3 < \beta < 1 \\ 1/2 & \text{if} \quad (\alpha = 1/2 \text{ and } \beta = 1) \text{ or } 1/2 < \alpha \end{cases}.$$

Since $d_H(f, f_{0,0}) = 0$ *if and only if* $f = f_{\alpha,\beta}$ *for some* $(\alpha, \beta) \in [0,1]^2$, *and in this case* $f^- = f_{0,0}$ *and* $f^+ = f_{1,1}$; *we deduce* $V(f_{\alpha,\beta}) = \left\{\frac{1}{3}, \frac{2}{5}, \frac{1}{2}\right\}$, *however*

$$S_{\frac{1}{3},\frac{1}{2}} = \left\{\tfrac{3}{8}, \tfrac{2}{5}, \tfrac{5}{12}, \tfrac{3}{7}, \tfrac{4}{9}\right\}.$$

Note also that for $\lambda \in [0,1]$ *(and* $f = f_{\alpha,\beta}$*) one has* $(1-\lambda) f^- + \lambda f^+ = f_{\lambda,\lambda}$ *so that*

$$\left\{\nu\left((1-\lambda) f^- + \lambda f^+\right) : \lambda \in [0,1]\right\} = \left\{\tfrac{1}{3}, \tfrac{1}{2}\right\} \neq V(f).$$

Another question that arises is whether in inclusion (1) we can replace S_{ν^-,ν^+} by a smaller set. By defining

$$V_{\nu^-,\nu^+} \equiv \left\{\nu(f) : \ f \in \mathcal{M} \quad \text{and} \quad \nu^- = \ \nu\left(f^-\right) < \nu(f) < \nu\left(f^+\right) = \nu^+\right\},$$

we obviously have $V(f) \subset \{\nu^-, \nu^+\} \cup V_{\nu^-,\nu^+}$ if $\nu^- = \nu(f^-)$ and $\nu^+ = \nu(f^+)$; and therefore the problem is whether $V_{\nu^-,\nu^+} = S_{\nu^-,\nu^+}$? This equality is verified in certain cases as shown in Examples 1.5 and 1.7, but surprisingly it is not true in general as shown by the following theorem which we shall prove in Section 2.

Theorem 1.9 *If* $\frac{p}{q} \in S_{\nu^-,\nu^+}$ *is such that* $\frac{p}{q}$ *is irreducible,* $\frac{p-1}{q} = \nu^-$, $\frac{p+1}{q} = \nu^+$ *and* q *is odd, then there is no* $f \in \mathcal{M}$ *with* $\nu(f^-) = \nu^-$, $\nu(f^+) = \nu^+$ *and* $\nu(f) = \frac{p}{q}$.

To be sure that the previous theorem is relevant we need an example of a function $f \in \mathcal{D}$ such that for a certain irreducible fraction $\frac{p}{q}$ with odd q, we have $\frac{p-1}{q} = \nu(f^-)$ and $\frac{p+1}{q} = \nu(f^+)$; which we shall see in the next example. In fact it would have been enough to give examples of irreducible fractions $\frac{p_-}{q_-}, \frac{p}{q}, \frac{p_+}{q_+}$ such that

$$\frac{p_+ - 1}{q_+} \leqslant \frac{p_-}{q_-} = \frac{p-1}{q} \quad \text{and} \quad \frac{p+1}{q} = \frac{p_+}{q_+} \leqslant \frac{p_- + 1}{q_-},$$

since in [1] it is shown that if $\frac{p_+-1}{q_+} \leqslant \frac{p_-}{q_-} < \frac{p_+}{q_+} \leqslant \frac{p_-+1}{q_-}$, then there exists $f \in \mathcal{M}$ with $\nu(f^-) = \frac{p_-}{q_-}$ and $\nu(f^+) = \frac{p_+}{q_+}$.

Example 1.10 *For $f \in \mathcal{D}$ defined by the following expression (see Figure 5),*

Figure 5: Graph of the functions f, $f(x) + \frac{1}{10}\left(\left\lfloor 1 + x - \frac{1}{5} \right\rfloor - \left\lceil x - \frac{1}{5} \right\rceil\right)$ and f^{+} from Example 1.10. Their rotation numbers are 1/5, 1/4 and 1/3, respectively.

$$f(x) = \tfrac{1}{5}\left(1 + \lceil 5x \rceil - \lceil x \rceil + \left\lceil x - \tfrac{1}{10} \right\rceil\right)$$

it is easy to verify that $\nu(f^{-}) = \frac{1}{5}$ and $\nu(f^{+}) = \frac{1}{3}$. Also $\frac{4-1}{15} = \frac{1}{5}$ and $\frac{4+1}{15} = \frac{1}{3}$; so $\frac{4}{15} \in S_{\frac{1}{5},\frac{1}{3}}$ and Theorem 1.9 shows that $\frac{4}{15} \notin V_{\frac{1}{5},\frac{1}{3}}$. Therefore $V_{\frac{1}{5},\frac{1}{3}} \neq S_{\frac{1}{5},\frac{1}{3}}$.

Although it is easy to see that, for example, $\frac{3}{11} \notin V(f)$, by constructing other examples $g \in \mathcal{M}$ with $\nu(g^{-}) = \frac{1}{5}$ and $\nu(g^{+}) = \frac{1}{3}$, it is possible to show that $V_{\frac{1}{5},\frac{1}{3}} = S_{\frac{1}{5},\frac{1}{3}} \setminus \left\{\frac{4}{15}\right\} = \left\{\frac{2}{9}, \frac{1}{4}, \frac{3}{11}, \frac{2}{7}, \frac{3}{10}\right\}$.

2 Proof of Theorem 1.9

Since each $f \in \mathcal{M}$ represents a map $\varphi : S^1 \to S^1$, we define an orbit of f as a set of the form

$$\left\{f^{k}(x_0) + m : \ k \in \mathbb{N}\ ,\ m \in \mathbb{Z}\right\},$$

where $x_0 \in \mathbb{R}$. This orbit is periodic if there exist a $p \in \mathbb{Z}$ and a $q \in \mathbb{Z}^{+}$ such that $f^{q}(x_0) = x_0 + p$.

We know from [1] that any $f \in \mathcal{D}$ has at least one periodic orbit. Also that if $f_0, f_1 \in \mathcal{D}$ are such that $d_H(f_0, f_1) = 0$ with $\nu(f_0) \neq \nu(f_1)$, then any periodic orbit of f_0 intersects all periodic orbits of f_1 (if this were not the case it would be possible to construct g by modifying only the discontinuities of f_0 in such a way that g maintains the periodic orbit of f_0 and also has one of f_1 in contradiction to the uniqueness of the rotation number).

In the proofs bellow, we will mainly use these facts and the following trivial property:

$$\text{if } d_H(f, g) = 0 \text{ and } x < y, \text{ then } f(x) \leqslant g(y).$$

Proposition 2.1 *Suppose that $f, f_1 \in \mathcal{D}$ are such that $d_H(f, f_1) = 0$ and write $\frac{p}{q} = \nu(f)$ and $\frac{p_1}{q_1} = \nu(f_1)$ as irreducible fractions. If $\frac{p+1}{q} = \frac{p_1}{q_1}$, then every periodic orbit of f_1 is contained in a periodic orbit of f and every periodic orbit of f contains a periodic orbit of f_1.*

Proof. Let x_0 be a point common to a periodic orbit of f and f_1. So that

$$f^q(x_0) = x_0 + p \quad \text{and} \quad f_1^{q_1}(x_0) = x_0 + p_1.$$

Let x_j be an increasing enumeration of the periodic orbit of f passing through x_0, that is,

$$\{x_j : j \in \mathbb{Z}\} = \left\{ f^k(x_0) + m : \ k \in \mathbb{N} \ , \ m \in \mathbb{Z} \right\},$$

with $x_j < x_{j+1}$ for all $j \in \mathbb{Z}$. Hence

$$x_{j+q} = x_j + 1 \quad \text{and} \quad f(x_j) = x_{j+p}.$$

We will prove the proposition by showing that for every $k \in \mathbb{N}$ we have

$$f_1^k(x_0) = x_{k(p+1)}. \tag{2}$$

Let us first see that this relation (2) is true when k is a multiple of qq_1. In fact (using $\frac{p+1}{q} = \frac{p_1}{q_1}$)

$$f_1^{nqq_1}(x_0) = x_0 + nqp_1 = x_0 + nq_1(p+1) = x_{nqq_1(p+1)}.$$

On the other hand, if $x_{k(p+1)} \leqslant f_1^k(x_0)$ is true for some $k \in \mathbb{Z}^+$, then

$$f\left(x_{(k-1)(p+1)}\right) = x_{k(p+1)-1} < x_{k(p+1)} \leqslant f_1^k(x_0) = f_1 \circ f_1^{k-1}(x_0)$$

and therefore we must have $x_{(k-1)(p+1)} \leqslant f_1^{k-1}(x_0)$. Then we conclude by descending induction that for every $k \in \mathbb{N}$ we have

$$x_{k(p+1)} \leqslant f_1^k(x_0).$$

Also, if $f_1^k(x_0) \leqslant x_{k(p+1)}$ is true for some $k \in \mathbb{Z}^+$, then, since $x_{k(p+1)} < x_{k(p+1)+1}$, we obtain

$$f_1^{k+1}(x_0) \leqslant f_1\left(x_{k(p+1)}\right) \leqslant f\left(x_{k(p+1)+1}\right) = x_{k(p+1)+1+p} = x_{(k+1)(p+1)},$$

which proves by induction that the relation

$$f_1^k(x_0) \leqslant x_{k(p+1)}$$

is also true for all $k \in \mathbb{N}$; so that the proposition is proved. ∎

Proposition 2.2 *Suppose that $f_0, f \in \mathcal{D}$ are such that $d_H(f_0, f) = 0$ and write $\frac{p_0}{q_0} = \nu(f_0)$ and $\frac{p}{q} = \nu(f)$ as irreducible fractions. If $\frac{p_0}{q_0} = \frac{p-1}{q}$, then every periodic orbit of f_0 is contained in a periodic orbit of f and every periodic orbit of f contains a periodic orbit of f_0.*

Proof. Similarly to the proof of Proposition 2.1, using the same notation for x_j, where now x_0 is a point common to a periodic orbit of f and f_0, we prove successively that $f_0^{nqq_0}(x_0) = x_{nqq_0(p-1)}$, that $f_0^k(x_0) \leqslant x_{k(p-1)}$ by descending induction and that $x_{k(p-1)} \leqslant f_0^k(x_0)$ by usual induction. We obtain

$$f_0^k(x_0) = x_{k(p-1)} \tag{3}$$

for all $k \in \mathbb{N}$, which was what we wanted to prove. ∎

Proposition 2.3 *Suppose that* $f \in \mathcal{D}$, $\frac{p}{q} = \nu(f)$ *is an irreducible fraction,* $\frac{p-1}{q} = \nu(f^-)$ *and* $\frac{p+1}{q} = \nu(f^+)$. *Then* p *is odd.*

Proof. Let x_0 be a point common to a periodic orbit of f^- and f^+. By the previous propositions, x_0 belongs to a periodic orbit of f which, as before, we denote by $\{x_j : j \in \mathbb{Z}\}$ with $x_j < x_{j+1}$. Using the relations (2) and (3) we have for every $k \in \mathbb{N}$

$$f^{-k}(x_0) = x_{k(p-1)} \quad \text{and} \quad f^{+k}(x_0) = x_{k(p+1)}.$$

Let us first note that $p \neq 0$; in fact Theorem 1.3 applied to f^- and f^+ shows in particular that $\nu(f^-)\nu(f^+) \geqslant 0$ and then $\nu(f^-) < \frac{p}{q} < \nu(f^+)$ implies $p \neq 0$.

If p were even, then $p-1$ and $p+1$ would be coprime of the same sign, so that there would exist k_0 and k_1 in $\mathbb{N}$ such that

$$k_0(p-1) - k_1(p+1) = 1.$$

Hence $x_{k_1(p+1)} < x_{k_0(p-1)}$ and therefore

$$x_{(k_1+1)(p+1)} = f^+\left(x_{k_1(p+1)}\right) \leqslant f^-\left(x_{k_0(p-1)}\right) = x_{(k_0+1)(p-1)},$$

in contradiction to

$$(k_0+1)(p-1) - (k_1+1)(p+1) = -1 < 0,$$

which implies $x_{(k_0+1)(p-1)} < x_{(k_1+1)(p+1)}$. Hence p must be odd. ■

We can now easily prove Theorem 1.9 by applying Proposition 2.3 to f and $f+1$, which have rotation numbers $\nu(f) = \frac{p}{q}$ and $\nu(f+1) = \frac{p+q}{q}$. We find that p and $p+q$ are odd, and therefore q is even.

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