

NONLOCAL HARNACK INEQUALITIES FOR NONLOCAL HEAT EQUATIONS

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ABSTRACT. By applying the De Giorgi-Nash-Moser theory, we obtain nonlocal Harnack inequalities for weak solutions of nonlocal parabolic equations given by an integro-differential operator L_K as follows;

$$\begin{cases} L_K u + \partial_t u = 0 & \text{in } \Omega \times (-T, 0] \\ u = g & \text{in } ((\mathbb{R}^n \setminus \Omega) \times (-T, 0]) \cup (\Omega \times \{t = -T\}) \end{cases}$$

where $g \in C(\mathbb{R}^n \times [-T, 0]) \cap L^\infty(\mathbb{R}^n \times (-T, 0]) \cap H_T^s(\mathbb{R}^n)$ and Ω is a bounded domain in \mathbb{R}^n with Lipschitz boundary. Moreover, we get nonlocal parabolic weak Harnack inequalities of the weak solutions.

CONTENTS

1. Introduction	1
2. Preliminaries	4
3. Maximum principle and comparison principle	7
4. Weak and viscosity solutions	8
5. Nonlocal weak Harnack inequality	14
6. Parabolic fractional Poincaré inequality	20
7. Nonlocal parabolic Harnack inequality	28
8. Appendix: Existence and uniqueness of weak solutions	39
References	44

1. INTRODUCTION

The study of fractional and nonlocal equations has recently been done not only in pure mathematical analysis area but also in the research that needs its concrete applications. The aim of this paper is to obtain nonlocal Harnack inequalities for weak solutions of nonlocal heat equations.

Let \mathcal{K}_0 be the collection of all positive symmetric kernels satisfying the uniformly ellipticity assumption

$$(1.1) \quad \frac{(1-s)\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{(1-s)\Lambda}{|y|^{n+2s}}, \quad 0 < s < 1.$$

Here the symmetricity means that $K(y) = K(-y)$ for all $y \in \mathbb{R}^n$. Then we consider the corresponding nonlocal operator L_K given by

$$(1.2) \quad L_K u(x, t) = \text{p.v.} \int_{\mathbb{R}^n} \mu_t(u, x, y) K(y) dy$$

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where $\mu_t(u, x, y) = 2u(x, t) - u(x + y, t) - u(x - y, t)$. Set $\mathfrak{L}_0 = \{L_K : K \in \mathcal{K}_0\}$. In particular, if $K(y) = c_{n,s}|y|^{-n-2s}$, $s \in (0, 1)$, where $c_{n,s}$ is the normalization constant comparable to $s(1-s)$ given by

$$c_{n,s} = \frac{1}{2} \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+2s}} d\xi,$$

then $L_K = (-\Delta)^s$ is the fractional Laplacian and it is well-known that

$$\lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$$

for any function u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

In this paper, we study the boundary value problem for the following nonlocal parabolic equations $\mathbf{NP}_{\Omega_I}(f, g, h)$

$$(1.3) \quad \begin{cases} L_K u + \partial_t u = f & \text{in } \Omega_I := \Omega \times I \\ u = g & \text{in } (\mathbb{R}^n \setminus \Omega) \times I \\ u(x, -T) = h(x) & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $I := (-T, 0]$ and Ω is a bounded domain in \mathbb{R}^n with Lipschitz boundary. More precisely speaking, by employing the De Giorgi–Nash–Moser theory, we obtain nonlocal parabolic Harnack inequalities for weak solutions of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ where $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ for $I_* := [-T, 0]$, and also we get nonlocal parabolic weak Harnack inequalities of the weak solutions.

Notations. We write the notations briefly for the readers as follows.

- For $r > 0$ and $s \in (0, 1)$, let us denote by $Q_r^0 := Q_r(x_0, t_0) = B_r^0 \times I_{r,s}(t_0)$ and $Q_r = Q_r(0, 0)$, where $B_r^0 = B_r(x_0)$, $B_r = B_r(0)$ and $I_{r,s}(t_0) = (t_0 - r^{2s}, t_0]$. Also, we denote by $\mathcal{Q}_r^0 = \mathcal{Q}_r(x_0, t_0) = B_r^0 \times (t_0 - (2 - \sigma)r^{2s}, t_0]$, $\mathcal{Q}_r = \mathcal{Q}_r(0, 0)$,

$$(Q_r^0)^+ := Q_r^+(x_0, t_0) = B_r^0 \times I_{r,s}^+(t_0), \quad Q_r^+(0, 0) := Q_r^+,$$

$$(Q_r^0)^- := Q_r^-(x_0, t_0) = B_r^0 \times I_{r,s}^-(t_0), \quad Q_r^-(0, 0) := Q_r^-,$$

where $\sigma \in (0, 1/2)$ is a constant to be given in Section 6,

$$I_{r,s}^+(t_0) = (t_0 - \sigma r^{2s}, t_0] \text{ and } I_{r,s}^-(t_0) = \left(t_0 - \left(\frac{1}{2} + \sigma \right) r^{2s}, t_0 - \frac{1}{2} r^{2s} \right].$$

For simplicity, we denote by $I_{r,s}(0) = I_{r,s}$, $I_{r,s}^+(0) = I_{r,s}^+$ and $I_{r,s}^-(0) = I_{r,s}^-$.

- For two quantities a and b , we write $a \lesssim b$ (resp. $a \gtrsim b$) if there is a universal constant $C > 0$ (depending only on λ, Λ, n, s and ϵ) such that $a \leq Cb$ (resp. $b \leq Ca$).

- For $a, b \in \mathbb{R}$, we denote by $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

- Let \mathcal{F}_T^n and \mathcal{F}^n be the families of all real-valued Lebesgue measurable functions on $\mathbb{R}^n \times (-T, 0]$ and \mathbb{R}^n , respectively. For $u \in \mathcal{F}_T^n$, we write $[\mathbf{u}(t)](x) := u(x, t)$ and $[L_K \mathbf{u}(t)](x) = L_K u(x, t)$. Let $H_T^s(\mathbb{R}^n)$ denote the function space consisting of all functions $u \in \mathcal{F}_T^n$ such that $\mathbf{u}(t) \in H^s(\mathbb{R}^n)$ for all $t \in I_*$.

- For $(x_0, t_0) \in \Omega_I$ and $r > 0$, we now define the *nonlocal parabolic tail* of the function u in $Q_r^0 \subset \Omega_I$ by

$$(1.4) \quad \mathcal{T}_r(u; (x_0, t_0)) = \frac{2s}{|S^{n-1}|} r^{2s} \sup_{t \in I_{r,s}(t_0)} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|u(y, t)|}{|y - x_0|^{n+2s}} dy.$$

The first Harnack-type inequality for globally nonnegative weak solutions of local heat equations given on $\Omega \times I$ was obtained independently by Hadamard [H] and

Pini [P]. After this, the major influential contribution to the study in this direction (in fact, for local parabolic equations of divergence type given in $\Omega \times I$) was made by J. Moser [M1, M2]. Interestingly, the phenomenon that the classical Harnack inequality no longer works for nonlocal elliptic operators was recently observed by Kassmann [K1]. This unexpected fact motivated lots of mathematicians to study the so-called *nonlocal Harnack inequality*.

We now state our main results which are called *nonlocal Harnack inequalities* and *weak Harnack inequalities* for weak solutions of nonlocal heat equations as follows. Their proofs can be obtained from Theorem 7.4, 7.5, Corollary 7.6 and Appendix.

Theorem 1.1. *Let $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n) \cap H_T^s(\mathbb{R}^n)$. If $u \in H^1(L; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ with $u \geq 0$ in $Q_R^0 \subset \Omega_I$, then there exists a constant $c > 0$ depending only on n, s, λ and Λ such that*

$$(1.5) \quad \sup_{(Q_r^0)^-} u \leq c \inf_{(Q_r^0)^+} u + c \left(\frac{r}{R} \right)^{2s} \mathcal{T}_r(u^-; (x_0, t_0))$$

for any $r \in (0, R/5)$.

We can easily obtain the following nonlocal parabolic Harnack inequalities for a nonnegative weak solutions of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ as a natural by-product of Theorem 1.1. By the way, it is interesting that the result has no nonlocal parabolic tail. That means that the result coincides with that of local parabolic case.

Corollary 1.2. *Let $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n) \cap H_T^s(\mathbb{R}^n)$. If $u \in H^1(L; X_g(\Omega))$ is any nonnegative weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$, then there exists a constant $c > 0$ depending only on n, s, λ and Λ such that*

$$(1.6) \quad \sup_{(Q_r^0)^-} u \leq c \inf_{(Q_r^0)^+} u$$

for any $r \in (0, R/5)$.

Theorem 1.3. *Let $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n) \cap H_T^s(\mathbb{R}^n)$. If $u \in H^1(L; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ with $u \geq 0$ in $Q_R^0 \subset \Omega_I$, then we have the estimate*

$$(1.7) \quad \left(\frac{1}{2|(Q_r^0)^+|} \int_{(Q_r^0)^+} u^p dx dt \right)^{\frac{1}{p}} \leq \inf_{(Q_r^0)^+} u + \frac{4}{3} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_r(u^-; (x_0, t_0))$$

for any $p \in (0, 1)$ and $r \in (0, R)$.

Remark. (a) In case that $\Omega = \mathbb{R}^n$, using the De Giorgi method, Caffarelli, Chan and Vasseur proved in [CCV] that any weak solution to the equation (1.3) with initial data h in $L^2(\mathbb{R}^n)$ is uniformly bounded and Hölder continuous.

(b) The elliptic result of this problem was obtained by Di Castro, Kuusi and Palatucci [DKP]. As a matter of fact, when $p \in (1, \infty)$, they proved nonlocal Harnack inequalities for elliptic nonlocal p -Laplacian equations there. Also, they obtained Hölder regularity in [DKP1].

(c) Using the Moser's iteration method, Felsinger and Kassmann obtained weak parabolic Harnack inequality and Hölder regularity in [FK]. Also, despite of failure for getting the classical Harnack inequality as mentioned above, the first attempt to obtain the nonlocal Harnack inequalities of the form (1.5) for the fractional elliptic equations was tried by Kassmann (see [K2]).

(d) The Hölder continuity of weak solutions of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ with $u \geq 0$ in $Q_R^0 \subset \Omega_I$ and $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n) \cap H_T^s(\mathbb{R}^n)$ was obtained in [K].

(e) In [BBK], Barlow, Bass and Kumagai gave a probabilistic proof for parabolic Harnack inequality by using the connection between stochastic processes and equations similar to the nonlocal equation (1.3).

(f) In [BSV], Bonforte, Sire and Vázquez established an optimal existence and uniqueness theory for the Cauchy problem for the fractional heat equations given in $\mathbb{R}^n \times I$.

The paper is organized as follows. In Section 2, we furnish the function spaces and the definition of weak solutions of the nonlocal parabolic equations given in (1.3), and also give two well-known useful lemmas. The maximum principle and comparison principle of weak solutions for the nonlocal heat equations are obtained in Section 3. In Section 4, we obtain a relation between weak solutions and viscosity solutions of the nonlocal heat equations, which makes its weak solutions possible to enjoy the previous nice results on its viscosity solutions. In Section 5, we get nonlocal weak Harnack inequality for the nonlocal heat equation which is useful in proving its nonlocal parabolic Harnack inequality. It turned out that, in the elliptic case, Poincaré inequality was one of the crucial tools for the proof of classical Harnack inequality and no longer depends on the given partial differential equations. However, the fractional Poincaré inequality in the parabolic sense is not available for a general weak solution $u \in L^2(I, X_0(\Omega))$. In Section 6, we obtain parabolic fractional Poincaré inequality depending on the nonlocal heat equations. In Section 7, we establish the proof of nonlocal parabolic Harnack inequality by applying the De Giorgi-Nash-Moser theory [D, N, M]. Finally, in Appendix, we give the proof of the existence and uniqueness for weak solutions of the nonlocal heat equations which is based on the results for the weak formulation of the nonlocal eigenvalue problem of elliptic type [SV] and the Galerkin's method.

2. PRELIMINARIES

Let Y be a real Banach space with norm $\|\cdot\|$ and let \mathcal{F}_T^Y be the family of all measurable vector-valued functions $\mathbf{u} : I \rightarrow Y$. For $1 \leq p \leq \infty$, we introduce vector-valued function spaces $L^p(I; Y) = \{\mathbf{u} \in \mathcal{F}_T^Y : \|\mathbf{u}\|_{L^p(I; Y)} < \infty\}$, where

$$(2.1) \quad \begin{aligned} \|\mathbf{u}\|_{L^p(I; Y)} &:= \left(\int_{-T}^0 \|\mathbf{u}(t)\|^p dt \right)^{1/p} \quad \text{for } 1 \leq p < \infty, \\ \|\mathbf{u}\|_{L^\infty(I; Y)} &:= \text{ess sup}_{t \in I} \|\mathbf{u}(t)\| \quad \text{for } p = \infty. \end{aligned}$$

We also consider the function space $C(I; Y)$ consisting of all functions $u \in \mathcal{F}_T^n$ such that $\mathbf{u} : I \rightarrow Y$ a continuous vector-valued function satisfying

$$(2.2) \quad \|\mathbf{u}\|_{C(I; Y)} := \sup_{t \in I} \|\mathbf{u}(t)\| < \infty.$$

Let $\mathbf{u} \in L^1(I; Y)$ Then we say that $\mathbf{v} \in L^1(I; Y)$ is the *weak derivative* of \mathbf{u} and we write $\mathbf{v} = \mathbf{u}'$ if

$$\int_{-T}^0 \varphi'(t) \mathbf{u}(t) dt = - \int_{-T}^0 \varphi(t) \mathbf{v}(t) dt$$

for all testing functions $\varphi \in C_c^\infty(I)$.

Let Ω be a bounded domain in \mathbb{R}^n with Lipschitz boundary and let $K \in \mathcal{K}_0$. Let $X(\Omega)$ be the linear function space of all Lebesgue measurable functions $v \in \mathcal{F}^n$ such that $v|_\Omega \in L^2(\Omega)$ and

$$\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy < \infty$$

where $\mathbb{R}^{2n} := \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)$. We also set

$$(2.3) \quad X_0(\Omega) = \{v \in X(\Omega) : v = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

Since $C_0^2(\Omega) \subset X_0(\Omega)$, we see that $X(\Omega)$ and $X_0(\Omega)$ are not empty. Then we see that $(X(\Omega), \|\cdot\|_{X(\Omega)})$ is a normed space, where the norm $\|\cdot\|_{X(\Omega)}$ defined by

$$(2.4) \quad \|v\|_{X(\Omega)} := \|v\|_{L^2(\Omega)} + \left(\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2} < \infty, \quad v \in X(\Omega).$$

We denote by $H^s(\Omega)$ the usual fractional Sobolev space with the norm

$$(2.5) \quad \|v\|_{H^s(\Omega)} := \|v\|_{L^2(\Omega)} + [v]_{H^s(\Omega)} < \infty$$

where the seminorm $[\cdot]_{H^s(\Omega)}$ is defined by

$$[v]_{H^s(\Omega)} := [v]_{W^{s,2}(\Omega)} = \left(\iint_{\Omega \times \Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

Then it is well-known [SV] that

$$(2.6) \quad \|v\|_{H^s(\Omega)} \leq \|v\|_{H^s(\mathbb{R}^n)} \leq c(\lambda, s) \|v\|_{X(\Omega)}$$

for any $v \in X_0(\Omega)$, where $c(\lambda, s) = \max\{1, [\lambda(1-s)]^{-1/2}\}$. Also, there is a constant $c_0 > 1$ depending only on n, λ, σ and Ω such that

$$(2.7) \quad \|v\|_{X_0(\Omega)}^2 \leq \|v\|_X^2 \leq c_0 \|v\|_{X_0(\Omega)}^2$$

for any $v \in X_0(\Omega)$; that is,

$$(2.8) \quad \|v\|_{X_0(\Omega)} := \left(\iint_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}$$

is a norm on $X_0(\Omega)$ equivalent to (2.4). Moreover it is known that $(X_0(\Omega), \|\cdot\|_{X_0(\Omega)})$ is a Hilbert space with inner product

$$(2.9) \quad \langle u, v \rangle_{X_0(\Omega)} := \iint_{\mathbb{R}^{2n}} (u(x) - u(y))(v(x) - v(y)) d_K(x, y)$$

where $d_K(x, y) := K(x - y) dx dy$. Let $X_0^*(\Omega)$ be the dual space of $X_0(\Omega)$, i.e. the family of all bounded linear functionals on $X_0(\Omega)$. Then we see that $X_0(\Omega) \subset X(\Omega) \subset L^2(\Omega) \subset X_0^*(\Omega)$ and $(X_0^*(\Omega), \|\cdot\|_{X_0^*(\Omega)})$ is a normed space, where the norm $\|\cdot\|_{X_0^*(\Omega)}$ is given by

$$\|u\|_{X_0^*(\Omega)} := \sup\{u(v) : v \in X_0(\Omega), \|v\|_{X_0(\Omega)} \leq 1\}, \quad u \in X_0^*(\Omega).$$

In what follows, for a Banach space $(B, \|\cdot\|_B)$ with its dual space $(B^*, \|\cdot\|_{B^*})$, we consider a vector-valued Banach space

$$H^1(I; B) = \{u \in L^2(I; B) : u' \in L^2(I; B^*)\}$$

with the norm

$$\|u\|_{H^1(I;B)} = \left(\int_{-T}^0 \|u(t)\|_B^2 dt \right)^{\frac{1}{2}} + \left(\int_{-T}^0 \|u'(t)\|_{B^*}^2 dt \right)^{\frac{1}{2}} < \infty.$$

For $g \in H^s(\mathbb{R}^n)$, we consider the convex subsets of $H^s(\mathbb{R}^n)$ by

$$\begin{aligned} X_g^\pm(\Omega) &= \{v \in H^s(\mathbb{R}^n) : (g-v)^\pm \in X_0(\Omega)\}, \\ X_g(\Omega) &:= X_g^+(\Omega) \cap X_g^-(\Omega) = \{v \in H^s(\mathbb{R}^n) : g-v \in X_0(\Omega)\}. \end{aligned}$$

For $g \in H_T^s(\mathbb{R}^n)$, we define the *convex subsets* of the space $H_T^s(\mathbb{R}^n)$ by

$$\begin{aligned} H^1(I; X_g^\pm(\Omega)) &= \{u \in H_T^s(\mathbb{R}^n) : (g-u)_\pm \in H^1(I; X_0(\Omega))\}, \\ H^1(I; X_g(\Omega)) &:= H^1(I; X_g^+(\Omega)) \cap H^1(I; X_g^-(\Omega)) \\ &= \{u \in H_T^s(\mathbb{R}^n) : u-g \in H^1(I; X_0(\Omega))\}. \end{aligned}$$

Remark. If $u \in H^1(I; X_0(\Omega))$, then it is well-known that (a) $u \in C(I; L^2(\Omega))$ after being modified on a set of measure zero, (b) the function α defined by $\alpha(t) = \|u(t)\|_{L^2(\Omega)}^2$ is absolutely continuous, and moreover $\alpha'(t) = 2\langle u'(t), u(t) \rangle_{L^2(\Omega)}$ for a.e. $t \in I$, and (c) there is a constant $C > 0$ depending only on T such that

$$\sup_{t \in I} \|u(t)\|_{L^2(\Omega)} \leq C \|u\|_{H^1(I; X_0(\Omega))}.$$

In order to define weak solutions, we consider a bilinear form defined by

$$\langle u, v \rangle_K = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) K(x-y) dx dy \quad \text{for } u, v \in X(\Omega).$$

Definition 2.1. Let $g \in H_T^s(\mathbb{R}^n)$ and $f \in L^2(I; X_0^*(\Omega))$. Then we say that a function $u \in H^1(I; X_g^-(\Omega))$ ($u \in H^1(I; X_g^+(\Omega))$) is a weak subsolution (weak supersolution) of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(f, g, h)$ given in (1.3), if it satisfies

$$(2.10) \quad \langle u(t), \varphi \rangle_K + \langle u'(t) - f(t), \varphi \rangle \leq 0 \quad (\geq 0)$$

for any nonnegative $\varphi \in X_0(\Omega)$ and a.e. $t \in I$, and

$$(2.11) \quad u(-T) = h,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pair between $X_0(\Omega)$ and $X_0^*(\Omega)$. Also, we say that a function u is a weak solution of the equation $\mathbf{NP}_{\Omega_I}(f, g, h)$, if it is both a weak subsolution and a weak supersolution, i.e.

$$(2.12) \quad \langle u(t), \varphi \rangle_K + \langle u'(t) - f(t), \varphi \rangle = 0$$

for any $\varphi \in X_0(\Omega)$ and a.e. $t \in I$, and $u(-T) = h$.

In order to prove our results, we need two well-known lemmas to be given in the following (see [GT]).

Lemma 2.2. Let $\{N_k\}_{k=0}^\infty \subset \mathbb{R}$ be a sequence of positive numbers such that

$$N_{k+1} \leq d_0 e_0^k N_k^{1+\eta}$$

where $d_0, \eta > 0$ and $e_0 > 1$. If $N_0 \leq d_0^{-1/\eta} e_0^{-1/\eta^2}$, then we have that $N_k \leq e_0^{-k/\eta} N_0$ for any $k = 0, 1, \dots$ and moreover $\lim_{k \rightarrow \infty} N_k = 0$.

Lemma 2.3. *Let f be a nonnegative bounded function defined in $[t_0, t_1]$, where $0 \leq t_0 < t_1$. Suppose that there are nonnegative constants c_1, c_2, θ , and $\eta \in (0, 1)$ such that*

$$f(t) \leq c_1(\tau - t)^{-\theta} + c_2 + \eta f(\tau)$$

for any $t, \tau \in [t_1, t_2]$ with $t < \tau$. Then there exists a constant $c > 0$ (depending only on θ and η) such that

$$f(\rho) \leq c[c_1(R - \rho)^{-\theta} + c_2]$$

for any $\rho, R \in [t_1, t_2]$ with $\rho < R$.

3. MAXIMUM PRINCIPLE AND COMPARISON PRINCIPLE

In this section, we furnish Maximum Principle and Comparison Principle for weak solutions of the nonlocal parabolic equations $\mathbf{NP}_{\mathcal{O}_J}(0, g, g)$ where $\mathcal{O} \subset \mathbb{R}^n$ is a bounded open set and $J := [a, b) \subset I$ is a half-open interval. We denote by $\mathbb{R}_{I_*}^n := \mathbb{R}^n \times I_*$ for $I_* = [-T, 0]$.

Lemma 3.1. *If $u \in H^1(J; X_g^-(\mathcal{O}))$ is a weak subsolution of the nonlocal parabolic equation $\mathbf{NP}_{\mathcal{O}_J}(0, g, g)$ given in (1.3) and $u = g \leq 0$ in $\mathbb{R}_{I_*}^n \setminus \mathcal{O}_J$, then $u \leq 0$ in $\mathbb{R}^n \times I$.*

Proof. By the assumption, we see that $u^+ = 0$ in $(\mathbb{R}^n \setminus \mathcal{O}) \times J$, and thus $u^+ \in H^1(J; X_0(\mathcal{O}))$. Thus we can use u^+ as a testing function in the weak formulation. Observing that $u^+(x, t)u^-(x, t) = 0$ and $u^+(x, t)u^-(y, t) \geq 0$ for a.e. $x, y \in \mathbb{R}^n$ and $t \in J$, it follows from the fractional sobolev inequality that

$$\begin{aligned} 0 &\geq \langle \mathbf{u}(t), \mathbf{u}^+(t) \rangle_{X_0(\mathcal{O})} + \langle \partial_t \mathbf{u}(t), \mathbf{u}^+(t) \rangle_{L^2(\mathcal{O})} \\ &= \|\mathbf{u}^+(t)\|_{X_0(\mathcal{O})}^2 - \langle \mathbf{u}^-(t), \mathbf{u}^+(t) \rangle_{X_0(\mathcal{O})} \\ &\quad + \partial_t (\langle \mathbf{u}^+(t), \mathbf{u}^+(t) \rangle_{L^2(\mathcal{O})} - \langle \mathbf{u}^-(t), \mathbf{u}^+(t) \rangle_{L^2(\mathcal{O})}) \\ &\geq c \|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2 + \partial_t \|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2 \\ &\quad - \iint_{\mathbb{R}_{\mathcal{O}}^{2n}} \frac{(u^-(x, t) - u^-(y, t))(u^+(x, t) - u^+(y, t))}{|x - y|^{n+2s}} dx dy \\ &\geq c \|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2 + \partial_t \|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2 \\ &\quad + \iint_{\mathbb{R}_{\mathcal{O}}^{2n}} \frac{u^-(x, t)u^+(y, t) + u^+(x, t)u^-(y, t)}{|x - y|^{n+2s}} dx dy \\ &\geq c \|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2 + \partial_t \|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2. \end{aligned}$$

Thus, by Gronwall's inequality and the assumption on the initial values, we conclude that

$$\|\mathbf{u}^+(t)\|_{L^2(\mathcal{O})}^2 \leq e^{-c(t+T)} \|\mathbf{u}^+(-T)\|_{L^2(\mathcal{O})}^2 = 0.$$

This implies that $u^+ = 0$ in $\mathcal{O} \times J$. Hence we are done. \square

Corollary 3.2. *If $u \in H^1(J; X_g^+(\mathcal{O}))$ is a weak supersolution of the nonlocal parabolic equation $\mathbf{NP}_{\mathcal{O}_J}(0, g, g)$ given in (1.3) and $u = g \geq 0$ in $\mathbb{R}_{I_*}^n \setminus \mathcal{O}_J$, then $u \geq 0$ in $\mathbb{R}^n \times I$.*

Corollary 3.3. *If $u \in H^1(I; X_{g_1}^-(\mathcal{O}))$ is a weak subsolution of the nonlocal parabolic equation $\mathbf{NP}_{\mathcal{O}_J}(0, g_1, g_1)$ and $v \in H^1(I; X_{g_2}^+(\mathcal{O}))$ is a weak supersolution of the nonlocal parabolic equation $\mathbf{NP}_{\mathcal{O}_J}(0, g_2, g_2)$ such that $g_1 \leq g_2$ in $\mathbb{R}_{I_*}^n \setminus \mathcal{O}_J$, then $u \leq v$ in $\mathbb{R}^n \times I$.*

4. WEAK AND VISCOSITY SOLUTIONS

In this section, we get boundedness and continuity on \mathbb{R}^n of weak solutions of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ with boundary condition $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ where $\mathbb{R}_{I_*}^n := \mathbb{R}^n \times I_*$ for $I_* = [-T, 0]$ and we study a relation between weak solutions and viscosity solutions of the equation $\mathbf{NP}_{\Omega_I}(0, g, g)$. The latter one makes its weak solutions possible to enjoy the previous nice results on its viscosity solutions.

Let us define viscosity solutions. Let $\mathcal{P}(\mathbb{R}^{n+1})$ denote the class of all parabolic quadratic polynomials of the form

$$p(x, t) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j + \sum_{i=1}^n b_i x_i + ct + d,$$

where $a_{ij}, b_i, c, d \in \mathbb{R}$. A upper (lower) semicontinuous function $u : \mathbb{R}^n \times I \rightarrow \mathbb{R}$ is called a *viscosity subsolution* (*viscosity supersolution*) of the equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ and we write $L_K u + \partial_t u \leq 0$ (res. $L_K u + \partial_t u \geq 0$) on Ω_I in the viscosity sense, if for each $(x, t) \in \Omega_I$ there is a neighborhood $Q_r(x, t) \subset \Omega_I$ such that $L_K u + \partial_t u$ is well-defined and $L_K v(x, t) + \partial_t p(x, t) \leq 0$ (res. $L_K v(x, t) + \partial_t p(x, t) \geq 0$) for $v = p \mathbb{1}_{Q_r(x, t)} + u \mathbb{1}_{\mathbb{R}_I^n \setminus Q_r(x, t)}$ whenever $p \in \mathcal{P}(\mathbb{R}^{n+1})$ with $p(x, t) = u(x, t)$ and $p > u$ (res. $p < u$) on $Q_r(x, t) \setminus \{(x, t)\}$ exists. Moreover, a function $u : \mathbb{R}^n \times I \rightarrow \mathbb{R}$ is called a *viscosity solution*, if it is both a viscosity subsolution and a viscosity supersolution of the equation.

Theorem 4.1. *If $u \in H^1(I; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ for $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$, then $u \in L^\infty(\mathbb{R}_I^n)$.*

Proof. By multiplying u by a sufficiently small constant, we may assume that

$$(4.1) \quad \|u\|_{L^2(\Omega_I)}^2 \leq \delta$$

where $\delta > 0$ is some small constant to be determined later. Let $M = 2\|g\|_{L^\infty(\mathbb{R}_I^n)}$ and take any $a, b \in (0, 1)$ so that $a + b = 1$. For $k \in \mathbb{N}$, let $M_k = M(1 - 2^{-k})$, $u_k = u - M_k$, $w_k = u_k^+$ and $N_k = \|w_k\|_{L^2(\Omega_{I_k})}^2$, where $I_k = (-T_k, 0]$ for $T_k = (a + 2^{-k}b)T$ and $I_0 = I$ for $T_0 = T$. Then we see that, for any $k \in \mathbb{N}$, $M_{k+1} > M_k$ and $u_{k+1} < u_k$, and so $w_{k+1} \leq w_k$. Moreover, on $(\mathbb{R}^n \setminus \Omega) \times I$, we have that

$$u_{k+1} = g - M + 2^{-k-1}M \leq -\frac{M}{2} + 2^{-k-1}M \leq 0 \text{ for all } k = 0, 1, \dots$$

So we have that $w_{k+1} = 0$ on $(\mathbb{R}^n \setminus \Omega) \times I$. We now use $\varphi_{k+1} = w_{k+1} \eta_{k+1}^2$ as a testing function in the weak formulation of the equation, where $\eta_{k+1} \in C_c^\infty(-T_k, \infty)$ is a function such that $0 \leq \eta_{k+1} \leq 1$ in \mathbb{R} , $\eta_{k+1} = 1$ in $(-T_{k+1}, \infty)$ and $0 \leq \eta'_{k+1} \leq 2^{k+2}(bT)^{-1}$ in \mathbb{R} . Then we have that

$$\int_{-T_k}^\tau \int_{\Omega} (\partial_t u) \varphi_{k+1} dx dt + \mathbf{I}(u, \varphi_{k+1}) = 0$$

for any $\tau \in (-T_k, 0]$, where the bilinear operator is given by

$$\begin{aligned} & \mathbf{I}(u, \varphi_{k+1}) \\ &= \int_{-T_k}^\tau \eta_{k+1}^2(t) \iint_{\Omega \times \Omega} (u(x, t) - u(y, t)) (\varphi_{k+1}(x, t) - \varphi_{k+1}(y, t)) d_K(x, y) dt. \end{aligned}$$

The first term in the left-hand side of the above equality can be evaluated by

$$(4.2) \quad \begin{aligned} \int_{-T_k}^{\tau} \int_{\Omega} (\partial_t u) \varphi_{k+1} dx dt &= \int_{-T_k}^{\tau} \int_{\Omega} [\partial_t (w_{k+1}^2 \eta_{k+1}^2) - 2w_{k+1}^2 \eta_{k+1} \eta'_{k+1}] dx dt \\ &= \eta_{k+1}^2(\tau) \int_{\Omega} w_{k+1}^2(x, \tau) dx - 2 \int_{-T_k}^{\tau} \eta_{k+1}(t) \eta'_{k+1}(t) \int_{\Omega} w_{k+1}^2(x, t) dx dt. \end{aligned}$$

We next split $I(u, \varphi_{k+1})$ into two parts as follows;

$$(4.3) \quad \begin{aligned} &I(u, \varphi_{k+1}) \\ &= \int_{-T_k}^{\tau} \eta_{k+1}^2(t) \iint_{\Omega \times \Omega} (u(x, t) - u(y, t)) (\varphi_{k+1}(x, t) - \varphi_{k+1}(y, t)) d_K(x, y) dt \\ &\quad + 2 \int_{-T_k}^{\tau} \eta_{k+1}^2(t) \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} (u(x, t) - u(y, t)) \varphi_{k+1}(x, t) d_K(x, y) dt \\ &:= I_1 + 2I_2. \end{aligned}$$

For the estimate of I_1 , we first observe that

$$(4.4) \quad \begin{aligned} &(u(x, t) - u(y, t)) (\varphi_{k+1}(x, t) - \varphi_{k+1}(y, t)) \\ &\geq (w_{k+1}(x, t) - w_{k+1}(y, t)) (\varphi_{k+1}(x, t) - \varphi_{k+1}(y, t)) \end{aligned}$$

whenever $(x, t), (y, t) \in \Omega_I$; indeed, without loss of generality we may assume that $u(x, t) \geq u(y, t)$. Then it can easily be checked by considering two possible cases (i) $u(x, t) \geq u(y, t) > M_{k+1}$, (ii) $u(x, t) > M_{k+1}$, $u(y, t) \leq M_{k+1}$, and (iii) $M_{k+1} \geq u(x, t) \geq u(y, t)$. For the estimate of I_2 , we note that

$$\begin{aligned} (u(x, t) - u(y, t)) \varphi_{k+1}(x, t) &\geq -(u(y, t) - u(x, t))_+ (u(x, t) - M_{k+1})_+ \eta_{k+1}^2(t) \\ &\geq -(u(y, t) - M_{k+1})_+ (u(x, t) - M_{k+1})_+ \eta_{k+1}^2(t) \\ &= -w_{k+1}(y, t) w_{k+1}(x, t) \eta_{k+1}^2(t) \end{aligned}$$

and thus we have that

$$(4.5) \quad I_2 \geq - \int_{-T_k}^{\tau} \eta_{k+1}^2(t) \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} w_{k+1}(y, t) w_{k+1}(x, t) d_K(x, y) dt = 0.$$

Since the following equality

$$(4.6) \quad \begin{aligned} &(w_{k+1}(x, t) - w_{k+1}(y, t)) (\varphi_{k+1}(x, t) - \varphi_{k+1}(y, t)) \\ &= (w_{k+1}(x, t) - w_{k+1}(y, t))^2 \eta_{k+1}^2(t) \end{aligned}$$

is always true, it follows from (4.3), (4.4), (4.5) and (4.6) that

$$(4.7) \quad I(u, \varphi_{k+1}) \geq \int_{-T}^{\tau} \eta_{k+1}^2(t) \iint_{\Omega \times \Omega} (w_{k+1}(x, t) - w_{k+1}(y, t))^2 d_K(x, y) dt.$$

Thus it follows from (4.2) and (4.7) that

$$(4.8) \quad \begin{aligned} &\sup_{t \in (-T_k, 0]} \eta_{k+1}^2(t) \|w_{k+1}(\cdot, t)\|_{L^2(\Omega)}^2 \\ &\quad + \int_{-T_k}^0 \eta_{k+1}^2(t) \iint_{\Omega \times \Omega} (w_{k+1}(x, t) - w_{k+1}(y, t))^2 d_K(x, y) dt \\ &\leq 2 \int_{-T_k}^0 \eta_{k+1}(t) \eta'_{k+1}(t) \|w_{k+1}(\cdot, t)\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Applying a well-known parabolic version of the fractional Sobolev inequality to (4.8), we obtain that

$$\begin{aligned}
& \|w_{k+1}\|_{L^{2\alpha}(\Omega_{I_{k+1}})}^{2\alpha} \leq \iint_{\Omega_{I_k}} |\eta_{k+1}(t)w_{k+1}(x,t)|^{2\alpha} dx dt \\
(4.9) \quad & \leq c(2\|\eta'_{k+1}\|_{L^\infty(\mathbb{R})} + 1)^\alpha \left(\int_{-T_k}^0 \eta_{k+1}(t) \int_{\Omega} w_{k+1}^2(x,t) dx dt \right)^\alpha \\
& \leq c \frac{2^{k\alpha}}{(bT)^\alpha} \|w_k\|_{L^2(\Omega_{I_k})}^{2\alpha},
\end{aligned}$$

where $\alpha = 1 + \frac{2s}{n}$. Since $\{w_{k+1} > 0\} \subset \{w_k > 2^{-(k+1)}M\}$, we have that

$$\begin{aligned}
(4.10) \quad N_k & \geq \iint_{\Omega_{I_k} \cap \{w_k > 2^{-(k+1)}M\}} w_k^2(x,t) dx dt \\
& \geq 2^{-2(k+1)}M^2 |\Omega_{I_k} \cap \{w_k > 2^{-(k+1)}M\}| \\
& \geq 2^{-2(k+1)}M^2 |\Omega_{I_k} \cap \{w_{k+1} > 0\}|.
\end{aligned}$$

Thus, by (4.9), (4.10) and Hölder's inequality, we have that

$$\begin{aligned}
N_{k+1} & \leq \left(\iint_{\Omega_{I_{k+1}}} w_{k+1}^{2\alpha}(x,t) dx dt \right)^{1/\alpha} |\Omega_{I_k} \cap \{w_k > 0\}|^{\frac{2s}{n+2s}} \\
& \leq \frac{c}{bTM^{\frac{4s}{n+2s}}} 2^{(1+\frac{4s}{n+2s})k} N_k^{1+\frac{2s}{n+2s}}.
\end{aligned}$$

Set $d_0 = \frac{c}{bTM^{\frac{4s}{n+2s}}} > 1$, $e_0 = 2^{1+\frac{4s}{n+2s}} > 1$ and $\eta = \frac{4s}{n+2s} > 0$. If we select $\delta > 0$ so that

$$\delta \leq d_0^{-1/\eta} e_0^{-1/\eta^2},$$

then by (4.1) we have that $N_0 = \|u^+\|_{L^2(\Omega_I)}^2 = \|u\|_{L^2(\Omega_I)}^2 \leq d_0^{-1/\eta} e_0^{-1/\eta^2}$. Thus, by Lemma 2.2, we have that

$$0 = \lim_{k \rightarrow \infty} N_k = \|(u - M)_+\|_{L^2(\Omega_{(-aT,0]})}^2,$$

and hence $u \leq M$ in $\Omega_{(-aT,0]}$. Also, by applying $-u$ instead of u , we have that $u \geq -M$ in $\Omega_{(-aT,0]}$. Thus we have that $|u| \leq M$ in $\Omega_{(-aT,0]}$ for any $a, b \in (0, 1)$ with $a + b = 1$. Therefore, taking $a \downarrow 1$, we obtain that

$$\|u\|_{L^\infty(\Omega_I)} \leq M,$$

and we conclude that

$$\|u\|_{L^\infty(\mathbb{R}_I^n)} \leq 2\|g\|_{L^\infty(\mathbb{R}_I^n)} < \infty.$$

Hence we complete the proof. \square

In the next theorem, we obtain the global continuity of weak solutions of the nonlocal heat equations with certain boundary condition whose proof is based on the idea of that of the elliptic case [SV1].

Theorem 4.2. *If $u \in H^1(I; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ for $g \in C(\mathbb{R}_I^n) \cap L^\infty(\mathbb{R}_I^n)$, then $u \in C(\mathbb{R}_I^n)$.*

Proof. For a contrapositive proof, we assume that there exists some $(x_0, t_0) \in \mathbb{R}_{I_*}^n$ and sequence $(x_k, t_k) \in \mathbb{R}_{I_*}^n$ with $\lim_{k \rightarrow \infty} (x_k, t_k) = (x_0, t_0)$ such that

$$|u(x_k, t_k) - u(x_0, t_0)| \geq \eta_0$$

for some $\eta_0 > 0$. Without loss of generality, we may assume that

$$(4.11) \quad u(x_k, t_k) - u(x_0, t_0) \geq \eta_0;$$

for, the other case can be done in a similar way. Then we first claim that

$$(4.12) \quad (x_0, t_0) \in \partial_p \Omega_I.$$

Indeed, $(x_0, t_0) \in \overline{\Omega} \times I_*$, because u is continuous in $\mathbb{R}_{I_*}^n \setminus \Omega_I$. Moreover, it is impossible that $(x_0, t_0) \in \Omega \times I$, because we see from the local interior regularity results [FK] that u is continuous in any compact domain contained in Ω_I . This implies (4.12).

For $\varepsilon \in (0, 1]$, let Ω^ε be a smooth ε -neighborhood of Ω , i.e. a set with smooth boundary such that

$$(4.13) \quad \bigcup_{x \in \Omega} B_{\varepsilon/2}(x) \subset \Omega^\varepsilon \subset \bigcup_{x \in \Omega} B_\varepsilon(x).$$

If we consider the function

$$g_\varepsilon(x, t) = (1 - \varphi_\varepsilon(x))u(x, t) + \varphi_\varepsilon(x)(t + 2T)$$

where the function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$ satisfies $\varphi_\varepsilon = 1$ in $\Omega_{\varepsilon/4}$ and $\varphi_\varepsilon = 0$ in $\mathbb{R}^n \setminus \Omega_{\varepsilon/2}$, then we see that $g_\varepsilon \in C(\mathbb{R}_{I_*}^n)$ and $g_\varepsilon = u$ in $\mathbb{R}_{I_*}^n \setminus \Omega_I^\varepsilon$. Let $u_\varepsilon \in H^1(I; X_{g_\varepsilon}(\Omega^\varepsilon))$ be a weak solution of the nonlocal parabolic equation

$$(4.14) \quad \begin{cases} \mathsf{L}_K u_\varepsilon + \partial_t u_\varepsilon = 0 & \text{in } \Omega_I^\varepsilon, \\ u_\varepsilon = g_\varepsilon = u & \text{in } \mathbb{R}_{I_*}^n \setminus \Omega_I^\varepsilon. \end{cases}$$

Then we see that $u_\varepsilon \in L^\infty(\mathbb{R}_{I_*}^n)$. From Theorem 1.2 [FK], there are a constant $\alpha \in (0, \sigma_0)$ (depending only on λ, Λ, n and $\sigma_0 \in (0, 2)$) such that

$$(4.15) \quad [u_\varepsilon]_{C^\alpha(Q)} \leq \frac{\|u_\varepsilon\|_{L^\infty(\mathbb{R}_{I_*}^n)}}{\eta_\varepsilon^\alpha}$$

for any $Q \Subset \Omega_I^\varepsilon$, where $\eta_\varepsilon = \eta_\varepsilon(Q) > 0$ is some constant depending on Q and Ω_I^ε . We consider the modulus of continuity ϱ of u_ε in Ω_I^ε defined by

$$\varrho(\beta) = \sup_{Q \Subset \Omega_I^\varepsilon} \sup_{\substack{(x,t), (y,\tau) \in Q \\ (|x-y|^{2s} + |t-\tau|)^{\alpha/2s} < \eta_\varepsilon^\alpha \beta / \|u_\varepsilon\|_{L^\infty(\mathbb{R}_{I_*}^n)}}} |u_\varepsilon(x, t) - u_\varepsilon(y, \tau)|.$$

Here we note that ϱ no longer depend on ε ; indeed, if $(x, t), (y, \tau) \in Q$ for some $Q \Subset \Omega_I^\varepsilon$ with $(|x-y|^{2s} + |t-\tau|)^{\alpha/2s} < \eta_\varepsilon^\alpha \beta / \|u_\varepsilon\|_{L^\infty(\mathbb{R}_{I_*}^n)}$, then by (4.15) we have the estimate

$$\begin{aligned} & |u_\varepsilon(x, t) - u_\varepsilon(y, \tau)| \\ & \leq \frac{|u_\varepsilon(x, t) - u_\varepsilon(y, t)|}{|x-y|^\alpha} |x-y|^\alpha + \frac{|u_\varepsilon(y, t) - u_\varepsilon(y, \tau)|}{|t-\tau|^{\alpha/2s}} |t-\tau|^{\alpha/2s} < 2\beta, \end{aligned}$$

and thus we have that

$$(4.16) \quad \varrho(\beta) < 2\beta.$$

Let ρ be the modulus of continuity of u in the compact subset $(\overline{\Omega^1})_{I_*} \setminus \Omega_I$ of $\mathbb{R}_{I_*}^n \setminus \Omega_I$ defined by

$$\rho(\beta) = \sup_{\substack{(x,t),(y,\tau) \in (\overline{\Omega^1})_{I_*} \setminus \Omega_I \\ (|x-y|^{2s} + |t-\tau|)^{1/2s} < \beta}} |u(x,t) - u(y,\tau)|.$$

Set $\vartheta_\varepsilon = \varepsilon + \varrho(\varepsilon) + \rho(\varepsilon)$. By (4.15) and the continuity of u in $\mathbb{R}_{I_*}^n \setminus \Omega_I$, we see that

$$(4.17) \quad \lim_{\varepsilon \downarrow 0} \vartheta_\varepsilon = 0.$$

Furthermore, we have that

$$(4.18) \quad u_\varepsilon + \vartheta_\varepsilon > u \quad \text{in } \mathbb{R}_J^n \setminus \Omega_I.$$

Indeed, let us take any $(x,t) \in \mathbb{R}_{I_*}^n \setminus \Omega_I$. If $(x,t) \in [(\mathbb{R}^n \setminus \Omega_\varepsilon) \times I] \cup (\mathbb{R}^n \times \{-T\})$, then we see that $u_\varepsilon(x,t) = u(x,t)$, and so (4.18) works well. If $(x,t) \in (\Omega_\varepsilon \setminus \Omega) \times I$, then there is some $y \in \partial\Omega_\varepsilon \subset \mathbb{R}^n \setminus \Omega_\varepsilon$ such that $|x-y| \leq \varepsilon$. Thus we have that $u_\varepsilon(y,t) = u(y,t)$, $|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \leq \varrho(\varepsilon)$ and $|u(x,t) - u(y,t)| \leq \rho(\varepsilon)$. Hence we obtain that

$$\begin{aligned} u_\varepsilon(x,t) - u(x,t) &\geq u_\varepsilon(y,t) - u(y,t) - (\varrho(\varepsilon) + \rho(\varepsilon)) \\ &= -(\varrho(\varepsilon) + \rho(\varepsilon)) = -\vartheta_\varepsilon + \varepsilon > -\vartheta_\varepsilon, \end{aligned}$$

which gives (4.18).

If we set $w_\varepsilon = u_\varepsilon + \vartheta_\varepsilon - u$, then we show that

$$w_\varepsilon \geq 0 \quad \text{in } \mathbb{R}_I^n.$$

Indeed, w_ε is a weak supersolution of the nonlocal equation $L_K w_\varepsilon + \partial_t w_\varepsilon = 0$ in Ω_I with boundary condition (4.18), and thus it follows from Corollary 3.2.

Since $u \leq u_\varepsilon + \vartheta_\varepsilon$ in \mathbb{R}_I^n , by (4.11) we have that

$$(4.19) \quad \begin{aligned} \eta + u(x_0, t_0) &\leq u(x_k, t_k) \leq u_\varepsilon(x_k, t_k) + \vartheta_\varepsilon \\ &\leq u_\varepsilon(x_0, t_0) + 2\vartheta_\varepsilon \end{aligned}$$

for a fixed $\varepsilon \in (0, 1]$ and a sufficiently large k . By (4.13), we see that there is some $(y_\varepsilon, \tau_\varepsilon) \in \mathbb{R}_{I_*}^n \setminus \Omega_I^\varepsilon$ such that $(|y_\varepsilon - x_0|^{2s} + |\tau_\varepsilon - t_0|)^{1/2s} \leq \varepsilon$. Since $u_\varepsilon(y_\varepsilon, \tau_\varepsilon) = u(y_\varepsilon, \tau_\varepsilon)$, it follows from (4.19) that

$$(4.20) \quad \eta + u(x_0, t_0) \leq u(y_\varepsilon, \tau_\varepsilon) + u_\varepsilon(x_0, t_0) - u_\varepsilon(y_\varepsilon, \tau_\varepsilon) + 2\vartheta_\varepsilon \leq u(y_\varepsilon, \tau_\varepsilon) + 3\vartheta_\varepsilon$$

for all sufficiently small $\varepsilon \in (0, 1]$. Since $(y_\varepsilon, \tau_\varepsilon), (x_0, t_0) \in (\overline{\Omega^1})_{I_*} \setminus \Omega_I$ and $\Omega_I^\varepsilon \subset \Omega_I^1$ for $\varepsilon \in (0, 1]$, by the continuity of u in $\mathbb{R}_{I_*}^n \setminus \Omega_I$ we have that

$$(4.21) \quad u(y_\varepsilon, \tau_\varepsilon) - u(x_0, t_0) \leq \rho(\varepsilon) \leq \vartheta_\varepsilon.$$

Thus by (4.20) and (4.21) we obtain that

$$\eta + u(x_0, t_0) \leq u(x_0, t_0) + 4\vartheta_\varepsilon,$$

and so $\eta \leq 4\vartheta_\varepsilon$. Taking $\varepsilon \downarrow 0$, we have that $\eta \leq 0$ by (4.15), which gives a contradiction. Therefore we conclude that $u \in C(\mathbb{R}_I^n)$. \square

Theorem 4.3. *If $u \in H^1(I; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ for $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$, then u is a viscosity solution of $\mathbf{NP}_{\Omega_I}(0, g, g)$.*

Proof. First, we show that any weak subsolution u of the nonlocal equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ for $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ is its viscosity subsolution. Take any $(x_0, t_0) \in \Omega_I$. For a contrapositive proof, by continuity property we may assume that there are some $r > 0$ with $Q_r(x_0, t_0) \subset \Omega_I$ and $p \in \mathcal{P}(\mathbb{R}^{n+1})$ such that $u(x_0, t_0) = p(x_0, t_0)$, $u(x, t) < p(x, t)$ and

$$(4.22) \quad \mathbf{L}_K v(x, t) + \partial_t p(x, t) > 0$$

for any $(x, t) \in Q_r(x_0, t_0) \setminus \{(x_0, t_0)\}$, where $v = p \mathbb{1}_{Q_r(x_0, t_0)} + u \mathbb{1}_{\mathbb{R}^n \setminus Q_r(x_0, t_0)}$. Then we see that $v \in C^2(Q_r(x_0, t_0)) \cap L^\infty(\mathbb{R}_I^n)$ by Theorem 4.1. Take any nonnegative testing function $\phi \in C_c^\infty(Q_r(x_0, t_0))$. Then we have the estimate

$$(4.23) \quad \begin{aligned} & \int_{\mathbb{R}^n} (\mathbf{L}_K \mathbf{v}(t)) \phi \, dx \\ &= \int_{B_r(x_0)} \left(\int_{|y| < r - |x - x_0|} + \int_{|y| \geq r - |x - x_0|} \right) \mu_t(u, x, y) K(y) \, dy \phi(x, t) \, dx \\ &\leq \frac{c \|\nabla p\|_{L^\infty(Q_r(x_0, t_0))}}{1 - s} \int_{B_r(x_0)} (r - |x - x_0|)^{2(1-s)} \phi(x, t) \, dx \\ &\quad + \frac{c \|v\|_{L^\infty(\mathbb{R}_I^n)}}{s} \int_{B_r(x_0)} \frac{\phi(x, t)}{(r - |x - x_0|)^{2s}} \, dx < \infty, \end{aligned}$$

because $\overline{\{x \in B_r(x_0) : \phi(x, t) \neq 0\}} \subset B_r(x_0)$ for any $t \in I_{r,s}(t_0)$. This implies that the integral in (4.23) is well-defined. Since $\mathbf{v}(t) - \mathbf{u}(t) = v(\cdot, t) - u(\cdot, t) \in X_0(B_r(x_0))$, $\mathbf{u}(t) - \mathbf{g}(t) = u(\cdot, t) - g(\cdot, t) \in X_0(\Omega)$ and $\mathbf{g}(t) = g(\cdot, t) \in H^s(\mathbb{R}^n)$ for all $t \in I_{r,s}(t_0)$, it thus follows from Fubini's theorem, the change of variables, Lemma 5.1 and 5.2 [CK] that

$$(4.24) \quad \begin{aligned} \int_{\mathbb{R}^n} (\mathbf{L}_K \mathbf{v}(t)) \phi \, dx &= \int_{\mathbb{R}^n} \mathbf{L}_K(\mathbf{v}(t) - \mathbf{u}(t)) \phi \, dx \\ &\quad + \int_{\mathbb{R}^n} \mathbf{L}_K(\mathbf{u}(t) - \mathbf{g}(t)) \phi \, dx + \int_{\mathbb{R}^n} (\mathbf{L}_K \mathbf{g}(t)) \phi \, dx \\ &= \langle \mathbf{v}(t) - \mathbf{u}(t), \phi \rangle_K + \langle \mathbf{u}(t) - \mathbf{g}(t), \phi \rangle_K + \langle \mathbf{g}(t), \phi \rangle_K \\ &= \langle \mathbf{v}(t), \phi \rangle_K. \end{aligned}$$

Thus, by (4.22) and (4.24), we have that

$$\begin{aligned} \langle \mathbf{v}(t), \phi \rangle_K + \int_{\mathbb{R}^n} \mathbf{v}'(t) \phi \, dx &= \langle \mathbf{v}(t), \phi \rangle_K + \int_{\mathbb{R}^n} (\partial_t v) \phi \, dx \\ &= \int_{B_r(x_0)} (\mathbf{L}_K \mathbf{v}(t) + \partial_t p) \phi \, dx \geq 0 \end{aligned}$$

for all $t \in I_{r,s}(t_0)$. Thus v is its weak supersolution on $Q_r(x_0, t_0)$, and so is $v + m$ on $Q_r(x_0, t_0)$ where

$$m = \inf_{\partial_p Q_r(x_0, t_0)} (u - p) < 0.$$

By comparison principle (Corollary 3.3) on $Q_r(x_0, t_0)$, we have that $u \leq v + m$ on $Q_r(x_0, t_0)$. This gives a contradiction, because $u(x_0, t_0) \leq v(x_0, t_0) + m < p(x_0, t_0)$.

Similarly, we can show that any weak subsolution of the nonlocal equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ for $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ is its viscosity subsolution. Therefore we complete the proof. \square

Lemma 4.4. *If $u \in H^1(I; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ for $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$, then there is a universal constant $C > 0$ such that*

$$[u]_{C_x^{0,1}}(Q_r(x, t)) \leq \frac{C}{(R-r)^{n+2s}} \|u\|_{L^\infty(\mathbb{R}_I^n)}$$

for any $r \in (0, R)$, where $Q_R(x, t) \subset \Omega_I$.

Remark. We apply Theorem 3.4 [KL] and Theorem 5.2 [KL1] in this proof. Looking over its proof scrupulously, we easily see that the Hölder estimate holds for all $s \in (0, 1)$ as follows; there are universal constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$(4.25) \quad \|u\|_{C^\alpha(Q_r(x, t))} \leq C \left(\|u\|_{C(Q_R(x, t))} + \frac{1}{(R-r)^{n+2s}} \|u\|_{L^\infty(\mathbb{R}_I^n)} \right).$$

for any $r \in (0, R)$ and $Q_R(x, t) \subset \Omega_I$.

Proof. Take any $r \in (0, R)$. For $k \in \mathbb{N} \cup \{0\}$, set $r_k = r + 2^{-k}(R-r)$. Then we see that $r_0 = R > r_1 > r_2 > \dots > r$ and $R - r_k \geq (R-r)/2$ for all $k \in \mathbb{N}$.

Since u is a viscosity solution by Theorem 4.3, by (4.25) there are universal constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$(4.26) \quad \|u\|_{C^\alpha(Q_{r_1}(x, t))} \leq C \left(\|u\|_{C(Q_R(x, t))} + \frac{1}{(R-r)^{n+2s}} \|u\|_{L^\infty(\mathbb{R}_I^n)} \right).$$

If we take $v = u \mathbb{1}_{Q_R(x, t)}$ with $Q_R(x, t) \subset \Omega_I$ as in the proof of Theorem 3.4 [KL], we obtain the estimate (4.26). Hence the required result follows from the standard telescopic sum argument [CC]. \square

5. NONLOCAL WEAK HARNACK INEQUALITY

In this section, we shall prove nonlocal weak Harnack inequalities with nonlocal parabolic tail term for weak subsolutions of the nonlocal parabolic equation (1.3). This result plays a crucial role in establishing nonlocal parabolic Harnack inequality for weak subsolutions of the nonlocal parabolic equation (1.3).

To do this, first of all, we need the following *nonlocal Caccioppoli type inequality*.

Theorem 5.1. *Let $\eta \in C_c^\infty(t_0 - r^{2s}, \infty]$ be a function such that $0 \leq \eta \leq 1$ and $0 \leq \eta' \leq c/r^{2s}$ in \mathbb{R} . If $u \in H^1(I; X_g^-(\Omega))$ is a weak subsolution of the nonlocal parabolic equation $\mathbf{NPE}(0, g, g)$ given in (1.3) and $Q_{2r}^0 \subset \Omega_I$ where $g \in H_T^s(\mathbb{R}^n)$ and $f \leq 0$ in Ω_I , and $w = (u - M)_\pm$ for $M \in \mathbb{R}$, then for any nonnegative $\zeta \in C_c^\infty(B_r^0)$ we have the following estimate*

$$\begin{aligned} & \sup_{t \in I_{r,s}(t_0)} \eta^2(t) \|w(\cdot, t) \zeta\|_{L^2(B_r^0)}^2 \\ & + \int_{t_0 - r^{2s}}^{t_0} \eta^2(t) \iint_{B_r^0 \times B_r^0} (\zeta(x)w(x, t) - \zeta(y)w(y, t))^2 d_K(x, y) dt \\ & \leq 2 \int_{t_0 - r^{2s}}^{t_0} \eta(t) \eta'(t) \|w(\cdot, t) \zeta\|_{L^2(B_r^0)}^2 dt \\ & + \int_{t_0 - r^{2s}}^{t_0} \eta^2(t) \iint_{B_r^0 \times B_r^0} [w(x, t) \vee w(y, t)]^2 (\zeta(x) - \zeta(y))^2 d_K(x, y) dt \\ & + 2 \left(\sup_{(x, t) \in \text{supp}(\zeta) \times I_{r,s}(t_0)} \int_{\mathbb{R}^n \setminus B_r^0} w(y, t) K(x - y) dy \right) \|w \zeta^2\|_{L^1(Q_r^0)}. \end{aligned}$$

Proof. For simplicity of the proof, we may assume that $(x_0, t_0) = (0, 0)$. Let $w = (u - M)_+$ and take any $\zeta \in C_c^\infty(B_r)$. We use $\varphi = w\zeta^2\eta^2$ as a testing function in the weak formulation of the equation. Then we have that

$$\int_{-r^{2s}}^\tau \int_{B_r} (\partial_t u) \varphi \, dx \, dt + \mathbf{I}(u, \varphi) \leq 0$$

for any $\tau \in I_{r,s}(t_0)$, where the bilinear operator is given by

$$\mathbf{I}(u, \varphi) = \int_{-r^{2s}}^\tau \eta^2(t) \iint_{\Omega \times \Omega} (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t)) \, d_K(x, y) \, dt.$$

The first term in the left-hand side of the above inequality can be evaluated by

$$\begin{aligned} \int_{-r^{2s}}^\tau \int_{B_r} (\partial_t u) \varphi \, dx \, dt &= \int_{-r^{2s}}^\tau \int_{B_r} [\partial_t (w^2 \zeta^2 \eta^2) - 2w^2 \zeta^2 \eta \eta_t] \, dx \, dt \\ (5.1) \quad &= \eta^2(\tau) \int_{B_r} w^2(x, \tau) \zeta^2(x) \, dx - 2 \int_{-r^{2s}}^\tau \eta(t) \eta'(t) \int_{B_r} w^2(x, t) \zeta^2(x) \, dx \, dt. \end{aligned}$$

We next split $\mathbf{I}(u, \varphi)$ into two parts as follows;

$$\begin{aligned} \mathbf{I}(u, \varphi) &= \int_{-r^{2s}}^\tau \eta^2(t) \iint_{B_r \times B_r} (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t)) \, d_K(x, y) \, dt \\ (5.2) \quad &+ 2 \int_{-r^{2s}}^\tau \eta^2(t) \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} (u(x, t) - u(y, t)) \varphi(x, t) \, d_K(x, y) \, dt \\ &:= I_1 + 2I_2. \end{aligned}$$

For the estimate of I_1 , we first observe that

$$(5.3) \quad (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t)) \geq (w(x, t) - w(y, t))(\varphi(x, t) - \varphi(y, t))$$

whenever $(x, t), (y, t) \in Q_r$; indeed, it can easily be checked by considering three possible cases (i) $u(x, t), u(y, t) > M$, (ii) $u(x, t) > M, u(y, t) \leq M$, and (iii) $u(x, t) \leq M, u(y, t) > M$. For the estimate of I_2 , we note that

$$\begin{aligned} (u(x, t) - u(y, t))\varphi(x, t) &\geq -(u(y, t) - u(x, t))_+(u(x, t) - M)_+\zeta^2(x)\eta^2(t) \\ &\geq -(u(y, t) - M)_+(u(x, t) - M)_+\zeta^2(x)\eta^2(t) \\ &= -w(y, t)w(x, t)\zeta^2(x)\eta^2(t) \end{aligned}$$

and thus we have that

$$\begin{aligned} I_2 &\geq - \int_{-r^{2s}}^\tau \eta^2(t) \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} w(y, t)w(x, t)\zeta^2(x) \, d_K(x, y) \, dt \\ (5.4) \quad &\geq - \left(\sup_{(x, t) \in \text{supp}(\zeta) \times I_{r,s}} w(y, t) K(x - y) \, dy \right) \iint_{Q_r} w(x, t)\zeta^2(x) \, dx \, dt. \end{aligned}$$

Since the following equality

$$\begin{aligned} (w(x, t) - w(y, t))(\varphi(x, t) - \varphi(y, t)) &= \eta^2(t)(\zeta(x)w(x, t) - \zeta(y)w(y, t))^2 \\ &\quad - \eta^2(t)w(x, t)w(y, t)(\zeta(x) - \zeta(y))^2 \end{aligned}$$

is always true, it follows from (5.2), (5.3) and (5.4) that

$$\begin{aligned}
(5.5) \quad I(u, \varphi) &\geq \int_{-r^{2s}}^{\tau} \eta^2(t) \iint_{B_r \times B_r} (\zeta(x)w(x, t) - \zeta(y)w(y, t))^2 d_K(x, y) dt \\
&- \int_{-r^{2s}}^{\tau} \eta^2(t) \iint_{B_r \times B_r} [w(x, t) \vee w(y, t)]^2 (\zeta(x) - \zeta(y))^2 d_K(x, y) dt \\
&- 2 \left(\sup_{(x, t) \in \text{supp}(\zeta) \times I_{r, s}} \int_{\mathbb{R}^n \setminus B_r} w(y, t) K(x - y) dy \right) \iint_{Q_r} w(x, t) \zeta^2(x) dx dt.
\end{aligned}$$

Hence the required inequality can be obtained from (5.1) and (5.5). \square

Next, we obtain nonlocal weak Harnack inequality with nonlocal parabolic tail term for weak subsolutions of the nonlocal parabolic equation (1.3) in the following theorems.

Theorem 5.2. *If $u \in H^1(I; X_g^-(\Omega))$ is a weak subsolution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(f, g, g)$ given in (1.3) and $Q_{2r}^0 \subset \Omega_I$ where $g \in H_T^s(\mathbb{R}^n)$ and $f \leq 0$ in Ω_I , then there is a constant $C_0 = C_0(n, s, \lambda, \Lambda) > 0$ such that*

$$\sup_{Q_r^0} u \leq \delta \mathcal{T}_r(u^+; (x_0, t_0)) + C_0 \delta^{-\frac{\alpha n}{4s}} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} [u^+]^2 dx dt \right)^{\frac{1}{2}}$$

for any $\delta \in (0, 1]$, where $\alpha = 1 + \frac{2s}{n}$.

Proof. Let $w = (u - M)_\pm$ for $M \in \mathbb{R}$. Using symmetry and the elementary inequality

$$[w(x, t) \vee w(y, t)]^2 \leq 2w^2(x, t) + 2w^2(y, t),$$

by Theorem 5.1 and the mean value theorem we have that

$$\begin{aligned}
(5.6) \quad &\sup_{t \in I_{r, s}(t_0)} \eta^2(t) \|w(\cdot, t)\zeta\|_{L^2(B_r^0)}^2 \\
&+ \int_{t_0 - r^{2s}}^{t_0} \eta^2(t) \iint_{B_r^0 \times B_r^0} (\zeta(x)w(x, t) - \zeta(y)w(y, t))^2 d_K(x, y) dt \\
&\leq c (\|\eta_t\|_{L^\infty(\mathbb{R})} + \|\nabla \zeta\|_{L^\infty(B_r^0)}^2) \|w\|_{L^2(Q_r^0)}^2 + \mathcal{A}(w, \zeta, t_0, r, s) \|w\|_{L^1(Q_r^0)}
\end{aligned}$$

where $\mathcal{A}(w, \zeta, t_0, r, s)$ is the value given by

$$\mathcal{A}(w, \zeta, t_0, r, s) = 2 \sup_{(x, t) \in \text{supp}(\zeta) \times I_{r, s}(t_0)} \int_{\mathbb{R}^n \setminus B_r^0} w(y, t) K(x - y) dy.$$

Applying a well-known parabolic version of the fractional Sobolev inequality to (5.6) and observing $|Q_r^0|/|Q_r^0| = 2 - \sigma$, we obtain that

$$\begin{aligned}
(5.7) \quad &\left(\frac{1}{|Q_r^0|} \iint_{Q_r^0} |w\zeta|^{2\alpha} dx dt \right)^{\frac{1}{\alpha}} \\
&\leq C_0 r^{2s} (\|\eta'\|_{L^\infty(\mathbb{R})} + \|\nabla \zeta\|_{L^\infty(B_r^0)}^2 r^{2-2s} + r^{-2s}) \left(\frac{1}{|Q_r^0|} \iint_{Q_r^0} |w|^2 dx dt \right) \\
&\quad + C_0 r^{2s} \mathcal{A}(w, \zeta, t_0, r, s) \frac{1}{|Q_r^0|} \iint_{Q_r^0} w dx dt
\end{aligned}$$

where $\alpha = 1 + \frac{2s}{n}$. For $k = 0, 1, 2, \dots$, we set

$$r_k = (1 + 2^{-k})r, \quad r_k^* = \frac{r_k + r_{k+1}}{2}, \quad M_k = M + (1 - 2^{-k})M_*, \quad M_k^* = \frac{M_k + M_{k+1}}{2},$$

$w_k = (u - M_k)_+$ and $w_k^* = (u - M_k^*)_+$ for a constant $M_* > 0$ to be determined later. In (5.6), for $k = 0, 1, \dots$, we choose a function $\eta_k \in C_c^\infty(t_0 - (r_k^*)^{2s}, \infty)$ with $\eta_k|_{[t_0 - r_{k+1}^{2s}, t_0]} \equiv 1$ such that $0 \leq \eta_k \leq 1$ and $0 \leq \eta_k' \leq 2^{(k+2)2s} r^{-2s}$ in \mathbb{R} , and a function $\zeta_k \in C_c^\infty(B_{r_k}^0)$ with $\zeta_k|_{B_{r_{k+1}}^0} \equiv 1$ such that $0 \leq \zeta_k \leq 1$ and $|\nabla \zeta_k| \leq 2^{k+2}/r$ in \mathbb{R}^n . For $k = 0, 1, 2, \dots$, we set

$$N_k = \left(\frac{1}{|\mathcal{Q}_{r_k}^0|} \iint_{\mathcal{Q}_{r_k}^0} |w_k|^2 dx dt \right)^{\frac{1}{2}}.$$

Since $w_k^* \geq w_{k+1}$ and $w_k^*(x, t) \geq M_{k+1} - M_k^* = 2^{-k-2} M_*$ whenever $u(x, t) \geq M_{k+1}$, we then have that

$$(5.8) \quad \begin{aligned} N_{k+1} &\leq c \left(\frac{1}{|\mathcal{Q}_{r_k}^0|} \iint_{\mathcal{Q}_{r_{k+1}}^0} \frac{w_{k+1}^2 (w_k^*)^{2(\alpha-1)}}{(M_{k+1} - M_k^*)^{2(\alpha-1)}} dx dt \right)^{\frac{1}{2}} \\ &\leq c \left(\frac{2^k}{M_*} \right)^{\alpha-1} \left(\frac{1}{|\mathcal{Q}_{r_k}^0|} \iint_{\mathcal{Q}_{r_k}^0} |w_k^* \zeta_k|^{2\alpha} dx dt \right)^{\frac{1}{2}} \end{aligned}$$

Since $\zeta_k \in C_c^\infty(B_{r_k}^0)$, $w_k^* \leq w_0$ for all k and

$$|y - x| \geq |y - x_0| - |x - x_0| \geq \left(1 - \frac{r_k^*}{r_k}\right) |y - x_0| \geq 2^{-k-2} |y - x_0|$$

for any $x \in B_{r_k}^0$ and $y \in \mathbb{R}^n \setminus B_{r_k}^0$, we easily obtain that

$$(5.9) \quad \mathcal{A}(w_k^*, \zeta_k, t_0, r_k, s) \leq c 2^{k(n+2s)} r^{-2s} \mathcal{T}_r(w_0; (x_0, t_0)).$$

Since $0 \leq w_k^* \leq w_k$ and $w_k(x, t) \geq M_k^* - M_k = 2^{-k-2} M_*$ if $u(x, t) \geq M_k^*$, it follows from (5.7), (5.8) and (5.9) that

$$\begin{aligned} \left(\frac{2^k}{M_*} \right)^{-\frac{2(\alpha-1)}{\alpha}} N_{k+1}^{2/\alpha} &\leq \frac{c 2^{2k}}{|\mathcal{Q}_{r_k}^0|} \iint_{\mathcal{Q}_{r_k}^0} |w_k^*|^2 dx dt \\ &\quad + c 2^{k(n+2s)} \left(\frac{r_k}{r} \right)^{2s} \mathcal{T}_r(w_0; (x_0, t_0)) \frac{1}{|\mathcal{Q}_{r_k}^0|} \iint_{\mathcal{Q}_{r_k}^0} w_k^* dx dt \\ &\leq c 2^{2k} N_k^2 + c 2^{k(n+2s)} \mathcal{T}_r(w_0; (x_0, t_0)) \frac{1}{|\mathcal{Q}_{r_k}^0|} \iint_{\mathcal{Q}_{r_k}^0} \frac{w_k^* w_k}{M_k^* - M_k} dx dt \\ &\leq c \left(2^{2k} + \frac{2^{k(n+2s+1)}}{M_*} \mathcal{T}_r(w_0; (x_0, t_0)) \right) N_k^2. \end{aligned}$$

Taking M^* in the above so that

$$M_* \geq \delta \mathcal{T}_r(w_0; (x_0, t_0)) \quad \text{for } \delta \in (0, 1],$$

we obtain that

$$(5.10) \quad \frac{N_{k+1}}{M_*} \leq d_0 a^k \left(\frac{N_k}{M_*} \right)^{1 + \frac{2s}{n}}$$

where $d_0 = c^{\frac{\alpha}{2}} \delta^{-\frac{\alpha}{2}}$ and $a = 2^{\frac{\alpha}{2}(n+2s+1) + \frac{2s}{n}}$. If $N_0 \leq d_0^{-\frac{n}{2s}} a^{-\frac{n^2}{4s^2}} M_*$, then we set

$$M_* = \delta \mathcal{T}_r(w_0; (x_0, t_0)) + c_0 \delta^{-\frac{\alpha n}{4s}} a^{\frac{n^2}{4s^2}} N_0$$

where $c_0 = c^{\frac{\alpha n}{4s}}$. By Lemma 2.2, we conclude that

$$\begin{aligned} \sup_{Q_r^0} u &\leq M + M_* \\ &\leq M + \delta \mathcal{T}_r(w_0; (x_0, t_0)) + c_0^{\frac{\alpha n}{4s}} \delta^{-\frac{\alpha n}{4s}} a^{\frac{n^2}{4s^2}} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} (u - M)_+^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, choosing $M = 0$ in the above estimate, we obtain the required result. \square

The third one is a lemma which furnishes a precise relation between the nonlocal parabolic tails of the positive and negative part of the weak subsolutions.

Lemma 5.3. *If $u \in H^1(I; X_g^-(\Omega))$ is a weak subsolution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(f, g, g)$ in (1.3) such that $u \geq 0$ in $Q_R^0 \subset \Omega_I$ where $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ and $f \leq 0$ in Ω_I , then we have the estimate*

$$\mathcal{T}_r(u^+; (x_0, t_0)) \lesssim \sup_{Q_r^0} u + \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0))$$

for any r with $0 < r < R$.

Proof. Without loss of generality, we may assume that $(x_0, t_0) = (0, 0)$ and $\mathcal{T}_r(u^+; (0, 0)) \neq 0$. Fix any $\varepsilon > 0$ with $\varepsilon \leq \mathcal{T}_r(u^+; (0, 0))/2$. Then it follows from the definition of supremum, the uniform continuity of u^+ on a big enough closed ball (via Theorem 4.2) and the Lebesgue's dominated convergence theorem (via Theorem 4.1) that there are some $\tau \in (-r^{2s}, 0]$ and $\epsilon_0 > 0$ with $[\tau - \epsilon_0 r^{2s}, \tau] \subset (-r^{2s}, 0]$ such that

$$(5.11) \quad r^{2s} \int_{\tau - \epsilon_0 r^{2s}}^{\tau} \int_{\mathbb{R}^n \setminus B_r} \frac{u^+(y, t)}{|y|^{n+2s}} dx \geq \mathcal{T}_r(u^+; (0, 0)) - \varepsilon \geq \frac{1}{2} \mathcal{T}_r(u^+; (0, 0)).$$

Let $M = \sup_{Q_r} u$ and set $\varphi(x, t) = (u(x, t) - 2M)\zeta^2(x)\eta^2(t)$ where $\zeta \in C_c^\infty(B_{3r/4})$ is a function satisfying that $\zeta|_{B_{r/2}} \equiv 1$, $0 \leq \zeta \leq 1$ and $|\nabla \zeta| \leq c/r$ in \mathbb{R}^n , and $\eta \in C_c^\infty(-r^{2s}, 0]$ is a function such that $\eta = 1$ in $[\tau - \epsilon_0 r^{2s}, \tau]$, $0 \leq \eta \leq 1$ and $|\eta'| \leq c/r^{2s}$ in \mathbb{R} . Then we have that

$$\begin{aligned} 0 &\geq - \iint_{\Omega \times I} u \partial_t \varphi dx dt \\ (5.12) \quad &+ \int_{-r^{2s}}^0 \iint_{B_r \times B_r} (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t)) d_K(x, y) dt \\ &+ 2 \int_{-r^{2s}}^0 \eta^2(t) \int_{B_r^c} \int_{B_r} (u(x, t) - u(y, t))(u(x, t) - 2M)\zeta^2(x) d_K(x, y) dt \\ &:= J_1 + J_2 + 2J_3. \end{aligned}$$

Since the fact that

$$-2M \leq w(x, t) := u(x, t) - 2M \leq -M$$

for any $(x, t) \in Q_r$ and the following elementary equality

$$(\beta - \alpha)(B^2\beta - A^2\alpha) = (B\beta - A\alpha)^2 - \alpha\beta(B - A)^2 \text{ for any } \alpha, \beta \in \mathbb{R} \text{ and } A, B \geq 0$$

leads us to obtain the estimate

$$(w(x, t) - w(y, t))(w(x, t)\zeta^2(x) - w(y, t)\zeta^2(y)) \geq -4M^2(\zeta(x) - \zeta(y))^2$$

for any $(x, t), (y, t) \in Q_r$, it follows from simple calculation that

$$\begin{aligned}
(5.13) \quad J_1 + J_2 &\geq \left[-\frac{\eta^2(t)}{2} \int_{B_r} u^2(x, t) \zeta^2(x) dx \right]_{t=-r^{2s}}^{t=0} \\
&\quad - 2 \int_{-r^{2s}}^0 \eta(t) \eta'(t) \int_{B_r} [u(x, t)(u(x, t) - 2M) - u^2(x, t)] dx dt \\
&\quad - 4M^2 \int_{-r^{2s}}^0 \eta^2(t) \iint_{B_r \times B_r} (\zeta(x) - \zeta(y))^2 d_K(x, y) dt \\
&\gtrsim -M^2 r^{-2s} |Q_r|.
\end{aligned}$$

The lower estimate on J_3 can be splitted as follows;

$$\begin{aligned}
J_3 &\geq \int_{-r^{2s}}^0 \eta^2(t) \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} M(u(y, t) - M)_+ \zeta^2(x) d_K(x, y) dt \\
&\quad - 2M \int_{-r^{2s}}^0 \eta^2(t) \int_{E_M^t} \int_{B_r} (u(x, t) - u(y, t))_+ \zeta^2(x) d_K(x, y) dt \\
&:= J_3^1 - J_3^2,
\end{aligned}$$

where $E_M^t = \{y \in \mathbb{R}^n \setminus B_r : u(y, t) < M\}$. Since $(u(y, t) - M)_+ \geq u^+(y, t) - M$, by (5.11) the lower estimate on J_3^1 can here be obtained as

$$(5.14) \quad J_3^1 \geq d_2 M r^{-2s} |Q_r| \mathcal{T}_r(u^+; (0, 0)) - d_3 M^2 r^{-2s} |Q_r|$$

with universal constants $d_2, d_3 > 0$. If $(x, t) \in Q_r$ and $y \in E_M^t$, then we observe that

$$\begin{aligned}
(u(x, t) - u(y, t))_+ &\leq |u(x, t) - M| + |M - u(y, t)| \\
&\leq M + (M + u^-(y, t) - u^+(y, t)) \\
&\leq 2M + u^-(y, t)
\end{aligned}$$

because $u^+(y, t) < M + u^-(y, t)$ for any $y \in E_M^t$. Since $u^-(y, t) = 0$ for all $(y, t) \in Q_R$, the upper estimate on J_3^2 can thus be achieved by

$$\begin{aligned}
(5.15) \quad J_3^2 &\leq 2M^2 \int_{-r^{2s}}^0 \eta^2(t) \int_{\mathbb{R}^n \setminus B_r} \int_{B_r} \zeta^2(x) d_K(x, y) \\
&\quad + 2M \int_{-r^{2s}}^0 \eta^2(t) \int_{B_R \setminus B_r} \int_{B_r} (2M + u^-(y, t)) \zeta^2(x) d_K(x, y) \\
&\quad + 2M \int_{-r^{2s}}^0 \eta^2(t) \int_{\mathbb{R}^n \setminus B_R} \int_{B_r} (2M + u^-(y, t)) \zeta^2(x) d_K(x, y) \\
&\leq d_4 M^2 r^{-2s} |Q_r| + d_5 M R^{-2s} |Q_r| \mathcal{T}_R(u^-; (0, 0))
\end{aligned}$$

with universal constants $d_4, d_5 > 0$. Thus, by (5.14) and (5.15), we have that

$$\begin{aligned}
(5.16) \quad J_3 &\geq -dM^2 r^{-2s} |Q_r| - dMR^{-2s} |Q_r| \mathcal{T}_R(u^-; (0, 0)) \\
&\quad + eMr^{-2s} |Q_r| \mathcal{T}_r(u^+; (0, 0))
\end{aligned}$$

where $d, e > 0$ are some universal constants depending only on n, s, λ and Λ . Hence the estimates (5.12), (5.13) and (5.16) give the required estimate. \square

Next we obtain a weak Harnack inequality for nonnegative weak subsolutions of the nonlocal parabolic equation (1.3) by employing Theorem 5.2 and Lemma 5.3.

It is interesting that this estimate no longer depends on the nonlocal parabolic tail term, but its proof is pretty simple.

Theorem 5.4. *If $u \in H^1(I; X_g^-(\Omega))$ is a nonnegative weak subsolution of the non-local parabolic equation $\mathbf{NP}_{\Omega_I}(f, g, g)$ given in (1.3) where $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ and $f \leq 0$ in Ω_I , then we have the estimate*

$$\sup_{Q_r^0} u \leq C \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^2(x, t) dx dt \right)^{1/2}$$

for any $r > 0$ with $Q_{2r}^0 \subset \Omega_I$.

Proof. We choose some $\delta \in (0, 1]$ so that $1 - \delta d_0 > 0$ and take any $r > 0$ with $Q_{2r}^0 \subset \Omega_I$. Then it follows from Theorem 5.2 and Lemma 5.3 that

$$\begin{aligned} \sup_{Q_r^0} u &\leq \delta d_0 \left[\sup_{Q_r^0} u + \mathcal{T}_r(u^-; (x_0, t_0)) \right] \\ &\quad + C_0 \delta^{-\frac{\alpha n}{4s}} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^2(x, t) dx dt \right)^{\frac{1}{2}} \end{aligned}$$

Since $\mathcal{T}_r(u^-; (x_0, t_0)) = 0$, we can easily derive the required result by taking

$$C = \frac{C_0 \delta^{-\frac{\alpha n}{4s}}}{1 - \delta d_0}.$$

Hence we complete the proof. \square

6. PARABOLIC FRACTIONAL POINCARÉ INEQUALITY

Let $n \geq 1$, $p \geq 1$, $s \in (0, 1)$ and $sp < n$. For a ball $B \subset \mathbb{R}^n$, let u_B denote the average of $u \in W^{s,p}(B)$ over B , i.e.

$$u_B = \frac{1}{|B|} \int_B u(y) dy.$$

Then it is well-known [BBM, MS] that

$$(6.1) \quad \|u - u_B\|_{L^p(B)}^p \leq \frac{c_{n,p}(1-s)|B|^{\frac{sp}{n}}}{(n-sp)^{p-1}} \|u\|_{W^{s,p}(B)}^p$$

with a universal constant $c_{n,p} > 0$ depending only on n and p , which is called the *fractional Poincaré inequality*. We note that this estimate no longer depends on partial differential equations. However, the inequality as (6.1) in the parabolic sense is not available for a general function $u(x, t) \in L^2(I; X_0(\Omega))$.

If $u \in L^\infty(\mathbb{R}_I^n)$, then we have that $\mathcal{T}_R(u; (x_0, t_0)) \leq \|u\|_{L^\infty(\mathbb{R}_I^n)}$. So, if we could assume that $\|u\|_{L^\infty(\mathbb{R}_I^n)} = 1$, we see that

$$\kappa := \frac{1}{2} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) < 1 \quad \text{for any } r \in (0, R),$$

where $R > 0$ satisfies $Q_R^0 \subset \Omega_I$. For $Q_r^0 := Q_r(x_0, t_0)$, we denote by

$$u_{Q_r^0} = \frac{1}{|Q_r^0|} \iint_{Q_r^0} u(x, t) dx dt.$$

For $x_0 \in \mathbb{R}^n$ and $\kappa \in (0, 1)$, we consider a radial function $\theta \in C_c^\infty(\mathbb{R}^n)$ with values in $[0, \kappa]$ such that $\theta|_{B_r(x_0)} \equiv \sqrt{\kappa}$, $\theta|_{\mathbb{R}^n \setminus B_{2r}(x_0)} \equiv 0$ and $|\nabla \theta| \leq c\sqrt{\kappa}/r$ in \mathbb{R}^n . For simplicity, we write $d\theta(x, t) = \theta(x) dx dt$, $d\theta(x) = \theta(x) dx$,

$$\theta(Q_r^0) = \iint_{Q_r^0} d\theta(x, t) \quad \text{and} \quad \theta(B_r^0) = \int_{B_r^0} d\theta(x).$$

Also, we denote by

$$u_{Q_r^0}^\theta = \frac{1}{\theta(Q_r^0)} \iint_{Q_r^0} u(x, t) d\theta(x, t) \quad \text{and} \quad u_{B_r^0}^\theta(t) = \frac{1}{\theta(B_r^0)} \int_{B_r^0} u(x, t) d\theta(x).$$

Now we establish a parabolic version of the fractional Poincaré inequality for weak solutions for nonlocal parabolic equation (1.3) as follows.

Theorem 6.1. *Let $u \in H^1(I; X_g(\Omega))$ be an weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ such that $u \geq \kappa > 0$ in $Q_R^0 \subset \Omega_I$ where $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$ and*

$$\kappa = \frac{1}{2} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)).$$

If $v(x, t) = \ln\left(\frac{d}{u(x, t)}\right)$ for $d > 0$ and there is a $\nu \in (0, 1)$ such that

$$(6.2) \quad |Q_r^0 \cap \{u \geq d + \kappa\}| \geq \nu |Q_r^0|$$

for some r with $0 < 5r < R$, then for each $\sigma \in (0, \nu \wedge 2/5)$ there exists a constant $C_0 = C_0(n, s, \lambda, \Lambda, \nu, \sigma) > 0$ as follows;

(a) *there is some $\tau_0 \in [t_0 - r^{2s}, t_0 - \sigma r^{2s}]$ such that*

$$(6.3) \quad \sup_{b \in [t_0 - \sigma r^{2s}, t_0]} \int_{\tau_0}^b \iint_{B_{2r}^0 \times B_{2r}^0} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy dt \leq C_0 r^n,$$

(b) *we have the estimate*

$$(6.4) \quad \begin{aligned} & \iint_{(Q_r^0)^+} |v(x, t) - v_{(Q_r^0)^+}|^2 dx dt \\ & \leq C_0 r^{2s} \int_{t_0 - \sigma r^{2s}}^{t_0} \iint_{B_{2r}^0 \times B_{2r}^0} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy dt \\ & \quad + C_0 |(Q_r^0)^+| \end{aligned}$$

where $(Q_r^0)^+ := B_r^0 \times [t_0 - \sigma r^{2s}, t_0] = B_r(x_0) \times [t_0 - \sigma r^{2s}, t_0]$ for $r > 0$.

Remark. The reason why $\sigma \leq 2/5$ follows from the inequality (7.42) below.

We now state three lemmas which is useful for the proof of Theorem 6.1. The first one is an elliptic version of *weighted Poincaré inequality* which was obtained by [FK] as follows.

Lemma 6.2. *If $u \in W^{s,2}(B_{2r}^0)$ for $s \in (0, 1)$ and $r > 0$, then there is a constant $c = c(n, s, \Lambda) > 0$ such that*

$$\int_{B_{2r}^0} |u(x) - u_{B_{2r}^0}^\theta|^2 \theta(x) dx \leq c r^{2s} \iint_{B_{2r}^0 \times B_{2r}^0} |u(x) - u(y)|^2 (\theta(x) \wedge \theta(y)) d_K(x, y)$$

where $d_K(x, y) = K(x - y) dx dy$.

The second one is the following lemma whose detailed proof can be found in [L].

Lemma 6.3. *If $u \in L^2((Q_r^0)^+)$ for $r > 0$, then we have that*

$$\iint_{(Q_r^0)^+} |u(x, t) - u_{(Q_r^0)^+}|^2 dx dt \leq 4 \iint_{(Q_r^0)^+} |u(x, t) - h|^2 dx dt$$

for any $h \in \mathbb{R}$.

Lemma 6.4. *For $d > 0$, the functions $r_{d,\epsilon}(t) = \ln^+ \left(\frac{d}{t} \right) ((d-t)_+ + \epsilon)^{-2}$ and $q_{d,\epsilon}(t) = \ln^+ \left(\frac{d}{t} \right) (d-t)_+ ((d-t)_+ + \epsilon)^{-3}$ are decreasing on $(0, \infty)$ for $\epsilon \geq (2e^{-\frac{1}{2}} - 1)d$ and are Lipschitz continuous in (κ, ∞) for $\kappa \in (0, 1)$.*

Proof. Since $r_{d,\epsilon}(t) = 0$ on $t \geq d$, we have only to consider $0 < t < d$. Then we have that

$$r'_{d,\epsilon}(t) = -(d + \epsilon - t)^{-3} \left(\frac{d + \epsilon - t}{t} - 2 \ln \left(\frac{d}{t} \right) \right) := -(d + \epsilon - t)^{-3} h(t).$$

Since $h'(t) = t^{-2}(2t - (d + \epsilon))$ and

$$h \left(\frac{d + \epsilon}{2} \right) = 1 - 2 \ln \left(\frac{2d}{d + \epsilon} \right) \geq 0$$

for $\epsilon \geq (2e^{-\frac{1}{2}} - 1)d$, we see that $h(t) \geq 0$, and so $r'_{d,\epsilon}(t) \leq 0$. Note that $q_{d,\epsilon}(t) = r_{d,\epsilon}(t)p_{d,\epsilon}(t)$ where

$$(6.5) \quad p_{d,\epsilon}(t) = \frac{(d-t)_+}{(d-t)_+ + \epsilon}.$$

Since it is easy to check that $p_{d,\epsilon}(t)$ is decreasing on $(0, \infty)$ for any $\epsilon > 0$, we can easily conclude the first required result. Also the second required result can be easily obtained. Hence we are done. \square

[Proof of Theorem 6.1] Without loss of generality, we may assume that $(x_0, t_0) = (0, 0)$. Take any $a, b \in I_{r,s}$ with $a < b$ and choose $r > 0$ so that $B_{5r} \subset B_R$ where $Q_R \subset \Omega_I$. Then we have two possible cases; either $r > 1$ or $r \leq 1$.

$$(6.6) \quad \begin{cases} B_{4r+h_0} \subset B_R & \text{for } r > 1 \\ r \leq r^{n+2s} & \text{for } r \leq 1, \end{cases}$$

where $h_0 = r^{\frac{1}{n+2s}}$. We use the function

$$\varphi(x, t) = \frac{\theta_0^2(x, t)}{\kappa u(x, t)}$$

as a testing function where $\theta_0(x, t) = \theta(x/2)h(x, t)$ and

$$h(x, t) = \frac{\sqrt{\kappa} (2 - u(x, t))_+}{(2 - u(x, t))_+ + \epsilon_0}$$

for $\epsilon_0 = 2(2e^{-\frac{1}{2}} - 1)$. From (1.3), we obtain that

$$(6.7) \quad \int_{\Omega} \partial_t u(x, t) \varphi(x, t) dx + \mathcal{I}_t(u, \varphi) = 0$$

for a.e. $t \in I$, where the bilinear operator \mathcal{I}_t is given by

$$\mathcal{I}_t(u, \varphi) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t)) d_K(x, y).$$

Since $v(x, t) - v(y, t) = V(x, t) - V(y, t)$ and $v(x, t) - v_{Q_r^+} = V(x, t) - V_{Q_r^+}$ where

$$V(x, t) = \ln\left(\frac{d\|u\|_{L^\infty(\mathbb{R}_I^n)}}{u(x, t)}\right),$$

without loss of generality, we may assume that $\|u\|_{L^\infty(\mathbb{R}_I^n)} = 1$; for, $u/\|u\|_{L^\infty(\mathbb{R}_I^n)}$ is also an weak solution of the equation $\mathbf{NP}_{\Omega_I}(0, g, g)$, and we use it in place of u . Now the integral of the first term on $[a, b] \subset I_{r, s}$ in (6.7) can be estimated as

$$\begin{aligned} \int_a^b \int_{\Omega} \partial_t u(x, t) \varphi(x, t) dx dt &= - \int_a^b \int_{\Omega} \partial_t \left[\frac{1}{\kappa} \ln\left(\frac{2}{u(x, t)}\right) \right] \theta_0^2(x, t) dx dt \\ &= - \int_{\Omega} \int_a^b \partial_t v_0(x, t) h^2(x, t) dt \theta_1^2(x) dx \\ (6.8) \quad &= - \int_{\Omega} \int_a^b \partial_t [v_0(x, t) h^2(x, t)] dt \theta_1^2(x) dx \\ &\quad + \int_{\Omega} \int_a^b v_0(x, t) \partial_t h^2(x, t) dt \theta_1^2(x) dx \\ &:= (w_{B_{4r}}^{\theta_1^2}(a) - w_{B_{4r}}^{\theta_1^2}(b)) \theta_1^2(B_{4r}) + \mathcal{I} \end{aligned}$$

where $\theta_1(x) = \theta(x/2)$, $w(x, t) = v_0(x, t) h^2(x, t)$ for $v_0(x, t) = \frac{1}{\kappa} \ln^+(2/u(x, t))$ and

$$\mathcal{I} = \int_{\Omega} \int_a^b v_0(x, t) \partial_t h^2(x, t) dt \theta_1^2(x) dx.$$

Here we observe that

$$(6.9) \quad -\mathcal{I} = \int_a^b \int_{\Omega} \partial_t u(x, t) \psi(x, t) \theta_1^2(x) dx dt = - \int_a^b \mathcal{I}_t(u, \psi \theta_1^2) dt$$

where

$$\psi(x, t) = \frac{\epsilon_0 (2 - u(x, t))_+ v_0(x, t)}{[(2 - u(x, t))_+ + \epsilon_0]^3}.$$

Since $\ln^+ \eta \leq (\eta - 1)_+$ for all $\eta > 0$, we have that

$$(6.10) \quad \psi(x, t) \leq \frac{\epsilon_0 (2 - u(x, t))_+^2}{u(x, t) [(2 - u(x, t))_+ + \epsilon_0]^3} \leq \frac{1}{\kappa}.$$

We note that

$$\begin{aligned} &(u(x, t) - u(y, t)) (\psi(x, t) \theta_1^2(x) - \psi(y, t) \theta_1^2(y)) \\ &= (u(x, t) - u(y, t)) (\psi(x, t) - \psi(y, t)) \theta_1^2(x) \\ &\quad + (u(x, t) - u(y, t)) (\theta_1^2(x) - \theta_1^2(y)) \psi(y, t). \end{aligned}$$

By Lemma 6.4, we have that

$$(6.11) \quad \mathcal{I}_t(u, \psi) \leq 0 \quad \text{for any } t \in I,$$

where

$$\mathcal{I}_t(u, \psi) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x, t) - u(y, t)) (\psi(x, t) - \psi(y, t)) \theta_1^2(x) dK(x, y).$$

Indeed, assuming without loss of generality that $u(x, t) \geq u(y, t)$, it follows from Lemma 6.4 that

$$(u(x, t) - u(y, t)) (\psi(x, t) - \psi(y, t)) \leq 0$$

for any $x, y \in \Omega$ and $t \in I$. Thus we conclude that $\mathcal{I}_t(u, \psi) \leq 0$.

Also, we can derive the estimate $\mathcal{J}_t(u, \theta_1^2) \leq cr^{n-2s}$ where

$$\mathcal{J}_t(u, \theta_1) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x, t) - u(y, t)) (\theta_1^2(x) - \theta_1^2(y)) \psi(y, t) d_K(x, y).$$

Indeed, by the mean value theorem, we have that

$$\begin{aligned} u(x, t) - u(y, t) &= \int_0^1 \langle \nabla u(x + \tau(y - x), t), y - x \rangle d\tau, \\ \theta_1^2(x, t) - \theta_1^2(y, t) &= \int_0^1 \theta \left(\frac{x + \tau(y - x)}{2} \right) \left\langle \nabla \theta \left(\frac{x + \tau(y - x)}{2} \right), y - x \right\rangle d\tau, \end{aligned}$$

and so it follows from the definition of θ_1 , (6.6), (6.10) and Lemma 4.4 that

$$\begin{aligned} (6.12) \quad \mathcal{J}_t(u, \theta_1^2) &\leq \frac{c}{r^2} \|u\|_{L^\infty(\mathbb{R}_T^n)} \iint_{B_{5r} \times B_{5r}} \frac{|x - y|^2}{|x - y|^{n+2s}} dx dy \\ &\quad + 2 \|u\|_{L^\infty(\mathbb{R}_T^n)} \iint_{(\mathbb{R}^n \setminus B_{5r}) \times B_{4r}} \frac{1}{|x - y|^{n+2s}} dx dy \leq cr^{n-2s}, \end{aligned}$$

because $\|u\|_{L^\infty(\mathbb{R}_T^n)} = 1$. Thus, by (6.9), (6.11) and (6.12), we obtain that

$$\begin{aligned} (6.13) \quad -\mathcal{I} &= - \int_a^b \mathcal{I}_t(u, \psi \theta_1^2) dt = - \int_a^b \mathcal{I}_t(u, \psi) dt - \int_a^b \mathcal{J}_t(u, \theta_1^2) dt \\ &\geq - \int_a^b \mathcal{J}_t(u, \theta_1^2) dt \geq -cr^n, \end{aligned}$$

and thus it follows from (6.8) and (6.13) that

$$(6.14) \quad (w_{B_{4r}}^{\theta_1^2}(a) - w_{B_{4r}}^{\theta_1^2}(b)) \theta_1^2(B_{4r}) + cr^n \geq - \int_a^b \mathcal{I}_t(u, \varphi) dt.$$

We note that

$$\begin{aligned} \mathcal{I}_t(u, \varphi) &= 2 \iint_{(\mathbb{R}^n \setminus B_{4r}) \times B_{4r}} (u(x, t) - u(y, t)) \varphi(x, t) d_K(x, y) \\ &\quad + \iint_{B_{4r} \times B_{4r}} (u(x, t) - u(y, t)) (\varphi(x, t) - \varphi(y, t)) d_K(x, y) \\ &:= I_t(u, \varphi) + J_t(u, \varphi). \end{aligned}$$

Since $h^2(y, t) \geq e/16$ for any $(y, t) \in B_{2r} \times [a, b]$, as in the proof of Lemma 1.3 [DKP1] we have the following estimates

$$\begin{aligned} (6.15) \quad I_t(u, \varphi) &\leq cr^{n-2s} + c\kappa^{-1}r^n \int_{\mathbb{R}^n \setminus B_R} \frac{u^-(y, t)}{|y|^{n+2s}} dy, \\ J_t(u, \varphi) &\leq -\frac{1}{c\kappa} \iint_{B_{4r} \times B_{4r}} \left[\ln \left(\frac{d}{u(x, t)} \right) - \ln \left(\frac{d}{u(y, t)} \right) \right]^2 \theta_0^2(y, t) d_K(x, y) \\ &\quad + c \iint_{B_{4r} \times B_{4r}} (\theta_0(x, t) - \theta_0(y, t))^2 d_K(x, y) \\ &\leq -\frac{e}{16c} \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy + c J_t(\theta_0, \theta_0). \end{aligned}$$

For the estimate of $I_t(\theta_0, \theta_0)$, we note that

$$(6.16) \quad (\theta_0(x, t) - \theta_0(y, t))^2 \leq 2(\theta(x) - \theta(y))^2 h^2(x, t) + 2\theta^2(y)(h(x, t) - h(y, t))^2.$$

Since $p_{2,\epsilon_0}(t)$ is Lipschitz continuous in (κ, ∞) and $c_0 = \sup_{\kappa < t < \infty} |p'_{2,\epsilon_0}(t)| \leq 1/\epsilon_0$ by (6.5), it follows from (6.6), (6.16), the mean value theorem and Lemma 4.4 that

$$\begin{aligned}
(6.17) \quad J_t(\theta_0, \theta_0) &\leq c r^{n-2s} + c \iint_{(\mathbb{R}^n \setminus B_{5r}) \times B_{4r}} \frac{1}{|x-y|^{n+2s}} dx dy \\
&+ \frac{c c_0}{r^2} \int_{B_{5r}} \int_{B_{5r}} \frac{\int_0^1 |\nabla u(x + \tau(y-x), t)|^2 |x-y|^2 d\tau}{|x-y|^{n+2s}} dx dy \\
&\leq c(\|u\|_{L^\infty(\mathbb{R}^n)}^2 + 2) r^{n-2s} \leq c r^{n-2s}.
\end{aligned}$$

By (6.14), (6.15) and (6.17), we have that

$$\begin{aligned}
(6.18) \quad w_{B_{4r}}^{\theta_1^2}(b) \theta_1^2(B_{4r}) &+ \frac{e}{16c} \int_a^b \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x-y|^{n+2s}} dx dy dt \\
&\leq c r^n + c \kappa^{-1} r^n \int_a^b \int_{\mathbb{R}^n \setminus B_R} \frac{u^-(y, t)}{|y|^{n+2s}} dy dt + w_{B_{4r}}^{\theta_1^2}(a) \theta_1^2(B_{4r}).
\end{aligned}$$

For $t \in I$ and $h < d/2$, we denote by

$$\omega(t) = |\{x \in B_r : u(x, t) \geq d + \kappa\}| \quad \text{and} \quad E_t^h = \{x \in B_r : u(x, t) \geq h + \kappa\}.$$

By the assumption (6.2), we have that

$$\int_{-r^{2s}}^0 \omega(t) dt \geq \nu |Q_r| = \nu r^{2s} |B_r|.$$

Also, for $\sigma \in (0, \nu)$, it is obvious that

$$\int_{-\sigma r^{2s}}^0 \omega(t) dt \leq \sigma r^{2s} |B_r|.$$

Thus these two inequalities yield that

$$(6.19) \quad \int_{-r^{2s}}^{-\sigma r^{2s}} \omega(t) dt \geq (\nu - \sigma) r^{2s} |B_r|.$$

By the mean value theorem, there is some $\tau_0 \in [-r^{2s}, -\sigma r^{2s}]$ such that

$$\omega(\tau_0) \geq \frac{\nu - \sigma}{1 - \sigma} |B_r| = \frac{\nu - \sigma}{2^n(1 - \sigma)} |B_{2r}|.$$

From now on, we take $a = \tau_0$ and $b \in [-\sigma r^{2s}, 0)$ in (6.8). Since the function $\gamma(t) := \ln(2/t) \left(\frac{(2-t)_+}{(2-t)_+ + \epsilon_0}\right)^2 = r_{2,\epsilon_0}(t) (2-t)_+^2$ is decreasing in $(0, \infty)$ by Lemma 6.4, we have that

$$\begin{aligned}
(6.20) \quad w_{B_{4r}}^{\theta_1^2}(b) \theta_1^2(B_{4r}) &= \int_{B_{4r}} w(x, b) \theta_1^2(x) dx = \int_{B_{2r}} w(x, b) \theta_1^2(x) dx \\
&\geq \int_{B_{2r} \setminus E_b^h} w(x, b) \theta_1^2(x) dx \geq \theta_1^2(B_{2r} \setminus E_b^h) \gamma(2h).
\end{aligned}$$

Also we have that

$$\begin{aligned}
(6.21) \quad w_{B_{4r}}^{\theta_1^2}(\tau_0) \theta_1^2(B_{4r}) &= \int_{B_{2r}} w(x, \tau_0) \theta_1^2(x) dx \leq \int_{B_{2r}} w(x, \tau_0) dx \\
&= \int_{B_{2r} \setminus E_{\tau_0}^d} w(x, \tau_0) dx + \int_{E_{\tau_0}^d} w(x, \tau_0) dx \\
&\leq (|B_{2r}| - \omega(\tau_0)/2) \gamma(h) + \omega(\tau_0)(-\gamma(h)/2 + \gamma(d)) \\
&\leq \left(1 - \frac{\nu - \sigma}{2^{n+1}(1 - \sigma)}\right) \gamma(h) |B_{2r}| - \frac{\gamma(h) - 2\gamma(d)}{2} \omega(\tau_0).
\end{aligned}$$

From (6.20) and (6.21), there is a constant $c_1 = c_1(n, s, \lambda, \Lambda) > 0$ such that

$$\begin{aligned}
(6.22) \quad \theta_1^2(B_{2r} \setminus E_b^h) &\leq \frac{c_1 \left(1 + \kappa^{-1} \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; \mathbf{0})\right) + \left(1 - \frac{\nu - \sigma}{2^{n+1}(1 - \sigma)}\right) \gamma(h)}{\gamma(2h)} |B_{2r}| \\
&\quad - \frac{\gamma(h) - 2\gamma(d)}{2\gamma(2h)} \omega(\tau_0)
\end{aligned}$$

for any $h \in (0, d/2)$. Here we may choose a sufficiently small $h \in (0, d/2)$ so that

$$\theta_1^2(B_{2r} \setminus E_b^h) \leq \vartheta_0 |B_{2r}|$$

with some constant $\vartheta_0 \in (0, 1)$. Thus it follows from (6.18), (6.21) and (6.22) that

$$(6.23) \quad \int_{\tau_0}^b \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy dt \leq c_2 r^n$$

for any $b \in [-\sigma r^{2s}, 0)$.

On the other hand, for any $b \in [-\sigma r^{2s}, 0)$ and $B_{5r}^0 \subset B_R^0$, we use the function

$$\varphi_0(x, t) = (v_{B_{2r}}^\theta(\tau_0) - v_{B_{2r}}^\theta(b)) \frac{\theta(x)}{\sqrt{\kappa} u(x, t)}$$

as another testing function. From (1.3), we obtain that

$$(6.24) \quad \int_{\Omega} \partial_t u(x, t) \varphi_0(x, t) dx + \mathcal{I}_t(u, \varphi_0) = 0$$

for a.e. $t \in I$. As in (6.8), the integral of the first term on $[\tau_0, b]$ in (6.24) can be estimated as

$$\begin{aligned}
(6.25) \quad \int_{\tau_0}^b \int_{\Omega} \partial_t u(x, t) \varphi_0(x, t) dx dt &= \frac{(v_{B_{2r}}^\theta(\tau_0) - v_{B_{2r}}^\theta(b))^2}{\sqrt{\kappa}} \int_{B_{2r}} \theta(x) dx \\
&\geq c_3 r^n (v_{B_{2r}}^\theta(\tau_0) - v_{B_{2r}}^\theta(b))^2.
\end{aligned}$$

Also we have that

$$-\mathcal{I}_t(u, \varphi_0) = (v_{B_{2r}}^\theta(b) - v_{B_{2r}}^\theta(\tau_0)) \mathcal{I}_t(u, \varphi_1) \quad \text{for } \varphi_1(x, t) = \frac{\theta(x)}{\sqrt{\kappa} u(x, t)}.$$

Thus, by (6.24) and (6.25), applying the mean value theorem and Lemma 4.4 as in (6.17) we have that

$$\begin{aligned}
(6.26) \quad c_3 r^n (v_{B_{2r}}^\theta(\tau_0) - v_{B_{2r}}^\theta(b))^2 &\leq \int_{\tau_0}^b [-\mathcal{I}_t(u, \varphi_0)] dt \\
&\leq |v_{B_{2r}}^\theta(b) - v_{B_{2r}}^\theta(\tau_0)| \int_{\tau_0}^b |\mathcal{I}_t(u, \varphi_1)| dt \\
&\leq c_4 r^n |v_{B_{2r}}^\theta(b) - v_{B_{2r}}^\theta(\tau_0)|.
\end{aligned}$$

This implies that, for any $\sigma \in (0, \nu)$, there is a constant $c_5 > 0$ depending only on $n, s, \lambda, \Lambda, \nu$ and σ such that

$$(6.27) \quad \sup_{b \in [-\sigma r^{2s}, 0)} |v_{B_{2r}}^\theta(b) - v_{B_{2r}}^\theta(\tau_0)| \leq c_5.$$

Finally, by Lemma 6.3, we obtain that

$$\begin{aligned}
(6.28) \quad \iint_{Q_r^+} |v(x, t) - v_{Q_r^+}|^2 dx dt &\leq 4 \iint_{Q_{2r}^+} |v(x, t) - v_{Q_{2r}^+}|^2 dx dt \\
&\leq 16 \iint_{Q_r^+} |v(x, t) - v_{B_{2r}}^\theta(t)|^2 dx dt + 16 \iint_{Q_{2r}^+} |v_{B_{2r}}^\theta(t) - v_{Q_{2r}^+}|^2 dx dt.
\end{aligned}$$

We observe that $\kappa \leq u \leq 1$ in B_{2r} and $\gamma(t) = \ln(d/t)$ is Lipschitz continuous in $[\kappa, \infty)$, and so we see that $v(\cdot, t) \in H^s(B_{2r})$ for any $t \in [-r^{2s}, 0)$. Since θ is a nonnegative function with $\theta \equiv \kappa$ on B_r , by Lemma 6.2 we have that

$$\begin{aligned}
(6.29) \quad \sqrt{\kappa} \int_{B_r} |v(x, t) - v_{B_{2r}}^\theta(t)|^2 dx \\
&\leq c r^{2s} \iint_{B_{2r} \times B_{2r}} |v(x, t) - v(y, t)|^2 (\theta(x) \wedge \theta(y)) d_K(x, y) \\
&\leq c_6 \sqrt{\kappa} r^{2s} \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy.
\end{aligned}$$

Thus this gives that

$$\begin{aligned}
(6.30) \quad \iint_{Q_r^+} |v(x, t) - v_{B_{2r}}^\theta(t)|^2 dx dt \\
&\leq c_6 r^{2s} \int_{-\sigma r^{2s}}^0 \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy dt.
\end{aligned}$$

Since we know from simple calculation that

$$v_{Q_{2r}^+}^\theta = \frac{1}{|Q_{2r}^+|} \iint_{Q_{2r}^+} v_{B_{2r}}^\theta(\tau) dy d\tau,$$

we have that

$$v_{B_{2r}}^\theta(t) - v_{Q_{2r}^+}^\theta = \frac{1}{|Q_{2r}^+|} \iint_{Q_{2r}^+} (v_{B_{2r}}^\theta(t) - v_{B_{2r}}^\theta(\tau)) dy d\tau.$$

So it follows from Jensen's inequality and (6.27) that

$$|v_{B_{2r}}^\theta(t) - v_{Q_{2r}^+}^\theta|^2 \leq \frac{1}{|Q_{2r}^+|} \iint_{Q_{2r}^+} |v_{B_{2r}}^\theta(t) - v_{B_{2r}}^\theta(\tau)|^2 dy d\tau \leq 4c_5^2.$$

Thus we obtain that

$$(6.31) \quad \iint_{Q_r^+} |v_{B_{2r}}^\theta(t) - v_{Q_r^+}^\theta|^2 dx dt \leq 4c_5^2 |Q_r^+|.$$

By (6.28), (6.30) and (6.31), we conclude that

$$\begin{aligned} & \iint_{Q_r^+} |v(x, t) - v_{Q_r^+}|^2 dx dt \\ & \leq 16 c_6 r^{2s} \int_{-\sigma r^{2s}}^0 \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy dt + 64 c_5^2 |Q_r^+|. \end{aligned}$$

Therefore we complete the proof. \square

7. NONLOCAL PARABOLIC HARNACK INEQUALITY

Lemma 7.1. *Let $u \in H^1(I; X_g(\Omega))$ be a weak solution of the equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ such that $u \geq 0$ in $Q_R^0 \subset \Omega_I$ where $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$, and let $m \geq 0$. If there is some $\nu \in (0, 1)$ such that*

$$(7.1) \quad |(Q_r^0)^+ \cap \{u \geq m\}| \geq \nu |(Q_r^0)^+|$$

for some $r \in (0, R/5)$, then there is a constant $C_0 = C_0(n, s, \lambda, \Lambda) > 0$ such that

$$(7.2) \quad \left| (Q_r^0)^+ \cap \left\{ u \leq 2\delta m - \frac{1}{2} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \right\} \right| \leq \frac{C_0}{\nu} \ln^{-1} \left(\frac{1}{2\delta} \right) |(Q_r^0)^+|$$

for any $\delta \in (0, 1/4)$.

Proof. For simplicity in writing, we proceed the proof with $(x_0, t_0) = (0, 0) := \mathbf{0}$. We set $\tilde{u} = u + \kappa$ where

$$(7.3) \quad \kappa = \frac{1}{2} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; \mathbf{0}).$$

Then we see that the function \bar{u} given by

$$\bar{u}(x, t) = \frac{\tilde{u}(x, t)}{m + \kappa}$$

is a weak solution of the equation. Set $v = \ln(1/\bar{u})$ and take any $\sigma \in (0, \nu)$. Then it follows from Theorem 6.1 that

$$\int_a^b \iint_{B_{2r} \times B_{2r}} \frac{|v(x, t) - v(y, t)|^2}{|x - y|^{n+2s}} dx dy dt \leq C r^n$$

for any $a, b \in [-\sigma r^{2s}, 0)$, and also we have the estimate

$$\begin{aligned} & \iint_{Q_r^+} |v(x, t) - v_{Q_r^+}|^2 dx dt \\ (7.4) \quad & \leq C r^{2s} \int_{-\sigma r^{2s}}^0 \iint_{B_{2r} \times B_{2r}} |v(x, t) - v(y, t)|^2 dK(x, y) + C |Q_r^+| \\ & \leq C |Q_r^+|. \end{aligned}$$

Thus, by Schwarz's inequality and (7.4), we easily obtain that

$$(7.5) \quad \frac{1}{|Q_r^+|} \iint_{Q_r^+} |v(x, t) - v_{Q_r^+}| dx dt \leq C,$$

and so this yields that

$$(7.6) \quad \frac{1}{|Q_r^+|} \iint_{Q_r^+} ||v(x, t)| - |v|_{Q_r^+}| dx dt \leq C$$

because $||v(x, t)| - |v|_{Q_r^+}| \leq |v(x, t) - v|_{Q_r^+}|$. For any $\delta \in (0, 1/4)$, we define the function \bar{v} by

$$\bar{v} = (v \vee 0) \wedge \ln\left(\frac{1}{2\delta}\right).$$

Observing the fact that

$$\alpha \wedge \beta = \frac{\alpha + \beta - |\alpha - \beta|}{2} \quad \text{and} \quad \alpha \vee \beta = \frac{\alpha + \beta + |\alpha - \beta|}{2}$$

for any $\alpha, \beta \in \mathbb{R}$, we can easily derive from (7.5) and (7.6) that

$$(7.7) \quad \frac{1}{|Q_r^+|} \iint_{Q_r^+} |\bar{v}(x, t) - \bar{v}|_{Q_r^+}| dx dt \leq C.$$

By the definition of \bar{v} , we see that

$$Q_r^+ \cap \{\bar{v} = 0\} = Q_r^+ \cap \{\bar{u} \geq m + \kappa\} = Q_r^+ \cap \{u \geq m\}.$$

Thus it follows from (7.1) that

$$(7.8) \quad |Q_r^+ \cap \{\bar{v} = 0\}| \geq \nu |Q_r^+|.$$

From (7.8), we can derive the following estimate

$$(7.9) \quad \begin{aligned} \ln\left(\frac{1}{2\delta}\right) &= \frac{1}{|Q_r^+ \cap \{\bar{v} = 0\}|} \iint_{Q_r^+ \cap \{\bar{v} = 0\}} \left[\ln\left(\frac{1}{2\delta}\right) - \bar{v}(x, t) \right] dx dt \\ &\leq \frac{1}{\nu} \left[\ln\left(\frac{1}{2\delta}\right) - \bar{v}|_{Q_r^+} \right]. \end{aligned}$$

Integrating the inequality (7.9) on $Q_r^+ \cap \{\bar{v} = \ln(1/2\delta)\}$ and applying (7.1), we easily obtain that

$$\begin{aligned} &\left| Q_r^+ \cap \left\{ \bar{v} = \ln\left(\frac{1}{2\delta}\right) \right\} \right| \ln\left(\frac{1}{2\delta}\right) \\ &\leq \frac{1}{\nu} \iint_{Q_r^+} |\bar{v}(x, t) - \bar{v}|_{Q_r^+}| dx dt \leq \frac{C_0}{\nu} |Q_r^+|. \end{aligned}$$

This implies that

$$|Q_r^+ \cap \{\bar{u} \leq 2\delta m\}| \leq \frac{C_0}{\nu} \ln^{-1}\left(\frac{1}{2\delta}\right) |Q_r^+|.$$

Hence we complete the proof. \square

Lemma 7.2. *Let $u \in H^1(I; X_g(\Omega))$ be a weak solution of the equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ such that $u \geq 0$ in $Q_R^0 \subset \Omega_I$ where $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$, and let $m \geq 0$. If there is some $\nu \in (0, 1]$ such that*

$$|(Q_r^0)^+ \cap \{u \geq m\}| \geq \nu |(Q_r^0)^+|$$

for some $r \in (0, R/5)$, then there is a constant $\delta = \delta(n, s, \lambda, \Lambda) \in (0, 1/4)$ such that

$$\inf_{(Q_r^0)^+} u \geq \delta m - \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)).$$

Proof. For simplicity, we proceed the proof by setting $(x_0, t_0) = (0, 0) := \mathbf{0}$. Since $u \geq 0$ in Q_R , without loss of generality we may assume that

$$(7.10) \quad \delta m \geq \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; \mathbf{0}).$$

Set $w = (h - u)_+$ for $h \in (\delta m, 2\delta m)$. For $\varrho \in (r, 2r)$, we now choose a testing function $\varphi(x, t) = \theta^2(x)\eta^2(t)w(x, t)$ where $\theta \in C_c^\infty(B_\varrho)$ is a function satisfying that $0 \leq \theta \leq 1$ and $|\nabla\theta| \leq c/\varrho$ in \mathbb{R}^n , and $\eta \in C_c^\infty(-\sigma\varrho^{2s}, \infty]$ is a function such that $0 \leq \eta \leq 1$ and $|\eta'| \leq c/\varrho^{2s}$ in \mathbb{R} . Then we have that

$$(7.11) \quad \begin{aligned} 0 &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t))K(x - y) dx dy \\ &\quad + \int_{\Omega} \partial_t u(x, t) \varphi(x, t) dx \\ &= \int_{B_\varrho} \int_{B_\varrho} (u(x, t) - u(y, t))(\varphi(x, t) - \varphi(y, t))K(x - y) dx dy \\ &\quad + 2 \int_{\mathbb{R}^n \setminus B_\varrho} \int_{B_\varrho} (u(x, t) - u(y, t)) \varphi(x, t)K(x - y) dx dy \\ &\quad + \int_{B_\varrho} \partial_t u(x, t) \varphi(x, t) dx \\ &:= \mathcal{I}_1(t) + \mathcal{I}_2(t) + \mathcal{I}_3(t) \end{aligned}$$

for a.e. $t \in I$. Splitting $\mathcal{I}_2(t)$ into two parts yields that

$$\begin{aligned} \frac{1}{2} \mathcal{I}_2(t) &= \int_{B_\varrho^c \cap \{y: u(y, t) < 0\}} \int_{B_\varrho} (u(x, t) - u(y, t)) \varphi(x, t)K(x - y) dx dy \\ &\quad + \int_{B_\varrho^c \cap \{y: u(y, t) \geq 0\}} \int_{B_\varrho} (u(x, t) - u(y, t)) \varphi(x, t)K(x - y) dx dy \\ &= \mathcal{I}_2^1(t) + \mathcal{I}_2^2(t). \end{aligned}$$

Thus we have that

$$\begin{aligned} \mathcal{I}_2^1(t) &\leq h|B_\varrho \cap \{x : u(x, t) < h\}| \sup_{x \in \text{supp}(\theta)} \int_{B_\varrho^c} (h + u^-(y, t))K(x - y) dy, \\ \mathcal{I}_2^2(t) &\leq h^2|B_\varrho \cap \{x : u(x, t) < h\}| \sup_{x \in \text{supp}(\theta)} \int_{B_\varrho^c} K(x - y) dy, \end{aligned}$$

and this leads us to get that

$$(7.12) \quad \begin{aligned} &\int_{-\sigma\varrho^{2s}}^0 \mathcal{I}_2(t) dt \\ &\leq 4h|Q_\varrho^+ \cap \{u < h\}| \sup_{(x, t) \in \Gamma_\varrho^0} \int_{B_\varrho^c} (h + u^-(y, t))K(x - y) dy, \end{aligned}$$

where $\Gamma_\varrho^\theta = \text{supp}(\theta) \times [-\sigma\varrho^{2s}, 0)$. As in the proof of the Caccioppoli type estimate in Theorem 1.4 [DKP1], we have that

$$(7.13) \quad \begin{aligned} & \int_{-\sigma\varrho^{2s}}^0 \mathcal{I}_1(t) dt \\ & \leq -c \int_{-\sigma\varrho^{2s}}^0 \iint_{B_\varrho \times B_\varrho} |w(x, t)\theta(x) - w(y, t)\theta(y)|^2 K(x - y) dx dy dt \\ & \quad + c \int_{-\sigma\varrho^{2s}}^0 \iint_{B_\varrho \times B_\varrho} (w(x, t) \vee w(y, t))^2 |\theta(x) - \theta(y)|^2 K(x - y) dx dy dt. \end{aligned}$$

For any $\tau \in [-\sigma\varrho^{2s}, 0)$, we have the estimate

$$(7.14) \quad \begin{aligned} \int_{-\sigma\varrho^{2s}}^\tau \mathcal{I}_3(t) dt &= - \int_{B_\varrho} w^2(x, \tau) \theta^2(x) \eta^2(\tau) dx \\ & \quad + 2 \int_{-\sigma\varrho^{2s}}^\tau \int_{B_\varrho} w^2(x, t) \theta^2(x) \eta(t) \eta'(t) dx dt. \end{aligned}$$

From (7.11), (7.12), (7.13) and (7.14), we have that

$$(7.15) \quad \begin{aligned} & \sup_{t \in [-\sigma\varrho^{2s}, 0)} \eta^2(t) \|w(\cdot, t)\theta\|_{L^2(B_\varrho)}^2 + c \int_{-\sigma\varrho^{2s}}^0 \eta^2(t) [w(\cdot, t)\theta]_{H^s(B_\varrho)}^2 dt \\ & \leq 4h |Q_\varrho^+ \cap \{u < h\}| \sup_{(x, t) \in \Gamma_\varrho^\theta} \int_{B_\varrho^c} (h + u^-(y, t)) K(x - y) dy \\ & \quad + c \int_{-\sigma\varrho^{2s}}^0 \iint_{B_\varrho \times B_\varrho} (w(x, t) \vee w(y, t))^2 |\theta(x) - \theta(y)|^2 K(x - y) dx dy dt \\ & \quad + c\varrho^{-2s} \|w\|_{L^2(Q_\varrho^+)}^2 := 4h |Q_\varrho^+ \cap \{u < h\}| A_\varrho(u, \theta, h) + B_\varrho(w, \theta) + C_\varrho(w). \end{aligned}$$

We now apply Lemma 2.2 to the estimates (7.15). For $k = 0, 1, \dots$, we set

$$(7.16) \quad h_k = \delta m + 2^{-(k+1)} \delta m, \quad \varrho_k = r + 2^{-k} r, \quad \bar{\varrho}_k = \frac{\varrho_{k+1} + \varrho_k}{2}.$$

We observe that $r < \varrho_k, \bar{\varrho}_k < 2r$ and $h_k - h_{k+1} \geq 2^{-(k+3)} h_k$, and

$$h_0 = \frac{3}{2} \delta m \leq 2\delta m - \frac{1}{2} \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; \mathbf{0}).$$

Since we see that

$$\{u < h_0\} \subset \left\{ u \leq 2\delta m - \frac{1}{2} \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; \mathbf{0}) \right\},$$

by Lemma 7.1 we have that

$$(7.17) \quad \frac{|Q_r^+ \cap \{u < h_0\}|}{|Q_r^+|} \leq \frac{C_0}{\nu} \ln^{-1} \left(\frac{1}{2\delta} \right).$$

For $k = 0, 1, \dots$, we note that

$$(7.18) \quad \begin{aligned} w_k &= (h_k - u)_+ \geq (h_k - h_{k+1}) \mathbb{1}_{\{u < h_{k+1}\}} \\ & \geq 2^{-k-3} h_k \mathbb{1}_{\{u < h_{k+1}\}}. \end{aligned}$$

For $k = 0, 1, \dots$, set $Q_k = Q_{\varrho_k}^+$, and let $\theta_k \in C_c^\infty(B_{\bar{\varrho}_k})$ be a function with $\theta_k \equiv 1$ on $B_{\varrho_{k+1}}$ such that $0 \leq \theta_k \leq 1$ and $|\nabla \theta_k| \leq 2^{k+1}/r$ in \mathbb{R}^n and let $\eta_k \in C_c^\infty(-\sigma \varrho_k^{2s}, \infty]$ is a function with $\eta_k \equiv 1$ on $[-\sigma \varrho_{k+1}^{2s}, 0)$ such that $0 \leq \eta_k \leq 1$ and

$$|\eta'_k| \leq \frac{2^{k+1}}{\sigma \varrho_k^{2s}} \quad \text{in } \mathbb{R}.$$

By applying H\u00f3lder's inequality with

$$q = \frac{n}{n-2s} \quad \text{and} \quad q' = \frac{n}{2s},$$

and the fractional Sobolev's inequality with

$$\alpha = 1 + \frac{2s}{n} = \frac{1}{q'} + 1,$$

we can derive the following inequalities

$$\begin{aligned}
& \iint_{Q_{k+1}} |w_k|^{2\alpha} dx dt \leq \iint_{Q_k} |w_k \theta_k \eta_k|^2 |w_k \theta_k \eta_k|^{\frac{4s}{n}} dx dt \\
& \leq \int_{-\sigma \varrho_k^{2s}}^0 \left(\int_{B_{\varrho_k}} |w_k \theta_k \eta_k|^{2q} dx \right)^{\frac{1}{q}} \left(\int_{B_{\varrho_k}} |w_k \theta_k \eta_k|^2 dx \right)^{\frac{1}{q'}} dt \\
(7.19) \quad & \leq \gamma_{n,2} \left(\sup_{t \in [-\sigma \varrho_k^{2s}, 0)} \eta_k^2(t) \|w_k(\cdot, t) \theta_k\|_{L^2(B_{\varrho_k})}^2 \right)^{\frac{1}{q'}} \\
& \quad \times \left[(1-s) \int_{-\sigma \varrho_k^{2s}}^0 \eta_k^2(t) [w_k(\cdot, t) \theta_k]_{H^s(B_{\varrho_k})}^2 dt + \varrho_k^{-2s} \|w_k\|_{L^2(Q_k)}^2 \right] \\
& \leq C \left(4h_k |Q_k \cap \{u < h_k\}| A_{\varrho_k}(u, h_k) + B_{\varrho_k}(w_k, \theta_k) + C_{\varrho_k}(w_k) \right)^\alpha
\end{aligned}$$

where $C > 0$ is a universal constant depending only on n and s . From (7.10), (7.15) and the fact that

$$|y - x| \geq |y| - |x| \geq \left(1 - \frac{\bar{\varrho}_k}{\varrho_k}\right) |y| \geq 2^{-k-3} |y|$$

for any $y \in B_{\varrho_k}^c$ and $x \in B_{\bar{\varrho}_k}$, by (7.10) and (7.16) we get the estimate

$$\begin{aligned}
A_{\varrho_k}(u, \theta_k, h_k) &= \sup_{(x,t) \in \Gamma_{\varrho_k}^{\theta_k}} \int_{B_{\varrho_k}^c} (h_k + u^-(y, t)) K(x - y) dy \\
&\leq c 2^{k(n+2s)} \int_{B_{\varrho_k}^c} \frac{h_k + u^-(y, t)}{|y|^{n+2s}} dy \\
(7.20) \quad &\leq c 2^{k(n+2s)} h_k r^{-2s} + c 2^{k(n+2s)} \int_{B_R^c} \frac{u^-(y, t)}{|y|^{n+2s}} dy \\
&\leq c 2^{k(n+2s)} h_k r^{-2s} + c 2^{k(n+2s)} r^{-2s} \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; \mathbf{0}) \\
&\leq c 2^{k(n+2s)} h_k r^{-2s}.
\end{aligned}$$

Also we have the following estimates

$$\begin{aligned}
& B_{\varrho_k}(w_k, \theta_k) \\
& \leq c \int_{-\sigma \varrho_k^{2s}}^0 \iint_{B_{\varrho_k} \times B_{\varrho_k}} (w_k(x, t) \vee w_k(y, t))^2 \frac{|\theta_k(x) - \theta_k(y)|^2}{|x - y|^{n+2s}} dx dy dt \\
(7.21) \quad & \leq c h_k^2 \int_{-\sigma \varrho_k^{2s}}^0 \int_{B_{\varrho_k}} \int_{B_{\varrho_k} \cap \{u(\cdot, t) < h_k\}} \frac{\sup_{\mathbb{R}^n} |\nabla \theta_k|^2}{|x - y|^{n+2s-2}} dx dy dt \\
& \leq c h_k^2 \left(\frac{2^k}{r}\right)^2 \int_{-\sigma \varrho_k^{2s}}^0 \int_{B_{\varrho_k} \cap \{u(\cdot, t) < h_k\}} \left(\int_{B_{2\varrho_k}} \frac{1}{|y|^{n+2s-2}} dy \right) dx dt \\
& \leq c 2^{2k} h_k^2 r^{-2s} |Q_k \cap \{u < h_k\}|.
\end{aligned}$$

and $C_{\varrho_k}(w_k) \leq c h_k^2 r^{-2s} |Q_k \cap \{u < h_k\}|$. From (7.19), (7.20) and (7.21), we conclude that

$$\left(\iint_{Q_{k+1}} |w_k|^{2\alpha} dx dt \right)^{\frac{1}{\alpha}} \leq C 2^{k(n+2s+2)} h_k^2 r^{-2s} |Q_k \cap \{u < h_k\}|.$$

Since $|Q_{k+1}|^{-1/\alpha} \sim r^{-n}$ and $|Q_k| \sim r^{n+2s}$, this estimate and (7.18) yield that

$$\begin{aligned}
(7.22) \quad & (h_k - h_{k+1})^2 \left(\frac{|Q_{k+1} \cap \{u < h_{k+1}\}|}{|Q_{k+1}|} \right)^{\frac{1}{\alpha}} \\
& \leq \left(\frac{1}{|Q_{k+1}|} \iint_{Q_{k+1}} |w_k|^{2\alpha} dx dt \right)^{\frac{1}{\alpha}} \\
& \leq C 2^{k(n+2s+2)} h_k^2 \frac{|Q_k \cap \{u < h_k\}|}{|Q_k|}.
\end{aligned}$$

For $k = 0, 1, 2, \dots$, we set

$$N_k = \frac{|Q_k \cap \{u < h_k\}|}{|Q_k|}.$$

Then it follows from (7.22) that

$$N_{k+1}^{\frac{1}{\alpha}} \leq C \frac{2^{k(n+2s+2)} h_k^2}{(h_k - h_{k+1})^2} N_k \leq C 2^{k(n+2s+4)} N_k.$$

This leads to us to obtain that

$$(7.23) \quad N_{k+1} \leq C_1 2^{k\alpha(n+2s+4)} N_k^{1+\frac{2s}{n}}$$

where $C_1 = C^\alpha$. In addition, we see from (7.17) that

$$(7.24) \quad N_0 \leq \frac{C_0}{\nu} \ln^{-1} \left(\frac{1}{2\delta} \right).$$

We apply Lemma 2.2 with

$$d_0 = C_1, \quad a = 2^{n+2s+4} > 1, \quad \text{and} \quad \eta = \frac{2s}{n}.$$

If we choose a small δ depending only on n, s, λ, Λ and ν so that

$$(7.25) \quad 0 < \delta := \frac{1}{2} \exp \left(-\frac{C_0 C_1^{\frac{n}{2s}} 2^{\frac{n^2}{4s^2}(n+2s+4)}}{\nu} \right) < \frac{1}{4},$$

then $N_0 \leq C_1^{-\frac{n}{2s}} (2^{n+2s+4})^{-\frac{n^2}{4s^2}}$. Thus we conclude that $\lim_{k \rightarrow \infty} N_k = 0$. This implies that

$$\inf_{Q_r^+} u \geq \delta m.$$

Hence we complete the proof. \square

Next, we need a parabolic version of the Krylov-Safonov covering theorem [KS] which is a useful tool for the proof of Theorem 7.4.

For $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $r > 0$ and $\sigma \in (0, \nu)$ (where ν is the constant given in (6.3)), we see that the parabolic cylinders

$$Q_r^+(x, t) = B_r(x) \times [t - \sigma r^{2s}, t)$$

is given by $Q_r^+(x, t) = \{Y \in \mathbb{R}^n \times \mathbb{R} : d(Y, X) < r\}$ where d is the parabolic distance between $X = (x, t)$ and $Y = (y, \tau)$ by

$$(7.26) \quad d(X, Y) = \begin{cases} |x - y| \vee \left(\frac{|t - \tau|}{\sigma} \right)^{1/2s}, & \tau < t, \\ \infty, & \tau \geq t. \end{cases}$$

Then we note that If $E \subset \mathbb{R}^n \times \mathbb{R}$ is a bounded set and \mathfrak{C}_E is a collection of cylinders $Q_r^+(x, t)$ with $(x, t) \in E$, then it follows from Vitali's covering theorem that there is a countable pairwise disjoint subcollection $\mathcal{C}_E = \{Q_{r_k}^+(x_k, t_k)\}_{k \in \mathbb{N}}$ of \mathfrak{C}_E such that

$$E \subset \bigcup_{k \in \mathbb{N}} Q_{r_k}^+(x_k, t_k).$$

For $\varrho > 0$, $\gamma \in (0, 1)$ and a measurable subset E of a cylinder $(Q_r^0)^+ = Q_r^+(x_0, t_0)$, we define the set

$$(7.27) \quad E_\gamma^\varrho = \bigcup_{0 < \rho < \varrho} \{Q_{3\rho}^+(X) \cap (Q_r^0)^+ : |E \cap Q_{3\rho}^+(X)| > \gamma |Q_\rho^+(X)|, X \in (Q_r^0)^+\}.$$

The following nonlocal parabolic version of the Krylov-Safonov covering theorem no longer depends on the threshold radius ϱ . Its proof is based on that of [KSh].

Lemma 7.3. *If $\varrho > 0$, $\gamma \in (0, 1)$ and $E \subset Q_r^+(x_0, t_0)$ is a measurable set, then*

$$\text{either } |E_\gamma^\varrho| \geq \frac{2^{-(n+2s)}}{\gamma} |E| \text{ or } E_\gamma^\varrho = Q_r^+(x_0, t_0).$$

Proof. We define the maximal operator $\mathcal{M} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\mathcal{M}(y, \tau) = \sup_{Q_{3\rho}^+(x, t) \in \mathfrak{F}} \frac{|E \cap Q_{3\rho}^+(x, t)|}{|Q_\rho^+(x, t)|}$$

where \mathfrak{F} is the family of all cylinders $Q_{3\rho}^+(x, t)$ with $(x, t) \in Q_r^+(x_0, t_0)$, $0 < \rho < \varrho$ and $(y, \tau) \in Q_{3\rho}^+(x, t)$. Then we see that

$$(7.28) \quad E_\gamma^\varrho = \{(y, \tau) \in Q_r^+(x_0, t_0) : \mathcal{M}(y, \tau) > \gamma\}.$$

Indeed, if $Y = (y, \tau) \in Q_r^+(x_0, t_0)$, then there is a cylinder $Q_{3\rho}^+(x, t)$ with $(x, t) \in Q_r^+(x_0, t_0)$, $0 < \rho < \varrho$ and $|E \cap Q_{3\rho}^+(x, t)| > \gamma |Q_\rho^+(x, t)|$. Thus this gives that $Y \in E_\gamma^\varrho$. On the other hand, if $Y \in E_\gamma^\varrho$, then there is a cylinder $Q_{3\rho}^+(x, t)$ with $(x, t) \in Q_r^+(x_0, t_0)$ and $Y \in Q_{3\rho}^+(x, t)$, $0 < \rho < \varrho$ and $|E \cap Q_{3\rho}^+(x, t)| > \gamma |Q_\rho^+(x, t)|$. This implies that $\mathcal{M}(Y) > \gamma$.

Suppose that $E_\gamma^g \neq Q_r^+(x_0, t_0)$, i.e. $Q_r^+(x_0, t_0) \setminus E_\gamma^g \neq \emptyset$. Since E_γ^g is open with respect to the metric d , we see that

$$\rho_Y := \frac{1}{2} \sup\{\rho > 0 : Y \in Q_\rho^+(x, t) \subset E_\gamma^g, Q_{2\rho}^+(x, t) \cap (Q_r^+(x_0, t_0) \setminus E_\gamma^g) = \emptyset\} > 0$$

for each $Y = (y, \tau) \in E_\gamma^g$. So we may assume that $\rho_Y < \varrho/4$. Since each $Y \in E_\gamma^g$ yields a point $X_Y := (x_Y, t_Y)$ so that $Y \in Q_{\rho_Y}^+(X_Y) \subset E_\gamma^g$ and

$$Q_{5\rho_Y}^+(X_Y) \cap (Q_r^+(x_0, t_0) \setminus E_\gamma^g) \neq \emptyset,$$

the family $\mathcal{C} = \{Q_{\rho_Y}^+(X_Y) : Y \in E_\gamma^g\}$ covers E_γ^g . From Vitali's covering lemma, we can extract a countable family $\{Q_{\rho_k}^+(X_k)\}_{k \in \mathbb{N}}$ of pairwise disjoint parabolic cubes (where $X_k = X_{Y_k}$ and $\rho_k = \rho_{Y_k}$ for $k \in \mathbb{N}$) such that

$$E_\gamma^g \subset \bigcup_{k \in \mathbb{N}} Q_{\rho_k}^+(X_k).$$

Observe that $Q_{5\rho_k}^+(X_k) \cap (Q_r^+(x_0, t_0) \setminus E_\gamma^g) \neq \emptyset$ for all $k \in \mathbb{N}$. If $Y_k = (y_k, \tau_k) \in Q_{5\rho_k}^+(X_k) \cap (Q_r^+(x_0, t_0) \setminus E_\gamma^g)$, then we easily see that $\mathcal{M}(Y_k) \leq \gamma$ for $k \in \mathbb{N}$. Since $Y_k \in Q_{5\rho_k}^+(X_k)$ and $5\rho_k/3 < \varrho$, we have that

$$(7.29) \quad |E \cap Q_{5\rho_k}^+(X_k)| \leq \gamma |Q_{5\rho_k/3}^+(X_k)| \leq 2^{n+2s}\gamma |Q_{\rho_k}^+(X_k)|.$$

Moreover, by (7.28) we see that every density points of E belongs to E_γ^g , because

$$\liminf_{\rho \rightarrow 0} \frac{|E \cap Q_{3\rho}^+(X)|}{|Q_\rho^+(X)|} \geq 1 > \gamma$$

for any density point X of E . Hence this and (7.29) enables us to obtain that

$$|E| = |E \cap E_\gamma^g| \leq \sum_{k=1}^{\infty} |E \cap Q_{5\rho_k}^+(X_k)| \leq 2^{n+2s}\gamma \sum_{k=1}^{\infty} |Q_{\rho_k}^+(X_k)| \leq 2^{n+2s}\gamma |E_\gamma^g|. \quad \square$$

Theorem 7.4. *Let $g \in C(\mathbb{R}_{I_x}^n) \cap L^\infty(\mathbb{R}_I^n)$. If $u \in H^1(I; X_g(\Omega))$ is a weak solution of the nonlocal equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ with $u \geq 0$ in $Q_R^0 \subset \Omega_I$, then we have the estimate*

$$(7.30) \quad \left(\frac{1}{2|(Q_r^0)^+|} \int_{(Q_r^0)^+} u^p dx dt \right)^{\frac{1}{p}} \leq \inf_{(Q_r^0)^+} u + \frac{4}{3} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_r(u^-; (x_0, t_0))$$

for any $p \in (0, 1)$ and $r \in (0, R)$.

Proof. Take any $r \in (0, R)$. For simplicity, we may assume that $(x_0, t_0) = (0, 0)$. For $\alpha > 0$ and $k = 0, 1, 2, \dots$, we set

$$\mathcal{R}_\alpha^k = \left\{ (x, t) \in Q_r^+ : u(x, t) \geq \delta^k \alpha - \frac{2\kappa}{1-\delta} \right\}$$

where $\delta \in (0, 1/4)$ is the constant in Lemma 7.2 and $\kappa > 0$ is the constant given by

$$\kappa = \frac{1}{2} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (0, 0)).$$

Then we see that $\mathcal{R}_\alpha^{k-1} \subset \mathcal{R}_\alpha^k$ for all $k \in \mathbb{N}$. Let $(x, t) \in Q_r^+$ with $Q_{3\rho}^+(x, t) \cap Q_r^+ \subset E_\gamma^g$ where $E = \mathcal{R}_\alpha^{k-1}$. From (7.27), we see that

$$|\mathcal{R}_\alpha^{k-1} \cap Q_{3\rho}^+(x, t)| > \gamma |Q_\rho^+(x, t)| = \frac{\gamma}{3^{n+2s}} |Q_{3\rho}^+(x, t)|.$$

Applying Lemma 7.2 with $m = \delta^{k-1}\alpha - \frac{2\kappa}{1-\delta}$ and $\nu = \gamma 3^{-n-2s}$, we have that

$$u \geq \delta \left(\delta^{k-1}\alpha - \frac{2\kappa}{1-\delta} \right) - 2\kappa = \delta^k \alpha - \frac{2\kappa}{1-\delta} \quad \text{in } Q_{3\rho}^+(x, t) \cap Q_r^+$$

and so we obtain that $E_\gamma^\ell \subset \mathcal{R}_\alpha^k$. Thus it follows from Lemma 7.3 that either $\mathcal{R}_\alpha^k = Q_r^+$ or $|\mathcal{R}_\alpha^k| \geq \frac{2^{-n-2s}}{\gamma} |\mathcal{R}_\alpha^{k-1}|$ for any $k \in \mathbb{N}$. Here, without loss of generality, we assume that

$$(7.31) \quad \frac{1}{16} 2^{-n-2s} < \gamma < 2^{-n-2s}.$$

Then we claim that, if there is some $N \in \mathbb{N}$ such that

$$(7.32) \quad |\mathcal{R}_\alpha^0| > (2^{n+2s}\gamma)^N |Q_r^+|,$$

then we have that $\mathcal{R}_\alpha^N = Q_r^+$. Indeed, if $\mathcal{R}_\alpha^N \neq Q_r^+$, then $|\mathcal{R}_\alpha^N| \geq \frac{2^{-n-2s}}{\gamma} |\mathcal{R}_\alpha^{N-1}|$, and so $\mathcal{R}_\alpha^k \neq Q_r^+$ for any $k = 1, 2, \dots, N$. This implies that

$$|\mathcal{R}_\alpha^N| \geq \frac{1}{2^{n+2s}\gamma} |\mathcal{R}_\alpha^{N-1}| \geq \dots \geq \left(\frac{1}{2^{n+2s}\gamma} \right)^N |\mathcal{R}_\alpha^0| > |Q_r^+|,$$

which gives a contradiction. Thus the fact that $\mathcal{R}_\alpha^N = Q_r^+$ leads us to obtain that

$$(7.33) \quad u \geq \delta^N \alpha - \frac{2\kappa}{1-\delta} \quad \text{in } Q_r^+.$$

If N is the smallest integer satisfying (7.31), we see that

$$(7.34) \quad N > \frac{1}{\ln(2^{n+2s}\gamma)} \ln \left(\frac{|\mathcal{R}_\alpha^0|}{|Q_r^+|} \right).$$

From (7.32) and (7.33), we conclude that

$$(7.35) \quad \inf_{Q_r^+} u \geq \alpha \left(\frac{|\mathcal{R}_\alpha^0|}{|Q_r^+|} \right)^{1/\ell} - \frac{2\kappa}{1-\delta} \quad \text{with } \ell = \frac{\ln(2^{n+2s}\gamma)}{\ln \delta},$$

where δ and ℓ depend only on n, s, λ and Λ . This enables us to get that

$$(7.36) \quad \frac{|Q_r^+ \cap \{u \geq \alpha - \frac{2\kappa}{1-\delta}\}|}{|Q_r^+|} = \frac{|\mathcal{R}_\alpha^0|}{|Q_r^+|} \leq \alpha^{-\ell} \left(\inf_{Q_r^+} u + \frac{2\kappa}{1-\delta} \right)^\ell.$$

By standard analysis, we have that

$$(7.37) \quad \frac{1}{|Q_r^+|} \iint_{Q_r^+} u^p dx dt = p \int_0^\infty \alpha^{p-1} \frac{|Q_r^+ \cap \{u \geq \alpha\}|}{|Q_r^+|} d\alpha$$

for any $p > 0$. Thus it follows from (7.35) and (7.36) that

$$\begin{aligned} \frac{1}{|Q_r^+|} \iint_{Q_{2r}^+} u^p dx dt &\leq p \int_0^\infty \alpha^{p-1} \frac{|Q_r^+ \cap \{u \geq \alpha - \frac{2\kappa}{1-\delta}\}|}{|Q_r^+|} d\alpha \\ &\leq p \int_0^h \alpha^{p-1} d\alpha + p \left(\inf_{Q_r^+} u + \frac{2\kappa}{1-\delta} \right)^\ell \int_h^\infty \alpha^{p-1-\ell} d\alpha. \end{aligned}$$

If we take $h = \inf_{Q_r^+} u + \frac{2\kappa}{1-\delta}$ and $p = \ell/2$, then we conclude that

$$\frac{1}{|Q_r^+|} \iint_{Q_r^+} u^p dx dt \leq 2 \left(\inf_{Q_r^+} u + \frac{2\kappa}{1-\delta} \right)^p \leq 2 \left(\inf_{Q_r^+} u + \frac{4}{3} 2\kappa \right)^p$$

and $0 < p < 1$ by (7.30) because $\delta \in (0, 1/4)$. Since we can take any sufficiently large C_0, C_1 in (7.25), by (7.30) and (7.34) the result of Theorem 7.4 holds for all $p \in (0, 1)$ with the same universal constants 1 and $4/3$. Hence we are done. \square

Theorem 7.5. *Let $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$. If $u \in H^1(I; X_g(\Omega))$ is a weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ with $u \geq 0$ in $Q_R^0 \subset \Omega_I$, then there exists a constant $c > 0$ depending only on n, s, λ and Λ such that*

$$(7.38) \quad \sup_{(Q_r^0)^-} u \leq c \inf_{(Q_r^0)^+} u + c \left(\frac{r}{R}\right)^{2s} \mathcal{T}_r(u^-; (x_0, t_0))$$

for any $r > 0$ with $5r < R$.

We can easily derive the following nonlocal parabolic Harnack inequalities for a nonnegative weak solutions of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$ as a natural by-product of Theorem 7.5. By the way, the result has no nonlocal parabolic tail, which means that the result coincides with that of local parabolic case.

Corollary 7.6. *Let $g \in C(\mathbb{R}_{I_*}^n) \cap L^\infty(\mathbb{R}_I^n)$. If $u \in H^1(I; X_g(\Omega))$ is any nonnegative weak solution of the nonlocal parabolic equation $\mathbf{NP}_{\Omega_I}(0, g, g)$, then there exists a constant $c > 0$ depending only on n, s, λ and Λ such that*

$$(7.39) \quad \sup_{(Q_r^0)^-} u \leq c \inf_{(Q_r^0)^+} u$$

for any $r > 0$ with $5r < R$.

[Proof of Theorem 7.5(nonlocal Harnack inequality)] If $Q_{3r}^0 \subset Q_R^0 \subset \Omega_I$, then by Theorem 5.2 and Lemma 5.3 there is a constant $c_0 = c_0(n, s, \lambda, \Lambda) > 0$ such that

$$(7.40) \quad \begin{aligned} \sup_{Q_{r/2}^0} u &\leq \delta \mathcal{T}_r(u^+; (x_0, t_0)) + c_0 \delta^{-\gamma_s} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^2 dx dt \right)^{\frac{1}{2}} \\ &\leq c_0 \delta^{-\gamma_s} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^2 dx dt \right)^{\frac{1}{2}} \\ &\quad + c \delta \sup_{Q_{2r}^0} u + c \delta \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \end{aligned}$$

for any $\delta \in (0, 1]$, where $\gamma_s = \frac{n+2s}{4s}$. Thus it follows from a covering argument with $\frac{1}{2} \leq a < b \leq 2$ that

$$(7.41) \quad \begin{aligned} \sup_{Q_{ar}^0} u &\leq \frac{c \delta^{-\gamma_s}}{(b-a)^{\frac{n+2s}{2}}} \left(\frac{1}{|Q_{br}^0|} \iint_{Q_{br}^0} u^2 dx dt \right)^{\frac{1}{2}} \\ &\quad + c \delta \sup_{Q_{br}^0} u + c \delta \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \\ &\leq \frac{c \delta^{-\gamma_s}}{(b-a)^{\frac{n+2s}{2}}} \left(\sup_{Q_{br}^0} u \right)^{\frac{2-p}{2}} \left(\frac{1}{|Q_{br}^0|} \iint_{Q_{br}^0} u^p dx dt \right)^{\frac{1}{2}} \\ &\quad + c \delta \sup_{Q_{br}^0} u + c \delta \left(\frac{r}{R}\right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \end{aligned}$$

Taking $\delta = \frac{1}{2c}$ in (7.38) and applying Young's inequality with $\varepsilon = (b-a)^{(n+2s)(\frac{1}{2}-\frac{1}{p})}$ yield that

$$(7.42) \quad \sup_{Q_{ar}^0} u \leq \frac{1}{2} \sup_{Q_{br}^0} u + \frac{c}{(b-a)^{\frac{n+2s}{p}}} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^p dx dt \right)^{\frac{1}{p}} + c \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)).$$

Employing Lemma 2.3 in (7.39) leads us to obtain that

$$(7.43) \quad \sup_{Q_{er}^0} u \leq c \left[\frac{1}{(2-\varrho)^{\frac{n+2s}{p}}} \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^p dx dt \right)^{\frac{1}{p}} + \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \right]$$

for any $p \in (0, 2]$ and any $\varrho \in [1/2, 2)$. We note that

$$(7.44) \quad (A+B)^{\frac{1}{p}} \leq 2^{\frac{1}{p}} (A^{\frac{1}{p}} + B^{\frac{1}{p}})$$

for any $A, B \geq 0$ and

$$(7.45) \quad \frac{2}{|Q_{2r}^0|} \leq \frac{1}{2|(Q_{2r}^0)^+|} \Leftrightarrow \frac{2}{2-\sigma} \leq \frac{1}{2\sigma} \Leftrightarrow 0 < \sigma \leq \frac{2}{5}.$$

Taking $\varrho = 1$ in (7.40) it follows from (7.41), (7.42) and Theorem 7.4 that

$$(7.46) \quad \begin{aligned} \sup_{Q_r^0} u &\leq c \left(\frac{1}{|Q_{2r}^0|} \iint_{Q_{2r}^0} u^p dx dt \right)^{\frac{1}{p}} + c \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \\ &\leq c \left(\frac{2}{|Q_{2r}^0|} \iint_{(Q_{2r}^0)^+} u^p dx dt \right)^{\frac{1}{p}} \\ &\quad + c \left(\frac{2|Q_{2r}^0 \setminus (Q_{2r}^0)^+|}{|Q_{2r}^0|} \right)^{\frac{1}{p}} \sup_{Q_{2r}^0 \setminus (Q_{2r}^0)^+} u + c \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \\ &\leq c \left(\inf_{(Q_{2r}^0)^+} u + \frac{4}{3} \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \right) \\ &\quad + \frac{1}{2} \sup_{(Q_r^0)^-} u + c \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)) \end{aligned}$$

for a sufficiently small $p \in (0, 2]$ (by (7.25) and (7.34)) satisfying

$$c \left(\frac{2|Q_{2r}^0 \setminus (Q_{2r}^0)^+|}{|Q_{2r}^0|} \right)^{\frac{1}{p}} \sup_{Q_{2r}^0 \setminus (Q_{2r}^0)^+} u = c \left(\frac{2-2\sigma}{2-\sigma} \right)^{\frac{1}{p}} \sup_{Q_{2r}^0 \setminus (Q_{2r}^0)^+} u < \frac{1}{2} \sup_{(Q_r^0)^-} u.$$

This implies that

$$(7.47) \quad \sup_{(Q_r^0)^-} u \leq c \inf_{(Q_r^0)^+} u + c \left(\frac{r}{R} \right)^{2s} \mathcal{T}_R(u^-; (x_0, t_0)).$$

Hence we complete the proof. \square

8. APPENDIX: EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

The main goal of this appendix is to give the existence and uniqueness of weak solutions of the nonlocal parabolic equation $\mathbf{NE}_{\Omega_I}(f, 0, h)$ where $\mathfrak{f} \in L^2(I; L^2(\Omega))$ and $h \in L^2(\Omega)$ and even where $\mathfrak{f} \in L^2(I; X_0^*(\Omega))$ and $h \in L^2(\Omega)$, and moreover is to give those of weak solutions of the nonlocal parabolic equation $\mathbf{NE}_{\Omega_I}(f, g, g)$ where $\mathfrak{f} \in L^2(I; X_0^*(\Omega))$ and $g \in H_T^s(\mathbb{R}^n)$.

It is easy to check that the weak formulation of the following eigenvalue problem

$$(8.1) \quad \begin{cases} -L_K u = \alpha u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $n > 2s$, is given by

$$(8.2) \quad \begin{cases} \langle u, v \rangle_{X_0(\Omega)} = \alpha \langle u, v \rangle_{L^2(\Omega)}, \forall v \in X_0(\Omega), \\ u \in X_0(\Omega). \end{cases}$$

Then it is well-known [SV] that there exists a sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ of eigenvalues α_i of (8.2) with $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_i \leq \alpha_{i+1} \leq \dots$ and $\lim_{i \rightarrow \infty} \alpha_i = \infty$ such that the set $\{e_i\}_{i \in \mathbb{N}}$ of eigenfunctions e_i corresponding to α_i is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0 . Moreover, it turns out that $e_i \in \mathcal{Q}_{i+1}$ and

$$(8.3) \quad \alpha_{i+1} = \|e_{i+1}\|_{X_0(\Omega)}^2$$

for any $i \in \mathbb{N}$, where $\mathcal{Q}_{i+1} = \{u \in X_0(\Omega) : \langle u, e_j \rangle_{X_0(\Omega)} = 0, \forall j = 1, 2, \dots, i\}$.

We construct a weak solution of the nonlocal parabolic boundary value problem (1.3) by using the eigenfunctions of the nonlocal eigenvalue problem mentioned in (8.2), which is called *Galerkin's approximation*.

Theorem 8.1. *If $\mathfrak{f} \in L^2(I; L^2(\Omega))$ and $h \in L^2(\Omega)$, then there exists a weak solution $\mathbf{u} \in H^1(I; X_0(\Omega))$ of the nonlocal parabolic boundary value problem $\mathbf{NE}_{\Omega_I}(f, 0, h)$. Moreover, $\mathbf{u} \in C(I; X_0(\Omega))$ after being modified on a set of measure zero.*

Let $\{e_i\}_{i \in \mathbb{N}}$ be the set of eigenfunctions e_i corresponding to eigenvalues α_i of (8.2) that is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $X_0(\Omega)$. For $k \in \mathbb{N}$, we consider a function $\mathbf{u}_k : I \rightarrow X_0(\Omega)$ of the form

$$(8.4) \quad \mathbf{u}_k(t) = \sum_{i=1}^k c_k^i(t) e_i, \quad t \in I.$$

Our next step is to show the existence of the functions $\{c_k^i(t)\}_{i=1}^k$ for which

$$(8.5) \quad c_k^i(-T) = \langle h, e_i \rangle_{L^2(\Omega)}$$

and

$$(8.6) \quad \langle \mathbf{u}_k(t), e_i \rangle_{X_0(\Omega)} + \langle \mathbf{u}_k'(t), e_i \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), e_i \rangle_{L^2(\Omega)} = 0, \quad t \in I,$$

for any $i = 1, 2, \dots, k$.

Lemma 8.2. *For each $k \in \mathbb{N}$, there exists a unique function \mathbf{u}_k of the form (8.4) so that (8.5) and (8.6) hold.*

Proof. By applying (8.2) and (8.4), we reduce the weak formulation (8.5) and (8.6) of (1.3) to the ordinary differential equations

$$(8.7) \quad \begin{cases} \nu_i c_k^i(t) + c_k^{i'}(t) + \langle \mathfrak{f}(t), e_i \rangle_{L^2(\Omega)} = 0 \\ c_k^i(-T) = \langle h, e_i \rangle_{L^2(\Omega)} \end{cases}$$

for any $i = 1, 2, \dots, k$. From standard O.D.E. theory, the initial value problem has a unique solution $(c_k^1(t), c_k^2(t), \dots, c_k^k(t))$ which satisfies (8.7) for a.e. $t \in I$ and is absolutely continuous on I . Hence the functions \mathbf{u}_k defined by (8.4) solves (8.5) and (8.6). \square

Next we want to obtain a subsequence of the solutions \mathbf{u}_k of (8.5) and (8.6) which converges to a weak solution of (1.3). To get this, we need some uniform estimates which is called *energy estimates*.

Theorem 8.3. *If $\mathbf{f} \in L^2(I; L^2(\Omega))$ and $h \in L^2(\Omega)$, then the solutions \mathbf{u}_k obtained in Lemma 8.2 satisfy the following energy estimates; that is, there exists a constant $c > 0$ depending only on T, K and Ω such that*

$$\|\mathbf{u}_k\|_{L^\infty(I; L^2(\Omega))} + \|\mathbf{u}_k\|_{H^1(I; X_0(\Omega))} \leq c(\|\mathbf{f}\|_{L^2(I; L^2(\Omega))} + \|h\|_{L^2(\Omega)}) \text{ for all } k \in \mathbb{N}.$$

Proof. From (8.6), we can easily derive the equality

$$(8.8) \quad \langle \mathbf{u}_k(t), \mathbf{u}_k(t) \rangle_{X_0(\Omega)} + \langle \mathbf{u}'_k(t), \mathbf{u}_k(t) \rangle_{L^2(\Omega)} + \langle \mathbf{f}(t), \mathbf{u}_k(t) \rangle_{L^2(\Omega)} = 0 \text{ a.e. } t \in I.$$

Thus this yields the inequality

$$(8.9) \quad \frac{d}{dt} \|\mathbf{u}_k(t)\|_{L^2(\Omega)}^2 + 2\|\mathbf{u}_k(t)\|_{X_0(\Omega)}^2 \leq \|\mathbf{f}(t)\|_{L^2(\Omega)}^2 + \|\mathbf{u}_k(t)\|_{L^2(\Omega)}^2 \text{ for a.e. } t \in I.$$

So it follows from the fractional Sobolev inequality and Gronwall's inequality that

$$(8.10) \quad \begin{aligned} \|\mathbf{u}_k(t)\|_{L^2(\Omega)}^2 &\leq e^{-c(t+T)} \left(\|h\|_{L^2(\Omega)}^2 + \int_{-T}^t \|\mathbf{f}(\tau)\|_{L^2(\Omega)}^2 d\tau \right) \\ &\leq (\|h\|_{L^2(\Omega)}^2 + \|\mathbf{f}(t)\|_{L^2(I; L^2(\Omega))}^2) \end{aligned}$$

for a.e. $t \in I$; that is,

$$\|\mathbf{u}_k\|_{L^\infty(I; L^2(\Omega))}^2 \leq (\|h\|_{L^2(\Omega)}^2 + \|\mathbf{f}(t)\|_{L^2(I; L^2(\Omega))}^2).$$

By (8.9) and (8.10), we also obtain that

$$(8.11) \quad \|\mathbf{u}_k\|_{L^2(I; X_0(\Omega))}^2 = \int_{-T}^0 \|\mathbf{u}_k(\tau)\|_{X_0(\Omega)}^2 d\tau \leq c(\|h\|_{L^2(\Omega)}^2 + \|\mathbf{f}(t)\|_{L^2(I; L^2(\Omega))}^2)$$

We see that $X_0(\Omega) = \mathcal{P}_k \oplus \mathcal{Q}_{k+1}$ where $\mathcal{P}_k = \text{span}\{e_1, e_2, \dots, e_k\}$ and \mathcal{Q}_{k+1} is the space given in (8.3). Fix any $v \in X_0(\Omega)$ with $\|v\|_{X_0(\Omega)} \leq 1$. Then we write $v = v_1 + v_2$ for $v_1 \in \mathcal{P}_k$ and $v_2 \in \mathcal{Q}_{k+1}$. Since $\{e_i\}_{i \in \mathbb{N}}$ are orthogonal in $X_0(\Omega)$, we have that $\|v_1\|_{X_0(\Omega)} \leq \|v\|_{X_0(\Omega)} \leq 1$. As in (8.6), we deduce that

$$(8.12) \quad \langle \mathbf{u}_k(t), v_1 \rangle_{X_0(\Omega)} + \langle \mathbf{u}'_k(t), v_1 \rangle_{L^2(\Omega)} + \langle \mathbf{f}(t), v_1 \rangle_{L^2(\Omega)} = 0$$

for a.e. $t \in I$. Thus (8.4) and (8.12) imply that

$$\langle \mathbf{u}'_k(t), v \rangle_{L^2(\Omega)} = \langle \mathbf{u}'_k(t), v_1 \rangle_{L^2(\Omega)} = -\langle \mathbf{u}_k(t), v_1 \rangle_{X_0(\Omega)} - \langle \mathbf{f}(t), v_1 \rangle_{L^2(\Omega)}.$$

So it follows from Schwarz inequality on $X_0(\Omega)$ that

$$|\langle \mathbf{u}'_k(t), v \rangle_{L^2(\Omega)}| \leq c(\|\mathbf{u}_k(t)\|_{X_0(\Omega)} + \|\mathbf{f}(t)\|_{L^2(\Omega)}).$$

This gives that

$$\|\mathbf{u}'_k(t)\|_{X_0^*(\Omega)} \leq c(\|\mathbf{u}_k(t)\|_{X_0(\Omega)} + \|\mathbf{f}(t)\|_{L^2(\Omega)}).$$

Therefore by (8.11) we conclude that

$$(8.13) \quad \|\mathbf{u}'_k\|_{L^2(I; X_0^*(\Omega))}^2 \leq c(\|h\|_{L^2(D)}^2 + \|\mathbf{f}(t)\|_{L^2(D)}^2).$$

Hence we are done. \square

[Proof of Theorem 8.1.] By Theorem 8.3 and Alaoglu's Theorem, we see that there exist a subsequence $\{\mathbf{u}_{k_j}\}_{j \in \mathbb{N}} \subset \{\mathbf{u}_k\}_{k \in \mathbb{N}}$ and a function $\mathbf{u} \in L^2(I; X_0(\Omega))$ with $\mathbf{u}' \in L^2(I; X_0^*(\Omega))$ such that

$$(8.14) \quad \begin{cases} \mathbf{u}_{k_j} \rightarrow \mathbf{u} \text{ weakly in } L^2(I; X_0(\Omega)) \\ \mathbf{u}'_{k_j} \rightarrow \mathbf{u}' \text{ weakly in } L^2(I; X_0^*(\Omega)) \end{cases} \quad \text{as } j \rightarrow \infty.$$

Let $C_c^1(I; X_0(\Omega))$ be the set of all $\mathbf{v}_N(t) \in C^1(I; X_0(\Omega))$ of the form

$$\mathbf{v}_N(t) = \sum_{i=1}^N c_i(t) e_i$$

where $\{c_i\}_{i=1}^N \subset C_c^1(I)$ for some $N \in \mathbb{N}$. Then we easily see that $C_c^1(I; X_0(\Omega))$ is dense in $L^2(I; X_0(\Omega))$. Take any $\mathbf{v}_N(t) \in C^1(I; X_0(\Omega))$ and choose $k \geq N$. Then it follows from (8.6) and integrating with respect to t that

$$(8.15) \quad \int_{-T}^0 [\langle \mathbf{u}_{k_j}(t), \mathbf{v}_N(t) \rangle_{X_0(\Omega)} + \langle \mathbf{u}'_{k_j}(t), \mathbf{v}_N(t) \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), \mathbf{v}_N(t) \rangle_{L^2(\Omega)}] dt = 0.$$

Passing (8.15) to weak limits (8.14) and using density of $C_c^1(I; X_0(\Omega))$ in $L^2(I; X_0(\Omega))$, we obtain that

$$(8.16) \quad \int_{-T}^0 [\langle \mathbf{u}(t), \mathbf{v}(t) \rangle_{X_0(\Omega)} + \langle \mathbf{u}'(t), \mathbf{v}(t) \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)}] dt = 0.$$

for any $\mathbf{v}(t) \in L^2(I; X_0(\Omega))$. If we choose any elements of the form $\mathbf{v}(t) = \varphi(t)v$ where $\varphi \in C_c^1(I)$ and $v \in X_0(\Omega)$, then (8.16) becomes

$$(8.17) \quad \int_{-T}^0 [\langle \mathbf{u}(t), v \rangle_{X_0(\Omega)} + \langle \mathbf{u}'(t), v \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), v \rangle_{L^2(\Omega)}] \varphi(t) dt = 0$$

for any $\varphi \in C_c^1(I)$. This implies that

$$\langle \mathbf{u}(t), v \rangle_{X_0(\Omega)} + \langle \mathbf{u}'(t), v \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), v \rangle_{L^2(\Omega)} = 0 \quad \text{for a.e. } t \in I$$

for any $v \in X_0(\Omega)$. Moreover, by the remark just above Definition 2.1, we see that $\mathbf{u} \in C(I; L^2(\Omega))$ after being modified on a set of measure zero.

In order to show that \mathbf{u} is a weak solution of (1.3), we finally have only to prove that $\mathbf{u}(-T) = h$. By (8.16) and integration by parts, we have that

$$(8.18) \quad \begin{aligned} \int_{-T}^0 [\langle \mathbf{u}(t), \mathbf{v}(t) \rangle_{X_0(\Omega)} - \langle \mathbf{u}(t), \mathbf{v}'(t) \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)}] dt \\ = -\langle \mathbf{u}(-T), \mathbf{v}(-T) \rangle_{L^2(\Omega)}. \end{aligned}$$

for any $\mathbf{v}(t) \in L^2(I; X_0(\Omega))$ with $\mathbf{v}(0) = 0$. Similarly it follows from (8.15) and integration by parts that

$$(8.19) \quad \begin{aligned} \int_{-T}^0 [\langle \mathbf{u}_{k_j}(t), \mathbf{v}(t) \rangle_{X_0(\Omega)} - \langle \mathbf{u}_{k_j}(t), \mathbf{v}'(t) \rangle_{L^2(\Omega)} + \langle \mathfrak{f}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)}] dt \\ = -\langle \mathbf{u}_{k_j}(-T), \mathbf{v}(-T) \rangle_{L^2(\Omega)}. \end{aligned}$$

for any $\mathbf{v}(t) \in L^2(I; X_0(\Omega))$ with $\mathbf{v}(0) = 0$. By (8.4) and (8.5), we get that

$$(8.20) \quad \mathbf{u}_{k_j}(-T) = \sum_{i=1}^{k_j} \langle h, e_i \rangle_{L^2(\Omega)} e_i.$$

Since the orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of $L^2(\Omega)$ is complete, we see that

$$(8.21) \quad \lim_{j \rightarrow \infty} \|\mathbf{u}_{k_j}(-T) - h\|_{L^2(\Omega)} = 0.$$

Passing (8.19) to weak limits (8.14) and comparing it with (8.18) and (8.21), we conclude that $\mathbf{u}(-T) = h$. \square

We shall obtain an improved regularity result for a weak solution of (1.3) which is better than that of Theorem 8.1.

Theorem 8.4. *Let $\mathfrak{f} \in L^2(I; L^2(\Omega))$ and $h \in X_0(\Omega)$. If $\mathbf{u} \in H^1(I; X_0(\Omega))$ is a weak solution of the nonlocal parabolic boundary value problem $\mathbf{NE}_{\Omega_T}(f, 0, h)$, then we have that $\mathbf{u} \in L^2(I; H^s(\Omega)) \cap L^\infty(I; H^s(\Omega))$ and $\mathbf{u}' \in L^2(I; L^2(\Omega))$, and moreover there exists a constant $c_1 > 0$ depending only on T, K and Ω such that*

$$\begin{aligned} \|\mathbf{u}\|_{L^\infty(I; H^s(\Omega))} + \|\mathbf{u}\|_{L^2(I; H^s(\Omega))} + \|\mathbf{u}'\|_{L^2(I; L^2(\Omega))} \\ \leq c_1 (\|\mathfrak{f}\|_{L^2(I; L^2(\Omega))} + \|h\|_{X_0(\Omega)}). \end{aligned}$$

Proof. From (8.6), we have that for any $t \in I$,

$$\int_{-T}^t \left(\frac{d}{dt} \|\mathbf{u}_{k_j}(\tau)\|_{X_0(\Omega)}^2 + 2 \|\mathbf{u}'_{k_j}(\tau)\|_{L^2(\Omega)}^2 + 2 \langle \mathfrak{f}(\tau), \mathbf{u}'_{k_j}(\tau) \rangle_{L^2(\Omega)} \right) d\tau = 0.$$

From (8.20) and Bessel's inequality on the Hilbert space $X_0(\Omega)$, we can obtain that

$$\|\mathbf{u}_{k_j}(-T)\|_{X_0(\Omega)}^2 \leq \|h\|_{X_0(\Omega)}^2$$

for all $i \in \mathbb{N}$. Applying Schwarz inequality on $L^2(\Omega)$ and Cauchy's inequality, this gives that

$$(8.22) \quad \|\mathbf{u}_{k_j}(t)\|_{X_0(\Omega)}^2 + \frac{3}{2} \int_{-T}^t \|\mathbf{u}'_{k_j}(\tau)\|_{L^2(\Omega)}^2 d\tau \leq \|h\|_{X_0(\Omega)}^2 + 2 \|\mathfrak{f}\|_{L^2(I; L^2(\Omega))}^2$$

for any $t \in I$. From (8.22), we easily get that

$$(8.23) \quad \begin{aligned} \|\mathbf{u}_{k_j}\|_{L^\infty(I; X_0(\Omega))}^2 + \frac{3}{2} \|\mathbf{u}'_{k_j}\|_{L^2(I; L^2(\Omega))}^2 \\ \leq \|h\|_{X_0(\Omega)}^2 + 2 \|\mathfrak{f}\|_{L^2(I; L^2(\Omega))}^2. \end{aligned}$$

Since $C_c(I; X_0(\Omega))$ is dense in $L^p(I; X_0(\Omega))$ for all $p \in [1, \infty)$, we note that

$$(8.24) \quad \begin{aligned} \|\mathbf{u}_{k_j}\|_{L^\infty(I; X_0(\Omega))} \\ = \sup \left\{ \int_{-T}^0 \langle \mathbf{u}_{k_j}(t), \mathfrak{h}(t) \rangle_{X_0(\Omega)} dt : \|\mathfrak{h}\|_{L^1(I; X_0(\Omega))} \leq 1, \mathfrak{h} \in C_c(I; X_0(\Omega)) \right\} \end{aligned}$$

by the duality of $L^1(I; X_0(\Omega))$. So it follows from (8.14), (8.23) and (8.24) that

$$(8.25) \quad \int_{-T}^0 \langle \mathbf{u}_{k_j}(t), \mathfrak{h}(t) \rangle_{X_0(\Omega)} dt \leq \|h\|_{X_0(\Omega)}^2 + 2 \|\mathfrak{f}\|_{L^2(I; L^2(\Omega))}^2$$

and thus, we have that

$$(8.26) \quad \int_{-T}^0 \langle \mathbf{u}(t), \mathfrak{h}(t) \rangle_{X_0(\Omega)} dt \leq \|h\|_{X_0(\Omega)}^2 + 2 \|\mathfrak{f}\|_{L^2(I; L^2(\Omega))}^2$$

for any $\mathfrak{h} \in C_c(I; X_0(\Omega))$ with $\|\mathfrak{h}\|_{L^1(I; X_0(\Omega))} \leq 1$. Combining this with (2.6) and (2.7) imply that

$$(8.27) \quad \|\mathbf{u}\|_{L^\infty(I; H^s(\Omega))}^2 \leq \|\mathbf{u}\|_{L^\infty(I; X_0(\Omega))}^2 \leq \|h\|_{X_0(\Omega)}^2 + 2 \|\mathfrak{f}\|_{L^2(I; L^2(\Omega))}^2.$$

On the other hand, integrating (8.22) with respect to $t \in [-T, 0)$, we obtain that

$$(8.28) \quad \begin{aligned} \|\mathbf{u}_{k_j}\|_{L^2(I; X_0(\Omega))}^2 + \frac{3}{2} \|\mathbf{u}'_{k_j}\|_{L^2(I; L^2(\Omega))}^2 \\ \leq \|h\|_{X_0(\Omega)}^2 + 2 \|\mathbf{f}\|_{L^2(I; L^2(\Omega))}. \end{aligned}$$

By Riesz representation theorem, we know that

$$(8.29) \quad \begin{aligned} & \|\mathbf{u}'_{k_j}\|_{L^2(I; L^2(\Omega))}^2 \\ & = \sup \left\{ \int_{-T}^0 \langle \mathbf{u}'_{k_j}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt : \|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1, \boldsymbol{\eta} \in L^2(I; L^2(\Omega)) \right\}. \end{aligned}$$

By (8.28) and (8.29), we see that

$$(8.30) \quad \int_{-T}^0 \langle \mathbf{u}'_{k_j}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt \leq c(\|h\|_{X_0(\Omega)}^2 + \|\mathbf{f}\|_{L^2(I; L^2(\Omega))})$$

for all $\boldsymbol{\eta} \in L^2(I; L^2(\Omega))$ with $\|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1$. In particular, (8.30) holds for all $\boldsymbol{\eta} \in L^2(I; X_0(\Omega))$ with $\|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1$. Thus by (8.14), we obtain that

$$(8.31) \quad \begin{aligned} \lim_{j \rightarrow \infty} \int_{-T}^0 \langle \mathbf{u}'_{k_j}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt &= \int_{-T}^0 \langle \mathbf{u}'(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt \\ &\leq c(\|h\|_{X_0(\Omega)}^2 + \|\mathbf{f}\|_{L^2(I; L^2(\Omega))}) \end{aligned}$$

for any $\boldsymbol{\eta} \in L^2(I; X_0(\Omega))$ with $\|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1$. Also by Alaoglu's theorem and (8.28), there are a subsequence $\{\mathbf{u}'_{k_\ell}\}_{\ell \in \mathbb{N}} \subset \{\mathbf{u}'_{k_j}\}_{j \in \mathbb{N}}$ and a function $\mathbf{v} \in L^2(I; L^2(\Omega))$ such that

$$\mathbf{u}'_{k_\ell} \rightarrow \mathbf{v} \text{ weakly in } L^2(I; L^2(\Omega)) \text{ as } \ell \rightarrow \infty.$$

This and (8.30) yields that

$$(8.32) \quad \begin{aligned} \lim_{\ell \rightarrow \infty} \int_{-T}^0 \langle \mathbf{u}'_{k_\ell}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt &= \int_{-T}^0 \langle \mathbf{v}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt \\ &\leq c(\|h\|_{X_0(\Omega)}^2 + \|\mathbf{f}\|_{L^2(I; L^2(\Omega))}) \end{aligned}$$

for all $\boldsymbol{\eta} \in L^2(I; L^2(\Omega))$ with $\|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1$. Hence it follows from (8.31) and (8.32) that

$$\int_{-T}^0 \langle \mathbf{u}'(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt = \int_{-T}^0 \langle \mathbf{v}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt$$

for any $\boldsymbol{\eta} \in L^2(I; X_0(\Omega))$ with $\|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1$. This implies $\mathbf{u}' = \mathbf{v}$ a.e.. Thus by (8.32) we get that $\mathbf{u}' \in L^2(I; L^2(\Omega))$ and

$$\int_{-T}^0 \langle \mathbf{v}(t), \boldsymbol{\eta}(t) \rangle_{L^2(\Omega)} dt \leq c(\|h\|_{X_0(\Omega)}^2 + \|\mathbf{f}\|_{L^2(I; L^2(\Omega))})$$

for all $\boldsymbol{\eta} \in L^2(I; L^2(\Omega))$ with $\|\boldsymbol{\eta}\|_{L^2(I; L^2(\Omega))} \leq 1$. Hence we conclude that

$$\|\mathbf{u}'\|_{L^2(I; L^2(\Omega))} \leq c(\|h\|_{X_0(\Omega)}^2 + \|\mathbf{f}\|_{L^2(I; L^2(\Omega))}).$$

Finally, we can derive the estimate

$$\|\mathbf{u}\|_{L^2(I; H^s(\Omega))}^2 \leq \|\mathbf{u}\|_{L^2(I; X_0(\Omega))}^2 \leq c(\|h\|_{X_0(\Omega)}^2 + \|\mathbf{f}\|_{L^2(I; L^2(\Omega))})$$

from (2.6), (2.7), (8.14) and (8.28). Hence we complete the proof. \square

Even when $\mathbf{f} \in L^2(I; X_0^*(\Omega))$ and $h \in L^2(\Omega)$, using duality we can obtain the following estimates in the similar way as in Theorem 8.3.

Theorem 8.5. *If $\mathfrak{f} \in L^2(I; X_0^*(\Omega))$ and $h \in L^2(\Omega)$, then the solutions \mathbf{u}_k obtained in Lemma 8.2 satisfy the following energy estimates; that is, there exists a constant $c > 0$ depending only on T, K and Ω such that*

$$\|\mathbf{u}_k\|_{L^\infty(I; L^2(\Omega))} + \|\mathbf{u}_k\|_{H^1(I; X_0(\Omega))} \leq c(\|\mathfrak{f}\|_{L^2(I; X_0^*(\Omega))} + \|h\|_{L^2(\Omega)}) \text{ for all } k \in \mathbb{N}.$$

By applying Theorem 8.5, we can prove the following theorem in the similar way as in Theorem 8.1.

Theorem 8.6. *If $\mathfrak{f} \in L^2(I; X_0^*(\Omega))$ and $h \in L^2(\Omega)$, then there exists a weak solution $\mathbf{u} \in H^1(I; X_0(\Omega))$ of the nonlocal parabolic boundary value problem $\mathbf{NE}_{\Omega_I}(f, 0, h)$. Moreover, $\mathbf{u} \in C(I; X_0(\Omega))$ after being modified on a set of measure zero.*

Theorem 8.7. *If $\mathfrak{f} \in L^2(I; X_0^*(\Omega))$ and $g \in H_T^s(\mathbb{R}^n)$, then there is a weak solution $\mathbf{u} \in H^1(I; X_g(\Omega))$ of the nonlocal parabolic boundary value problem $\mathbf{NE}_{\Omega_I}(f, g, g)$. Moreover, $\mathbf{u} - \mathbf{g} \in C(I; X_0(\Omega))$ after being modified on a set of measure zero.*

Proof. By Theorem 8.6, there is a weak solution $\mathbf{v} \in H^1(I; X_0(\Omega))$ of the nonlocal parabolic boundary value problem $\mathbf{NE}_{\Omega_I}(f - L_K g - \partial_t g, 0, 0)$, because it is easy to check that $f - L_K g - \partial_t g \in L^2(I; X_0^*(\Omega))$. Then we easily see that $\mathbf{u} = \mathbf{v} + \mathbf{g}$ satisfies the equation $\mathbf{NE}_{\Omega_I}(f, g, g)$ and $\mathbf{u} \in H^1(I; X_g(\Omega))$. Hence we are done. \square

In the following theorem, we give the uniqueness of the weak solutions of the nonlocal parabolic equation given in (1.3). Its proof is quite simple and follows from a direct application of Gronwall's inequality.

Theorem 8.8. *A weak solution $\mathbf{u} \in H^1(I; X_g(\Omega))$ of the nonlocal parabolic boundary value problem $\mathbf{NE}_{\Omega_I}(f, g, h)$ is unique, if it exists.*

Proof. We have only to check that the only weak solution of the equation $\mathbf{NE}_{\Omega_I}(f, g, h)$ with $\mathfrak{f} = \mathbf{0}$ and $g = h = 0$ must be $\mathbf{u} \equiv \mathbf{0}$ a.e.. By (2.12), we have that

$$\frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(\Omega)}^2 \leq \frac{d}{dt} \|\mathbf{u}(t)\|_{L^2(\Omega)}^2 + 2\|\mathbf{u}(t)\|_{X_0(\Omega)}^2 = 0$$

for a.e. $t \in I$. Thus it follows from Gronwall's inequality that

$$\|\mathbf{u}(t)\|_{L^2(\Omega)}^2 \leq \|g\|_{L^2(\Omega)} = 0.$$

Hence we conclude that $\mathbf{u} \equiv \mathbf{0}$ a.e.. \square

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