Approximation of mild solutions of a semilinear fractional elliptic equation with random noise

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Abstract

We study for the first time the Cauchy problem for semilinear fractional elliptic equation. This paper is concerned with the Gaussian white noise model for the initial Cauchy data. We establish the ill-posedness of the problem. Then, under some assumption on the exact solution, we propose the Fourier truncation method for stabilizing the ill-posed problem. Some convergence rates between the exact solution and the regularized solution is established in L^2 and H^q norms.

1 Introduction

The theory of fractional differential equations has received much attention over the past twenty years, since they are important in describing the natural models such as diffusion processes, stochastic processes, finance and hydrology. We refer for instance to the books [\[9,](#page-13-0) [13,](#page-13-1) [15,](#page-13-2) [17\]](#page-13-3). In this paper, we consider the following Cauchy problem of fractional semi-linear elliptic equations:

$$
\frac{D^{\beta}\mathbf{u}(t,y)}{Dt^{\beta}} = \mathcal{A}\mathbf{u}(t,y) + G(t,y,\mathbf{u}(t,y)), \quad (t,y) \in \Omega := \Omega_1 \times \Omega_2,
$$
\n(1.1)

associated with the zero Dirichlet boundary condition in y and the initial data and nonhomogeneous initial velocity given by

$$
\mathbf{u}(0,y) = \mathbf{u}_0(y), \quad \left. \frac{d\mathbf{u}(t,y)}{dt} \right|_{t=0} = \mathbf{u}_1(y), \quad y \in \Omega_2.
$$
 (1.2)

In [\(1.1\)](#page-0-0), $\beta \in (1,2)$ is the fractional order and $\frac{D^{\beta}}{Dt^{\beta}}$ denotes the Caputo fractional derivative with respect to t , (see [\[8,](#page-13-4) [16\]](#page-13-5)),

$$
\frac{D^{\beta}\mathbf{u}(t,y)}{Dt^{\beta}} := \frac{1}{\Gamma(2-\beta)} \int_0^t (t-\eta)^{1-\beta} \frac{\partial^2 \mathbf{u}}{\partial \eta^2}(\eta,y) d\eta,
$$

where Γ is the Gamma function. The function $\mathbf{u} : \Omega_1 \to L^2(\Omega_2)$ denotes the distribution of a body where $\Omega_1 := (0, a) \subset \mathbb{R}$ and $\Omega_2 \subset \mathbb{R}^n$ are open, bounded and connected domains with a smooth

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boundary for $n \geq 2$ and $a > 0$, and A is the linear second-order differential operator with variable coefficients depending on y only:

$$
\mathcal{A}\mathbf{u}(t,y) = \mathcal{A}_y\mathbf{u}(t,y) = \sum_{i,j=1}^n \frac{\partial}{\partial y_i} \left(d_{i,j}(y) \frac{\partial \mathbf{u}(t,y)}{\partial y_j} \right) + d(y) \mathbf{u}(t,y).
$$

The basic requirement for the coefficients $d_{i,j}(y)$ and $d(y)$ is that A is a positive, self-adjoint operator in the Hilbert space $L^2(\Omega_2)$. Consequently, there exists an orthonormal basis of $L^2(\Omega_2)$, denoted by $\{\phi_p\}_{p\in\mathbb{N}^*}$, satisfying

$$
\phi_p \in H_0^1(\Omega_2) \cap C^\infty(\overline{\Omega_2}), \quad \mathcal{A}\phi_p(y) = \lambda_p \phi_p(y) \text{ for } y \in \Omega_2,
$$
\n(1.3)

and the corresponding discrete spectrum $\{\lambda_p\}_{p\in\mathbb{N}^*}$ satisfies

$$
0 < \lambda_1 \le \lambda_2 \le \dots \lim_{p \to \infty} \lambda_p = \infty. \tag{1.4}
$$

A related fractional elliptic equation with homogeneous source term, i.e, $G = 0$ in Eqs [\(1.1\)](#page-0-0)-[\(1.2\)](#page-0-1) has been introduced in section 4.2 in [\[7\]](#page-13-6) where the authors established the ill-posedness of the problem in the sense of Hadamard $[6]$. This means that a solution of Problem $(1.1)-(1.2)$ $(1.1)-(1.2)$ corresponding to the data does not always exist, and in the case of existence, it does not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions will have large errors. Hence, one has to resort to a regularization. In [\[7\]](#page-13-6), the authors did not mention the regularization results for this problem.

If we replace the operator $\mathcal A$ by $-\mathcal A$ in equation [\(1.1\)](#page-0-0) then we get the fractional wave equation which is studied in $[8]$. As introduced in $[8]$, the kinds of the equation (1.1) have many applications in anamolous diffusion phenomenon and in heterogeneous media. Some more physical applications can be found in [\[8\]](#page-13-4).

Until now, to the best of our knowledge, there are no results concerning a regularization for the nonlinear problem $(1.1)-(1.2)$ $(1.1)-(1.2)$. Motivated by this reason, in this paper, we study the regularization results for [\(1.1\)](#page-0-0)-[\(1.2\)](#page-0-1). In addition, one usually meets the measurement in practice, i.e. we need to assume the presence of an approximation $(\mathbf{u}_0^{\epsilon}, \mathbf{u}_1^{\epsilon}) \in L^2(\Omega_2) \times L^2(\Omega_2)$. If the errors are generated from uncontrollable sources (or called external reason) as environment, wind, rain, humidity, etc, then the model is random. As we know, the problem with random data is more difficult than the deterministic case. Hence, we study the problem $(1.1)-(1.2)$ $(1.1)-(1.2)$ with the following random model

$$
\mathbf{u}_0^{\epsilon}(y) = \mathbf{u}_0(y) + \epsilon \xi(y), \quad \mathbf{u}_1^{\epsilon}(y) = \mathbf{u}_1(y) + \epsilon \xi(y) \tag{1.5}
$$

in which the constant $\varepsilon > 0$ represents the upper bound of the noise level in $L^2(\Omega_2)$. And ξ is a Gaussian white noise process. In practice, we only obtain finite errors as follows

$$
\langle \mathbf{u}_0^{\epsilon}, \phi_p \rangle = \langle \mathbf{u}_0, \phi_p \rangle + \epsilon \langle \xi, \phi_p \rangle, \quad \langle \mathbf{u}_1^{\epsilon}, \phi_p \rangle = \langle \mathbf{u}_1, \phi_j \rangle + \epsilon \langle \xi, \phi_p \rangle, \quad p = \overline{1, \mathbf{N}}.
$$
 (1.6)

where N is the natural number which is the number of steps for discrete observations. Our task here is to find a regularized solution (called the estimator) \mathbf{u}_{re} for **u** and then investigate the rate of convergence $\mathbf{E} \|\mathbf{u}_{re} - \mathbf{u}\|$, which is called the mean integrated square error (MISE). Here **E** denotes the expectation w.r.t. the distribution of the data in the model [\(1.5\)](#page-1-0).

If $G = 0$ in Eqs [\(1.1\)](#page-0-0) and $\mathbf{u}_1 = 0$ in [\(1.2\)](#page-0-1), using [\(2.10\)](#page-2-0), we can see that the solution to (1.1)-[\(1.2\)](#page-0-1) satisfies a following linear operator with random noise defined in [\(1.5\)](#page-1-0)

$$
\mathcal{K}\mathbf{u}(a,y) + "random noise" = \mathbf{u}_0(y),\tag{1.7}
$$

where $Kv = \sum_{p=1}^{\infty} \frac{1}{E_{\text{eq}}(p)}$ $\frac{1}{E_{\beta,1}(\lambda_p a^{\beta})} \langle v, \phi_p \rangle$. The linear random model [\(1.5\)](#page-1-0)-[\(1.7\)](#page-1-1) is one of many linear inverse problems in statistics which have been studied by well-known methods including spectral cut-off (or called truncation method) [\[1,](#page-13-8) [2,](#page-13-9) [12,](#page-13-10) [10\]](#page-13-11), the Tiknonov method [\[3\]](#page-13-12), iterative regularization methods [\[5\]](#page-13-13). For the nonlinear problem, we can not transform $(1.1)-(1.2)$ $(1.1)-(1.2)$ $(1.1)-(1.2)$ into (1.7) . Hence, previous techniques for solving (1.7) are not suitable for solving the nonlinear problem $(1.1)-(1.2)$ $(1.1)-(1.2)$. The main idea in this paper is to approximate the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ by an approximate data and use this function to establish a solution of a regularized problem by truncation method.

This paper is organized as follows. In section 2, we present a mild solution and show the illposedness of the solution to fractional semilinear elliptic equation. In section 3, we establish a regularized solution and investigate the convergence rates of the expectation of the difference for the solution and the regularized solution in L^2 and in the Sobolev spaces H^q for $q > 0$.

2 The mild solution of Cauchy problem for fractional elliptic equation

Suppose that problem [\(1.1\)](#page-0-0)-[\(1.2\)](#page-0-1) has a mild solution **u** which has the form $\mathbf{u}(t, y) = \sum_{p=1}^{\infty} \mathbf{u}_p(t) \phi_p(y)$. Then the function $\mathbf{u}_n(t)$ solves the following ordinary differential equation

$$
\begin{cases}\n\frac{D^{\beta} \mathbf{u}_p(t)}{Dt^{\beta}} - \lambda_p \mathbf{u}_p(t) &= \langle G(t, y, \mathbf{u}(t, \cdot)), \phi_p \rangle, \\
\mathbf{u}_p(0) &= \langle \mathbf{u}_0, \phi_p \rangle \\
\frac{d}{dt} \mathbf{u}_p(0) &= \langle \mathbf{u}_1, \phi_p \rangle\n\end{cases}
$$
\n(2.8)

By applying the method in $[8, 16]$ $[8, 16]$, we obtain the solution of (2.8) as follows

$$
\mathbf{u}_p(t) = E_{\beta,1}(\lambda_p t^{\beta}) \langle \mathbf{u}_0, \phi_p \rangle + t E_{\beta,2}(\lambda_p t^{\beta}) \langle \mathbf{u}_1, \phi_p \rangle + \int_0^t (t - \eta)^{\beta - 1} E_{\beta, \beta}(\lambda_p (t - \eta)^{\beta}) \langle G(t, \eta, \mathbf{u}(t, \cdot)), \phi_p \rangle d\eta
$$
(2.9)

and u is given by

$$
\mathbf{u}(t,y) = \sum_{p=1}^{\infty} \left[E_{\beta,1}(\lambda_p t^{\beta}) \langle \mathbf{u}_0, \phi_p \rangle + t E_{\beta,2}(\lambda_p t^{\beta}) \langle \mathbf{u}_1, \phi_p \rangle \right] \phi_p(y) + \sum_{p=1}^{\infty} \left[\int_0^t (t-\eta)^{\beta-1} E_{\beta,\beta}(\lambda_p (t-\eta)^{\beta}) \langle G(t,\eta, \mathbf{u}(t,\cdot)), \phi_p \rangle d\eta \right] \phi_p(y)
$$
(2.10)

Next we give some lemmas that will be useful in this paper.

Lemma 2.1. Let $0 < \beta_0 < \beta_1 < 2$ and $\beta \in [\beta_0, \beta_1]$. Then for $z \in \mathbb{R}, z \ge 0$ then

$$
\frac{\widetilde{C}}{\beta}e^{z^{\frac{1}{\beta}}} \le E_{\beta,1}(z) \le \frac{\overline{C}}{\beta}e^{z^{\frac{1}{\beta}}}.
$$
\n(2.11)

Proof. The proof can be found in [\[4\]](#page-13-14).

Now, we have the following Lemma

Lemma 2.2. Let $0 < \beta < 2$ and $t \in [0, a]$. Then there exists C_1, C_2, C_3 which does not depend on t, such that

$$
E_{\beta,1}(\lambda_p t^{\beta}) \le C_1 \exp\left(\lambda_p^{\frac{1}{\beta}} t\right) \tag{2.12}
$$

$$
tE_{\beta,2}(\lambda_p t^{\beta}) \le C_2 \left(1 + \lambda_p^{\frac{-1}{\beta}}\right) \exp\left(\lambda_p^{\frac{1}{\beta}} t\right) \tag{2.13}
$$

$$
t^{\beta - 1} E_{\beta, \beta}(\lambda_p t^{\beta}) \le C_3 \exp\left(\lambda_p^{\frac{1}{\beta}} t\right) \tag{2.14}
$$

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\Box
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Proof. Applying Proposition 2.5 in [\[14\]](#page-13-15), we obtain

$$
E_{\beta,\gamma}(wt^{\beta}) \le C_{\beta,\gamma}\left(1+w^{\frac{1-\gamma}{\beta}}\right)\left(1+t^{1-\gamma}\right)\exp\left(w^{\frac{1}{\beta}}t\right), \quad w \ge 0, t \ge 0. \tag{2.15}
$$

Let $w = \lambda_p$ and $\gamma = 1$ into [\(2.15\)](#page-3-0), we get

$$
E_{\beta,1}(\lambda_p t^{\beta}) \le 4C_{\beta,\gamma} \exp\left(w^{\frac{1}{\beta}} t\right) = C_1 \exp\left(\lambda_p^{\frac{1}{\beta}} x\right). \tag{2.16}
$$

Let $w = \lambda_p$ and $\gamma = 2$ into [\(2.15\)](#page-3-0), we get

$$
E_{\beta,2}(\lambda_p t^{\beta}) \le C_{\beta,\gamma} \left(1 + \lambda_p^{\frac{-1}{\beta}}\right) \left(1 + t^{-1}\right) \exp\left(\lambda_p^{\frac{1}{\beta}} t\right). \tag{2.17}
$$

Multiplying both sides of the latter inequality with x , we obtain

$$
xE_{\beta,2}(\lambda_p t^{\beta}) \le C_{\beta,\gamma} \left(1 + \lambda_p^{\frac{-1}{\beta}}\right) \left(1 + a\right) \exp\left(w^{\frac{1}{\beta}} t\right) = C_2 \left(1 + \lambda_p^{\frac{-1}{\beta}}\right) \exp\left(\lambda_p^{\frac{1}{\beta}} t\right). \tag{2.18}
$$

Let $w = \lambda_p$ and $\gamma = \beta$ into [\(2.15\)](#page-3-0), we get

$$
E_{\beta,\beta}(\lambda_p t^{\beta}) \le C_{\beta,\gamma} \left(1 + \lambda_p^{\frac{1-\beta}{\beta}}\right) \left(1 + t^{1-\beta}\right) \exp\left(\lambda_p^{\frac{1}{\beta}} t\right). \tag{2.19}
$$

Multipying bothsides of the latter inequality to $t^{\beta-1}$ and noting that $\beta > 1$, we obtain

$$
t^{\beta-1} E_{\beta,\beta}(\lambda_p t^{\beta}) \leq C_{\beta,\gamma} \left(1 + \lambda_p^{\frac{1-\beta}{\beta}}\right) \left(1 + t^{\beta-1}\right) \exp\left(w^{\frac{1}{\beta}} t\right)
$$

$$
\leq \underbrace{C_{\beta,\gamma} \left(1 + a^{\beta-1}\right) \left(1 + \lambda_1^{\frac{1-\beta}{\beta}}\right)}_{C_3} \exp\left(\lambda_p^{\frac{1}{\beta}} t\right). \tag{2.20}
$$

 \Box

2.1 The ill-posedness of problem [\(1.1\)](#page-0-0)-[\(1.2\)](#page-0-1) with random noise

In this section, we show that the problem $(1.1)-(1.2)$ in a special case with random noise is ill-posed in the sense of Hadamard.

Theorem 2.1. Problem $(1.1)-(1.2)$ $(1.1)-(1.2)$ $(1.1)-(1.2)$ is ill-posed in the sense of Hadamard.

Proof. Now, we give an example which shows that Problem $(1.1)-(1.2)$ $(1.1)-(1.2)$ has a unique solution and its solution is not stable. For simple computation, we assume that $\Omega_2 = (0, \pi)$, $\mathcal{A} = -\Delta$ where Δ is the Laplacian operator, and the function $\mathbf{u}_1 = 0$. It immediately follows that $\lambda_{\mathbf{N}} = \mathbf{N}^2$.

Let us consider the following parabolic equation

$$
\begin{cases}\n\frac{D^{\beta} \mathbf{V}_{\mathbf{N}(\epsilon)}(t, y)}{dt^{\beta}} = \mathcal{A} \mathbf{V}_{\mathbf{N}(\epsilon)}(t, y) + \overline{G}(t, y, \mathbf{V}_{\mathbf{N}(\epsilon)}(t, y)), & (t, y) \in \Omega := \Omega_1 \times \Omega_2 \\
\mathbf{V}_{\mathbf{N}(\epsilon)}(t, 0) = \mathbf{V}_{\mathbf{N}(\epsilon)}(t, \pi) = 0, & (2.21) \\
\mathbf{V}_{\mathbf{N}(\epsilon)}(0, y) = \mathbf{U}_{\mathbf{N}(\epsilon)}(y), & \frac{d \mathbf{V}_{\mathbf{N}(\epsilon)}(0, y)}{dt} = 0\n\end{cases}
$$

where \overline{G} is given by

$$
\overline{G}(t, y, v(t, y)) = \sum_{p=1}^{\infty} \frac{\exp\left(\lambda_p^{\frac{1}{\beta}}(t-a)\right)}{2aC_3} \left\langle v(t, \cdot), \phi_p \right\rangle \phi_p(y) \tag{2.22}
$$

for any $v \in L^2(\Omega_2)$, and $\phi_p(y) = \sqrt{\frac{2}{\pi}}$ $\frac{2}{\pi} \sin(py)$ and C_3 is defined in Lemma [\(2.2\)](#page-2-2). Let $\mathbf{U}_{\mathbf{N}(\epsilon)} \in \mathcal{L}^2(\Omega_2)$ be such that

$$
\mathbf{U}_{\mathbf{N}(\epsilon)}(y) = \sum_{p=1}^{\mathbf{N}(\epsilon)} \langle \mathbf{u}_0^{\epsilon}, \phi_p \rangle \phi_p(y) \tag{2.23}
$$

where \mathbf{u}_0^{ϵ} is defined by

$$
\langle \mathbf{u}_0^{\epsilon}, \phi_j \rangle = \epsilon \langle \xi, \phi_j \rangle, \quad j = \overline{1, \mathbf{N}(\epsilon)}. \tag{2.24}
$$

By the usual MISE (mean integrated squared error) decomposition which involves a variance term and a bias term (see p.9, [\[11\]](#page-13-16)), we get

$$
\mathbf{E} \|\mathbf{U}_{\mathbf{N}(\epsilon)}\|_{L^2(\Omega)}^2 = \mathbf{E} \Big(\sum_{j=1}^{\mathbf{N}(\delta)} \langle \mathbf{u}_0^{\epsilon}, \phi_j \rangle^2 \Big) = \epsilon^2 \mathbf{E} \Big(\sum_{j=1}^{\mathbf{N}(\epsilon)} \xi_j^2 \Big) = \epsilon^2 \mathbf{N}(\epsilon). \tag{2.25}
$$

The solution of Problem [\(2.21\)](#page-3-1) is given by Fourier series

$$
\mathbf{V}_{\mathbf{N}(\epsilon)}(t, y) = \sum_{p=1}^{\infty} \left[E_{\beta,1} \left(\lambda_p t^{\beta} \right) \langle \mathbf{U}_{\mathbf{N}(\epsilon)}, \phi_p \rangle + \int_0^t (t - \eta)^{\beta - 1} E_{\beta, \beta} \left(\lambda_p (t - \eta)^{\beta} \right) \langle \overline{G} \left(\eta, \cdot, \mathbf{V}_{\mathbf{N}(\epsilon)} \left(\eta, \cdot \right) \right), \phi_p \rangle d\eta \right] \phi_p(y).
$$
\n(2.26)

We show that Problem [\(2.26\)](#page-4-0) has a unique solution $\mathbf{V}_{\mathbf{N}(\epsilon)} \in C([0, a]; L^2(\Omega_2))$. Let us consider

$$
\mathcal{H}v: \n= \sum_{p=1}^{\infty} \left[E_{\beta,1} \left(\lambda_p t^{\beta} \right) \langle \mathbf{U}_{\mathbf{N}(\epsilon)}, \phi_p \rangle + \int_0^t (t - \eta)^{\beta - 1} E_{\beta, \beta} \left(\lambda_p (t - \eta)^{\beta} \right) \langle \overline{G} \left(\eta, \cdot, v \left(\eta, \cdot \right) \right), \phi_p \rangle d\eta \right] \phi_p(y).
$$
\n(2.27)

For any $v_1, v_2 \in C([0, a]; L^2(\Omega_2))$, using Hölder inequality and Lemma (2.2) , we have for all $t \in [0, a]$

$$
\|\mathcal{H}v_{1}(t) - \mathcal{H}v_{2}(t)\|^{2} = \sum_{p=1}^{\infty} \left[\int_{0}^{t} (t - \eta)^{\beta - 1} E_{\beta,\beta} \left(\lambda_{p}(t - \eta)^{\beta} \right) \langle \overline{G}(\eta, \cdot, v_{1}(\eta, \cdot)) - \overline{G}(\eta, \cdot, v_{2}(\eta, \cdot)), \phi_{p} \rangle d\eta \right]^{2}
$$

\n
$$
\leq a \sum_{p=1}^{\infty} \int_{0}^{t} \left| (t - \eta)^{\beta - 1} E_{\beta,\beta} \left(\lambda_{p}(t - \eta)^{\beta} \right) \right|^{2} \left| \langle \overline{G}(\eta, \cdot, v_{1}(\xi, \cdot)) - \overline{G}(\eta, \cdot, v_{2}(\eta, \cdot)), \phi_{p} \rangle^{2} d\eta \right|
$$

\n
$$
\leq \frac{1}{4a} \sum_{p=1}^{\infty} \int_{0}^{t} \exp\left(2\lambda_{p}^{\frac{1}{\beta}}(t - a) \right) \langle v_{1}(\eta) - v_{2}(\eta), \phi_{p} \rangle^{2} d\eta
$$

\n
$$
\leq \frac{1}{4} ||v_{1} - v_{2}||_{C([0,a];L^{2}(\Omega_{2}))}^{2}.
$$
\n(2.28)

Hence, we obtain that

$$
\|\mathcal{H}v_1 - \mathcal{H}v_2\||_{C([0,a];L^2(\Omega_2))} \le \frac{1}{2} \|v_1 - v_2\|_{C([0,a];L^2(\Omega_2))}.
$$
\n(2.29)

This implies that H is a contraction. Using the Banach fixed-point theorem, we conclude that the equation $\mathcal{H}(w) = w$ has a unique solution $\mathbf{V}_{\mathbf{N}(\epsilon)} \in C([0,a];L^2(\Omega_2))$. Using the inequality

$$
a^{2} + b^{2} \geq \frac{1}{2}(a - b)^{2}, \quad a, b \in \mathbb{R}, \text{ we have the following estimate}
$$
\n
$$
\left\| \mathbf{V}_{\mathbf{N}(\epsilon)} \right\|_{L^{2}(\Omega_{2})}^{2} \geq \frac{1}{2} \left\| \sum_{p=1}^{\infty} E_{\beta,1} \left(\lambda_{p} t^{\beta} \right) \langle \mathbf{U}_{\mathbf{N}(\epsilon)}, \phi_{p} \rangle \phi_{p}(y) \right\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
- \underbrace{\left\| \sum_{p=1}^{\infty} \left[\int_{0}^{t} (t - \eta)^{\beta - 1} E_{\beta,\beta} \left(\lambda_{p}(t - \eta)^{\beta} \right) \langle \overline{G} \left(\eta, \cdot, \mathbf{V}_{\mathbf{N}(\epsilon)} \left(\eta, \cdot \right) \right), \phi_{p} \rangle d\eta \right] \phi_{p}(y) \left\|_{L^{2}(\Omega_{2})}^{2} \right\|}_{L^{2}(\Omega_{2})}.
$$
\n(2.30)

First, using Hölder's inequality and Lemma (2.2) , we get

$$
I_{2} = \sum_{p=1}^{\infty} \left[\int_{0}^{t} (t - \xi)^{\beta - 1} E_{\beta, \beta} \left(\lambda_{p} (t - \eta)^{\beta} \right) \langle \overline{G} \left(\eta, \cdot, \mathbf{V}_{\mathbf{N}(\epsilon)} \left(\eta, \cdot \right) \right), \phi_{p} \rangle d\eta \right]^{2}
$$

\n
$$
\leq a \sum_{p=1}^{\infty} \int_{0}^{t} \left| (t - \eta)^{\beta - 1} E_{\beta, \beta} \left(\lambda_{p} (t - \eta)^{\beta} \right) \right|^{2} \langle \overline{G} \left(\eta, \cdot, \mathbf{V}_{\mathbf{N}(\epsilon)} \left(\eta, \cdot \right) \right), \phi_{p} \rangle^{2} d\eta
$$

\n
$$
\leq \frac{1}{4a} \sum_{p=1}^{\infty} \int_{0}^{t} \exp \left(2\lambda_{p}^{\frac{1}{\beta}} (t - a) \right) \langle \mathbf{V}_{\mathbf{N}(\epsilon)}, \phi_{p} \rangle^{2} d\eta
$$

\n
$$
\leq \frac{1}{4} ||\mathbf{V}_{\mathbf{N}(\epsilon)}||_{C([0,a];L^{2}(\Omega_{2}))}^{2}.
$$
\n(2.31)

And using Lemma [2.1,](#page-2-3) we have the lower bound for I_1 as follows

$$
\mathbf{E}I_{1} = \frac{1}{2} \sum_{p=1}^{\infty} \left| E_{\beta,1} \left(\lambda_{p} t^{\beta} \right) \right|^{2} \left| \mathbf{E} \left\langle \mathbf{U}_{\mathbf{N}(\epsilon)}, \phi_{p} \right\rangle^{2}
$$

$$
= \frac{1}{2} \sum_{p=1}^{\mathbf{N}(\epsilon)} \epsilon^{2} \left| E_{\beta,1} \left(\lambda_{p} t^{\beta} \right) \right|^{2} \geq \frac{\tilde{C}}{2\beta} \epsilon^{2} \exp \left(2t |\lambda_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}} \right). \tag{2.32}
$$

Combining [\(2.30\)](#page-5-0), [\(2.31\)](#page-5-1), [\(2.32\)](#page-5-2), we obtain

$$
\mathbf{E} \left\| \mathbf{V}_{\mathbf{N}(\epsilon)} \right\|_{L^2(\Omega_2)}^2 + \frac{1}{4} \mathbf{E} \|\mathbf{V}_{\mathbf{N}(\epsilon)}\|_{C([0,a];L^2(\Omega_2))}^2 \ge \frac{\tilde{C}}{2\beta} \epsilon^2 \exp\left(2t|\mathbf{N}(\epsilon)|^{\frac{2}{\beta}}\right). \tag{2.33}
$$

By taking supremum of both sides on $[0, a]$, we get

$$
\mathbf{E} \|\mathbf{V}_{\mathbf{N}(\epsilon)}\|_{C([0,a];L^2(\Omega_2))}^2 \ge \frac{2\overline{C}}{5} \sup_{0 \le t \le a} \epsilon^2 \exp\left(2t|\mathbf{N}(\epsilon)|^{\frac{2}{\beta}}\right) = \frac{2\widetilde{C}\epsilon^2}{5\beta} \exp\left(2a|\mathbf{N}(\epsilon)|^{\frac{2}{\beta}}\right). \tag{2.34}
$$

Let us choose $\mathbf{N} := \mathbf{N}(\epsilon) = \left[\left(\frac{2}{a} \ln(\frac{1}{\epsilon}) \right)^{\frac{\beta}{2}} \right] + 1$, where [*z*] is the greatest integer less than or equal to z . Then using (2.33) , we obtain

$$
\mathbf{E} \|\mathbf{U}_{\mathbf{N}(\epsilon)}\|_{L^2(\Omega_2)}^2 = \epsilon^2 \mathbf{N}(\epsilon) \le \epsilon^2 \Big(\frac{2}{a} \ln(\frac{1}{\epsilon})\Big)^{\frac{\beta}{2}} + \epsilon^2 \to 0, \text{ when } \epsilon \to 0.
$$
 (2.35)

and by (2.34) , we get

$$
\mathbf{E} \|\mathbf{V}_{\mathbf{N}(\epsilon)}\|_{C([0,a];L^2(\Omega_2))}^2 \ge \frac{2\tilde{C}}{5\beta\epsilon^2} \to +\infty, \text{ when } \epsilon \to 0.
$$
 (2.36)

From [\(2.35\)](#page-5-5) and [\(2.36\)](#page-5-6), the expectation of input data $\mathbf{U}_{\mathbf{N}(\epsilon)}$ tends to zero, while the expectation of output data $V_{N(\epsilon)}$ tends to infinity. Hence, we can conclude that Problem [\(1.1\)](#page-0-0)-[\(1.2\)](#page-0-1) is ill-posed in the sense of Hadamard. \Box

3 Regularization and error estimate

Next we prove the following lemma

Lemma 3.1. Let $\overline{U}_{\mathbf{N}(\epsilon)}^{0}, \ \overline{U}_{\mathbf{N}(\epsilon)}^{1} \in L^{2}(\Omega_{2})$ be such that

$$
\overline{U}_{\mathbf{N}(\epsilon)}^{0}(y) = \sum_{p=1}^{\mathbf{N}(\epsilon)} \langle \mathbf{u}_{0}^{\epsilon}, \phi_{p} \rangle \phi_{p}(y), \quad \overline{U}_{\mathbf{N}(\epsilon)}^{1}(y) = \sum_{p=1}^{\mathbf{N}(\epsilon)} \langle \mathbf{u}_{1}^{\epsilon}, \phi_{p} \rangle \phi_{p}(y)
$$
\n(3.37)

Suppose that $\mathbf{u}_0, \mathbf{u}_1 \in H^{2\gamma}(\Omega_2)$. Then we have the following estimates

$$
\mathbf{E} \|\overline{U}_{\mathbf{N}(\epsilon)}^{0} - \mathbf{u}_{0}\|_{L^{2}(\Omega_{2})}^{2} \leq \epsilon^{2} \mathbf{N}(\epsilon) + \frac{1}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}} \|\mathbf{u}_{0}\|_{H^{2\gamma}(\Omega_{2})}^{2}
$$
\n
$$
\mathbf{E} \|\overline{U}_{\mathbf{N}(\epsilon)}^{1} - \mathbf{u}_{1}\|_{L^{2}(\Omega_{2})}^{2} \leq \epsilon^{2} \mathbf{N}(\epsilon) + \frac{1}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}} \|\mathbf{u}_{1}\|_{H^{2\gamma}(\Omega_{2})}^{2}
$$
\n(3.38)

for any $\gamma \geq 0$. Here N depends on ϵ and satisfies that $\lim_{\epsilon \to 0} \mathbf{N}(\epsilon) = +\infty$ and $\lim_{\epsilon \to 0} \epsilon^2 \mathbf{N}(\epsilon) = 0$.

Proof. For the following proof, we consider the genuine model (1.6) . By the usual MISE decomposition which involves a variance term and a bias term, we get

$$
\mathbf{E} \|\overline{U}_{\mathbf{N}(\epsilon)}^{0} - \mathbf{u}_{0}\|_{L^{2}(\Omega_{2})}^{2} = \mathbf{E} \Big(\sum_{p=1}^{\mathbf{N}(\epsilon)} \langle \mathbf{u}_{0}^{\epsilon} - \mathbf{u}_{0}, \phi_{p} \rangle^{2}\Big) + \sum_{p \geq \mathbf{N}(\epsilon)+1} \langle \mathbf{u}_{0}, \phi_{p} \rangle^{2}
$$

$$
= \epsilon^{2} \mathbf{E} \Big(\sum_{p=1}^{\mathbf{N}(\epsilon)} \xi_{j}^{2}\Big) + \sum_{p \geq \mathbf{N}(\epsilon)+1} \lambda_{p}^{-2\gamma} \lambda_{p}^{2\gamma} \langle \mathbf{u}_{0}, \phi_{p} \rangle^{2}. \tag{3.39}
$$

 \Box

Since $\xi_j \stackrel{iid}{\sim} N(0, 1)$, it follows that $\mathbf{E}\xi_j^2 = 1$, so the proof is completed.

In this paper, we apply the truncation method to establish a regularized solution as follows

$$
\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,y) = \sum_{p=1}^{\infty} \mathcal{R}(\lambda_p, \mathbf{N}(\epsilon)) \left[E_{\beta,1} \left(\lambda_p t^{\beta} \right) \left\langle \overline{U}_{\mathbf{N}(\epsilon)}^0, \phi_p \right\rangle + t E_{\beta,2} \left(\lambda_p t^{\beta} \right) \left\langle \overline{U}_{\mathbf{N}(\epsilon)}^1, \phi_p \right\rangle \right. \\
\left. + \int_0^t (t-\eta)^{\beta-1} E_{\beta,\beta} \left(\lambda_p (t-\eta)^{\beta} \right) \left\langle G \left(\eta, \cdot, \mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon} \left(\eta, \cdot \right) \right), \phi_p \right\rangle d\eta \right] \phi_p(y), \quad (t,y) \in \Omega. \tag{3.40}
$$

Here $\mathcal{R}(\lambda_p, \mathbf{N}(\epsilon)) = 1$ if $\lambda_p \leq B_{\mathbf{N}(\epsilon)}$ and is zero if $\lambda_p > B_{\mathbf{N}(\epsilon)}$ and $B_{\mathbf{N}(\epsilon)}$ is called a parameter of regularization which will be chosen later.

Our main result is as follows

Theorem 3.1. The integral equation [\(3.40\)](#page-6-0) has a unique solution $\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon} \in C([0,a];L^2(\Omega_2))$. Suppose that $\mathbf{u}_0, \mathbf{u}_1 \in H^{\gamma}(\Omega_2)$ that satisfy

$$
\|\mathbf{u}_0\|_{H^{2\gamma}(\Omega_2)} + \|\mathbf{u}_1\|_{H^{2\gamma}(\Omega_2)} \leq \mathcal{M}_0.
$$

Assume that problem $(1.1)-(1.2)$ $(1.1)-(1.2)$ $(1.1)-(1.2)$ has a unique mild solution **u** which satisfies that

$$
\sum_{p=1}^{\infty} \lambda_p^{\mu} \exp\left(2(a-t)\lambda_p^{\frac{1}{\beta}}\right) \langle \mathbf{u}(t,\cdot), \phi_p \rangle^2 \le \mathcal{M}, \quad t \in [0, a], \tag{3.41}
$$

for some positive constants $\mu, \mathcal{M}.$ Assume that $B_{\mathbf{N}(\epsilon)}$ satisfy

$$
\lim_{\epsilon \to 0} B_{\mathbf{N}(\epsilon)} = +\infty, \ \lim_{\epsilon \to 0} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}a\right) \epsilon^2 \mathbf{N}(\epsilon) = \lim_{\epsilon \to 0} \frac{\exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}a\right)}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}} = 0. \tag{3.42}
$$

Then the following estimate holds

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq 2C_{1} \exp \left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right) \left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right) + 2D_{1} \exp \left(-2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \lambda_{\mathbf{N}(\epsilon)}^{-\mu} \mathcal{M}^{2}.
$$
\n(3.43)

Remark 3.1. From the theorem above, it is easy to see that $\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{L^{2}(\Omega_{2})}^{2}$ is of order

$$
\max\left[\lambda_{\mathbf{N}(\epsilon)}^{-\mu}\exp\Big(-2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\Big),\epsilon^2\mathbf{N}(\epsilon)e^{2a|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}},\frac{e^{2a|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right].
$$
\n(3.44)

We give one example for the choice of $\mathbf{N}(\epsilon)$ which satisfies the condition [\(3.42\)](#page-7-0). It is well-known that $\lambda_{\mathbf{N}(\epsilon)} \sim (\mathbf{N}(\epsilon))^{\frac{2}{d}}$, we can choose $\mathbf{N}(\epsilon)$ such that $\mathbf{N}(\epsilon) = [\epsilon^{\frac{-2b}{2m+1}}]$ for some $b > 0$ and

$$
e^{ka|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}} = (\mathbf{N}(\epsilon))^m, \ \ 0 < m < \frac{2\gamma}{d}.
$$

Then, we get

$$
B_{\mathbf{N}(\epsilon)} = \left(\frac{m}{ka}\log(\mathbf{N}(\epsilon))\right)^{\beta}
$$

Then the error $\mathbf{E} \|\mathbf{u}^{\epsilon}_{\mathbf{N}(\epsilon)}(t,.) - \mathbf{u}(t,.)\|_{L^2(\Omega_2)}^2$ is of order

$$
\epsilon^{\frac{4bm(a-x)}{(2m+1)a}} \max\left(\epsilon^{2-2b}, \epsilon^{\frac{2b(4\gamma-2md)}{(2m+1)d}}, \epsilon^{\frac{4b\mu}{(2m+1)d}}\right). \tag{3.45}
$$

Proof of Theorem [3.1.](#page-6-1) We divide the proof into some smaller parts.

Part 1. The existence and uniqueness of the solution of the nonlinear integral equation (3.40) . For $v \in C([0, a]; L^2(\Omega_2)))$, we put

$$
\mathcal{F}(v)(t,y) = \sum_{p=1}^{\infty} \mathcal{R}(\lambda_p, \mathbf{N}(\epsilon)) \left[E_{\beta,1} \left(\lambda_p t^{\beta} \right) \left\langle \overline{U}_{\mathbf{N}(\epsilon)}^0, \phi_p \right\rangle + t E_{\beta,2} \left(\lambda_p t^{\beta} \right) \left\langle \overline{U}_{\mathbf{N}(\epsilon)}^1, \phi_p \right\rangle \right. \\ \left. + \int_0^t (t - \eta)^{\beta - 1} E_{\beta,\beta} \left(\lambda_p (t - \eta)^{\beta} \right) \left\langle G \left(\eta, \cdot, v \left(\eta, \cdot \right) \right), \phi_p \right\rangle d\eta \right] \phi_p(y). \tag{3.46}
$$

We will prove by induction that if $v_1, v_2 \in C([0, a]; L^2(\Omega_2)))$ then

$$
\left\| \mathcal{F}^m(w_1)(t,.) - \mathcal{F}^m(w_2)(t,.) \right\|_{L^2(\Omega_2)} \le \left(\frac{K^2 a^2 A_1^2 \lambda_1^{-\frac{2-2\beta}{\beta}} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}} a\right)}{\beta^2} \right)^m \frac{t^m}{m!} \|w_1 - w_2\|_{C([0,a];L^2(\Omega_2))}. \tag{3.47}
$$

For $m = 1$, we have by using Lemma [2.2](#page-2-2) and the fact that G is Lipchitz

$$
\|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|_{L^2(\Omega_2)}^2
$$
\n
$$
\leq t \sum_{p=1}^{\infty} |\mathcal{R}(\lambda_p, \mathbf{N}(\epsilon))|^2 \int_0^t (t - \eta)^{2\beta - 2} |E_{\beta, \beta}(\lambda_p(t - \eta)^{\beta})|^2 \langle G(\eta, \cdot, v_1(\eta, \cdot)) - G(\eta, \cdot, v_2(\eta, \cdot)), \phi_p \rangle^2 d\eta
$$
\n
$$
\leq \frac{a^2 A_1^2}{\beta^2} \lambda_1^{\frac{2 - 2\beta}{\beta}} \int_0^t \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(t - \eta)\right) ||G(\eta, \cdot, v_1(\eta, \cdot)) - G(\eta, \cdot, v_2(\eta, \cdot))||_{L^2(\Omega_2)}^2 d\eta
$$
\n
$$
\leq \frac{K^2 a^2 A_1^2 \lambda_1^{\frac{2 - 2\beta}{\beta}}}{\beta^2} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}} a\right) ||v_1 - v_2||_{C([0, a]; L^2(\Omega_2))}^2.
$$
\n(3.48)

Assume that [\(3.47\)](#page-7-1) holds for $m = p$. We show that (3.47) holds for $m = p + 1$. In fact, we have

$$
\|\mathcal{F}^{p+1}(v_{1}) - \mathcal{F}^{p+1}(v_{2})\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq t \sum_{p=1}^{\infty} |\mathcal{R}(\lambda_{p}, \mathbf{N}(\epsilon))|^{2} \int_{0}^{t} (t - \eta)^{2\beta - 2} |E_{\beta, \beta}(\lambda_{p}(t - \eta)^{\beta})|^{2} \langle G(\eta, \cdot, \mathcal{F}^{p}(v_{1})(\eta, \cdot)) - G(\eta, \cdot, \mathcal{F}^{p}(v_{2})(\eta, \cdot)), \phi_{p}\rangle^{2} d\eta
$$
\n
$$
\leq \frac{a^{2} A_{1}^{2} \lambda_{1}^{\frac{2-2\beta}{\beta}}}{\beta^{2}} \int_{0}^{t} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(t - \eta)\right) ||G(\eta, \cdot, \mathcal{F}^{p}(v_{1})(\eta, \cdot)) - G(\eta, \cdot, \mathcal{F}^{p}(v_{2})(\eta, \cdot))||_{L^{2}(\Omega_{1})}^{2} d\eta
$$
\n
$$
\leq \frac{K^{2} a^{2} A_{1}^{2} \lambda_{1}^{\frac{2-2\beta}{\beta}} t}{\beta^{2}} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}} a\right) ||\mathcal{F}^{p}(v_{1}) - \mathcal{F}^{p}(v_{2})||_{C([0,a];L^{2}(\Omega_{2}))}^{2}
$$
\n
$$
\leq \left(\frac{K^{2} a^{2} A_{1}^{2} \lambda_{1}^{\frac{2-2\beta}{\beta}} \exp\left(2\lambda_{\mathbf{N}(\epsilon)}^{\frac{1}{\beta}} a\right)}{\beta^{2}}\right)^{p+1} \frac{x^{p+1}}{(p+1)!} ||v_{1} - v_{2}||_{C([0,a];L^{2}(\Omega_{2}))}. \tag{3.49}
$$

Therefore, by induction, we have [\(3.47\)](#page-7-1) for all $w, v \in C([0, a]; L^2(\Omega_2))$. Since

$$
\lim_{m \to +\infty} \left(\frac{K^2 a^2 A_1^2 \lambda_1^{\frac{2-2\beta}{\beta}} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}a\right)}{\beta^2} \right)^m \frac{a^m}{m!} = 0
$$

there exists a positive integer m_0 such that \mathcal{F}^{m_0} is a contraction. It follows that the equation $\mathcal{F}^{m_0}w=w$ has a unique solution $u_{N(\epsilon)}^{\epsilon} \in C([0,a];L^2(\Omega_2))$. We claim that $\mathcal{F}(u_{N(\epsilon)}^{\epsilon})=u_{N(\epsilon)}^{\epsilon}$. In fact, since $\mathcal{F}^{m_0}(u_{N(\epsilon)}^{\epsilon}) = u_{N(\epsilon)}^{\epsilon}$, we know that $\mathcal{F}(\mathcal{F}^{m_0}(u_{N(\epsilon)}^{\epsilon})) = \mathcal{F}(u_{N(\epsilon)}^{\epsilon})$. This is equavilent to $\mathcal{F}^{m_0}\left(\mathcal{F}(u_{N(\epsilon)}^{\epsilon})\right)=\mathcal{F}(u_{N(\epsilon)}^{\epsilon}).$ Hence, $\mathcal{F}(u_{N(\epsilon)}^{\epsilon})$ is a fixed point of \mathcal{F}^{m_0} . Moreover, as noted above, $u_{N(\epsilon)}^{\epsilon}$ is a fixed point of \mathcal{F}^{m_0} .

Part 2. Estimate the expectation of the error between the exact solution \bf{u} and the regularized solution $\mathbf{u}^{\epsilon}_{\mathbf{N}(\epsilon)}$.

Let us consider the following integral equation

$$
\mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,y) = \sum_{p=1}^{\infty} \mathcal{R}(\lambda_p, \mathbf{N}) \left[E_{\beta,1} \left(\lambda_p t^{\beta} \right) \langle \mathbf{u}_0, \phi_p \rangle + t E_{\beta,2} \left(\lambda_p t^{\beta} \right) \langle \mathbf{u}_1, \phi_p \rangle \right.+ \int_0^t (t-\eta)^{\beta-1} E_{\beta,\beta} \left(\lambda_p (x-\eta)^{\beta} \right) \langle G \left(\eta, \cdot, \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon} (\eta, \cdot) \right), \phi_p \rangle d\eta \right] \phi_p(y), \quad (t,y) \in \Omega,(3.50)
$$

Combining (3.40) and (3.50) and taking the expectation of both sides of the norm in L^2 , we get

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq 3\mathbf{E} \Biggl(\sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} |E_{\beta,1}(\lambda_{p}t^{\beta})|^{2} \langle \overline{U}_{\mathbf{N}(\epsilon)}^{0} - \mathbf{u}_{0}, \phi_{p} \rangle^{2}\Biggr)
$$
\n
$$
+ 3\mathbf{E} \Biggl(\sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} |tE_{\beta,2}(\lambda_{p}t^{\beta})|^{2} \langle \overline{U}_{\mathbf{N}(\epsilon)}^{1} - \mathbf{u}_{1}, \phi_{p} \rangle^{2}\Biggr)
$$
\n
$$
+ 3\mathbf{E} \Biggl(\sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} \left[\int_{0}^{t} (t-\eta)^{\beta-1} E_{\beta,\beta}(\lambda_{p}(t-\eta)^{\beta}) \langle G(\eta,\cdot,\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,\cdot)) - G(\eta,\cdot,\mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,\cdot)) \rangle, \phi_{p} \rangle d\eta \right]^{2} \Biggr).
$$
\n(3.51)

Where above we have used the inequality $(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2$ for real numbers a, b, c. Using Lema [3.1](#page-6-2) and the Hölder inequality, we deduce that

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq \frac{3A_{1}^{2}}{\beta^{2}} \exp \left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right) \mathbf{E} \|\overline{U}_{\mathbf{N}(\epsilon)}^{0} - \mathbf{u}_{0}\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
+ \frac{3A_{1}^{2}}{\beta^{2}} \lambda_{1}^{-\frac{2}{\beta}} \exp \left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right) \mathbf{E} \|\overline{U}_{\mathbf{N}(\epsilon)}^{1} - \mathbf{u}_{1}\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
+ \frac{3k^{2}aA_{1}^{2}}{\beta^{2}} \lambda_{1}^{-\frac{2-2\beta}{\beta}} \int_{0}^{t} \exp \left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(t-\eta)\right) \mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.)\|_{L^{2}(\Omega_{2})}^{2} d\eta. \tag{3.52}
$$

Multiplying both sides with $\exp(-2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t)$, we obtain

$$
\exp\left(-2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right)\mathbf{E}\|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq \frac{3A_{1}^{2}}{\beta^{2}}\mathbf{E}\|\overline{U}_{\mathbf{N}(\epsilon)}^{0} - \mathbf{u}_{0}\|_{L^{2}(\Omega_{y})}^{2} + \frac{3A_{1}^{2}}{\beta^{2}}\lambda_{1}^{-\frac{2}{\beta}}\mathbf{E}\|\overline{U}_{\mathbf{N}(\epsilon)}^{1} - \mathbf{u}_{1}\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
+ \frac{3k^{2}aA_{1}^{2}}{\beta^{2}}\lambda_{1}^{\frac{2-2\beta}{\beta}}\int_{0}^{t} \exp\left(-2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\eta\right)\mathbf{E}\|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.)\|_{L^{2}(\Omega_{2})}^{2}d\eta. \tag{3.53}
$$

Applying Gronwall's inequality, we get

$$
\exp\left(-2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right)\mathbf{E}\|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq \frac{3A_{1}^{2}}{\beta^{2}}\max\left(1, \lambda_{1}^{\frac{2-2\beta}{\beta}}\right)\exp\left(\frac{3k^{2}aA_{1}^{2}}{\beta^{2}}\lambda_{1}^{\frac{2-2\beta}{\beta}}\right)\left(\mathbf{E}\|\overline{U}_{\mathbf{N}(\epsilon)}^{0} - \mathbf{u}_{0}\|_{L^{2}(\Omega_{2})}^{2} + \mathbf{E}\|\overline{U}_{\mathbf{N}(\epsilon)}^{1} - \mathbf{u}_{1}\|_{L^{2}(\Omega_{2})}^{2}\right)
$$
\n
$$
\leq \frac{3A_{1}^{2}}{\beta^{2}}\max\left(1, \lambda_{1}^{\frac{2-2\beta}{\beta}}\right)\exp\left(\frac{3k^{2}aA_{1}^{2}}{\beta^{2}}\lambda_{1}^{\frac{2-2\beta}{\beta}}\right)\left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\|\mathbf{u}_{0}\|_{H^{2\gamma}(\Omega_{2})}^{2} + \|\mathbf{u}_{1}\|_{H^{2\gamma}(\Omega_{y})}^{2}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right)
$$
\n
$$
C_{1}:=C_{1}(\beta, A_{1}, a, k, \lambda_{1})
$$
\n
$$
\leq C_{1}\left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right). \tag{3.54}
$$

Now, we continue to estimate $\|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^2(\Omega_2)}$. Indeed, using Hölder inequality, globally

Lipschitzp roperty of G , and equations (2.10) and (2.26) we get

$$
\| \mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) \|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq 2 \sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} \left[\int_{0}^{t} (t - \eta)^{\beta - 1} E_{\beta,\beta} \left(\lambda_{p}(t - \eta)^{\beta} \right) \left\langle G\left(\eta, \cdot, \mathbf{u}(\eta, \cdot)\right) - G\left(\eta, \cdot, \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}\left(\eta, \cdot\right) \right), \phi_{p} \right\rangle d\eta \right]^{2}
$$
\n
$$
+ 2 \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \left\langle \mathbf{u}(t, y), \phi_{p} \right\rangle^{2}
$$
\n
$$
\leq 2 \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \lambda_{p}^{-\mu} \exp\left(-2(a - t) \lambda_{p}^{\frac{1}{\beta}} \right) \lambda_{p}^{\mu} \left(2(a - t) \lambda_{p}^{\frac{1}{\beta}} \right) \left\langle \mathbf{u}(t, y), \phi_{p} \right\rangle^{2}
$$
\n
$$
+ \frac{2k^{2}aA_{1}^{2}}{\beta^{2}} \lambda_{1}^{\frac{2-2\beta}{\beta}} \int_{0}^{t} \exp\left(2B_{\mathbf{N}(\epsilon)}^{\frac{1}{\alpha}}(t - \eta) \right) || \mathbf{u}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.) ||_{L^{2}(\Omega_{2})}^{2} d\eta
$$
\n
$$
\leq |B_{\mathbf{N}(\epsilon)}|^{-\mu} \exp\left(-2(a - t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}} \right) \mathcal{M}^{2}
$$
\n
$$
+ \frac{2k^{2}aA_{1}^{2}}{\beta^{2}} \lambda_{1}^{\frac{2-2\beta}{\beta}} \int_{0}^{t} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(t - \eta) \right) || \mathbf{u}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.) ||_{L
$$

Multiplying both sides with $\exp\left(2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right)$, we obtain

$$
\exp\left(2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq |B_{\mathbf{N}(\epsilon)}|^{-\mu} \mathcal{M}^{2} + \frac{2k^{2}aA_{1}^{2}}{\beta^{2}} \lambda_{1}^{\frac{2-2\beta}{\beta}} \int_{0}^{t} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(a-\eta)\right) \|\mathbf{u}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.)\|_{L^{2}(\Omega_{2})}^{2} d\eta. \tag{3.55}
$$

Gronwall's inequality implies that

$$
\exp\left(2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2} \leq \underbrace{\exp\left(\frac{2k^{2}aA_{1}^{2}t}{\beta^{2}}\lambda_{1}^{\frac{2-2\beta}{\beta}}\right)}_{D_{1}:=D_{1}(k,a,A_{1},\beta)} |B_{\mathbf{N}(\epsilon)}|^{-\mu} \mathcal{M}^{2}.
$$
 (3.56)

This together with the estimate [\(3.54\)](#page-9-0) leads to

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq 2\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2} + 2\|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}
$$
\n
$$
\leq 2C_{1} \exp (2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t) \left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right) + 2D_{1} \exp \left(-2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right)|B_{\mathbf{N}(\epsilon)}|^{-\mu} \mathcal{M}^{2}
$$
\n(3.57)

which completes our proof.

The next result provides an error estimate in the Sobolev space $H^q(\Omega_2)$ which is equipped with a norm defined by

$$
||g||_{H^q(\Omega_2)}^2 = \sum_{p=1}^{\infty} \lambda_p^q \langle g, \phi_p \rangle^2.
$$
\n(3.58)

To estimate the error in the H^q norm, we need stronger assumption on solution **u**.

 \Box

Theorem 3.2. Suppose that the problem $(1.1)-(1.2)$ $(1.1)-(1.2)$ $(1.1)-(1.2)$ has unique solution **u** such that

$$
\sum_{p=1}^{\infty} \exp\left(2(a-t+r)\lambda_p^{\frac{1}{\beta}}\right) \langle \mathbf{u}(t,y), \phi_p \rangle^2 \le \mathcal{M}_1, \quad t \in [0, a], \tag{3.59}
$$

for any $r > 0$. Let $\mathbf{N}(\epsilon), B_{\mathbf{N}(\epsilon)}$ be as in Theorem [\(3.1\)](#page-6-1). Then the following estimate holds

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2}
$$
\n
$$
\leq 4|B_{\mathbf{N}(\epsilon)}|^{q} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right)C_{1}\left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right) + \mathcal{M}_{1}^{2}(2D_{1}+1)|B_{\mathbf{N}(\epsilon)}|^{q} \exp\left(-2(a-t+r)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \tag{3.60}
$$

Remark 3.2. In physical modelling and engineering, the estimation on a Hilbert scale space, for example $H^{q}(\Omega)$ is important. Furthermore, the problem of estimating the error in this space more difficult than $L^2(\Omega)$. Hence, the above theorem is a new and interesting result.

Proof. First, we have

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathcal{Q}_{B_{\mathbf{N}(\epsilon)}} \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2} = \mathbf{E} \left(\sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} \lambda_{j}^{q} \left\langle \mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.) , \phi_{p}(y) \right\rangle^{2} \right)
$$

$$
\leq |B_{\mathbf{N}(\epsilon)}|^{q} \mathbf{E} \left(\sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} \left\langle \mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.) , \phi_{p}(y) \right\rangle^{2} \right)
$$

$$
\leq |B_{\mathbf{N}(\epsilon)}|^{q} \mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{L^{2}(\Omega_{2})}^{2}.
$$
 (3.61)

where $\mathcal{Q}_{B_{\mathbf{N}(\epsilon)}}\mathbf{u}(t,.)=\sum_{\lambda_p\leq B_{\mathbf{N}(\epsilon)}}\langle \mathbf{u}(t,.)\rangle \phi_p(y)\rangle \phi_p(y)$. Under the assumption [\(3.59\)](#page-11-0), we get

$$
\begin{split}\n&\|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2} \\
&\leq 2 \sum_{\lambda_{p} \leq B_{\mathbf{N}(\epsilon)}} \left[\int_{0}^{t} (t - \eta)^{\beta - 1} E_{\beta,\beta} \left(\lambda_{p}(t - \eta)^{\beta} \right) \left\langle G\left(\eta, \cdot, \mathbf{u}(\eta, \cdot)\right) - G\left(\eta, \cdot, \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}\left(\eta, \cdot\right) \right), \phi_{p} \right\rangle d\eta \right]^{2} \\
&+ 2 \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \left\langle \mathbf{u}(t, y), \phi_{p} \right\rangle^{2} \\
&\leq 2 \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \exp \left(-2(a - t + r) \lambda_{p}^{\frac{1}{\beta}} \right) \exp \left(2(a - t + r) \lambda_{p}^{\frac{1}{\beta}} \right) \left\langle \mathbf{u}(t, y), \phi_{p} \right\rangle^{2} \\
&+ \frac{2k^{2} a A_{1}^{2}}{\beta^{2}} \lambda_{1}^{\frac{2 - 2\beta}{\beta}} \int_{0}^{t} \exp \left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(t - \eta) \right) ||\mathbf{u}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.)||_{L^{2}(\Omega_{2})}^{2} d\eta \\
&\leq \exp \left(-2(a - t + r) |B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}} \right) \mathcal{M}_{1}^{2} \\
&+ \frac{2k^{2} a A_{1}^{2}}{\beta^{2}} \lambda_{1}^{\frac{2 - 2\beta}{\beta}} \int_{0}^{t} \exp \left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(t - \eta) \right) ||\mathbf{u}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.)||_{L^{2}(\Omega_{2})}^{2} d\eta \\
&\leq \mathcal{M} \mathbf{N} \mathbf{N} \mathbf{N} \mathbf{N} \mathbf
$$

Multiplying both sides with $\exp\left(2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right)$, we obtain

$$
\exp\left(2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq \exp\left(-2r|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \mathcal{M}_{1}^{2}
$$
\n
$$
+\frac{2k^{2}aA_{1}^{2}}{\beta^{2}} \lambda_{1}^{\frac{2-2\beta}{\beta}} \int_{0}^{t} \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}(a-\eta)\right) \|\mathbf{u}(\eta,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(\eta,.)\|_{L^{2}(\Omega_{2})}^{2} d\eta. \tag{3.62}
$$

Then Gronwall's inequality implies that

$$
\exp\left(2(a-t)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2} \le D_{1} \exp\left(-2r|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \mathcal{M}_{1}^{2} \tag{3.63}
$$

This latter estimate together with the estimate [\(3.54\)](#page-9-0) leads to

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{L^{2}(\Omega_{2})}^{2}
$$
\n
$$
\leq 2\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}^{2} + 2\|\mathbf{u}(t,.) - \mathbf{v}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.)\|_{L^{2}(\Omega_{2})}
$$
\n
$$
\leq \exp\left(2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t\right) \left[2C_{1}\left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right) + 2D_{1}\exp\left(-2(r+a)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right)\mathcal{M}_{1}^{2}\right].
$$
\n(3.64)

It follows from [\(3.61\)](#page-11-1) that

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathcal{Q}_{B_{\mathbf{N}(\epsilon)}} \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2}
$$
\n
$$
\leq |B_{\mathbf{N}(\epsilon)}|^{q} \exp \left(2B_{\mathbf{N}(\epsilon)}^{\frac{1}{\alpha}} t\right) \left[2C_{1}\left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right) + 2D_{1} \exp \left(-2(r+a)\lambda_{\mathbf{N}(\epsilon)}^{\frac{1}{\beta}}\right)\mathcal{M}_{1}^{2}\right].
$$
\n(3.65)

On the other hand, consider the function

$$
\mathcal{G}(z) = z^q e^{-Dz}, \quad D > 0,\tag{3.66}
$$

 $\overline{ }$

 \Box

From the derivative of G is $\mathcal{G}'(z) = z^{q-1}e^{-Dz}(q-Dz)$, we know that G is strictly decreasing when $Dz \geq q$. Since $\lim_{\epsilon \to 0} B_{\mathbf{N}(\epsilon)} = +\infty$, we see that if ϵ small enough then $2rB_{\mathbf{N}(\epsilon)} \geq q$. Replacing $D = 2(a - t + r)$, $z = B_{\mathbf{N}(\epsilon)}$ into [\(3.66\)](#page-12-0), we obtain for $\lambda_p > B_{\mathbf{N}(\epsilon)}$

$$
\mathcal{G}(\lambda_p) = \lambda_p^q \exp\left(-2(a-t+r)\lambda_p^{\frac{1}{\beta}}\right) \leq \mathcal{G}(B_{\mathbf{N}(\epsilon)}) = |B_{\mathbf{N}(\epsilon)}|^q \exp\left(-2(a-t+r)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right)
$$

The latter equality leads to

$$
\begin{split} \|\mathbf{u}(t,.) - \mathcal{Q}_{B_{\mathbf{N}(\epsilon)}} \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2} &= \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \lambda_{p}^{q} \langle \mathbf{u}(t,y), \phi_{p}(y) \rangle^{2} \\ &= \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \mathcal{G}(\lambda_{p}) \exp\left(2(a-t+r)\lambda_{p}^{\frac{1}{\beta}}\right) \langle \mathbf{u}(t,y), \phi_{p}(y) \rangle^{2} \\ &\leq \mathcal{G}(B_{\mathbf{N}(\epsilon)}) \sum_{\lambda_{p} > B_{\mathbf{N}(\epsilon)}} \exp\left(2(a-t+r)\lambda_{p}^{\frac{1}{\beta}}\right) \langle \mathbf{u}(t,y), \phi_{p}(y) \rangle^{2} \\ &\leq \mathcal{M}_{1}^{2} |B_{\mathbf{N}(\epsilon)}|^{q} \exp\left(-2(a-t+r)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \end{split} \tag{3.67}
$$

where we use the assumption (3.59) for the latter inequality. Combining (3.61) , (3.64) and (3.67) , we deduce that

$$
\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2}
$$
\n
$$
\leq 2\mathbf{E} \|\mathbf{u}_{\mathbf{N}(\epsilon)}^{\epsilon}(t,.) - \mathcal{Q}_{B_{\mathbf{N}(\epsilon)}} \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2} + 2\|\mathbf{u}(t,.) - \mathcal{Q}_{B_{\mathbf{N}(\epsilon)}} \mathbf{u}(t,.)\|_{H^{q}(\Omega_{2})}^{2}
$$
\n
$$
\leq 4|B_{\mathbf{N}(\epsilon)}|^{q} \exp (2|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}t)C_{1}\left(2\epsilon^{2}\mathbf{N}(\epsilon) + \frac{\mathcal{M}_{0}}{\lambda_{\mathbf{N}(\epsilon)}^{2\gamma}}\right) + \mathcal{M}_{1}^{2}(2D_{1} + 1)|B_{\mathbf{N}(\epsilon)}|^{q} \exp \left(-2(a - t + r)|B_{\mathbf{N}(\epsilon)}|^{\frac{1}{\beta}}\right) \tag{3.68}
$$

which completes the proof.

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